

# Study of $1^{--}$ P-wave charmoniumlike and bottomoniumlike tetraquark spectroscopy

Zheng Zhao<sup>1,\*</sup>, Attaphon Kaewsnod<sup>1</sup>, Kai Xu<sup>1</sup>, Nattapat Tagsinsit<sup>1</sup>, Xuyang Liu,<sup>1,2</sup>  
Ayut Limphirat<sup>1</sup>, and Yupeng Yan<sup>1,†</sup>

<sup>1</sup>*School of Physics and Center of Excellence in High Energy Physics and Astrophysics, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand*

<sup>2</sup>*School of Physics, Liaoning University, Shenyang 110036, China*



(Received 9 September 2025; accepted 21 September 2026; published 13 October 2026)

The masses of  $1^{--}$  P-wave charmoniumlike and bottomoniumlike tetraquark states are calculated in a constituent quark model where the Cornell-like potential and Breit-Fermi interaction are employed. All model parameters were imported from previous work, and predetermined by studying the low-lying conventional S- and P-wave light, charmed, bottom, charmonium, and bottomonium meson mass spectra. The lowest  $1^{--}$  tetraquark mass is predicted to be around 4.15 GeV. The decay widths of  $1^{--}$  P-wave tetraquark states are calculated for possible two-body strong decay channels within the rearrangement mechanism, including  $\omega\chi_{cJ}$  and  $\eta J/\psi$  for charmoniumlike tetraquarks, and  $\omega\chi_{bJ}$  for bottomoniumlike tetraquarks. The theoretical results are compared with the selected exotic states, also known as Y states, and tentative assignments are suggested. This study suggests that  $\psi(4230)$ ,  $\psi(4360)$ ,  $\psi(4660)$ , and  $\Upsilon(10753)$  may be P-wave tetraquark states and that multiple states might exist around 4.36 GeV.

DOI: 10.1103/k24h-d2mp

## I. INTRODUCTION

The study of exotic hadrons has gained significant attention in recent years, particularly in the context of the quark model. While the conventional meson and baryon spectra are well described by quantum chromodynamics (QCD) in terms of quark-antiquark and three-quark configurations, respectively, experimental discoveries of unconventional states challenge this simple classification. Among these exotic candidates, compact tetraquarks, bound states of two quarks and two antiquarks, offer a compelling framework to explain certain resonances that do not fit within the conventional meson spectrum [1,2].

Charmoniumlike tetraquarks, which contain a charm and anticharm quark pair along with an additional light quark pair, are particularly intriguing due to their potential connection to the so-called Y states observed in electron-positron annihilation experiments. In the latest Particle Data Group (PDG) [3] naming scheme, the Y states have been renamed as  $\psi$  states because they share the same quantum numbers ( $J^{PC} = 1^{--}$ ) as the  $J/\psi$  family.

Y(4230), an exotic state confirmed by multiple experimental collaborations and considered part of the broader family of charmoniumlike states, was first detected by the BABAR Collaboration in the  $e^+e^- \rightarrow \pi^+\pi^-J/\psi$  channel [4]. Y(4360) was later observed by the Belle Collaboration in the  $e^+e^- \rightarrow \gamma\pi^+\pi^-\psi(2S)$  process [5]. Subsequently, Y(4660) was discovered by BABAR in the initial state radiation process  $e^+e^- \rightarrow \gamma_{\text{ISR}}\pi^+\pi^-\psi(2S)$  [6]. More recently, Y(4484) was observed by the BESIII Collaboration in the  $K^+K^-J/\psi$  channel with a statistical significance greater than  $8\sigma$  [7]. Additionally, Y(4544) was reported by BESIII in the  $\omega\chi_{c1}$  channel [8] with a significance of  $5.8\sigma$ . Notably, the mass of Y(4544) is significantly higher than that of Y(4484), suggesting possible differences in their underlying structure.

The  $\Upsilon(10753)$  bottomoniumlike state was observed by the Belle Collaboration in the  $e^+e^- \rightarrow \pi^+\pi^-\Upsilon(nS)$  process [9]. Recently, the Belle II Collaboration reported another observation that strongly supports the existence of the radiative transition process  $\Upsilon(10753) \rightarrow \omega\chi_{bJ}(1P)$  [10]. The energy dependence of the Born cross sections for  $\Upsilon(10753)$  in the  $\omega\chi_{bJ}(1P)$  channel [10] was found to be consistent with the shape of the  $\Upsilon(10753)$  state in the  $\pi^+\pi^-\Upsilon(nS)$  channel [9]. Belle II concluded that the internal structure of  $\Upsilon(10753)$  may differ from that of  $\Upsilon(10860)$ , where the latter is well understood as a predominantly conventional bottomonium state.

The exotic states exhibit properties that cannot be easily accommodated with conventional quarkonium expectations,

\*Contact author: zhaozheng1022@hotmail.com

†Contact author: yupeng@sut.ac.th

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP<sup>3</sup>.

suggesting the need for alternative interpretations. Various theoretical pictures for charmoniumlike exotic states, including charmonium core admixed with coupled channels [11–15], charmonium hybrids [16–20], hidden-charm molecular states [21–25], and hidden-charm tetraquark states [26–32], have been proposed to explore their internal structure and mass spectra, providing valuable insights into their nature.

Since the inception of the quark model [33], the tetraquark picture has been widely employed and further developed to understand the internal structure of exotic states [34,35], not only in charmoniumlike systems. For exotic states in the four-heavy-quark region, one may refer to the fully-heavy tetraquark picture [36–53]. Similarly, for light exotic states, the fully-light tetraquark picture has also been considered [54–60].

In the previous work,  $Y(4230)$  and the  $Y(10753)$  could not be accommodated within a conventional heavy quarkonium picture including  $S - D$  mixing [61]. In this work, we focus on studying the mass spectrum and decay properties of the  $1^{--}$  P-wave charmoniumlike and bottomoniumlike compact tetraquark states (which we refer to as  $Y_c$  and  $Y_b$  for convenience) within a constituent quark model (CQM) framework.

The paper is organized as follows. In Sec. II, all the possible configurations of color, spin, and spatial degrees of freedom of tetraquark states are introduced, and a constituent quark model developed from our previous work [54,62] is briefly reviewed. In Sec. III,  $1^{--}$  P-wave tetraquark mass spectra and decay branching ratios are evaluated. The theoretical results are compared with experimental data of  $Y$  states, and tentative assignments for  $1^{--}$  tetraquarks are suggested. A summary is given in Sec. IV. The details of the tetraquark spatial wave function and the cross terms between different configurations from one-gluon-exchange interaction are shown in the Appendixes.

## II. THEORETICAL MODEL

### A. Quark configurations of tetraquarks

The construction of tetraquark states follows the principle that a tetraquark state must be a color singlet, meaning the color wave function of the tetraquark must be a  $[222]_1$  singlet within the  $SU_c(3)$  group. In this work, we consider only charmoniumlike and bottomoniumlike  $(q_1 Q_2 \bar{q}_3 \bar{Q}_4)$  states, where  $q_1$  and  $Q_2$  are a light quark and a heavy quark, and  $\bar{q}_3$  and  $\bar{Q}_4$  are a light antiquark and a heavy antiquark, respectively.

The permutation symmetry of the two-quark cluster  $(qQ)$  in tetraquark states is described by the Young tableaux  $[2]_6$  and  $[11]_3$  of the  $SU_c(3)$  group, while the color configuration of the two-antiquark cluster  $(\bar{q}\bar{Q})$  consists of a  $[211]_3$  triplet and a  $[22]_6$  antisextet.

Thus, a  $[222]_1$  color singlet for tetraquark states demands the following configurations:  $[2]_6(q_1 Q_2) \otimes [22]_6(\bar{q}_3 \bar{Q}_4)$

and  $[11]_3(q_1 Q_2) \otimes [211]_3(\bar{q}_3 \bar{Q}_4)$ , which correspond to the color sextet-antisextet ( $6_c \otimes \bar{6}_c$ ) and triplet-antitriplet ( $\bar{3}_c \otimes 3_c$ ) configurations, respectively. The explicit color wave function for each tetraquark color configuration can be found in the previous work [62].

The  $J^{PC} = 1^{--}$   $Y$  states can be described by  $L = 1$  tetraquark configurations with  $S = 0$  or  $S = 2$ . Thus, two possible spin combinations for tetraquarks are considered as follows:  $[\psi_{[s=1]}^{qQ} \otimes \psi_{[s=1]}^{\bar{q}\bar{Q}}]_{S=0,2}$  and  $\psi_{[s=0]}^{qQ} \otimes \psi_{[s=0]}^{\bar{q}\bar{Q}}$ .

The complete basis is constructed by coupling the harmonic-oscillator (HO) wave functions, taking the general form,

$$\psi_{NL} = \sum_{n_{\chi_i}, l_{\chi_i}} A(n_{\chi_1}, n_{\chi_2}, n_{\chi_3}, l_{\chi_1}, l_{\chi_2}, l_{\chi_3}) \times \psi_{n_{\chi_1} l_{\chi_1}}(\vec{\chi}_1) \otimes \psi_{n_{\chi_2} l_{\chi_2}}(\vec{\chi}_2) \otimes \psi_{n_{\chi_3} l_{\chi_3}}(\vec{\chi}_3), \quad (1)$$

where  $\psi_{n_{\chi_i} l_{\chi_i}}$  are HO wave functions. The sum  $n_{\chi_i}, l_{\chi_i}$  is over  $n_{\chi_1}, n_{\chi_2}, n_{\chi_3}, l_{\chi_1}, l_{\chi_2}, l_{\chi_3}$ . The relative Jacobi coordinates  $\vec{\chi}_1, \vec{\chi}_2$ , and  $\vec{\chi}_3$  are defined as

$$\begin{aligned} \vec{\chi}_1 &= \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2), \\ \vec{\chi}_2 &= \frac{1}{\sqrt{2}}(\vec{r}_3 - \vec{r}_4), \\ \vec{\chi}_3 &= \frac{m_u \vec{r}_1 + m_Q \vec{r}_2}{m_u + m_Q} - \frac{m_u \vec{r}_3 + m_Q \vec{r}_4}{m_u + m_Q}, \end{aligned} \quad (2)$$

where  $\vec{r}_j$  denotes the coordinate of the  $j$ th quark. The reduced quark masses are defined as

$$\begin{aligned} u_1 &= \frac{2m_u m_Q}{m_u + m_Q}, \\ u_2 &= \frac{2m_u m_Q}{m_u + m_Q}, \\ u_3 &= \frac{m_u + m_Q}{2}, \end{aligned} \quad (3)$$

where  $m_u$  and  $m_Q$  are the masses of the light and heavy quarks, respectively.  $N$  and  $L$  are, respectively, the total principal quantum number and total angular momentum, and  $L = l_{\chi_1} + l_{\chi_2} + l_{\chi_3} = 1$  for a P-wave tetraquark and  $N = (2n_{\chi_1} + l_{\chi_1}) + (2n_{\chi_2} + l_{\chi_2}) + (2n_{\chi_3} + l_{\chi_3})$ .

We stress that the Jacobi coordinates in Eq. (2) are introduced for convenience when constructing the complete HO basis and evaluating matrix elements, and do not impose a dynamical assumption on the internal structure of the tetraquark. Other commonly used coordinate sets, often referred to as H- and K-type, are related to Eq. (2) by linear transformations of the same four-body relative coordinates. Likewise, different color coupling schemes associated with these coordinate choices are connected by basis transformations (e.g.,  $\bar{3}_c - 3_c$  and  $6_c - \bar{6}_c$  can be

written as linear combinations of  $1_c - 1_{\bar{c}}$  and  $8_c - 8_{\bar{c}}$ . Therefore, in a sufficiently large basis, the physical observables are independent of the chosen coordinate representation [63].

### B. Hamiltonian

We study meson and tetraquark systems within the nonrelativistic quark model [64–66]. The Hamiltonian has been widely employed in recent years [37,53] and takes the form

$$\begin{aligned} H &= H_0 + H_{so}, \\ H_0 &= \sum_{k=1}^N \left( \frac{1}{2} M_k^{ave} + \frac{p_k^2}{2m_k} \right) + \sum_{i<j}^N \left( -\frac{3}{16} \vec{\lambda}_i \cdot \vec{\lambda}_j \right) V_{cen}(r_{ij}), \\ H_{so} &= \sum_{i<j}^N \left( -\frac{3}{16} \vec{\lambda}_i \cdot \vec{\lambda}_j \right) (V_{so}(r_{ij})), \end{aligned} \quad (4)$$

where  $M_k^{ave}$  denotes the spin-averaged mass given by  $\frac{1}{4} M_{PS} + \frac{3}{4} M_V$ .  $m_k$  are the constituent quark masses.  $\vec{\lambda}_i$  is the quark color operator in SU(3). The Cornell-like potential and one-gluon-exchange potentials are employed in the central potential, taking the form

$$V_{cen}(r_{ij}) = A_{ij} r_{ij} - \frac{B_{ij}}{r_{ij}} + V_{ss}(r_{ij}), \quad (5)$$

where the spin-spin interaction  $V_{ss}(r_{ij})$  originates from the one-gluon-exchange potential and takes the form

$$V_{ss} = \frac{1}{6m_i m_j} \Delta V_V(r) \vec{\sigma}_i \cdot \vec{\sigma}_j = \frac{2B_{ij}\sigma_{ij}^3}{3m_i m_j \sqrt{\pi}} e^{-\sigma_{ij}^2 r_{ij}^2} \vec{\sigma}_i \cdot \vec{\sigma}_j. \quad (6)$$

The spin-orbital interaction  $V_{so}(r_{ij})$  takes the form

$$\begin{aligned} V_{so} &= \frac{1}{r_{ij}} \frac{dV_V}{d(r_{ij})} \frac{1}{4} \left[ \left( \frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{4}{m_i m_j} \right) \vec{L}_{ij} \cdot \vec{S}_{ij} \right. \\ &\quad \left. + \left( \frac{1}{m_i^2} - \frac{1}{m_j^2} \right) \vec{L}_{ij} \cdot (\vec{s}_i - \vec{s}_j) \right] \\ &\quad - \frac{1}{r_{ij}} \frac{dV_S}{d(r_{ij})} \left( \frac{\vec{L}_{ij} \cdot \vec{s}_i}{2m_i^2} + \frac{\vec{L}_{ij} \cdot \vec{s}_j}{m_j^2} \right) \\ &= \left( \frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{4}{m_i m_j} \right) \left( \frac{-B_{ij}\sigma_{ij}}{2\sqrt{\pi}} \right) \frac{e^{-\sigma_{ij}^2 r_{ij}^2}}{r_{ij}^3} \vec{L}_{ij} \cdot \vec{S}_{ij} \\ &\quad + \left( \frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{4}{m_i m_j} \right) \left( \frac{-B_{ij}}{4} \right) \frac{Erf[\sigma_{ij} r_{ij}]}{r_{ij}^3} \vec{L}_{ij} \cdot \vec{S}_{ij} \\ &\quad + \left( \frac{-A_{ij}}{2} \right) \frac{1}{r_{ij}} \left( \frac{\vec{L}_{ij} \cdot \vec{s}_i}{m_i^2} + \frac{\vec{L}_{ij} \cdot \vec{s}_j}{m_j^2} \right), \end{aligned} \quad (7)$$

where  $\vec{\sigma}_i$  are the quark spin operators in SU(2). Note that we have employed  $V_V(r) = -BErf[\sigma r]/r$  and  $V_S(r) = Ar$ , taken from Ref. [65].  $m_i$  and  $m_j$  are the constituent quark masses of the  $i$ th and  $j$ th quarks.  $\vec{s}_i$  represents the spin operator of the  $i$ th quark.  $\vec{S}_{ij} = \vec{s}_i + \vec{s}_j$  is the spin operator for the  $(ij)$ th quark pair.  $\vec{L}_{ij}$  is the relative orbital operator, taking the form

$$\vec{L}_{ij} = \vec{r}_{ij} \times \vec{p}_{ij} = \vec{r}_{ij} \times \frac{m_i \vec{p}_i - m_j \vec{p}_j}{m_i + m_j}. \quad (8)$$

In line with the previous works,  $A_{ij}$ ,  $B_{ij}$ , and  $\sigma_{ij}$  are proposed to be mass-dependent coupling parameters, taking the form

$$A_{ij} = a + b m_{ij}, \quad B_{ij} = B_0 \sqrt{\frac{1}{m_{ij}}}, \quad \sigma_{ij} = \sigma_0 m_{ij}, \quad (9)$$

where  $m_{ij}$  is the reduced mass of the  $i$ th and  $j$ th quarks, defined as  $m_{ij} = \frac{2m_i m_j}{m_i + m_j}$ .  $a$ ,  $b$ ,  $B_0$ , and  $\sigma_0$  are constants. For a more detailed discussion, one may refer to Ref. [62]. The hyperfine coefficient  $\sigma_{ij}$  is also proposed to be mass-dependent [65].

The central potential  $V_0$  and the spin-spin interaction  $V_{ss}$  are treated as the leading effects, while the remaining spin-orbital interaction  $V_{so}$  is considered a perturbation that shifts the mass spectrum. The Schrödinger equation is solved for mesons and tetraquarks within the Hamiltonian  $H_0$ , yielding the eigenvalue  $E_0$  and eigenstates  $\psi_0$ . The mass spectrum is then calculated by adding the diagonalized  $H_{so}$  matrix to the basis of the eigenstates obtained in the previous step, which takes the form

$$E = E_0 + \langle \psi_0 | H_{so} | \psi_0 \rangle. \quad (10)$$

By importing the coupling constants  $a$ ,  $b$ , and  $B_0$ , as well as the constituent quark masses  $m_u$ ,  $m_c$ , and  $m_b$  from previous work [62], the mass spectra of S- and P-wave light, charmed, bottom, charmonium, and bottomonium conventional mesons are calculated in the Hamiltonian in Eq. (4). A comparison of the theoretical results, presented in Table I, with experimental data from the PDG [3], yields a model coupling parameter  $\sigma_0 = 0.7$ .

The present calculation is performed in a compact tetraquark CQM basis and does not include an explicit coupling to open two-meson continuum channels. Coupled-channel effects, which may induce additional mass shifts, especially for states close to open-flavor thresholds, will be considered in future extensions of this framework.

### C. Two-body strong decay

The two-body strong decays of the Y tetraquark states are studied within the rearrangement mechanism, as shown in Fig. 1.

TABLE I. S- and P-wave meson states applied to fit the model parameters. The last column shows the deviation between the experimental and theoretical mean values,  $D = 100 \cdot (M^{\text{exp}} - M^{\text{cal}})/M^{\text{exp}}$ .  $M^{\text{exp}}$  is taken from PDG [3].

| Meson       | $J^{P(C)}$ | nL | $M^{\text{exp}}$ (MeV) | $M^{\text{cal}}$ (MeV) | D (%) |
|-------------|------------|----|------------------------|------------------------|-------|
| $\eta_b$    | $0^{-+}$   | 1S | 9399                   | 9394                   | 0.1   |
|             |            | 2S | 9999                   | 10024                  | -0.3  |
| $\Upsilon$  | $1^{--}$   | 1S | 9460                   | 9467                   | -0.1  |
|             |            | 2S | 10023                  | 10054                  | -0.3  |
| $h_b$       | $1^{+-}$   | 1P | 9899                   | 9922                   | -0.2  |
|             |            | 2P | 10260                  | 10320                  | -0.6  |
| $\chi_{b0}$ | $0^{++}$   | 1P | 9859                   | 9896                   | -0.4  |
|             |            | 2P | 10232                  | 10301                  | -0.7  |
| $\chi_{b1}$ | $1^{++}$   | 1P | 9893                   | 9911                   | -0.2  |
|             |            | 2P | 10255                  | 10312                  | -0.6  |
| $\chi_{b2}$ | $2^{++}$   | 1P | 9912                   | 9935                   | -0.2  |
|             |            | 2P | 10269                  | 10332                  | -0.6  |
| $\eta_c$    | $0^{-+}$   | 1S | 2984                   | 2962                   | 0.7   |
|             |            | 2S | 3638                   | 3614                   | 0.7   |
| $\psi$      | $1^{--}$   | 1S | 3097                   | 3090                   | 0.2   |
|             |            | 2S | 3686                   | 3655                   | 0.8   |
|             |            | 3S | 4040                   | 4027                   | 0.3   |
| $h_c$       | $1^{+-}$   | 1P | 3525                   | 3516                   | 0.2   |
|             |            | 2P | 3860                   | 3876                   | -0.4  |
| $\chi_{c0}$ | $0^{++}$   | 1P | 3415                   | 3464                   | -1.4  |
|             |            | 2P | 3510                   | 3500                   | 0.3   |
| $\chi_{c1}$ | $1^{++}$   | 1P | 3556                   | 3553                   | 0.1   |
| $\chi_{c2}$ | $2^{++}$   | 1P | 3930                   | 3932                   | -0.1  |
| $B^0$       | $0^-$      | 1S | 5279                   | 5289                   | -0.2  |
| $B^*$       | $1^-$      | 1S | 5325                   | 5337                   | -0.2  |
| $B_1$       | $1^+$      | 1P | 5721                   | 5800                   | -1.4  |
| $B_2^*$     | $2^+$      | 1P | 5747                   | 5814                   | -1.2  |
| $D^0$       | $0^-$      | 1S | 1865                   | 1906                   | -2.2  |
|             |            | 2S | 2549                   | 2590                   | -1.6  |
| $D^{*0}$    | $1^-$      | 1S | 2007                   | 2021                   | -0.7  |
|             |            | 2S | 2627                   | 2622                   | 0.2   |
| $D_1^0$     | $1^+$      | 1P | 2420                   | 2465                   | -1.9  |
| $D_0^{*0}$  | $0^+$      | 1P | 2343                   | 2438                   | -4.1  |
| $D_1^{*0}$  | $1^+$      | 1P | 2412                   | 2465                   | -2.2  |
| $D_2^{*0}$  | $2^+$      | 1P | 2460                   | 2508                   | -2.0  |
|             |            | 2S | 1450                   | 1395                   | 3.8   |
| $\rho$      | $1^{--}$   | 1S | 770                    | 766                    | 0.5   |
| $h_1(1170)$ | $1^{+-}$   | 1P | 1170                   | 1202                   | -2.7  |
| $b_1(1235)$ | $1^{+-}$   | 1P | 1235                   |                        | 2.3   |
| $f_0(1370)$ | $0^{++}$   | 1P | 1200–1500              | 1191                   | ...   |
| $f_1(1285)$ | $1^{++}$   | 1P | 1280                   | 1223                   | 4.4   |
| $a_1(1260)$ | $1^{++}$   | 1P | 1230                   |                        | 0.6   |
| $f_2(1270)$ | $2^{++}$   | 1P | 1275                   | 1277                   | -0.2  |
| $a_2(1320)$ | $2^{++}$   | 1P | 1318                   |                        | 3.1   |

$$T_{cs} = \langle \psi_f^{cs} | \psi_i^{cs} \rangle = \langle \psi_{M_1 M_2}^c | \psi_Y^c \rangle \langle \psi_{M_1 M_2}^s | \psi_Y^s \rangle, \quad (11)$$

where  $\psi_Y^c$  and  $\psi_Y^s$  are color and spin wave functions of the initial Y tetraquark states in the mass spectrum.  $\psi_{M_1 M_2}^c$  and  $\psi_{M_1 M_2}^s$  are the color and spin wave functions of the final

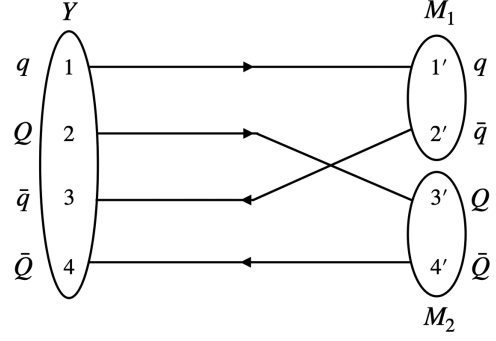


FIG. 1. Quark rearranged diagram for Y tetraquark in the hidden-charm decay modes.

states of all possible two-body strong decay channels:  $\omega\chi_{c0}$ ,  $\omega\chi_{c1}$ ,  $\omega\chi_{c2}$ ,  $\eta J/\psi$ ,  $\omega\chi_{b0}$ ,  $\omega\chi_{b1}$ , and  $\omega\chi_{b2}$ .

In the work, the ratio of  $|T_{cs}|^2$  is approximated as the branching ratio, assuming that the spatial contributions from different decay channels are similar.

### III. RESULTS AND DISCUSSION

#### A. Results

The predetermined and imported parameters are applied to predict the masses of Y tetraquark states in the Hamiltonian in Eq. (4) including the color-dependent central potential  $V_{cen}$ , which may mix different color-spin configurations.

Because of the cross terms from the color operator  $(\vec{\lambda}_i \cdot \vec{\lambda}_j)$  and the spin operator  $(\vec{\sigma}_i \cdot \vec{\sigma}_j)$ , listed in Appendix B, eigenstates of the Hamiltonian for  $S = 0$  are linear combinations of  $\psi_{3\bar{3}}^c \psi_{(0\bar{0}0)}^{S=0}$ ,  $\psi_{3\bar{3}}^c \psi_{(1\bar{0}1)}^{S=0}$ ,  $\psi_{6\bar{6}}^c \psi_{(0\bar{0}0)}^{S=0}$ , and  $\psi_{6\bar{6}}^c \psi_{(1\bar{0}1)}^{S=0}$ . Eigenstates for  $S = 2$  are linear combinations of  $\psi_{3\bar{3}}^c \psi_{(1\bar{0}1)}^{S=2}$  and  $\psi_{6\bar{6}}^c \psi_{(1\bar{0}1)}^{S=2}$ . There is no configuration mixing between  $S = 0$  and  $S = 2$  states, as no cross terms are found. En, where  $n = 1, 2, 3, \dots$ , represents the  $n$ th eigenstate of the Hamiltonian in Eq. (4) in the notation used later in this work, incorporating color-spin configuration mixing.

The theoretical masses and branching ratios for the E1–E15  $S = 0$  charmoniumlike  $Y_c$  tetraquarks with various mixed configurations are listed in Table II. Similarly, the masses and branching ratios of the E1 to E8 mixed  $S = 0$  bottomoniumlike  $Y_b$  tetraquarks are listed in Table III. Likewise, the masses of the E1 to E6 mixed  $S = 2$  charmoniumlike  $Y_c$  and bottomoniumlike  $Y_b$  tetraquarks are provided in Table IV.

For  $S = 0$   $Y_c$  tetraquarks, the spin-color factor for the two-body strong decay into  $\eta J/\psi$  is close to zero. For  $S = 2$   $Y_c$  and  $Y_b$  tetraquarks, the spin-color factors in  $\omega\chi_{c0}$ ,  $\omega\chi_{c1}$ ,  $\omega\chi_{c2}$ ,  $\omega\chi_{b0}$ ,  $\omega\chi_{b1}$ ,  $\omega\chi_{b2}$  decay channels are all equal to zero.

Since the diquark-type color couplings  $\bar{3}_c - 3_c$  and  $6_c - \bar{6}_c$  are linearly related to the meson–meson couplings

TABLE II. Mass spectrum and decay branching ratios for the  $S = 0$   $Y_c$  tetraquark states. Threshold masses are given in square brackets for each decay channel, and the ratios are normalized to the  $E2 \rightarrow \omega\chi_{c0}$  channel (set to 1). All masses are in MeV.

| En  | Mass | $\omega\chi_{c0}[4196]$ | $\omega\chi_{c1}[4292]$ | $\omega\chi_{c2}[4338]$ |
|-----|------|-------------------------|-------------------------|-------------------------|
| E1  | 4139 | ...                     | ...                     | ...                     |
| E2  | 4219 | 1                       | ...                     | ...                     |
| E3  | 4300 | 15                      | 45                      | ...                     |
| E4  | 4313 | 1.8                     | 5                       | ...                     |
| E5  | 4388 | 1.7                     | 5                       | 8.5                     |
| E6  | 4408 | 0.01                    | 0.03                    | 0.05                    |
| E7  | 4480 | 4.8                     | 14                      | 24                      |
| E8  | 4498 | 0.3                     | 0.7                     | 1.3                     |
| E9  | 4516 | 6                       | 18                      | 30                      |
| E10 | 4531 | 1.4                     | 4                       | 7                       |
| E11 | 4567 | 6                       | 17                      | 28                      |
| E12 | 4575 | 0.3                     | 1                       | 1.6                     |
| E13 | 4587 | 11                      | 33                      | 55                      |
| E14 | 4595 | 3                       | 10                      | 17                      |
| E15 | 4625 | 56                      | 169                     | 282                     |

TABLE III. Mass spectrum and decay branching ratios for the  $S = 0$   $Y_b$  tetraquark states. Threshold masses are given in square brackets for each decay channel, and the ratios are normalized to the  $E4 \rightarrow \omega\chi_{b0}$  channel (set to 1). All masses are in MeV.

| En | Mass  | $\omega\chi_{b0}[10641]$ | $\omega\chi_{b1}[10674]$ | $\omega\chi_{b2}[10694]$ |
|----|-------|--------------------------|--------------------------|--------------------------|
| E1 | 10504 | ...                      | ...                      | ...                      |
| E2 | 10548 | ...                      | ...                      | ...                      |
| E3 | 10565 | ...                      | ...                      | ...                      |
| E4 | 10679 | 1                        | 3                        | ...                      |
| E5 | 10754 | 1.4                      | 4                        | 7                        |
| E6 | 10774 | 0.1                      | 0.3                      | 0.5                      |
| E7 | 10791 | 0.2                      | 0.7                      | 1.2                      |
| E8 | 10810 | 0.2                      | 0.5                      | 0.8                      |

TABLE IV. Mass spectrum of mixed  $S = 2$   $Y_c$  and  $Y_b$  tetraquark states. Mass values are in MeV.

| En    | E1    | E2    | E3    | E4    | E5    | E6    |
|-------|-------|-------|-------|-------|-------|-------|
| $Y_c$ | 4284  | 4350  | 4463  | 4570  | 4578  | 4617  |
| $Y_b$ | 10574 | 10657 | 10797 | 10830 | 10855 | 10940 |

$1_c - 1_c$  and  $8_c - 8_c$ , an eigenstate may occasionally acquire a noticeable  $1_c - 1_c$  admixture. Such a component can resemble a weakly interacting two-cluster configuration and may therefore exhibit enhanced sensitivity to changes of the relative-coordinate basis. We therefore apply a real-basis-scaling stability test [67] to examine the stability of each eigenstate against variations of the basis length parameter and, in particular, to identify possible strongly basis-dependent solutions associated with the  $\vec{\chi}_3$  relative coordinate. Here  $\vec{\chi}_3$  denotes the relative coordinate between

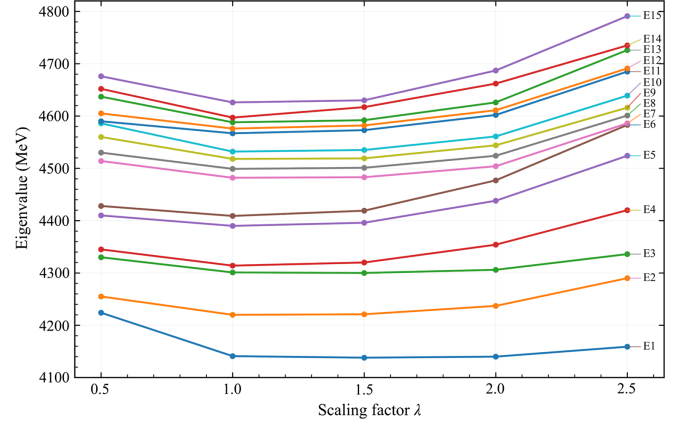


FIG. 2. Real-basis-scaling stability trajectories of the mixed  $S = 0$   $Y_c$  tetraquark levels. The eigenvalues  $E_n(\lambda)$  (E1–E15) are obtained by rescaling only the length parameter associated with the  $\vec{\chi}_3$  coordinate,  $b_{\chi_3} \rightarrow \lambda b_{\chi_3}$ , while keeping the Hamiltonian fixed. The scaling is employed here as a basis-stability diagnostic within the compact tetraquark model space.

the two subclusters defined in Eq. (2). The length parameter associated with the  $\vec{\chi}_3$  coordinate is rescaled by a factor,  $b_{\chi_3} \rightarrow \lambda b_{\chi_3}$ , while the basis structure and Hamiltonian are kept fixed [68–70].

The present calculation is performed within a compact tetraquark CQM basis and does not explicitly include two-meson scattering channels. Therefore, the present real-scaling analysis is used as a basis-stability diagnostic, rather than as a resonance-width extraction procedure based on identifying avoided crossing points [71]. In the standard stabilization treatment, an avoided crossing arises when a scale-dependent continuumlike eigenvalue approaches and interacts with a comparatively stable localized eigenvalue. Such behavior requires an explicit representation of the corresponding scattering continuum, which is not included in the present compact tetraquark Hamiltonian.

We present, as an example, in Fig. 2 the eigenvalue trajectories of the  $S = 0$   $Y_c$  levels, E1–E15, over a representative range of  $\lambda$  obtained by rescaling the  $\vec{\chi}_3$  coordinate. The low-lying levels E1–E10 display clear flat valleys over the range of  $\lambda$ , with variations generally limited to a few tens of MeV, indicating good stability against the rescaling of the  $\chi_3$  coordinate. The higher levels show a more noticeable upward shift as  $\lambda$  increases, especially E14 and E15, indicating a stronger sensitivity to the basis truncation. The eigenvalues at  $\lambda = 1.0$ , which correspond to the mass spectrum results presented in the work, are located near the bottoms of the valleys. Within the considered model space and scaling range, no eigenvalue shows a clearly identifiable strongly-scale-dependent continuumlike trajectory.

As an additional numerical check, we apply the complex-scaling method to the mixed  $S = 0$   $Y_c$  tetraquark system. The complex rotation is introduced through  $r \rightarrow r e^{i\theta}$ , with  $\theta = 6^\circ$  [63]. Under complex scaling, in

TABLE V. Complex eigenenergies for different basis dimensions, with units in MeV.  $N$  represents the total principal quantum number of the complete basis, taking the form  $N = (2n_{\chi_1} + l_{\chi_1}) + (2n_{\chi_2} + l_{\chi_2}) + (2n_{\chi_3} + l_{\chi_3})$ .

| En  | $N = 3$        | $N = 5$        | $N = 7$        | $N = 9$       | $N = 11$      | $N = 13$      |
|-----|----------------|----------------|----------------|---------------|---------------|---------------|
| E1  | 4174.3 − 18.5i | 4153.6 − 12.2i | 4146.2 − 7.0i  | 4142.4 − 4.4i | 4140.4 − 2.7i | 4139.2 − 1.8i |
| E2  | 4246.5 − 10.2i | 4229.0 − 9.5i  | 4224.1 − 4.7i  | 4221.3 − 2.9i | 4219.8 − 1.6i | 4219.0 − 1.0i |
| E3  | 4315.7 − 4.0i  | 4304.8 − 3.8i  | 4302.6 − 2.0i  | 4301.4 − 1.4i | 4300.7 − 0.8i | 4300.4 − 0.5i |
| E4  | 4338.6 − 2.8i  | 4322.7 − 5.7i  | 4318.5 − 3.3i  | 4315.6 − 2.4i | 4314.1 − 1.6i | 4313.1 − 1.1i |
| E5  | 4418.4 − 5.8i  | 4399.5 − 6.9i  | 4394.3 − 4.1i  | 4391.2 − 3.0i | 4389.4 − 2.2i | 4388.3 − 1.7i |
| E6  | 4435.7 − 3.8i  | 4418.5 − 5.5i  | 4413.3 − 3.6i  | 4410.5 − 2.7i | 4408.8 − 2.0i | 4407.7 − 1.6i |
| E7  | 4504.2 − 9.6i  | 4491.7 − 7.6i  | 4485.0 − 5.0i  | 4482.5 − 3.0i | 4481.1 − 2.0i | 4480.3 − 1.4i |
| E8  | 4528.7 − 8.2i  | 4514.8 − 6.2i  | 4503.9 − 4.3i  | 4501.0 − 3.5i | 4498.9 − 2.8i | 4497.7 − 1.7i |
| E9  | 4551.9 − 5.3i  | 4539.0 − 5.5i  | 4521.7 − 2.0i  | 4519.0 − 3.2i | 4517.1 − 2.9i | 4516.2 − 1.8i |
| E10 | 4577.5 + 8.1i  | 4547.3 + 6.9i  | 4535.1 − 3.3i  | 4533.0 − 2.3i | 4531.9 − 1.6i | 4531.2 − 1.3i |
| E11 | 4593.5 + 0.5i  | 4580.0 − 0.4i  | 4571.0 − 2.3i  | 4568.9 − 1.8i | 4567.4 − 1.5i | 4566.6 − 1.0i |
| E12 | 4601.2 + 10.6i | 4584.7 − 2.2i  | 4579.1 − 2.0i  | 4577.4 − 1.4i | 4576.1 − 1.1i | 4575.4 − 0.9i |
| E13 | 4605.7 − 1.6i  | 4592.8 − 3.5i  | 4589.7 − 2.4i  | 4588.5 − 1.8i | 4587.7 − 1.5i | 4587.1 − 1.3i |
| E14 | 4635.9 + 18.2i | 4627.2 + 9.4i  | 4599.3 − 6.4i  | 4598.9 − 3.2i | 4595.9 − 2.8i | 4594.6 − 2.2i |
| E15 | 4708.1 + 54.9i | 4658.8 + 29.8i | 4628.1 + 11.5i | 4624.1 + 2.4i | 4624.0 − 1.7i | 4624.6 − 0.5i |

principle, the eigenenergies of bound states shall remain on the real-energy axis and are independent of the rotation angle in the complete-basis limit. By contrast, continuum states are rotated into the complex-energy plane and generally acquire nonzero imaginary components that vary with the rotation angle  $\theta$ . In the present calculation, the complex-scaling analysis is therefore used as an additional diagnostic of the numerical stability of the discrete solutions obtained from the compact Hamiltonian.

The complex eigenenergies calculated with different basis dimensions are listed in Table V. With increasing basis dimension, the real parts of the eigenenergies gradually converge to the corresponding values obtained without complex rotation, while the imaginary parts systematically decrease toward zero. The small imaginary components appearing in a finite basis are attributed to basis truncation

and numerical uncertainties rather than to physical decay widths. In the infinite basis limit, the imaginary parts are expected to vanish. No eigenvalue exhibits the characteristic complex-plane motion expected for a continuum-like state.

Taken together, the real- and complex-scaling analyses support the interpretation that the calculated levels are stable discrete solutions of the present compact tetraquark Hamiltonian. The conclusion does not constitute a definitive classification of these states as bound states or resonances of the physical coupled-channel system. Such a classification would require an explicit treatment of the open two-hadron continuum channels, which is beyond the scope of the present work.

Experimental data on  $Y_c$  and  $Y_b$  state candidates from different decay processes are listed in Table VI and will be

TABLE VI. Masses and widths of  $Y_c$  and  $Y_b$  state candidates from the cited sources, with units in MeV. PDG represents the names of established states in the PDG, while Assignment refers to the names used in this work. The symbol  $\psi$  in the third column represents both  $J/\psi$  and  $\psi(2S)$ .

| PDG               | Assignment         | $\pi^+\pi^-\psi$                                 | $K^+K^-J/\psi$ | $\omega\chi_{cJ}$ | $\pi^+\pi^-h_c$ | $\eta J/\psi$   | $\pi^+\pi^-\Upsilon(nS)$ | $\omega\chi_{bJ}$ |
|-------------------|--------------------|--------------------------------------------------|----------------|-------------------|-----------------|-----------------|--------------------------|-------------------|
| $\psi(4230)$      | Y(4230)            | 4220, 46 [72]<br>4221, 42 [76]<br>4234, 18 [77]  | 4225, 73 [7]   | 4218, 28 [73]     | 4218, 66.0 [74] | 4219, 82.7 [75] |                          |                   |
| $\psi(4360)$      | Y(4300)<br>Y(4340) | 4298, 127 [76]<br>4340, 94 [6]<br>4347, 103 [78] |                |                   |                 |                 |                          |                   |
|                   | Y(4390)            | 4390, 143 [77]                                   |                |                   | 4392, 140 [74]  | 4382, 136 [75]  |                          |                   |
| ...               | Y(4484)            |                                                  | 4484, 111 [7]  |                   |                 |                 |                          |                   |
| ...               | Y(4544)            |                                                  |                | 4544, 116 [8]     |                 |                 |                          |                   |
| $\psi(4660)$      | Y(4660)            | 4651, 155 [77]<br>4652, 68 [78]<br>4669, 104 [6] | 4708, 126 [79] |                   |                 |                 |                          |                   |
| $\Upsilon(10753)$ | Y(10753)           |                                                  |                |                   |                 |                 | 10753, 36 [9]            | Yes [10]          |

TABLE VII. Theoretical mass and branching ratio predictions, along with assignments for  $S = 0$  and  $S = 2$   $Y_c$  tetraquark states. The mass results are given in units of MeV.  $Y(4390)$  and  $Y'(4390)$  denote the states observed in the  $\pi^+\pi^-h_c$  and  $\pi^+\pi^-\psi$  processes, respectively.

| Mass              | 4139     | 4219    | 4284    | 4300    | 4313 | 4350    | 4388    |
|-------------------|----------|---------|---------|---------|------|---------|---------|
| $\omega\chi_{c0}$ | ...      | 1       | ...     | 15      | 1.8  | ...     | 1.7     |
| data              | X(4160)  | Y(4230) | Y(4300) | ...     | ...  | Y(4340) | Y(4390) |
| Mass              | 4408     | 4463    | 4480    | 4498    | 4516 | 4531    | 4567    |
| $\omega\chi_{c0}$ | 0.01     | ...     | 4.8     | 0.3     | 6    | 1.4     | 6       |
| data              | Y'(4390) | ...     | ...     | Y(4484) | ...  | ...     | Y(4544) |
| Mass              | 4570     | 4575    | 4578    | 4587    | 4595 | 4617    | 4625    |
| $\omega\chi_{c0}$ | ...      | 0.3     | ...     | 11      | 3    | ...     | 56      |
| data              | ...      | ...     | ...     | ...     | ...  | Y(4660) | ...     |

TABLE VIII. Theoretical mass, branching ratio predictions, and assignments for  $S = 0$   $Y_b$  tetraquark states. Mass values are in MeV.

| En                | E1       | E2    | E3    | E4    | E5    | E6    | E7    | E8    |
|-------------------|----------|-------|-------|-------|-------|-------|-------|-------|
| mass              | 10504    | 10548 | 10565 | 10679 | 10754 | 10774 | 10791 | 10810 |
| $\omega\chi_{b0}$ | ...      | ...   | ...   | 1     | 1.4   | 0.1   | 0.2   | 0.2   |
| $\omega\chi_{b1}$ | ...      | ...   | ...   | 3     | 4     | 0.3   | 0.7   | 0.5   |
| $\omega\chi_{b2}$ | ...      | ...   | ...   | ...   | 7     | 0.5   | 1.2   | 0.8   |
| data              | Y(10753) |       |       |       |       |       |       |       |

reviewed and discussed separately below. The assignments of  $S = 0$  and  $S = 2$   $Y_c$  tetraquarks, along with their theoretical masses and two-body decay branching ratios to  $\omega\chi_{c0}$  are presented in Table VII. Similarly, the assignments of  $S = 0$   $Y_b$  tetraquarks, along with their theoretical masses and two-body decay branching ratios to  $\omega\chi_{b0}$ ,  $\omega\chi_{b1}$ , and  $\omega\chi_{b2}$ , are provided in Table VIII.

### B. Y(4230)

Over the past two decades, Y(4230) has been established as an exotic state observed and confirmed by several experimental collaborations, and is considered part of the broader family of charmoniumlike states [3]. Y(4260), the original name of Y(4230), was observed by BABAR in the  $\pi^+\pi^-J/\psi$  invariant-mass spectrum [4], and was subsequently confirmed by the CLEO Collaboration [80] and the Belle Collaboration [81] in the same process. With improved statistical precision, BESIII identified an asymmetry in the cross section, which caused the peak position to move to a lower mass [82]. Y(4260) was therefore renamed Y(4230).

Unlike conventional quarkonium, which consists of a quark-antiquark pair, Y(4230) is classified as an exotic hadron due to its unconventional properties, including anomalous decay channels and unexpected production mechanisms. In the previous work, Y(4230) could not

be accommodated in a conventional charmonium picture including  $S - D$  mixing [61].

The exact internal structure of Y(4230) remains a subject of ongoing research, with theoretical models proposing interpretations such as tetraquark states [26,27,32], molecular bound states of charmed mesons [21,22], or hybrid mesons [19,20].

Y(4230) has been observed in several channels, but primarily via electron-positron collisions, where the processes include  $e^+e^- \rightarrow \pi^+\pi^-\psi$  [72,76,77],  $\eta J/\psi$  [75],  $\omega\chi_{c0}$  [83],  $\pi^+\pi^-h_c$  [74], and  $K^+K^-J/\psi$  [7], which are listed in Table VI. Observations in different channels provide clues for identifying whether certain resonances correspond to distinct states.

In the first observation of Y(4230) in the invariant mass of  $\omega\chi_{c0}$  by BESIII in 2014 [83], the resonance parameters were found to be inconsistent with the line shape of Y(4260) observed in  $\pi^+\pi^-J/\psi$  [4] which was the first observed Y state by BABAR in 2005. However, Y(4260) structure in the  $\pi^+\pi^-J/\psi$  channel was later superseded in PDG by an updated BABAR analysis [84], in which the presence of two interfering resonances was not excluded. Similarly, Y(4230) observed in the  $\omega\chi_{c0}$  channel [83] was superseded by an updated BESIII analysis in 2019 [73]. In this updated BESIII study, no solid conclusion could be drawn regarding whether Y(4230) observed in the  $\pi^+\pi^-J/\psi$  and  $\omega\chi_{c0}$  channels corresponds to the same state.

Y(4230) was also observed in the  $K^+K^-J/\psi$  process [7], and the resonance parameters are consistent with those of Y(4230) observed in the  $\pi^+\pi^-J/\psi$  channel, which suggests that the signals seen in  $K^+K^-J/\psi$  and  $\pi^+\pi^-J/\psi$  may correspond to the same state. Later, Y(4230) was observed in the process  $\pi^+\pi^-h_c$  [74], whose resonance parameters are consistent with those of the resonance observed in  $\omega\chi_{c0}$  [83], which supports the possibility that the structures seen in  $\omega\chi_{c0}$  and  $\pi^+\pi^-h_c$  correspond to the same state.

The number of states in this mass region remains to be further determined experimentally [3]. This study predicts only one state in the mass region. As shown in Table VII, we assign Y(4230), observed in the  $\pi^+\pi^-J/\psi$ ,  $K^+K^-J/\psi$ ,  $\omega\chi_{c0}$ , and  $\pi^+\pi^-h_c$  channels, to the E2 tetraquark state, with a corresponding theoretical mass of 4219 MeV.

### C. Y(4360)

Y(4360) was first observed in the process  $e^+e^- \rightarrow \gamma\pi^+\pi^-\psi(2S)$  [5]. With a mass of approximately 4.36 GeV, Y(4360) resides in the energy region above the open-charm threshold, similar to other Y-states. Y(4360) is of particular interest due to its unusual decay patterns, including prominent decays into  $\psi(2S)$  rather than into lower charmonium states like  $J/\psi$ , which contrasts with expectations from conventional quarkonium models. This anomalous behavior suggests that Y(4360) could be an exotic hadron.

As a PDG established state, in PDG data listed in Table VI, the reported masses of  $\psi(4360)$  range from

4.3 to 4.4 GeV, which may indicate three distinct states within this broad mass range. The first, second, and third states are around 4.30, 4.34, and 4.39 GeV, and are denoted here as Y(4300), Y(4340), and Y(4390), respectively.

Y(4300) and Y(4340) observed in the  $\pi^+\pi^-J/\psi$  and  $\pi^+\pi^-\psi(2S)$  processes are assigned as  $S = 2$  tetraquark states, with corresponding theoretical masses of 4284 and 4350 MeV, respectively.

Y(4390) was observed in  $\pi^+\pi^-J/\psi$  [77] and  $\pi^+\pi^-h_c$  [74] channels. In the  $\pi^+\pi^-h_c$  process [74], Y(4230) was also observed. The resonance parameters of Y(4230) in the  $\pi^+\pi^-h_c$  process were consistent with those of Y(4230) observed in the  $\omega\chi_{c0}$  channel [83], which may suggest that Y(4390) could also be observed in the  $\omega\chi_{c0}$  channel.

The experimentally observed Y(4390) in the  $\pi^+\pi^-h_c$  channel is assigned to the state with a theoretical mass of 4388 MeV in Table VII, considering the large decay ratio to  $\omega\chi_{c0}$ . Y(4390) could potentially be observed in the  $\omega\chi_{c0}$  channel, which may motivate future experimental searches.

In Ref. [74], the line shapes of Y(4390) observed in the  $\pi^+\pi^-h_c$  process [74] are inconsistent with those in  $\pi^+\pi^-J/\psi$  and  $\pi^+\pi^-\psi(2S)$  by BABAR [4,6,84,85] and by Belle [5,78,81,86]. One may support the existence of two Y states around 4390 MeV, which are tentatively named Y(4390) for the one from  $\pi^+\pi^-h_c$ , and Y'(4390) for the one from  $\pi^+\pi^-J/\psi$ .

In Table VII, the experimentally observed Y'(4390) in the  $\pi^+\pi^-\psi$  channel is assigned to the state with a theoretical mass of 4408 MeV, due to the small decay ratio to  $\omega\chi_{c0}$ .

#### D. Y(4660)

Y(4660) was first discovered in the initial state radiation process  $e^+e^- \rightarrow \gamma_{\text{ISR}}\pi^+\pi^-\psi(2S)$  [6]. With a mass of approximately 4.66 GeV, Y(4660) lies well above the open-charm threshold, adding complexity to the spectrum of charmoniumlike states. Y(4660) exhibits intriguing decay patterns, most notably decaying into  $\psi(2S)$  rather than  $J/\psi$ , a feature shared with other exotic states like Y(4360). This has led to speculation that Y(4660) could be a tetraquark [11,32,87], a  $S - D$  wave mixture state [88], or a hybrid meson [19,20].

As shown in Table VII, the experimentally observed Y(4660) has been reported mainly in the  $\pi^+\pi^-\psi$  channel and may be naturally grouped with Y(4300), Y(4340), and Y'(4390), which are seen in the same process. Y(4660) may be assigned to a tetraquark state with a small decay ratio to  $\omega\chi_{cJ}$  and a corresponding theoretical mass of 4617 MeV.

#### E. Other $Y_c$ states

The X(4160) was first observed by the Belle Collaboration in double-charmonium production,  $e^+e^- \rightarrow J/\psi X(4160)$  with  $X(4160) \rightarrow D^+D^-$  [89]. Later, a related state was observed by the LHCb Collaboration in the decay  $B^+ \rightarrow J/\psi\phi K^+$  [90], reconstructed from the invariant mass

spectrum of  $J/\psi\phi$ . The two states observed around 4.15 GeV are grouped together as a single entry in the PDG [3], according to their similar masses and decay widths. Whether these observations correspond to the same state still requires further experimental confirmation. The nature of the X(4160) therefore remains uncertain and requires further theoretical and experimental investigation.

The production rate of the X(4160) in double-charmonium processes, along with its decay patterns into the  $D^+D^-$  and  $J/\psi\phi$  channels, does not align well with expectations for conventional  $c\bar{c}$  states [3,91]. However, some studies have still considered interpretations of the X(4160) as a higher charmonium state [92]. The X(4160) is more likely interpreted as an exotic hadron, with several theoretical scenarios proposed. These include a compact tetraquark configuration, either hidden-charm  $c\bar{c}q\bar{q}$  or  $c\bar{c}s\bar{s}$ , supported by QCD sum rules [93], as well as a meson-meson molecular state, in particular a  $D_s^*\bar{D}_s^*$  bound state [94].

Despite these scenarios, the quantum numbers of the X(4160) remain unclear. The Belle Collaboration did not measure its quantum numbers [89], while LHCb analyses favor an assignment of  $J^{PC} = 2^{-+}$  [90]. Given its measured mass near 4.16 GeV, the X(4160) may also be interpreted as a tetraquark candidate, based on mass consistency with theoretical predictions for hidden-charm exotics in this region [93]. However, further experimental confirmation is still required. The X(4160) is tentatively assigned as a tetraquark state based on mass agreement, with a corresponding theoretical mass of 4139 MeV.

Recently, Y(4484) was observed in the  $K^+K^-J/\psi$  process by BESIII for the first time with a statistical significance greater than  $8\sigma$  [7]. Together with Y(4230) observed in the same process, the line shape is consistent with that of the established Y(4230) in the  $\pi^+\pi^-J/\psi$  channel. In Table VII, Y(4484) is grouped with Y'(4390) from the  $\pi^+\pi^-J/\psi$  channel and is assigned as a tetraquark state based on mass agreement, with a corresponding theoretical mass of 4498 MeV.

The newly observed Y(4544) was seen in the  $\omega\chi_{c1}$  channel by BESIII [8], and the structure was observed for the first time with a significance of  $5.8\sigma$ . Y(4544) has a significantly higher mass than Y(4484). One may support the view that Y(4484) and Y(4544) are different states rather than combining them into a single Y(4500). In the assignment, Y(4544) is grouped with Y(4230) from  $\omega\chi_{c0}$  channel and assigned to be a tetraquark state considering the mass agreement and the large ratio to  $\omega\chi_{cJ}$  with a corresponding theoretical mass of 4567 MeV.

The theoretical branching ratios for  $1^{--}$  Y tetraquark states decaying into  $\eta J/\psi$  are close to zero. Thus, the Y(4230) state and the Y(4360) state observed in  $\eta J/\psi$  [75] cannot be accommodated in the current tetraquark picture, and these two states could correspond to other structures, for example, hadronic molecule states [21–25], or hybrid mesons [16–20].

### F. Y(10753)

The Y(10753) bottomoniumlike state was first observed by Belle in the  $e^+e^- \rightarrow \pi^+\pi^-\Upsilon(nS)$  process [9]. Recently, the Belle II Collaboration reported another observation that strongly supports the existence of the radiative transition process  $\Upsilon(10753) \rightarrow \omega\chi_{bJ}(1P)$  [10]. The energy dependence of the Born cross sections for Y(10753) in the  $\omega\chi_{bJ}(1P)$  channel [10] was found to be consistent with the shape of the Y(10753) state in the  $\pi^+\pi^-\Upsilon(nS)$  channel [9]. Belle II concluded that the internal structure of Y(10753) may differ from that of  $\Upsilon(10860)$ , where the latter is well understood as a predominantly conventional bottomonium state.

In previous work, Y(10753) could not be accommodated within a conventional bottomonium picture, including  $S-D$  mixing [61]. After the experimental discovery, a P-wave hidden-bottom tetraquark interpretation was proposed in the first set of theoretical pictures [95]. For a tetraquark mixing interpretation, one may refer to Ref. [96].

As shown in Table VIII, the experimentally observed Y(10753) in  $\pi^+\pi^-\Upsilon(nS)$  and  $\omega\chi_{bJ}(1P)$  channels is assigned to the E5 tetraquark state, considering the mass agreement and the large ratio to  $\omega\chi_{bJ}$ .

## IV. SUMMARY

The masses of P-wave charmoniumlike tetraquark states are calculated using a constituent quark model that incorporates a Cornell-like potential and the Breit-Fermi interaction. All model parameters were predetermined by reproducing the mass spectra of low-lying S- and P-wave light, charmed, and charmonium mesons. The theoretical results for P-wave tetraquarks are compared with selected exotic states, known as Y states, and a tentative assignment is proposed.

The PDG-established state around 4.23 GeV may be treated as a single state at this stage. In the assignment, Y(4230), observed in the  $\pi^+\pi^-J/\psi$ ,  $K^+K^-J/\psi$ ,  $\omega\chi_{c0}$  and  $\pi^+\pi^-h_c$  channels, is assigned as an  $S=0$   $1^{--}$  tetraquark state.

The PDG established state around 4.36 GeV may be split into four distinct states. In the assignment, Y(4300) and Y(4340) observed only in the  $\pi^+\pi^-J/\psi$  process are assigned as  $1^{--}$  tetraquark states, respectively. Y(4390), observed in the  $\pi^+\pi^-h_c$  channel, is assigned as a  $1^{--}$  tetraquark state with a large decay ratio to  $\omega\chi_{cJ}$ , while Y'(4390), observed in the  $\pi^+\pi^-J/\psi$  channel, is assigned as a tetraquark state with a small decay ratio to  $\omega\chi_{cJ}$ . Y(4390) could potentially be observed in the  $\omega\chi_{c0}$  channel, which may motivate future experimental searches.

The PDG established state Y(4660), observed in the  $\pi^+\pi^-J/\psi$  channel and grouped with Y'(4390) from the same process, is assigned as a  $1^{--}$  tetraquark state.

Y(4484), observed in the  $K^+K^-J/\psi$  channel by BESIII, and Y(4544), newly observed in the  $\omega\chi_{c1}$  channel by BESIII, are tentatively assigned as  $1^{--}$  tetraquark states because their masses agree with our predictions.

Y(10753), observed in  $\pi^+\pi^-\Upsilon(nS)$  and  $\omega\chi_{bJ}(1P)$  channels by Belle, is assigned as the E5  $S=0$   $1^{--}$  bottomoniumlike tetraquark state.

## ACKNOWLEDGMENTS

This research has received funding support from the National Science Research and Innovation Fund (NSRF) via the Program Management Unit for Human Resources & Institutional Development, Research and Innovation [Grant No. B50G670107]. A. L. and Y. Y. acknowledge the support of (i) Suranaree University of Technology, (ii) Thailand Science Research and Innovation (TSRI), and (iii) NSRF, Project No. 195242. Z. Z. is additionally supported by SUT (Full-Time 66/15/2024).

## DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

## APPENDIX A: TETRAQUARK SPATIAL WAVE FUNCTION

The total spatial wave function of the tetraquark, coupling the  $\chi_1$ ,  $\chi_2$ , and  $\chi_3$  HO wave functions, takes the general form

$$\psi_{NL} = \sum_{n_{\chi_i}, l_{\chi_i}} A(n_{\chi_1}, n_{\chi_2}, n_{\chi_3}, l_{\chi_1}, l_{\chi_2}, l_{\chi_3}) \times \psi_{n_{\chi_1} l_{\chi_1}}(\vec{\chi}_1) \otimes \psi_{n_{\chi_2} l_{\chi_2}}(\vec{\chi}_2) \otimes \psi_{n_{\chi_3} l_{\chi_3}}(\vec{\chi}_3), \quad (\text{A1})$$

where  $\psi_{n_{\chi_i} l_{\chi_i}}$  are HO wave functions. The sum  $n_{\chi_i}, l_{\chi_i}$  is over  $n_{\chi_1}, n_{\chi_2}, n_{\chi_3}, l_{\chi_1}, l_{\chi_2}, l_{\chi_3}$ .  $N$  and  $L$  are, respectively, the total principal quantum number and total angular momentum, and  $L = l_{\chi_1} + l_{\chi_2} + l_{\chi_3} = 1$  while  $l_{\chi_1}, l_{\chi_2}, l_{\chi_3} \leq 1$  for P-wave tetraquark and  $N = (2n_{\chi_1} + l_{\chi_1}) + (2n_{\chi_2} + l_{\chi_2}) + (2n_{\chi_3} + l_{\chi_3})$ .

The complete bases of the tetraquarks are listed in Table IX up to  $N = 3$  for reference.

TABLE IX. The complete bases of tetraquark with quantum number,  $N = 2n + L$  and  $L = 1$ .

|             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |
|-------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\psi_{11}$ | $\psi_{0,0}(\vec{\chi}_1)\psi_{0,0}(\vec{\chi}_2)\psi_{0,1}(\vec{\chi}_3), \psi_{0,0}(\vec{\chi}_1)\psi_{0,1}(\vec{\chi}_2)\psi_{0,0}(\vec{\chi}_3),$<br>$\psi_{0,1}(\vec{\chi}_1)\psi_{0,0}(\vec{\chi}_2)\psi_{0,0}(\vec{\chi}_3)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
| $\psi_{31}$ | $\psi_{1,0}(\vec{\chi}_1)\psi_{0,0}(\vec{\chi}_2)\psi_{0,1}(\vec{\chi}_3), \psi_{1,0}(\vec{\chi}_1)\psi_{0,1}(\vec{\chi}_2)\psi_{0,0}(\vec{\chi}_3),$<br>$\psi_{1,1}(\vec{\chi}_1)\psi_{0,0}(\vec{\chi}_2)\psi_{0,0}(\vec{\chi}_3), \psi_{0,0}(\vec{\chi}_1)\psi_{1,0}(\vec{\chi}_2)\psi_{0,1}(\vec{\chi}_3),$<br>$\psi_{0,0}(\vec{\chi}_1)\psi_{1,1}(\vec{\chi}_2)\psi_{0,0}(\vec{\chi}_3), \psi_{0,1}(\vec{\chi}_1)\psi_{1,0}(\vec{\chi}_2)\psi_{0,0}(\vec{\chi}_3),$<br>$\psi_{0,0}(\vec{\chi}_1)\psi_{0,0}(\vec{\chi}_2)\psi_{1,1}(\vec{\chi}_3), \psi_{0,0}(\vec{\chi}_1)\psi_{0,1}(\vec{\chi}_2)\psi_{1,0}(\vec{\chi}_3),$<br>$\psi_{0,1}(\vec{\chi}_1)\psi_{0,0}(\vec{\chi}_2)\psi_{1,0}(\vec{\chi}_3)$ |

TABLE X. Expectation values of the color-spin operator for the  $S = 0$  tetraquark state.  $\langle \psi_c^{k1} \psi_s^{h1} | \vec{\lambda}_i \vec{\lambda}_j \vec{\sigma}_i \vec{\sigma}_j | \psi_c^{k2} \psi_s^{h2} \rangle$ .

| $\vec{\lambda}_i \vec{\lambda}_j \vec{\sigma}_i \vec{\sigma}_j$ | $ \psi_{\bar{3}\bar{3}}^c \psi_{(0\bar{0})}^{S=0}\rangle$                                  | $ \psi_{\bar{3}\bar{3}}^c \psi_{(1\bar{0})}^{S=0}\rangle$                                  | $ \psi_{\bar{6}\bar{6}}^c \psi_{(0\bar{0})}^{S=0}\rangle$                                      | $ \psi_{\bar{6}\bar{6}}^c \psi_{(1\bar{0})}^{S=0}\rangle$                                      |
|-----------------------------------------------------------------|--------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|
| $ \psi_{\bar{3}\bar{3}}^c \psi_{(0\bar{0})}^{S=0}\rangle$       | (8, 0, 0, 0, 0, 8)                                                                         | $(0, \frac{4}{\sqrt{3}}, -\frac{4}{\sqrt{3}}, -\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}, 0)$ | (0, 0, 0, 0, 0, 0)                                                                             | $(0, 2\sqrt{6}, 2\sqrt{6}, 2\sqrt{6}, 2\sqrt{6}, 0)$                                           |
| $ \psi_{\bar{3}\bar{3}}^c \psi_{(1\bar{0})}^{S=0}\rangle$       | $(0, \frac{4}{\sqrt{3}}, -\frac{4}{\sqrt{3}}, -\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}, 0)$ | $(-\frac{8}{3}, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}, -\frac{8}{3})$                      | $(0, 2\sqrt{6}, 2\sqrt{6}, 2\sqrt{6}, 2\sqrt{6}, 0)$                                           | $(0, 4\sqrt{2}, -4\sqrt{2}, -4\sqrt{2}, 4\sqrt{2}, 0)$                                         |
| $ \psi_{\bar{6}\bar{6}}^c \psi_{(0\bar{0})}^{S=0}\rangle$       | (0, 0, 0, 0, 0, 0)                                                                         | $(0, 2\sqrt{6}, 2\sqrt{6}, 2\sqrt{6}, 2\sqrt{6}, 0)$                                       | $(-4, 0, 0, 0, 0, -4)$                                                                         | $(0, \frac{10}{\sqrt{3}}, -\frac{10}{\sqrt{3}}, -\frac{10}{\sqrt{3}}, \frac{10}{\sqrt{3}}, 0)$ |
| $ \psi_{\bar{6}\bar{6}}^c \psi_{(1\bar{0})}^{S=0}\rangle$       | $(0, 2\sqrt{6}, 2\sqrt{6}, 2\sqrt{6}, 2\sqrt{6}, 0)$                                       | $(0, 4\sqrt{2}, -4\sqrt{2}, -4\sqrt{2}, 4\sqrt{2}, 0)$                                     | $(0, \frac{10}{\sqrt{3}}, -\frac{10}{\sqrt{3}}, -\frac{10}{\sqrt{3}}, \frac{10}{\sqrt{3}}, 0)$ | $(\frac{4}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{4}{3})$           |

TABLE XI. Expectation values of the color-spin operator for the  $S = 2$  tetraquark state.  $\langle \psi_c^{k1} \psi_s^{h1} | \vec{\lambda}_i \vec{\lambda}_j \vec{\sigma}_i \vec{\sigma}_j | \psi_c^{k2} \psi_s^{h2} \rangle$ .

| $\vec{\lambda}_i \vec{\lambda}_j \vec{\sigma}_i \vec{\sigma}_j$ | $ \psi_{\bar{3}\bar{3}}^c \psi_{(1\bar{0})}^{S=2}\rangle$             | $ \psi_{\bar{6}\bar{6}}^c \psi_{(1\bar{0})}^{S=2}\rangle$                                |
|-----------------------------------------------------------------|-----------------------------------------------------------------------|------------------------------------------------------------------------------------------|
| $ \psi_{\bar{3}\bar{3}}^c \psi_{(1\bar{0})}^{S=2}\rangle$       | $(-\frac{8}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, -\frac{8}{3})$ | $(0, 4\sqrt{2}, -4\sqrt{2}, -4\sqrt{2}, 4\sqrt{2}, 0)$                                   |
| $ \psi_{\bar{6}\bar{6}}^c \psi_{(1\bar{0})}^{S=2}\rangle$       | $(0, 4\sqrt{2}, -4\sqrt{2}, -4\sqrt{2}, 4\sqrt{2}, 0)$                | $(\frac{4}{3}, -\frac{10}{3}, -\frac{10}{3}, -\frac{10}{3}, -\frac{10}{3}, \frac{4}{3})$ |

## APPENDIX B: CROSS TERMS BETWEEN DIFFERENT CONFIGURATIONS FROM CENTRAL POTENTIAL

The color operator  $\vec{\lambda}_i \cdot \vec{\lambda}_j$  in the Hamiltonian in Eq. (4), along with the mass-dependent coefficient in the Cornell-like potential, mixes different color configurations, leading to nonzero cross terms for charmoniumlike and bottomoniumlike tetraquarks.

Similarly, the color-spin operator  $\vec{\lambda}_i \cdot \vec{\lambda}_j \vec{\sigma}_i \cdot \vec{\sigma}_j$ , along with the mass-dependent factor in the spin-spin interaction,

$$\vec{\lambda}_i \cdot \vec{\lambda}_j V_{ss} = \frac{2B_{ij}\sigma_{ij}^3}{3m_i m_j \sqrt{\pi}} e^{-\sigma_{ij}^2 r_{ij}^2} \vec{\lambda}_i \cdot \vec{\lambda}_j \vec{\sigma}_i \cdot \vec{\sigma}_j, \quad (\text{B1})$$

mixes different color-spin configurations, also leading to

nonzero cross terms for charmoniumlike and bottomoniumlike tetraquarks.

For reference, the expectation values of  $O_{ij} = \vec{\lambda}_i \cdot \vec{\lambda}_j \vec{\sigma}_i \cdot \vec{\sigma}_j$  for  $S = 0$  and  $S = 2$  states are listed in Tables X and XI, respectively, with all components given in the format  $(O_{12}, O_{13}, O_{23}, O_{14}, O_{24}, O_{34})$ . The mass-dependent factor in Eq. (B1) induces nonzero cross terms between different color-spin configurations.

In conclusion, the  $S = 0$  charmoniumlike and bottomoniumlike tetraquarks are linear combinations of  $\psi_{\bar{3}\bar{3}}^c \psi_{(0\bar{0})}^{S=0}$ ,  $\psi_{\bar{3}\bar{3}}^c \psi_{(1\bar{0})}^{S=0}$ ,  $\psi_{\bar{6}\bar{6}}^c \psi_{(0\bar{0})}^{S=0}$ , and  $\psi_{\bar{6}\bar{6}}^c \psi_{(1\bar{0})}^{S=0}$ . Similarly, the  $S = 2$  charmoniumlike and bottomoniumlike tetraquarks are linear combinations of  $\psi_{\bar{3}\bar{3}}^c \psi_{(1\bar{0})}^{S=2}$  and  $\psi_{\bar{6}\bar{6}}^c \psi_{(1\bar{0})}^{S=2}$ . There is no configuration mixing between  $S = 0$  and  $S = 2$  states, as no cross terms are found.

[1] Y.-R. Liu, H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, *Prog. Part. Nucl. Phys.* **107**, 237 (2019).  
[2] H.-X. Chen, W. Chen, X. Liu, Y.-R. Liu, and S.-L. Zhu, *Rep. Prog. Phys.* **86**, 026201 (2023).  
[3] S. Navas *et al.* (Particle Data Group Collaboration), *Phys. Rev. D* **110**, 030001 (2024).  
[4] B. Aubert *et al.* (BABAR Collaboration), *Phys. Rev. Lett.* **95**, 142001 (2005).  
[5] X. L. Wang *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **99**, 142002 (2007).  
[6] J. P. Lees *et al.* (BABAR Collaboration), *Phys. Rev. D* **89**, 111103 (2014).  
[7] M. Ablikim *et al.* (BESIII Collaboration), *Chin. Phys. C* **46**, 111002 (2022).

[8] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. Lett.* **132**, 161901 (2024).  
[9] R. Mizuk *et al.* (Belle Collaboration), *J. High Energy Phys.* **10** (2019) 220.  
[10] I. Adachi *et al.* (Belle-II Collaboration), *Phys. Rev. Lett.* **130**, 091902 (2023).  
[11] Y. Lu, M. N. Anwar, and B.-S. Zou, *Phys. Rev. D* **96**, 114022 (2017).  
[12] B.-Q. Li and K.-T. Chao, *Phys. Rev. D* **79**, 094004 (2009).  
[13] M. Shah, A. Parmar, and P. C. Vinodkumar, *Phys. Rev. D* **86**, 034015 (2012).  
[14] J.-Z. Wang, D.-Y. Chen, X. Liu, and T. Matsuki, *Phys. Rev. D* **99**, 114003 (2019).  
[15] J.-Z. Wang and X. Liu, *Phys. Rev. D* **107**, 054016 (2023).

- [16] L. Liu, G. Moir, M. Peardon, S. M. Ryan, C. E. Thomas, P. Vilaseca, J. J. Dudek, R. G. Edwards, B. Joo, and D. G. Richards (Hadron Spectrum Collaboration), *J. High Energy Phys.* **07** (2012) 126.
- [17] X.-Q. Luo and Y. Liu, *Phys. Rev. D* **74**, 034502 (2006).
- [18] S.-L. Zhu, *Phys. Lett. B* **625**, 212 (2005).
- [19] R. Oncala and J. Soto, *Phys. Rev. D* **96**, 014004 (2017).
- [20] M. Berwein, N. Brambilla, J. Tarrús Castellà, and A. Vairo, *Phys. Rev. D* **92**, 114019 (2015).
- [21] T.-W. Chiu and T.-H. Hsieh (TWQCD Collaboration), *Phys. Rev. D* **73**, 094510 (2006).
- [22] W. Qin, S.-R. Xue, and Q. Zhao, *Phys. Rev. D* **94**, 054035 (2016).
- [23] G.-J. Ding, *Phys. Rev. D* **79**, 014001 (2009).
- [24] F. Close and C. Downum, *Phys. Rev. Lett.* **102**, 242003 (2009).
- [25] F. Close, C. Downum, and C. E. Thomas, *Phys. Rev. D* **81**, 074033 (2010).
- [26] L. Maiani, V. Riquer, F. Piccinini, and A. D. Polosa, *Phys. Rev. D* **72**, 031502 (2005).
- [27] A. Ali, L. Maiani, A. V. Borisov, I. Ahmed, M. Jamil Aslam, A. Y. Parkhomenko, A. D. Polosa, and A. Rehman, *Eur. Phys. J. C* **78**, 29 (2018).
- [28] N. V. Drenska, R. Faccini, and A. D. Polosa, *Phys. Rev. D* **79**, 077502 (2009).
- [29] D. Ebert, R. N. Faustov, and V. O. Galkin, *Eur. Phys. J. C* **58**, 399 (2008).
- [30] Z.-G. Wang, *Nucl. Phys.* **B1007**, 116661 (2024).
- [31] Z.-G. Wang, *Nucl. Phys.* **B1005**, 116580 (2024).
- [32] Z.-G. Wang, *Eur. Phys. J. C* **79**, 29 (2019).
- [33] M. Gell-Mann, *Phys. Lett.* **8**, 214 (1964).
- [34] R. L. Jaffe, *Phys. Rev. D* **15**, 267 (1977).
- [35] R. L. Jaffe, *Phys. Rev. D* **15**, 281 (1977).
- [36] X.-W. Bai, Y. Huang, and W.-L. Sang, *Phys. Rev. D* **111**, 054006 (2025).
- [37] W.-L. Wu and S.-L. Zhu, *Phys. Rev. D* **111**, 034044 (2025).
- [38] R. Mistry and A. Majethiya, *Chin. J. Phys. (Taipei)* **91**, 932 (2024).
- [39] W.-Y. Liu and H.-X. Chen, *Universe* **11**, 36 (2025).
- [40] X.-W. Bai, F. Feng, C.-M. Gan, Y. Huang, W.-L. Sang, and H.-F. Zhang, *J. High Energy Phys.* **09** (2024) 002.
- [41] Y. Wu, X. Liu, J. Ping, H. Huang, and Y. Tan, *Eur. Phys. J. C* **85**, 147 (2025).
- [42] Z.-Z. Chen, X.-L. Chen, P.-F. Yang, and W. Chen, *Phys. Rev. D* **109**, 094011 (2024).
- [43] W.-L. Wu, Y.-K. Chen, L. Meng, and S.-L. Zhu, *Phys. Rev. D* **109**, 054034 (2024).
- [44] H.-F. Zhang, X.-M. Mo, and Y.-P. Yan, *Phys. Rev. D* **110**, 096021 (2024).
- [45] G.-L. Yu, Z.-Y. Li, Z.-G. Wang, J. Lu, and M. Yan, *Eur. Phys. J. C* **83**, 416 (2023).
- [46] J. Zhang, J.-B. Wang, G. Li, C.-S. An, C.-R. Deng, and J.-J. Xie, *Eur. Phys. J. C* **82**, 1126 (2022).
- [47] H. Mutuk, *Phys. Lett. B* **834**, 137404 (2022).
- [48] G.-J. Wang, Q. Meng, and M. Oka, *Phys. Rev. D* **106**, 096005 (2022).
- [49] H.-X. Chen, Y.-X. Yan, and W. Chen, *Phys. Rev. D* **106**, 094019 (2022).
- [50] J. Hu, B.-R. He, and J.-L. Ping, *Eur. Phys. J. C* **83**, 559 (2023).
- [51] F.-X. Liu, M.-S. Liu, X.-H. Zhong, and Q. Zhao, *Phys. Rev. D* **104**, 116029 (2021).
- [52] Q.-N. Wang, Z.-Y. Yang, and W. Chen, *Phys. Rev. D* **104**, 114037 (2021).
- [53] G.-J. Wang, L. Meng, M. Oka, and S.-L. Zhu, *Phys. Rev. D* **104**, 036016 (2021).
- [54] Z. Zhao, K. Xu, A. Kaewsnod, X. Liu, A. Limphirat, and Y. Yan, *Phys. Rev. D* **105**, 036001 (2022).
- [55] S. S. Agaev, K. Azizi, and H. Sundu, *Phys. Lett. B* **789**, 405 (2019).
- [56] D. Ebert, R. N. Faustov, and V. O. Galkin, *Eur. Phys. J. C* **60**, 273 (2009).
- [57] Q.-N. Wang, D.-K. Lian, and W. Chen, *Phys. Rev. D* **110**, 034022 (2024).
- [58] Q. Xin and Z.-G. Wang, *Chin. Phys. C* **48**, 033104 (2024).
- [59] E. Santopinto and G. Galata, *Phys. Rev. C* **75**, 045206 (2007).
- [60] C. Lodha and A. K. Rai, *Few Body Syst.* **65**, 99 (2024).
- [61] Z. Zhao, K. Xu, A. Limphirat, W. Sreethawong, N. Tagsinsit, A. Kaewsnod, X. Liu, K. Khosonthongkee, S. Cheedket, and Y. Yan, *Phys. Rev. D* **109**, 016012 (2024).
- [62] Z. Zhao, K. Xu, A. Kaewsnod, X. Liu, A. Limphirat, and Y. Yan, *Phys. Rev. D* **103**, 116027 (2021).
- [63] G. Yang, J. Ping, and J. Segovia, *Phys. Rev. D* **102**, 054023 (2020).
- [64] S. Godfrey and N. Isgur, *Phys. Rev. D* **32**, 189 (1985).
- [65] F. Schoberl, *Nuovo Cimento A* **94**, 79 (1986).
- [66] T. Barnes, S. Godfrey, and E. S. Swanson, *Phys. Rev. D* **72**, 054026 (2005).
- [67] J.-M. Richard, A. Valcarce, and J. Vijande, *Few Body Syst.* **65**, 71 (2024).
- [68] E. Hiyama, A. Hosaka, M. Oka, and J.-M. Richard, *Phys. Rev. C* **98**, 045208 (2018).
- [69] X. Hu and J. Ping, *Phys. Rev. D* **106**, 054028 (2022).
- [70] Y. Tan, Y. Wu, H. Huang, and J. Ping, *Universe* **10**, 17 (2024).
- [71] J. Simons, *J. Chem. Phys.* **75**, 2465 (1981).
- [72] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. D* **102**, 012009 (2020).
- [73] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. D* **99**, 091103 (2019).
- [74] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. Lett.* **118**, 092002 (2017).
- [75] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. D* **102**, 031101 (2020).
- [76] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. D* **106**, 072001 (2022).
- [77] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. D* **104**, 052012 (2021).
- [78] X. L. Wang *et al.* (Belle Collaboration), *Phys. Rev. D* **91**, 112007 (2015).
- [79] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. Lett.* **131**, 211902 (2023).
- [80] Q. He *et al.* (CLEO Collaboration), *Phys. Rev. D* **74**, 091104 (2006).
- [81] C. Z. Yuan *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **99**, 182004 (2007).
- [82] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. Lett.* **118**, 092001 (2017).
- [83] M. Ablikim *et al.* (BESIII Collaboration), *Phys. Rev. Lett.* **114**, 092003 (2015).

- [84] J. P. Lees *et al.* (BABAR Collaboration), *Phys. Rev. D* **86**, 051102 (2012).
- [85] B. Aubert *et al.* (BABAR Collaboration), *Phys. Rev. Lett.* **98**, 212001 (2007).
- [86] Z. Q. Liu *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **110**, 252002 (2013); **111**, 019901(E) (2013).
- [87] R. F. Lebed and A. D. Polosa, *Phys. Rev. D* **93**, 094024 (2016).
- [88] T. Bhavsar, M. Shah, and P. C. Vinodkumar, *Eur. Phys. J. C* **78**, 227 (2018).
- [89] P. Pakhlov *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **100**, 202001 (2008).
- [90] R. Aaij *et al.* (LHCb Collaboration), *Phys. Rev. Lett.* **127**, 082001 (2021).
- [91] K.-T. Chao, *Phys. Lett. B* **661**, 348 (2008).
- [92] Y.-c. Yang, Z. Xia, and J. Ping, *Phys. Rev. D* **81**, 094003 (2010).
- [93] W. Chen, H.-X. Chen, X. Liu, T. G. Steele, and S.-L. Zhu, *Phys. Rev. D* **96**, 114017 (2017).
- [94] E. Wang, J.-J. Xie, L.-S. Geng, and E. Oset, *Phys. Rev. D* **97**, 014017 (2018).
- [95] Z.-G. Wang, *Chin. Phys. C* **43**, 123102 (2019).
- [96] A. Ali, L. Maiani, A. Y. Parkhomenko, and W. Wang, *Phys. Lett. B* **802**, 135217 (2020).