

Anomaly of continuous symmetries from topological defect network

Qiang Jia^{1,2}, Ran Luo³, Jiahua Tian¹, Yi-Nan Wang^{3,4,5} and Yi Zhang⁶

¹*School of Physics, East China Normal University, Shanghai, People's Republic of China, 200241*

²*Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 34141, Korea*

³*School of Physics, Peking University, Beijing, People's Republic of China*

⁴*Center for High Energy Physics, Peking University, Beijing, People's Republic of China*

⁵*Peng Huanwu Center for Fundamental Theory, Hefei, Anhui 230026, People's Republic of China,*

⁶*Kavli IPMU (WPI), UTIAS, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan*



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We show that the 't Hooft anomaly of a quantum field theory with continuous flavor symmetry can be detected from rearrangements of the topological defect webs implementing the global symmetry in general spacetime dimension, which is concretized in 2D by the *F-moves* of the defect lines. Via dualizing the defects to flat background gauge field configurations, we characterize the 't Hooft anomaly by various cohomological data of the symmetry group, where the cohomology of Lie groups with discrete topology plays the central role. We find that an extra dimension emerges naturally as a consequence of the mathematical description of the 't Hooft anomaly in the case of flat gauging.

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Introduction. A global symmetry of a quantum field theory (QFT) has a 't Hooft anomaly when gauging it is obstructed [1]. The robustness of the 't Hooft anomaly across UV to IR makes it a particularly effective tool to study the properties of QFT even in the strongly coupled region [2,3]. It appears as a nontrivial phase $e^{2\pi i A(A,g)}$ acquired by the background-coupled partition function $Z[A]$ under $A \rightarrow g^{-1}Ag + g^{-1}dg$, where $A(A,g)$ cannot be canceled by local counterterms built from A . Traditionally for continuous G , it can be evaluated using the *descent equations* for small gauge transformations [4–8] and more generally shown to be the η -invariant when g is disconnected from the identity [9–12].

Recent years have witnessed a revival of the study of 't Hooft anomalies in both condensed matter and high-energy physics [13–17]. Thanks to the reformulation of the symmetry action as linking symmetry operator and charged defect in spacetime [18–23], the action of finite group symmetries is now conveniently studied on equal footing with that of continuous symmetries. However, whereas the 't Hooft anomaly of discrete group symmetry can be computed directly as the *F-symbol* associated with rearrangements of webs of defects, which are often described by elements of the group cohomology [e.g., $H^{d+1}(G, U(1))$] for a 0-form symmetry group G in d -spacetime dimensions [24–27], an analogous algorithm to compute the 't Hooft

anomaly for continuous group symmetries in *flat gauging* has long remained elusive.

We present such an algorithm for continuous symmetries, placing finite and continuous symmetries on parallel footing. We first show that a web of defects can be dualized to be a flat background field configuration. This effectively establishes the equivalence between the insertion of defects and coupling the conserved current to flat background fields. Via the dualization, we show in full generality how the 't Hooft anomaly manifests itself as the phase factor arising from rearrangements of the topological defects in the manner hinted in Ref. [21]. We then compute the anomaly concretely via *F-moves* of the *topological defect lines* in 2D QFT [16,17,28–30] and show that the anomaly is characterized by cohomological data of G .

Very importantly, unlike the case of a finite group G where the group cohomology $H^3(G, U(1)) \cong H^3(BG, U(1))$ is unambiguously defined, one should be careful about the exact definition for Lie group cohomology. We emphasize that for a Lie group G , the correct version should be $H^3(G^\delta, U(1))$, which is defined to be the group cohomology when G is equipped with discrete topology [31]. Despite that $H^3(G^\delta, U(1))$ is a highly complicated mathematical object, there exists an injection map from the group $H^4(BG, \mathbb{Z})$, where the anomaly polynomial for G lives, to $H^3(G^\delta, U(1))$. Thus, the usual anomaly polynomial $\sim F \wedge F$ can be elegantly embedded in this unified framework.

We find that the unifying mathematical structure underlying these various cohomology groups of G suggests that the QFT with a 't Hooft anomaly naturally lives on the boundary of an *anomaly theory* in one higher dimension.

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The generalization of the above cohomological characterization of 't Hooft anomaly to higher-dimensional spacetime is immediate in our formalism.

For detailed examples, we discuss the cases of a simple G and $U(1)$ separately. For a compact simply connected and simple Lie group G , the $3d$ bulk anomaly theory is the non-Abelian Chern-Simons invariant characterized by an integer $k \in \mathbb{Z}$ called the *level*. It has a local expression as the Chern-Simons three-form $I_3 = CS_k[A] = \frac{k}{2} \text{tr}(A \wedge F - \frac{1}{3} A \wedge A \wedge A)$. We explicitly uplift the topological defect networks across the F-move to a three-dimensional bulk, and show that after assigning $F = 0$, the cubic term in $CS_k[A]$ precisely generates the de Rham cohomology group $H_{\text{dR}}^3(G, \mathbb{R})$ for G .

For a nonsimply connected Lie group such as $G = U(1)$, we point out that the field strength $F = dA$ would not vanish at the junction point for this case. This is consistent with flat gauging, whose precise condition is trivial holonomy $\exp(\oint_C A)$ around any topologically trivial loop C . Such a nonvanishing F is precisely the source of the nontrivial $U(1)$ phase factor across the F-move, which we compute explicitly from the integration of ϵF over the $2D$ spacetime, where ϵ is the finite gauge transformation parameter generating the F-move. The phase factor [32]

$$\omega(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) = e^{i\alpha_1(\alpha_2 + \alpha_3 - [\alpha_2 + \alpha_3])/(2\pi)}, \quad (1)$$

with $\alpha_i \in [0, 2\pi)$ $[r] = r \bmod 2\pi\mathbb{Z}$, is consistent with the $N \rightarrow \infty$ limit of the generator of $H^3(\mathbb{Z}_N, U(1))$.

We would also like to comment that Eq. (1) is perfectly consistent with the newly proposed categorical formulation of continuous symmetry [35]. Namely, if one pretends that the definition of the symmetry category $\mathbf{Vec}_G^{\omega}$ and Drinfeld center $Z(\mathbf{Vec}_G^{\omega})$ can be generalized to the case of $G = U(1)$, all the category theoretical data are consistent with the mathematical results in Ref. [36], which are confirmed in Ref. [35].

Our results can be applied to all compact connected Lie groups. From the structure theorem [37], compact connected Lie groups are finitely covered by direct products of tori and simple non-Abelian groups, such that their classification reduces to that of simple groups. The restriction to a compact connected and simple Lie group G or $U(1)$ is sufficient in this context.

Finally, although the detailed examples are discussed for the $d = 2$ case, our framework applies to continuous 0-form symmetries in general d spacetime dimensions as well.

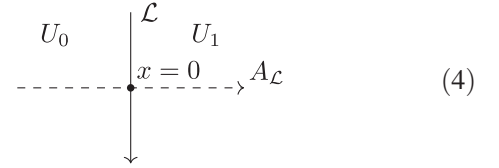
Dualization of the defect web. The vacuum expectation value of a topological defect web $\mathcal{W}$ of a (Lagrangian) QFT $\mathcal{T}_G$ in d -dimensional spacetime M_d can be computed as

$$\langle \mathcal{W} \rangle := \int \mathcal{D}\psi e^{iS[\psi] + \int_{M_d} \text{tr}(A_{\mathcal{W}} \wedge \star J)}, \quad (2)$$

where ψ stands for the set of local operators of $\mathcal{T}_G$ and $J(\psi)$ is the conserved current of the continuous global G -symmetry. $A_{\mathcal{W}}$ is the ordinary flat gauge field dual to $\mathcal{W}$. To illustrate the dualization, we consider a defect line $\mathcal{L}$ labeled by $g_\alpha = e^{i\alpha} \in G$ along $x = 0$ in $\mathbb{R}^2$, meaning that a particle charged under the global G -symmetry is transformed under a representation of g_α when crossing $x = 0$ from left to right; see Eq. (4). We define the dualization $A_{\mathcal{L}}$ of $\mathcal{L}$ to be the background gauge field such that $\mathcal{P}e^{\int_{\ell} A_{\mathcal{L}}} = e^{i\alpha}$ for an arbitrary path ℓ crossing $x = 0$ once, which in turn means that $A_{\mathcal{L}}$ must be *flat*. Given the condition, we can write

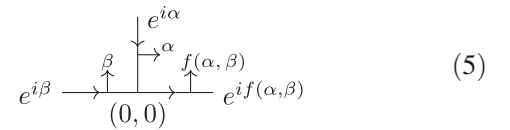
$$A_{\mathcal{L}} = i\alpha \delta(x) dx = e^{-i\alpha H(x)} d e^{i\alpha H(x)}, \quad (3)$$

where $H(x)$ is the Heaviside function, and we can indeed check that $dA_{\mathcal{L}} + A_{\mathcal{L}} \wedge A_{\mathcal{L}} = 0$. Equivalently, $\mathcal{L}$ defines the gauge configuration $A_{\mathcal{L}}|_{U_i} = g_i^{-1} dg_i$ for $g_0 = e$ and $g_1 = e^{i\alpha}$ on the following open cover of $\mathbb{R}^2$:



where the two patches $U_0 = (-\infty, \epsilon)$ and $U_1 = (-\epsilon, \infty)$ are glued along $(-\epsilon, \epsilon)$ for small positive ϵ . Here, we adopt the convention that the direction of the defect line rotates right-handedly with respect to the direction of the gauge transformation from U_0 to U_1 . Clearly, the above discussions can be generalized to a codimension-one web of $\mathcal{W}$ in general d dimensions.

The fundamental building block of any defect web (of lines) is the trivalent junction:



The background gauge field $A_{\mathcal{L}}$ corresponding to the group element $e^{i\alpha}$ is also labeled, with α . This trivalent junction can be smeared out to give $A = g^{-1} dg$ with $F = dA + A \wedge A = 0$ for a globally defined g on $\mathbb{R}^2$ for G with trivial $\pi_1(G)$, since all paths from identity to $e^{i\alpha} e^{i\beta}$ on the group manifold are homotopic. However, a globally defined g cannot be obtained by any kind of smearing for G with nontrivial $\pi_1(G)$, e.g., for $G = U(1)$, and there exist flat configurations of $U(1)$ gauge fields with nonzero [38] $F = dA$. This subtlety is crucial for the derivation of anomalies for Abelian groups, as in the examples of the finite gauge theory $G = \mathbb{Z}_N$ [39].

Now we clarify the notion of flat gauging of continuous G global symmetry, given by the following two equivalent conditions:

- (1) A flat gauging is equivalent to summing over all possible insertions of topological defect networks in the new partition function.
- (2) Summing over configurations of gauge fields A , where the holonomy of the gauge field around a topologically trivial loop,

$$\mathcal{P} \exp \left(\oint_C A \right) = e, \quad (6)$$

the identity element of G . This is coincident with the definition of flat gauge fields for finite groups such as $G = \mathbb{Z}_N$.

't Hooft anomaly from F-move and the Lie group cohomology. Having defined the dualization for continuous G , one can compare $\langle \mathcal{W} \rangle$ with $\langle \mathcal{W}' \rangle$ by coupling $\mathcal{T}_G$ to flat $A_{\mathcal{W}}$ - and $A_{\mathcal{W}'}$ -backgrounds, respectively. Supposing $A_{\mathcal{W}'} = e^{-i\Lambda} A_{\mathcal{W}} e^{i\Lambda} + e^{-i\Lambda} d e^{i\Lambda}$ for gauge transformation $\psi \rightarrow e^{i\Lambda} \psi$ parametrized by Λ , we have:

$$\langle \mathcal{W}' \rangle = \int \mathcal{D}\psi e^{iS[\psi] + \int_{M_d} \text{tr}(A_{\mathcal{W}'} \wedge \star J)} = e^{2\pi i \mathcal{A}[A_{\mathcal{W}'}; \Lambda]} \langle \mathcal{W} \rangle, \quad (7)$$

where $\mathcal{A}[A_{\mathcal{W}'}; \Lambda] \neq 0$ is the quantum anomaly. Therefore, given our dualization, the computation of anomaly arises from rearrangements of the defect webs amounts to finding the gauge transformation interpolating the dual flat field configurations.

As a reminder, the 't Hooft anomaly of $\mathbb{Z}_m$ -symmetry of a 2D bosonic theory is well known to be characterized by $H^3(\mathbb{Z}_m, U(1))$, the third group cohomology of $\mathbb{Z}_m$ with $U(1)$ -coefficient, in which the phase factor $e^{2\pi i a_1(a_2+a_3-\bar{a}_2+\bar{a}_3)/m^2}$ of the F-move of the defect webs in Fig. 1 lives [28]. In d -spacetime dimensions, the characterization is given by $H^{d+1}(G, U(1))$ for discrete G [16,26]. To fill the gap of a similar characterization for continuous G , we will focus on continuous G of interests, and compute their anomalies in the case of *flat gauging*, i.e., from coupling $\mathcal{T}_G$ to a *flat* background field that is the closest cousin of the discrete case where all background fields are automatically flat.

We show that the 't Hooft anomaly is encoded in the phase factor in $\langle \text{diagram} \rangle = e^{2\pi i \mathcal{A}[A_{\mathcal{A}}; \Lambda]} \langle \text{diagram} \rangle$ [cf., Eq. (7)] for the defect webs diagram and diagram living on M_2 in $\mathcal{T}_G$ with Lie group G . Here, Λ is the gauge transformation from $A_{\mathcal{A}}$ to $A_{\mathcal{A}'}$.

In the modern language [12], $\mathcal{A}[A_{\mathcal{A}}; \Lambda]$ can be calculated as (cf., Supplemental Material [40]):

FIG. 1. F-move of defect web for $G = \mathbb{Z}_m$ 0-form symmetry group in 2D. $a_i \in \{0, 1, \dots, m-1\}$ and $\bar{p}$ is defined as $p \bmod m$.

$$\mathcal{A}(A_{\mathcal{A}}; \Lambda) = \int_{M_3} CS_k[A(t)], \quad (8)$$

using the level- k Chern-Simons invariant CS_k where $t \in [0, 1]$ parametrizing a mapping cylinder in the space of gauge field configurations along which $A(t) := g(t)^{-1} A_{\mathcal{A}} g(t) + g(t)^{-1} dg(t)$ for $g(x, y, t=0) = 1 \in G$ and $g(x, y, t=1) = e^{i\Lambda(x, y)}$ interpolates from $A_{\mathcal{A}}$ to $A_{\mathcal{A}'}$. We use the normalization $CS_k[A] := \frac{k}{2} \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = \frac{k}{2} \text{tr}(A \wedge F - \frac{1}{3} A \wedge A \wedge A)$ on a local patch of M_3 . For simplicity, we restrict to theories whose Chern-Simons invariants used in anomaly computations originate from integral classes in $H^4(BG, \mathbb{Z})$ (cf., Supplemental Material [40]). This condition constrains the allowed minimal level k , which depends on G . We do not discuss these cases individually and omit the subscript k for brevity [50].

The interpolation from $A_{\mathcal{A}}$ to $A_{\mathcal{A}'}$ is shown in the left panel of Fig. 2: by our convention, the front and back configurations are equivalent to diagram and diagram , respectively. Since $\mathcal{A}[A_{\mathcal{A}}; \Lambda]$ is clearly independent from small deformations of the mapping cylinder, i.e., the details of g , as long as boundary configurations $A_{\mathcal{A}}$ and $A_{\mathcal{A}'}$ are fixed, we further glue the vertices of diagram with the ones with the same labels in diagram at the two copies $M_2 \times \{0\}$ and $M_2 \times \{1\}$ to arrive at the configuration in the middle of Fig. 2. This is the difference between the corresponding webs of defects and the middle configuration, and can be further deformed to be a decorated tetrahedron as in the rightmost figure of Fig. 2, where the directions of the defects and of the gauge transformations are set by our convention (4). This process can be understood as the anomaly inflow—very similar to the finite group case [16]—i.e., an extra three-simplex can be attached in the bulk to cancel the anomaly and the decoration of the tetrahedron in the bulk describes the pullback of the three-simplex $\langle e, g_1, (g_1 g_2), (g_1 g_2 g_3) \rangle$ in BG^δ to the bulk. Here, G^δ is the Lie group G with discrete topology and BG^δ is the Eilenberg-MacLane space $K(G, 1)$ classifying flat G -bundles, which we denote in the classifying map by $\varphi^\delta: M_3 \rightarrow BG^\delta$ [52].

Similar to the finite group case, in terms of the classifying space, the anomaly is obtained by evaluating a cohomology class $[\Omega] \in H^3(BG^\delta, \mathbb{R}/\mathbb{Z})$ on $\langle e, g_1, (g_1 g_2), (g_1 g_2 g_3) \rangle$. Here, we treat the coefficient $U(1) = \mathbb{R}/\mathbb{Z}$ as an additive group. After pullback with Ω with φ^δ and integrating over the 3D spacetime, we obtain the Dijkgraaf-Witten phase

$$\mathcal{A}(A_{\mathcal{A}}; \Lambda) = \int_{M_3} (\varphi^\delta)^*(\Omega), \quad (9)$$

in other words, we claim that Ω found by F-move pulls back to the Chern-Simons invariant CS . See the proof of Eq. (9) in the Supplemental Material [40].

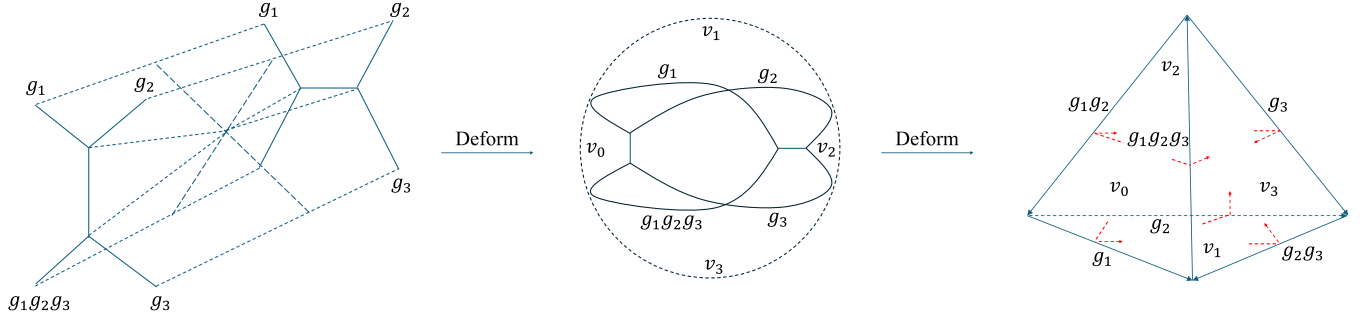


FIG. 2. Deformations of the combined $\nwarrow$ and $\nearrow$ configuration. The group multiplications in this figure are all taken from the right.

By investigating the local expression of the Chern-Simons invariant, we can get some other manifestations of anomalies. For this discussion, as G is simply connected, we can assign $F = 0$ for flat gauge field. In the M_3 bulk, our gauge field $A(t)$ is flat. On a local patch U_i , we have $A_i := A|_{U_i} = \tilde{g}_i^{-1} d\tilde{g}_i$ for some $\tilde{g}_i: U_i \rightarrow G$ and

$$\int_{U_i} -\frac{k}{6} \text{tr}(A_i \wedge A_i \wedge A_i) = \int_{U_i} k \tilde{g}_i^* \omega_3 = \int_{\tilde{g}_i(U_i)} k \omega_3, \quad (10)$$

where $\tilde{g}_i^*$ is the pullback from $\Omega^*(G)$ to $\Omega^*(U_i)$ and $\tilde{g}_i^* \omega_3 = \frac{i^2}{3!} \text{tr}(\tilde{g}_i^{-1} d\tilde{g}_i)^3$ with ω_3 being a *left-invariant three-form* on G . For a compact, connected, simply connected, and simple G , ω_3 generates the de Rham cohomology $H^3_{\text{dR}}(G, \mathbb{R}) \cong \mathbb{R}$; it is also the image of an integral class in the singular cohomology $H^3_{\text{singular}}(G, \mathbb{Z}) \cong \mathbb{Z}$ [53]. Furthermore, such G -bundle on any three-manifold is necessarily trivial. Hence, $\tilde{g}_i = \tilde{g}$ is globally defined, and the anomaly can be calculated as

$$\mathcal{A}(A_{\nwarrow}; \Lambda) = \int_{\tilde{g}(M_3)} k \omega_3. \quad (11)$$

The rhs of Eq. (11) now has another interpretation, namely, the action of ungauged 2D *Wess-Zumino-Witten* model. Indeed, this is a fact of anomaly matching [2,7,55]. The appearance of the left invariant three-form ω_3 as a *suspension* (given explicitly in the Supplemental Material [40]; the inverse is referred to as *transgression* [29,37]) of *CS* on the fiber G via the bundle fibration $G \rightarrow P \rightarrow M_3$ was discussed originally in Ref. [56]; see also discussions in Refs. [29,54,57,58] related to anomalies.

On the other hand, if we take $\langle X, Y \rangle = \text{tr}(XY)$ to be the Killing form of the Lie algebra $\mathfrak{g}$ of G , then

$$-\frac{k}{6} \text{tr}(A \wedge A \wedge A) = -\frac{k}{12} \langle A, [A, A] \rangle, \quad (12)$$

where we recognize that $\langle \cdot, [\cdot, \cdot] \rangle$ is the generator of the third *Lie algebra cohomology* $H^3(\mathfrak{g}, \mathbb{R})$ of the flavor algebra $\mathfrak{g}$. We see that the group $H^3(\mathfrak{g}, \mathbb{R})$ also classifies the local density of anomaly in flat gauging.

Since our G is taken to be compact, connected, and simple, we always have $H^3(\mathfrak{g}, \mathbb{R}) \cong H^3_{\text{dR}}(G, \mathbb{R}) \cong \mathbb{R}$. As the three-forms have integral periods, they are the image of the lattice $\mathbb{Z} \cong H^3_{\text{dR}}(G, \mathbb{Z})$ inside $H^3_{\text{dR}}(G, \mathbb{R})$. The coefficient exchanging map is in fact injective [59], as follows:

$$\begin{aligned} H^3_{\text{singular}}(G, \mathbb{Z}) &\cong H^3_{\text{dR}}(G, \mathbb{Z}) \hookrightarrow H^3_{\text{dR}}(G, \mathbb{R}) \\ &\cong H^3(\mathfrak{g}, \mathbb{R}). \end{aligned} \quad (13)$$

Recall that the suspension map

$$\tau: H^4(BG, \mathbb{Z}) \longrightarrow H^3_{\text{singular}}(G, \mathbb{Z}) \quad (14)$$

is also an isomorphism if G is in addition simply connected. For these groups, we have

$$H^3_{\text{singular}}(G, \mathbb{Z}) \cong H^4(BG, \mathbb{Z}) \xrightarrow{\kappa^\delta} H^3(BG^\delta, \mathbb{R}/\mathbb{Z}), \quad (15)$$

where the last map is also an injection [59]; see the Supplemental Material [40] for details.

The emergence of extra dimension. Moreover, there is a homomorphism $w: H^3(\mathfrak{g}, \mathbb{R}) \rightarrow H^3(B\bar{G}, \mathbb{R})$ where $B\bar{G}$ is the *classifying space of flat G -trivial bundles* [60]. One particularly interesting feature about $B\bar{G}$ is that a trivialization can be explicitly written down as an element of $\bar{G} = \{(g, \ell) \in G^\delta \times G^I | \ell(0) = g, \ell(1) = e\}$, where G^δ is G equipped with discrete topology together with the canonical map $\iota: G^\delta \rightarrow G$ and $G^I := \text{Map}(I, G)$ for $I := [0, 1]$.

More precisely, we consider a web of defects on a general 2D spacetime manifold M_2 which, by Poincaré duality, defines a flat G -bundle over M_2 determined up to isomorphism by the homotopy class of the *classifying map* $f^\delta: M_2 \rightarrow BG^\delta$ and a trivial bundle given by a null-homotopic map $f^n: M_2 \rightarrow BG$. Using the lift of $\iota, B\iota: BG^\delta \rightarrow BG$, the map $\tilde{f}: (x, t) \mapsto (f^\delta(x), \gamma_x(t)) \in BG^\delta \times BG$, where $\gamma_x(0) = B\iota(f^\delta(x))$ defines a homotopy from f^δ to f^n by the *mapping path space construction* [61]. Equivalently, $\tilde{f}$ is a map from $M_2 \times [0, 1]$ to $B\bar{G} \subset BG^\delta \times BG$. Therefore, we have

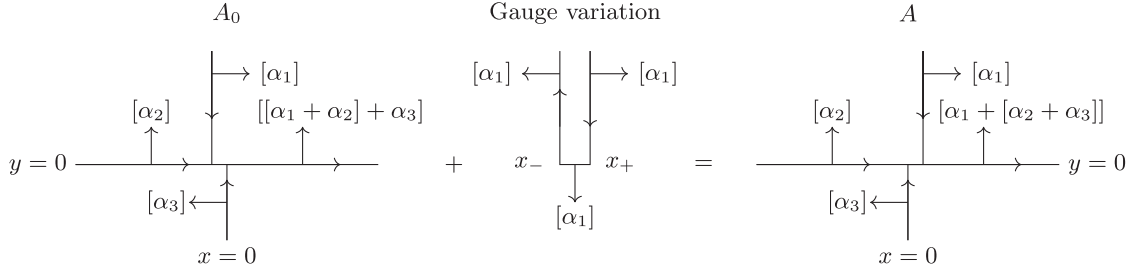


FIG. 3. F-move of defect lines in a theory with $U(1)$ flavor symmetry. A topological line carrying $e^{i\alpha} \in U(1)$ is labeled by $[\alpha] := \alpha \bmod 2\pi\mathbb{Z}$, with an arrow labeling the direction of gauge transformation in the same sense as in Eq. (5).

$$\int_{\tilde{f}(M_2 \times I)} w(\omega_3) = \int_{M_2 \times I} \tilde{f}^* \circ w(\omega_3), \quad (16)$$

where $\tilde{f}^*$ is the pullback of $\tilde{f}$.

A concrete open-cover realization of this mapping-path construction is given in the Supplemental Material [40]. It shows that the anomaly data can be captured by the Chern-Simons action of a flat gauge field on $M_2 \times I$, with nontrivial transition functions at $M_2 \times \{0\}$ and a trivialization at $M_2 \times \{1\}$ at the price of a nonzero flat connection. Thus, the characterization of the anomaly by $H^3(\mathfrak{g}, \mathbb{R})$ naturally introduces the extra interval I .

U(1) flavor symmetry and its 't Hooft anomaly. A mathematically cautious reader might have already noticed that the previous calculation leading to $H^3(G, \mathbb{R})$ fails for $U(1)$, since both $H^3(U(1), \mathbb{R})$ and $H^3(\mathfrak{u}(1), \mathbb{R})$ vanish. Moreover, the anomaly polynomial for $U(1)$ that generates $H^4(BU(1), \mathbb{Z}) \cong \mathbb{Z}$ is $CS[A] = A \wedge dA$, where the cubic term $\frac{1}{3}A \wedge A \wedge A$ is absent. To get a nonzero anomaly, we comment that there is a nonzero field strength at the junction point $(x, y) = (0, 0)$ in Eq. (5).

For the case of $G = U(1)$, we have $f(\alpha, \beta) = (\alpha + \beta) \bmod 2\pi\mathbb{Z} \equiv [\alpha + \beta]$. The flat gauge field dual to the trivalent junction is then

$$A = i\alpha\delta(x)H(y)dx + i(\beta H(-x) + [\alpha + \beta]H(x))\delta(y)dy, \quad (17)$$

and the corresponding field strength

$$F = dA = i([\alpha + \beta] - \alpha - \beta)\delta(x)\delta(y)dx \wedge dy \quad (18)$$

is nonvanishing at the junction point, but still satisfying the flatness condition (6).

Now to compute the 't Hooft anomaly of $U(1)$ flavor symmetry, we apply the previous computation to the configuration in Fig. 3.

It is easy to see that the leftmost and the rightmost configurations in Fig. 3 are topologically equivalent to $\nwarrow$ and $\nearrow$, respectively. Hence our task is to find their Poincaré duals $A_{\nwarrow}$ and $A_{\nearrow}$, and the gauge transformation parameter $d\epsilon \equiv id\Lambda = A_{\nearrow} - A_{\nwarrow}$. Actually, using the configuration in the middle of Fig. 3 and assuming $\alpha_1 < 2\pi$

hence $[\alpha_1] = \alpha_1$, one can show that (see the details in the Supplemental Material [40])

$$\epsilon = -i\alpha_1(H(x - x_-) - H(x - x_+))H(y). \quad (19)$$

This enables us to compute the 't Hooft anomaly to be

$$\mathcal{A}[A_{\nwarrow}; \epsilon] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \epsilon dA_{\nwarrow} = \frac{1}{(2\pi)^2} \alpha_1(\alpha_2 + \alpha_3 - [\alpha_2 + \alpha_3]). \quad (20)$$

Viewing

$$f(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) := e^{2\pi i \mathcal{A}[A_{\nwarrow}; \epsilon]} = e^{i\alpha_1(\alpha_2 + \alpha_3 - [\alpha_2 + \alpha_3])/(2\pi)} \quad (21)$$

as a map $f: G^3 \rightarrow U(1)$, one can check that $\delta f = 0$, where δ is the coboundary operator for the Lie group cohomology of $U(1)^\delta$ (see an explicit computation in the Supplemental Material [40]).

The map (21) is obviously not continuous, which aligns with the usage of group cohomology $H^3(U(1)^\delta, U(1))$ with discrete topology. Comparing to the phase factor in Fig. 1, we see that Eq. (21) is exactly the $m \rightarrow \infty$ limit of the generator of $H^3(\mathbb{Z}_m, U(1))$. One can further show that this map defines a nontrivial algebraic three-cocycle [33].

At the group level, the torsion subgroup $\mathbb{Q}/\mathbb{Z} \subset U(1)^\delta$ packages all finite subgroups $\mathbb{Z}_m \subset U(1)^\delta$, and $H^3(\mathbb{Q}/\mathbb{Z}, U(1)) = \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}}$ is the group of profinite integers. Thus the $\mathbb{Q}/\mathbb{Z}$ sector captures the anomalies for all $\mathbb{Z}_m$ subgroups simultaneously. An explicit representative of the corresponding $\hat{\mathbb{Z}}$ -valued family of three-cocycles, and its relation to $H^3(U(1)^\delta, U(1)) \supset H^4(BU(1), \mathbb{Z})$, is given in the Supplemental Material [40].

Higher dimensional generalizations. We have explicitly derived the connection between the F-move of topological defects and the anomaly polynomial for the flat gauging of the continuous symmetry group G , for the $d = 2$ case. Note that if we take the dual graph of the right picture in Fig. 2, we obtain a three-simplex $v_0 v_1 v_2 v_3$ whose vertices are v_i ($i = 0, \dots, 3$) while the edges correspond to elements of G . Such a picture can be easily generalized to $d = 2k$

spacetime dimensions, where the dual three-simplex $v_0 v_1 v_2 v_3$ is replaced by a dual $(2k+1)$ -simplex $v_0 v_1 \dots v_{2k+1}$. Taking the dual of $v_0 v_1 \dots v_{2k+1}$, one obtains the higher-dimensional generalization of F-move, as in Dijkgraaf-Witten theory [29].

The Lie algebra cohomology $H^3(\mathfrak{g}, \mathbb{R})$ and group cohomology $H^3(U(1)^\delta, U(1))$ should be straightforwardly generalized to $H^{2k+1}(\mathfrak{g}, \mathbb{R})$ and $H^{2k+1}(U(1)^\delta, U(1))$. We present the explicit generators for $H^{2k+1}(\mathfrak{g}, \mathbb{R})$ for semisimple Lie algebra $\mathfrak{g}$, $H^{2k+1}((U(1)^l)^\delta, U(1))$ for a product of $U(1)$ 0-form symmetry groups, and their correspondence with anomaly polynomials, in the Supplemental Material [40].

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