

Comment on “Exact vacuum solution with Hopf structure in general relativity”

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Harada’s recently announced Kerr-Schild solution of Einstein’s equations in general relativity [Phys. Rev. D **112**, 024020 (2025)] is identified as isometric with one of the well-known NUT solutions.

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In a recent paper [1], Harada derived a Kerr-Schild vacuum solution of Einstein’s equations, based on a Hopf fibering, in the form

$$\begin{aligned}
 ds^2 = & -2du dv + dx^2 + dy^2 \\
 & + H \left[dv - \frac{ux + by}{u^2 + b^2} dx - \frac{uy - bx}{u^2 + b^2} dy \right. \\
 & \left. + \frac{x^2 + y^2}{2(u^2 + b^2)} du \right]^2, \quad (1)
 \end{aligned}$$

where $H = Nu/(u^2 + b^2)$. Harada showed that this metric is of Petrov type D. All Petrov type D vacuum solutions are known [2]. The solutions formed one of the first applications [3] of the Cartan-Karlhede procedure for characterization of spacetimes described in [4], chapter 9. There is a useful table summarizing the results in [5]. The files for the CLASSI software (see [6,7]) used in creating that table are freely available with the software. Here I use those files to identify which of the known solutions (1) is.

To apply the classification methods, the first step is choice of a suitable canonical tetrad. Debever [8] showed that Kerr-Schild spacetimes with a geodesic, diverging, and shear-free null vector k_μ can be written (using the conventions of [4] and interchanging the names u and v) as

$$ds^2 = 2(d\zeta d\bar{\zeta} - du dv) + 2S(dv + \bar{Y}d\zeta + Yd\bar{\zeta} + Y\bar{Y}du)^2 \quad (2)$$

with $\zeta = (x + iy)/\sqrt{2}$. This is of the same form as (1) if $a = u + ib$, $Y = -\zeta/a$ and $S = H/2$. Following [4], a

suitable tetrad for (2) is

$$\begin{aligned}
 \omega^1 = & d\zeta + Ydu, & \omega^2 = & d\bar{\zeta} + \bar{Y}du, \\
 \omega^3 = & dv + \bar{Y}d\zeta + Yd\bar{\zeta} + Y\bar{Y}du, & \omega^4 = & du - S\omega^3. \quad (3)
 \end{aligned}$$

Using (3), CLASSI readily confirmed that (1) is a Petrov type D vacuum solution. In the coordinates of (1), Harada noted the Killing vectors ∂_v and $y\partial_x - x\partial_y$, but CLASSI reports the metric has four Killing vectors. This could have easily been foreseen since the invariants contain only one essential coordinate.

Harada’s solution has (in Newman-Penrose [9] notation) $\Psi_2 = -N/2(u + ib)^3$ as the only nonzero component of the Riemann tensor, and an invariant from the derivative of the Weyl tensor

$$C^{abcd;e} C_{abcd;e} = 45N^3 u/2(u^2 + b^2)(u + ib)^8. \quad (4)$$

Using Åman’s files and coordinates, Kinnersley type I metrics with $C = 0$ have the invariants $\Psi_2 = -m/(r + i\ell)^3$ and $C^{abcd;e} C_{abcd;e} = 180m^3 r/(r^2 + \ell^2)(r + i\ell)^8$. Clearly these agree with the Harada solution if we identify $r = u$, $b = \ell$, $N = 2m$.

The other cases in Åman’s table that have four Killing vectors have different invariants, not matching those of Harada’s solution. Specifically, Kinnersley Type IIA with $a = 0$ has $\Psi_2 = -(m + i\ell)/(r + i\ell)^3$, Kinnersley Type IV A with nonzero a has extra terms in $C^{abcd;e} C_{abcd;e}$, and Kinnersley Type IV B has a real Ψ_2 . There is a error in the table for Type IIBCD with $b = 0$, which has only two, not four, Killing vectors.

Hence from these comparisons one finds that Harada’s solution is locally isometric to one of the well-known Taub-NUT solutions, the one with a zero two-space curvature parameter. (This parameter is denoted μ_0 in [10] and ϵ in [11]).

Starting from the Debever form (2) of the metric enables a concise way to present coordinate transformations from (1)

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to standard forms of the NUT metric. In (2) substitute $\zeta = az$, $d\zeta = adz + zdu$. The metric then becomes

$$ds^2 = -2du\sigma + 2(u^2 + b^2)dzd\bar{z} + H\sigma^2 \quad (5)$$

where

$$\sigma = (dv - a\bar{z}dz - \bar{a}z d\bar{z} - z\bar{z}du). \quad (6)$$

Substituting $a = u + ib$ and $z = (P + iQ)/\sqrt{2}$ [so that $P = (ux + by)/(u^2 + b^2)$, $Q = (uy - bx)/(u^2 + b^2)$] into σ one finds

$$\begin{aligned} \sigma &= dv - u(PdP + QdQ) - \frac{1}{2}(P^2 + Q^2)du \\ &\quad + bPdQ - bQdP \\ &= dv - \frac{1}{2}d((P^2 + Q^2)u) + bPdQ - bQdP. \end{aligned} \quad (7)$$

Making the variable transformation $v = w + \frac{1}{2}((P^2 + Q^2)u)$ the metric becomes

$$ds^2 = -2du\sigma + (u^2 + b^2)(dP^2 + dQ^2) + H\sigma^2, \quad (8)$$

with $\sigma = dw + b(PdQ - QdP)$. This is the original NUT form for the case $\mu_0 = 0$, with renamed parameters and coordinates. [10] gives the Killing vectors Harada did not find as $\partial_P - bQ\partial_w$ and $\partial_Q + bP\partial_w$.

If one substitutes $w = V + \int du/H$ and introduces polar coordinates (ρ, ϕ) in the (P, Q) plane the metric becomes

$$-du^2/H + (u^2 + b^2)(d\rho^2 + \rho^2 d\phi^2) + H(dV + b\rho^2 d\phi)^2, \quad (9)$$

which (with $V = t$ and $H = -f$ there) is the form used in [11], where Sec. 12.3.2 analyzes properties such as the conformal structure. In the coordinates of (9) the Killing vectors that Harada did not find are $\cos\phi\partial_\rho - \sin\phi(1/\rho\partial_\phi + b\rho\partial_V)$ and $\sin\phi\partial_\rho + \cos\phi(1/\rho\partial_\phi + b\rho\partial_V)$.

I am grateful to their authors for informing me that the same identification had been given in [12] and [13].

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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