

Island of an acoustic black hole in Schwarzschild spacetime

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(Received 15 December 2025; accepted 20 February 2026; published 16 March 2026)

We study an analog information paradox in acoustic black holes which emerge from the superfluid surrounding a Schwarzschild black hole. The resulting acoustic black hole contains both acoustic horizons and optical horizons, with analog Hawking radiation, i.e., phonons, emitted from the outer acoustic horizon. By using the island formula, we calculate the entanglement entropy of analog Hawking radiation of the acoustic black hole in both nonextremal and extremal cases. In the nonextremal case, the entanglement entropy of phonons follows the Page curve due to the emergence of islands, and it is approximately proportional to the area of the acoustic horizon at late times. In the extremal case, the entanglement entropy of phonons diverges, leading to an ill-defined Page time. Our study verifies the unitarity of the analog gravity system and provides further insight into the connection between the entanglement entropy and the causal structure of spacetime.

DOI: [10.1103/k1gn-zbz2](https://doi.org/10.1103/k1gn-zbz2)

I. INTRODUCTION

The black hole information paradox, which argues that the evaporation of black holes may conflict with the unitarity in quantum mechanics [1–3], has been a central problem in gravitational physics for decades. The original calculations of Hawking suggest that black holes will emit thermal radiation with monotonically increasing entanglement entropy, which implies that information is lost as the black hole evaporates. However, the unitarity of quantum mechanics demands that the black hole and its radiation should remain in a pure state even at the end of evaporation. To avoid the information paradox in the semiclassical gravity, Page argued that during a unitary black hole evaporation process, the entanglement entropy of the radiation should increase first, then saturate, and eventually decrease, which satisfies the so-called Page curve [4]. Then, the key point to solve this paradox is to correctly calculate the entanglement entropy of the radiation which obeys the Page curve.

Recently, important progress has been made in solving the black hole information paradox at the semiclassical level, based on the holographic calculation of entanglement entropy, i.e., the Ryu-Takayanagi (RT) formula [5,6], and its quantum extension, i.e., the quantum extremal surface

(QES) formula [7]. The new approach proposed is called the island formula, which was shown to successfully reproduce the Page curve by calculating the entanglement entropy of Hawking radiation during black hole evaporation [8–11]. The formula was later confirmed by using the replica method for the gravitational path integral, where the emergence of new saddles, known as replica wormholes, leads to islands, which are regions located inside the black hole that belong to part of the entanglement wedge of the external radiation region [12–16]. The island formula explicitly states the transition in entanglement entropy that leads to the Page curve. Explicitly, in the initial no-island phase, the entropy grows monotonically with time. After a critical point (the Page time), a new island phase dominates and makes the entropy stable. This transition ensures that the total entanglement entropy of the system follows the Page curve. The island formula can be explicitly expressed as

$$S_R = \min \left\{ \text{ext} \left[\frac{\text{Area}(\partial I)}{4\tilde{G}_N} + S_{\text{matter}}^{\text{finite}}(I \cup R) \right] \right\}, \quad (1)$$

where R is the radiation region and ∂I is the boundary of the island, and the arealike divergences of the entanglement entropy of matter fields that depend on the short-distance cutoff have been absorbed into the renormalized Newton's constant $\tilde{G}_N$ [17–19]. So far, the island formula has been successfully applied to resolve the information paradox in two-dimensional gravity, and it has subsequently been generalized to other lower- and higher-dimensional black holes; see, e.g., Refs. [20–49].

In the present paper, we will instead investigate the information paradox in analog gravity. Our motivations are

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as follows. First, the typical feature of analog gravity is the emergence of curved geometry, especially the horizon, which can produce analog Hawking radiation composed of quantized phonons [50]. Therefore, it was originally proposed to mimic curved geometries and quantum effects in curved spacetime. For example, Unruh first proposed the acoustic black hole from nonrelativistic fluids in flat spacetime to mimic the near horizon geometry and Hawking radiation of black holes [51]. Since then, analog gravity has been successfully realized in various condensed matter systems such as the Bose-Einstein condensation [52], the optical media [53], and many others (see the review [50] on analog gravity models). The advantage of analog gravity is that it provides alternative experimental tools in the lab for simulating quantum effects of a curved spacetime like the Hawking radiation of black holes [54], which are inaccessible by current astrophysical observations due to technological limitations and precision requirements.¹ Second, the analog gravity is not merely an analogy. Using the fluid-gravity duality, it was first shown in [58] that the acoustic black hole formed in fluid at the radial cutoff surface can be mapped to a real black brane in asymptotically AdS spacetime at the dynamical level; the Hawking-like temperature of the acoustic black hole is connected to the Hawking temperature of the AdS black brane, and moreover, the phonon is dual to the sound channel of the quasinormal mode in the bulk AdS black brane. The dynamical connection between an acoustic black hole in fluid and a black hole has also been extended to the black D3 brane in flat spacetime [59]. Third, apart from systems in flat spacetime, analog gravity can also appear in curved spacetime; e.g., acoustic black holes have been extended in general curved spacetimes from relativistic Gross-Pitaevskii (GP) theory and Yang-Mills (YM) theory, where a real black hole is surrounded by an acoustic horizon, which is equivalent to a “dumb hole” from which neither light nor sound can escape [60]. Thus, the analog gravity in curved spacetime can be used to describe real black holes surrounded by environments such as some fluid accreting onto a black hole [61]. Later works have discussed effects such as black hole shadows, quasinormal modes, and analog Hawking radiation in this class of analog gravity [62–67].

Based on these specific properties of analog gravity, it is natural to ask whether the analog Hawking radiation in acoustic black holes can preserve unitarity as well. To answer this question, we will consider acoustic black holes formed in a four-dimensional Schwarzschild black hole background. Unlike acoustic black holes in flat spacetime, the Schwarzschild black hole background introduces a nontrivial causal structure characterized by multiple

distinct horizons, which can be compared with the results in the Schwarzschild background [31] and presents the universal characteristics of the island rule from the perspective of analog gravity. Due to the presence of multiple acoustic horizons, the acoustic black hole can be classified into nonextremal and extremal cases, similar to the Reissner-Nordström (RN) black hole, although the singularity remains spacelike in this case. We utilize the island formula to compute the entanglement entropy of analog Hawking radiation from acoustic black holes. Explicitly, by introducing the radiation region and island region associated with analogue Hawking radiation, we calculate the finite component of the entanglement entropy for phonons. In this setup, we discuss the impact of the emergence of islands related to the acoustic horizon on entanglement entropy, so as to reconstruct the Page curve. In the nonextremal case, we find that, in the absence of the island contribution, the entanglement entropy of radiative phonons exhibits a linear time dependence in the late-time regime. Subsequently, we calculate the entanglement entropy when the radiation region is close to and far from the acoustic horizon, and we reproduce the Page curve. In the extremal case, our analysis shows that, although a symmetric Penrose diagram exists, the vanishing surface gravity still leads to a divergence of entanglement entropy. This result is different from previous works on extremal black holes [41,42,44,45], which indicates that the divergence of entanglement entropy is caused by the Cauchy horizon terminating at a timelike singularity. Although this makes the Page time ill defined, the total entropy will be finite when an island exists. This suggests that information may be closely related to the causal structure of spacetime.

The paper is organized as follows. In Sec. II, we give a brief review on basic properties of acoustic black holes in curved spacetime. Then, in Secs. III and IV, we calculate the entanglement entropy of analog Hawking radiation using the island formula, both in the nonextremal and the extremal Schwarzschild acoustic black hole, respectively. The Page curve will be investigated in Sec. V. Finally, we give conclusions and discussions in Sec. VI.

II. REVIEW OF ACOUSTIC BLACK HOLE IN CURVED SPACETIME

In this section, we give a brief review of the acoustic black hole in Schwarzschild black hole spacetime [60]. The acoustic black hole in a general curved spacetime can be constructed from the relativistic GP theory; the action for a complex scalar field φ in this theory is

$$S = \int d^4x \sqrt{-g} \left(|\partial_\mu \varphi|^2 + m^2 |\varphi|^2 - \frac{b}{2} |\varphi|^4 \right), \quad (2)$$

where b is a constant and m^2 is a parameter depending on temperature as $m^2 \sim (T - T_c)$. The equation of motion for the scalar field φ is

¹There are also relevant studies discussing the simulation of physical phenomena such as cosmic inflation and particle pair production through analog gravity [55–57].

$$\square\varphi + m^2\varphi - b|\varphi|^2\varphi = 0. \quad (3)$$

One can choose a fixed static background spacetime ds_{bg}^2 as

$$ds_{\text{bg}}^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \quad (4)$$

By using the Madelung representation $\varphi = \sqrt{\rho(\vec{x}, t)}e^{i\theta(\vec{x}, t)}$ and considering perturbations via $\rho = \rho_0 + \rho_1$ and $\theta = \theta_0 + \theta_1$, Eq. (3) reduces to those for the background and phase fluctuations if we neglect the quantum

potential terms and require the equation of continuity $\partial_\mu(\sqrt{-g}\rho_0 v^\mu) = 0$ to hold,

$$b\rho_0 = m^2 - g^{\mu\nu}\partial_\mu\theta_0\partial_\nu\theta_0 = m^2 - v_\mu v^\mu, \quad (5)$$

$$\frac{1}{\sqrt{-\mathcal{G}}}\partial_\mu(\sqrt{-\mathcal{G}}\mathcal{G}^{\mu\nu}\partial_\nu\theta_1) = 0, \quad (6)$$

where $v_\mu = \{-\partial_0\theta_0, \partial_i\theta_0\}$, and the effective metric $\mathcal{G}_{\mu\nu}$ for the phase fluctuations can be obtained as

$$\mathcal{G}_{\mu\nu} = \frac{c_s}{\sqrt{c_s^2 - v_\mu v^\mu}} \begin{pmatrix} g_{tt}(c_s^2 - v_i v^i) & \vdots & -v_i v_t \\ \dots\dots\dots & \cdot & \dots\dots\dots \\ -v_i v_t & \vdots & g_{ij}(c_s^2 - v_\mu v^\mu) + v_i v_j \end{pmatrix}. \quad (7)$$

The speed of sound is defined as $c_s^2 \equiv \frac{b\rho_0}{2}$. The effective metric $\mathcal{G}_{\mu\nu}$ depends on the background spacetime and the four-velocity of the fluid. By considering $v_r \neq 0$, $v_t \neq 0$, $v_a = 0$ for $(a = \theta, \phi)$, and $g_{tt}g_{rr} = -1$, and taking the coordinate transformation $dt \rightarrow dt - \frac{v_r v_r}{g_{tt}(c_s^2 - v_r v^r)} dr$, the effective metric can be rewritten as

$$ds^2 = c_s \left(\frac{c_s^2 - v_r v^r}{\sqrt{c_s^2 - v_\mu v^\mu}} g_{tt} dt^2 + \frac{c_s^2 \sqrt{c_s^2 - v_\mu v^\mu}}{c_s^2 - v_r v^r} g_{rr} dr^2 + \sqrt{c_s^2 - v_\mu v^\mu} g_{\theta\theta} d\theta^2 + \sqrt{c_s^2 - v_\mu v^\mu} g_{\phi\phi} d\phi^2 \right). \quad (8)$$

Now, let us choose the background spacetime to be a Schwarzschild black hole,

$$ds_{\text{bg}}^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (9)$$

where $f(r) = 1 - \frac{r_h}{r}$ and r_h is the horizon radius. Consider a vortex that is falling radially from infinity towards the black hole. The radial velocity v_r should not be smaller than the escape velocity at Schwarzschild coordinate radius r , and it can be set as $v_r \sim \sqrt{2M\xi/r}$ with a tuning parameter ξ .² After rescaling $m^2 \rightarrow \frac{m^2}{2c_s^2}$ as well as $v_\mu v^\mu \rightarrow \frac{v_\mu v^\mu}{2c_s^2}$, Eq. (5) becomes $v_\mu v^\mu = m^2 - 1$. We work at the critical temperature T_c , such that the angular part of the metric takes the form of a two-dimensional sphere to simplify calculations and the mass parameter m^2 vanishes, leading to the condition $v_\mu v^\mu = -1$. Under these conditions, the effective metric can be rewritten as

²Note that the flow velocity is chosen as the escape velocity of an observer remaining stationary at the radius r [60,63,64]. Although in this case the flow velocity of the fluid does not satisfy the equation of continuity, the problem can be resolved by introducing an external source like $J\varphi^* + J^*\varphi$. However, this does not affect the main results of the transition of entanglement entropy induced by islands.

$$ds^2 = \sqrt{3}c_s^2 \left(-\mathcal{F}(r)dt^2 + \frac{1}{\mathcal{F}(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right),$$

$$\text{with } \mathcal{F}(r) = \left(1 - \frac{r_h}{r}\right) \left(1 - \xi \frac{r_h}{r} \left(1 - \frac{r_h}{r}\right)\right). \quad (10)$$

For convenience, we set $c_s^2 = 1/\sqrt{3}$.

The metric function $\mathcal{F}(r) = 0$ has three roots: the optical event horizon r_h and two acoustic horizons $r_\pm$, in which the acoustic horizons are given by

$$r_\pm = \frac{r_h}{2}(\xi \pm \sqrt{\xi^2 - 4\xi}), \quad (11)$$

where the condition $\xi \geq 4$ is required to ensure the existence of two acoustic horizons. These horizons define four distinct regions within the spacetime:

- (i) The region inside the optical horizon $r < r_h$.
- (ii) Two regions $r_h < r < r_-$ and $r_- < r < r_+$ where sound cannot escape.
- (iii) The region outside the outer acoustic horizon $r > r_+$ where both light and sound can escape.

For $\xi > 4$, the horizons are distinct, satisfying the relation $r_h < r_- < r_+$. When $\xi = 4$, the two acoustic horizons

coincide, resulting in a single horizon at $r_+ = r_- = 2r_h$. The acoustic black hole becomes extremal.

The Hawking temperature and the area entropy at the outer horizon are

$$T_H = \frac{1}{\beta} = \frac{\mathcal{F}'(r_+)}{4\pi} = \frac{(r_+ - r_h)(r_+ - r_-)}{4\pi r_+^3} = \frac{\kappa_+}{2\pi},$$

$$S = \frac{A}{4G_N} = \frac{\pi r_+^2}{G_N}, \quad (12)$$

where κ_+ is the surface gravity of the outer horizon r_+ . The surface gravity of different horizons could be derived from the tortoise coordinate,

$$\begin{aligned} r^* &= r + \frac{r_-^3}{(r_- - r_+)(r_- - r_h)} \log\left(\frac{r}{r_-} - 1\right) \\ &\quad + \frac{r_+^3}{(r_+ - r_-)(r_+ - r_h)} \log\left(\frac{r}{r_+} - 1\right) \\ &\quad + \frac{r_h^3}{(r_h - r_-)(r_h - r_+)} \log\left(\frac{r}{r_h} - 1\right) \\ &= r + \frac{1}{2\kappa_-} \log\left(\frac{r}{r_-} - 1\right) + \frac{1}{2\kappa_+} \log\left(\frac{r}{r_+} - 1\right) \\ &\quad + \frac{1}{2\kappa_h} \log\left(\frac{r}{r_h} - 1\right). \end{aligned} \quad (13)$$

We can further define the Kruskal coordinates as

$$ds^2 = -\Omega^2 dU dV + d\Omega^2, \quad (14)$$

$$U = -e^{-\kappa_+(t-r^*)}, \quad V = e^{\kappa_+(t+r^*)}, \quad (15)$$

with the conformal factor

$$\Omega^2 = \frac{\mathcal{F}(r)}{\kappa_+^2} e^{-2\kappa_+ r^*}. \quad (16)$$

In the extremal case $\xi = 4$, the acoustic horizon becomes $2r_h$ and the metric function $\mathcal{F}(r)$ becomes $\frac{(r-r_h)(r-r_+)^2}{r^3}$. The tortoise coordinate is now

$$\begin{aligned} r^* &= r - \frac{r_+^3}{(r - r_+)(r_+ - r_h)} + \frac{r_h^3}{(r_h - r_+)^2} \log\left(\frac{r - r_h}{r_h}\right) \\ &\quad + \frac{r_+^2(2r_+ - 3r_h)}{(r_h - r_+)^2} \log\left(\frac{r - r_+}{r_+}\right). \end{aligned} \quad (17)$$

The surface gravity $\kappa_+ = \kappa_-$ now vanishes, and the Hawking temperature $T_H = 0$. Therefore, we approach the result of the calculation by taking the limit $\kappa_+ \rightarrow 0$, as in previous studies [41,42,44,45].

III. ENTANGLEMENT ENTROPY IN NONEXTREMAL CASE

In this section, we will calculate the entanglement entropy of the analog Hawking radiation for the nonextremal Schwarzschild acoustic black hole. We will first discuss the entanglement entropy in the absence of islands, then analyze two cases that the radiation region is close to and far from the outer horizon when islands exist.

A. Without islands

In the absence of islands, there are two points of the entanglement regions located in the right wedge R_+ and the left wedge R_- . We set the coordinates of these points as $b_+ = (t_b, b)$ and $b_- = (-t_b + i\beta/2, b)$; see Fig. 1. It can be found that the Penrose diagram of an acoustic Schwarzschild black hole is similar to that of a RN black hole rotated by 90 degrees, with the addition of a region for the outer acoustic horizon. When the distance between the two points is large, we can calculate the entanglement entropy of the analog Hawking radiation composed of quantized phonons by assuming the s-wave approximation [31]

$$S_{\text{phonon}} = -I(R_+, R_-) = \frac{c}{3} \log d(b_+, b_-), \quad (18)$$

where c is the central charge and the mutual information is given by the distance between the boundaries $d(b_+, b_-)$. Through a conformal transformation, we obtain the total entropy in Kruskal coordinates

$$\begin{aligned} S_{\text{gen}} = S_{\text{phonon}} &= \frac{c}{6} \log [\Omega(b_+) \Omega(b_-) (U(b_-) - U(b_+)) \\ &\quad \times (V(b_+) - V(b_-))] \\ &= \frac{c}{6} \log \left[4 \frac{\mathcal{F}(b)}{\kappa_+^2} \cosh^2(\kappa_+ t_b) \right]. \end{aligned} \quad (19)$$

At late times $t_b \gg b > r_+$, the result is approximated as

$$\begin{aligned} S_{\text{gen}} &\simeq \frac{c}{3} \kappa_+ t_b \\ &= \frac{c}{12} \frac{(4 - \xi + \sqrt{(\xi - 4)\xi})}{r_h \xi} t_b. \end{aligned} \quad (20)$$

We find that the non-negativity of entropy also requires $\xi \geq 4$. In this nonisland phase, the entanglement entropy of radiation diverges with time, which is similar to Hawking's original calculations. In the following section, we will consider the entanglement entropy with the island, which should lead to the Page curve after the Page time.

B. Radiation region near the horizon

Now, we will calculate the entanglement entropy with islands. The boundary of the radiation region is assumed to be very close to the acoustic horizon, i.e., $b - r_+ \ll r_+$.

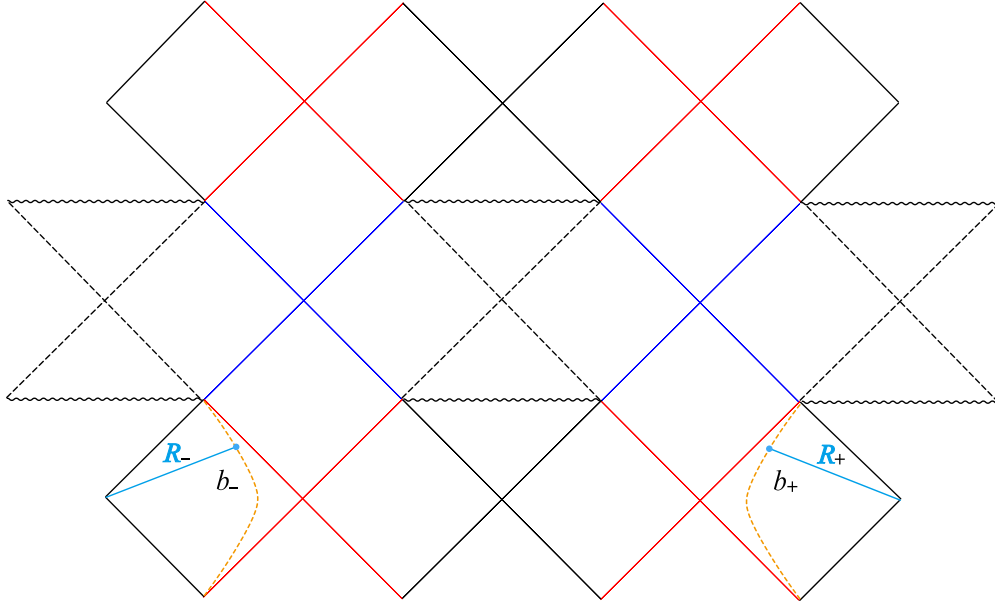


FIG. 1. Part of the Penrose diagram for a nonextremal Schwarzschild acoustic black hole without islands. The outer and inner acoustic horizons r_+ , r_- are denoted by red lines and blue lines. The optical horizon r_h is shown by the dashed line. The black line represents null infinity. In the absence of islands, the total entanglement entropy is determined by boundaries $b_{\pm}$ of the radiation region $R = R_+ \cup R_-$. The Penrose diagram can be infinitely continued in both the vertical and horizontal directions.

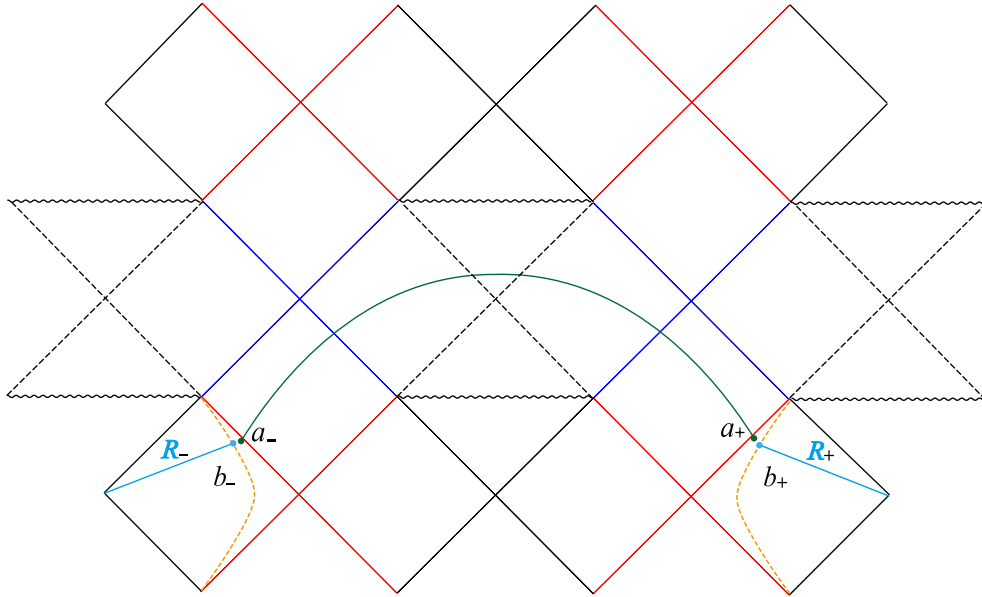


FIG. 2. Part of the Penrose diagram for a nonextremal Schwarzschild acoustic black hole with an island. The island extends between the right wedge and the left wedge (green line). The boundaries of the radiation region R are denoted by $b_{\pm}$, and the boundaries of the island I are denoted by $a_{\pm}$.

For the case of a single island, its boundaries are set as $a_+ = (t_a, a)$ and $a_- = (-t_a + i\beta/2, a)$; see Fig. 2. Since the distance between the island boundary and the radiation region boundary is sufficiently small, the finite part of entanglement entropy of the massless matter field can be evaluated as [68,69]

$$S_{\text{phonon}} = -2I(R_+, I) = -2\kappa c \frac{\text{Area}}{L^2}. \quad (21)$$

Here, κ is a dimensionless constant that should not be confused with the surface gravity. The factor 2 comes from the fact that the entanglement entropy is calculated for both

the left and right wedges. In the acoustic black hole, the geodesic distance between the boundary of region I and that of region R is analytically given by

$$L = \frac{1}{\sqrt{r(r-c)(r-r_+)}} \left\{ \frac{r(r-r_-)(r-r_+)}{\sqrt{r-r_h}} - \sqrt{\frac{(r-r_-)(r-r_+)r}{(r_h-r_-)r_+}} \right. \\ \times \left[(r_h-r_-)r_+E(\arcsin A_1|A_2) - (r_h^2-r_-r_+ + (r_-+r_+)r_h)F(\arcsin A_1|A_2) \right. \\ \left. \left. + (r_h-r_+)(r_h+r_-+r_+)\Pi(A_2r_+/r_h; \arcsin A_1|A_2) \right] \right\} \Big|_a^b, \quad (22)$$

where $F(\phi|m)$, $E(\phi|m)$, $\Pi(n; \phi|m)$ are the elliptic integrals of the first kind, the second kind, and the third kind, and $A_1 = \sqrt{\frac{(r_-r_h)(r-r_+)}{(r-r_h)(r_-r_+)}}$, $A_2 = \frac{(r_+-r_-)r_h}{(r_h-r_-)r_+}$. Taking the limit $a-r_+ \ll r_+$ and $b-r_+ \ll r_+$, we obtain the short-distance limit of the above integral as

$$L \simeq 2\sqrt{\frac{r_+^3}{(r_+-r_h)(r_+-r_-)}}(\sqrt{b-r_+} - \sqrt{a-r_+}) \\ = 2\sqrt{\frac{1}{2\kappa_+}}(\sqrt{b-r_+} - \sqrt{a-r_+}). \quad (23)$$

Thus, the total entanglement entropy of the radiation, i.e., the generalized entanglement entropy, is

$$S_{\text{gen}} = \frac{2\pi a^2}{G_N} - 8\kappa c \frac{\pi b^2}{L^2} \\ = \frac{2\pi a^2}{G_N} - \frac{4\pi b^2 \kappa c \kappa_+}{(\sqrt{b-r_+} - \sqrt{a-r_+})^2} \\ \simeq \frac{2\pi a^2}{G_N} - \frac{4\pi r_+ \kappa c \kappa_+}{(y-x)^2} \quad (24)$$

with new parameters $x = \sqrt{a-r_+} > 0$ and $y = \sqrt{b-r_+} > 0$. We take $x < y \ll 1$ as the new near-horizon condition. According to the island formula, the actual entanglement entropy requires taking the minimum value among all extremal cases. Since the result is independent of time, we only need to take the derivative with respect to the position of the island, which is equivalent to

$$\frac{x}{y} \left(1 - \frac{x}{y}\right)^3 = \frac{G_N \kappa c \kappa_+}{r_+ y^4}. \quad (25)$$

The above equation has an extremum point, provided that the right-hand side is smaller than the maximum value 27/256 of the function [35,42]. Moreover, since the right-hand side is sufficiently small, we can approximately solve for the position of the island as

$$a = r_+ + \frac{(r_+ G_N \kappa c \kappa_+)^2}{(b-r_+)^3}, \quad (26)$$

which shows that there is a modification of order $O(G_N^2)$ compared to the position of the acoustic horizon, indicating that the island is located just slightly outside the horizon. Substituting this into Eq. (24), we obtain

$$S_{\text{gen}} \simeq \frac{2\pi r_+^2}{G_N} - 4\pi \kappa c \frac{r_+^2 \kappa_+}{b-r_+}, \quad (27)$$

in which the first term is twice the area entropy at acoustic horizon r_+ , while the second term represents the effect of quantum corrections; the resulting entanglement entropy is now finite and no longer diverges with time.

C. Radiation region far from the horizon

Next, we will consider the case where the radiation region is far from the acoustic horizon. Since the s-wave approximation remains valid, the entanglement entropy in this case is

$$S_{\text{phonon}} = \frac{c}{3} \log \frac{d(a_+, a_-)d(b_+, b_-)d(a_+, b_+)d(a_-, b_-)}{d(a_+, b_-)d(a_-, b_+)}. \quad (28)$$

Working in the Kruskal coordinates, the generalized entanglement entropy is expressed as

$$S_{\text{gen}} = \frac{2\pi a^2}{G_N} + \frac{c}{6} \log \left[\frac{2^4 \mathcal{F}(a) \mathcal{F}(b)}{\kappa_+^4} \cosh^2(\kappa_+ t_a) \cosh^2(\kappa_+ t_b) \right] + \frac{c}{3} \log \left\{ \frac{\cosh[\kappa_+(r_+(a) - r_+(b))] - \cosh[\kappa_+(t_a - t_b)]}{\cosh[\kappa_+(r_+(a) - r_+(b))] + \cosh[\kappa_+(t_a + t_b)]} \right\}. \quad (29)$$

We can further use the following equation for simplification:

$$\cosh[\kappa_+(r_*(a) - r_*(b))] = \frac{1}{2} \left[\sqrt{\frac{K(a)}{K(b)}} e^{\kappa_+(a-b)} + \sqrt{\frac{K(b)}{K(a)}} e^{\kappa_+(b-a)} \right], \quad (30)$$

where $K(r) = (r - r_+)(r - r_-)^{\frac{\kappa_+}{\kappa_-}}(r - r_h)^{\frac{\kappa_+}{\kappa_-}}$. For $a \sim r_+$, the first term can be ignored since the second term dominates. Then, we can take the late-time limit [42]

$$\frac{1}{2} \sqrt{\frac{K(b)}{K(a)}} e^{\kappa_+(b-a)} \ll \cosh[\kappa_+(t_a + t_b)], \quad (31)$$

and the approximation

$$\cosh[\kappa_+(t_a - t_b)] \ll \frac{1}{2} \sqrt{\frac{K(b)}{K(a)}} e^{\kappa_+(b-a)}. \quad (32)$$

Under these conditions, the entanglement entropy can be simplified as

$$\begin{aligned} S_{\text{gen}} &= \frac{2\pi a^2}{G_N} + \frac{c}{6} \log \left[\frac{2^4 \mathcal{F}(a) \mathcal{F}(b)}{\kappa_+^4} \cosh^2(\kappa_+ t_a) \cosh^2(\kappa_+ t_b) \right] \\ &\quad - \frac{c}{3} \log \left[2 \sqrt{\frac{K(a)}{K(b)}} e^{\kappa_+(a-b)} \cosh[\kappa_+(t_a + t_b)] \right] - \frac{2c}{3} \sqrt{\frac{K(a)}{K(b)}} e^{\kappa_+(a-b)} \cosh[\kappa_+(t_a - t_b)] \\ &\approx \frac{2\pi a^2}{G_N} + \frac{c}{6} \log \left[\frac{\mathcal{F}(a) \mathcal{F}(b) K(b)}{K(a)} e^{2\kappa_+(b-a)} \right] - \frac{2c}{3} \sqrt{\frac{K(a)}{K(b)}} e^{\kappa_+(a-b)} \cosh[\kappa_+(t_a - t_b)]. \end{aligned} \quad (33)$$

To obtain the extremum, we first take the derivative with respect to the location a , which gives

$$a \simeq r_+ + \frac{(cG_N)^2 \left(\frac{r_+ - r_-}{b - r_h}\right)^{\frac{\kappa_+}{\kappa_-}} \left(\frac{r_+ - r_-}{b - r_-}\right)^{\frac{\kappa_+}{\kappa_-}}}{144\pi^2 (b - r_+) r_+^2} e^{2\kappa_+(r_+ - b)} \cosh^2[\kappa_+(t_a - t_b)]. \quad (34)$$

It is clear that the position of the island is also slightly outside the horizon, and the entanglement entropy can be approximated as

$$\begin{aligned} S_{\text{gen}} &\approx \frac{2\pi r_+^2}{G_N} + \frac{c}{6} \log \left[\frac{4\mathcal{F}(b)K(b)}{\kappa_+^4 r_+^3} (r_+ - r_h)^{1 - \frac{\kappa_+}{\kappa_-}} (r_+ - r_-)^{1 - \frac{\kappa_+}{\kappa_-}} e^{\kappa_+(b - r_+)} \right] \\ &\quad - \frac{c^2 G_N \left(\frac{r_+ - r_-}{b - r_h}\right)^{\frac{\kappa_+}{\kappa_-}} \left(\frac{r_+ - r_-}{b - r_-}\right)^{\frac{\kappa_+}{\kappa_-}}}{36\pi r_+ (r_+ - b)} (1 + 2e^{\kappa_+(r_+ - b)}) \cosh^2[\kappa_+(t_a - t_b)]. \end{aligned} \quad (35)$$

In addition, $\frac{\partial S_{\text{gen}}}{\partial t}$ gives $t_a = t_b$. Then, ignoring the higher order terms, the entanglement entropy of radiation with the island is

$$S_{\text{gen}} = \frac{2\pi r_+^2}{G_N} + \frac{c}{6} \log \left[\frac{4\mathcal{F}(b)K(b)}{\kappa_+^4 r_+^3} (r_+ - r_h)^{1 - \frac{\kappa_+}{\kappa_-}} (r_+ - r_-)^{1 - \frac{\kappa_+}{\kappa_-}} e^{\kappa_+(b - r_+)} \right], \quad (36)$$

which shows that when the radiation region is far from the acoustic horizon, the entanglement entropy of radiation eventually reaches a stable value. This outcome is consistent with the case where the radiation region is close to the acoustic horizon, as both results include an area term and a quantum correction term. In both cases, the entanglement entropy initially grows with time before transitioning to this stable value at the Page time. However, as we will show in the next section, the situation is different for extremal black holes. Although we can still compute the entanglement entropy with the existence of an island, the Page time cannot be well defined.

IV. ENTANGLEMENT ENTROPY IN THE EXTREMAL CASE

In this section, we will focus on the extremal acoustic black hole in Schwarzschild spacetime, where the two acoustic horizons coincide at $r_+ = r_- = 2r_h$. Previous studies [41,42,44] have shown that the extremal case is significantly different from the nonextremal case, particularly regarding the Penrose diagram. This is because the extremal geometry is not obtained by taking the continuous limit of the nonextremal case. We will adopt the same approach to calculate the entanglement entropy, both without and with islands. Furthermore, since the entanglement entropy depends on the specific position of the radiation region, we will begin with the case where the region is close to the horizon.

A. Without islands

In contrast to previous studies [41,42,44] where the Penrose diagram of the spacetime reduces to a one-sided structure due to a timelike singularity blocking the propagation of radiation, our case features a symmetric Penrose diagram because of the acoustic horizon. Consequently, as shown in Fig. 3, there are two boundaries of the radiation region, denoted as $b_+ = (t_b, b)$ and $b_- = (-t_b + i\beta/2, b)$, located at the right and left wedges. By using Eq. (18), the entanglement entropy of Hawking radiation is given by

$$S_{\text{gen}} = S_{\text{phonon}} = \lim_{\kappa_+ \rightarrow 0} \frac{c}{3} \log d(b_+, b_-) \\ = \lim_{\kappa_+ \rightarrow 0} \frac{c}{3} \log \left(\frac{2}{\kappa_+} \cosh(\kappa_+ t) \sqrt{\mathcal{F}(b)} \right), \quad (37)$$

in which r^* is defined in Eq. (17) and $\mathcal{F}(b) = (b - r_h)(b - r_+)^2/b^3$. It can be seen that the entanglement entropy diverges as $\kappa_+ \rightarrow 0$. While this divergence in the absence of islands implies that the Page time cannot be well defined, the presence of islands still allows us to derive an exact result for the entanglement entropy.

B. Radiation region near the horizon

We set the boundaries of islands as $a_+ = (t_a, a)$ and $a_- = (-t_a + i\beta/2, a)$, and consider the case where the radiation region is near the acoustic horizon first. Assuming the boundaries of the island are outside and near the horizon such that $a - r_+ \ll b - r_+ \ll r_+$, the geodesic distance is

$$L = \int_a^b \sqrt{\frac{r^3}{(r - r_h)(r - r_+)^2}} dr \simeq \frac{2r_+^{\frac{3}{2}} \log(\frac{x}{y})}{\sqrt{r_+ - r_h}}. \quad (38)$$

By using Eq. (21), the generalized entropy is

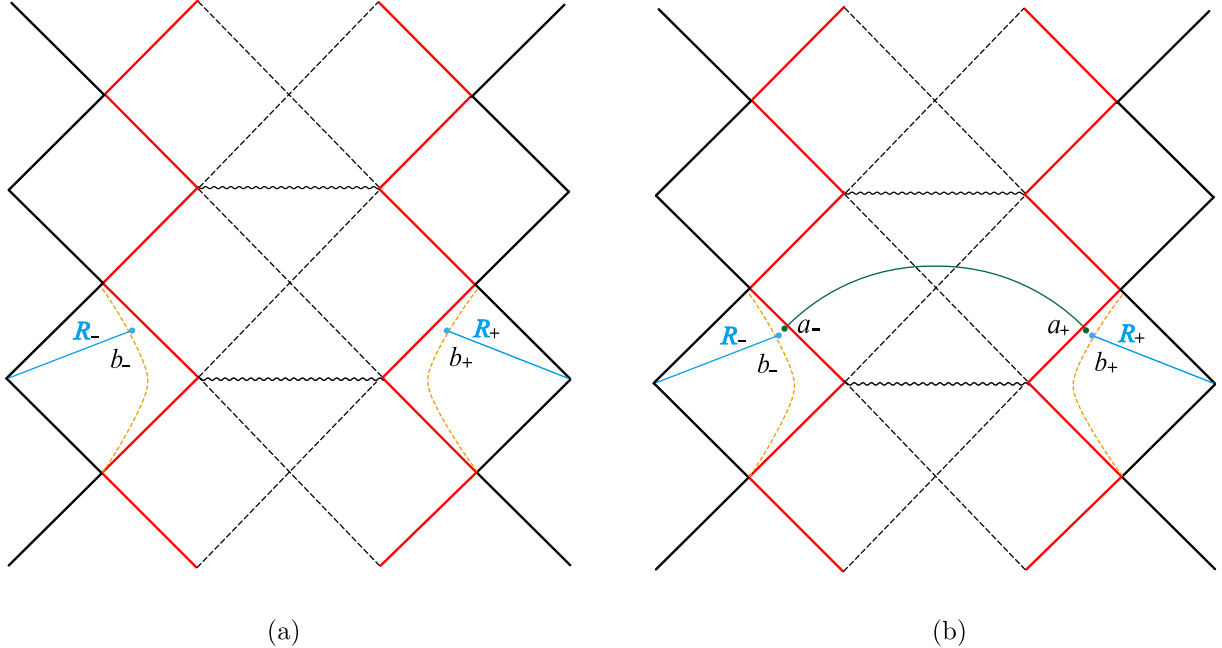


FIG. 3. Penrose diagram of the Schwarzschild acoustic black hole in the extremal case with and without islands. (a) Without island. (b) With island.

$$S_{\text{gen}} = \frac{2\pi a^2}{G_N} - c\kappa \frac{8\pi b^2}{L^2} = \frac{2\pi a^2}{G_N} - c\kappa \frac{2\pi b^2(r_+ - r_h)}{r_+^3 \log\left(\frac{x}{y}\right)^2}, \quad (39)$$

where the parameters $x = \sqrt{a - r_+} > 0$ and $y = \sqrt{b - r_+} > 0$, and $x \ll y \ll 1$. Before finding the extremal solutions of Eq. (39), we first determine whether an island solution exists by analyzing the relevant conditions. From the extremum condition $\frac{\partial S_{\text{gen}}}{\partial a}$, one obtains

$$\left(\frac{y^2}{x^2} \log\left(\frac{x^2}{y^2}\right)\right)^3 = \frac{2c\kappa b^2 G_N (r_+ - r_h)}{r_+^5 x^2}. \quad (40)$$

Further defining $z \equiv \frac{y^2}{x^2}$, the above equation becomes

$$z \left(\log \frac{1}{z}\right)^3 = \frac{2c\kappa b^2 G_N (r_+ - r_h)}{r_+^5 x^2} \equiv F_0. \quad (41)$$

To ensure the existence of the island solution, the constant F_0 must be less than the maximum value of the function $F = z(\log \frac{1}{z})^3$ in the interval $z \in (0, 1)$. In this case, the line of F_0 intersects the graph of the function F at two points (as shown in Fig. 4), which correspond to two solutions, z_1 and z_2 . In addition, z_1 corresponds to the island solution, which implies the constraint

$$\frac{27}{e^3} > \frac{2c\kappa b^2 G_N (r_+ - r_h)}{r_+^5 x^2}, \quad (42)$$

which is the special case of [42] for $d = 3$. When the constraint is violated, there would be no nontrivial island solution.

We demonstrate the existence of an island solution without explicitly solving for its position. Specifically, under the specified constraints and near-horizon conditions, we find that the entanglement entropy is approximately

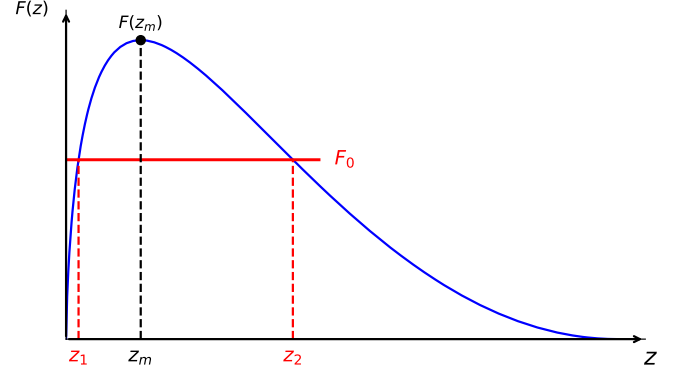


FIG. 4. Diagram of the function $F = z(\log \frac{1}{z})^3$ in the interval $z \in (0, 1)$. The maximal value $\frac{27}{e^3}$ is located at e^{-3} . When $F(e^{-3}) > F_0$, it is clear that the line $F = F_0$ intersects the function's curve at two points, which correspond to two solutions, z_1 and z_2 .

equal to the area entropy of the acoustic horizon,

$$\begin{aligned} S_{\text{gen}} &= \frac{2\pi a^2}{G_N} - c\kappa \frac{2\pi b^2(r_+ - r_h)}{r_+^3 \log\left(\frac{x}{y}\right)^2} \\ &> \frac{2\pi a^2}{G_N} - c\kappa \frac{2\pi b^2(r_+ - r_h)}{9r_+^3} \\ &> \frac{2\pi r_+^2}{G_N} \left(1 - c\kappa \frac{G_N b^2(r_+ - r_h)}{9r_+^5}\right) \\ &> \frac{2\pi r_+^2}{G_N} \left(1 - \frac{3}{2e^3} x^2\right) \simeq \frac{2\pi r_+^2}{G_N}. \end{aligned} \quad (43)$$

C. Radiation region far from the horizon

Unlike the case where the radiation region is close to the horizon, when the radiation region is far from the horizon, the position of the island can be directly solved. Using the formula Eq. (28), one can derive the entanglement entropy of quantized phonons by taking the limit $\kappa_+ \rightarrow 0$,

$$\begin{aligned} S_{\text{phonon}} &= \lim_{\kappa_+ \rightarrow 0} \frac{c}{3} \log \left\{ \frac{2\sqrt{\mathcal{F}(a)\mathcal{F}(b)}}{\kappa_+^2} [\cosh[\kappa_+(r_*(a) - r_*(b))] - \cosh[\kappa_+(t_a - t_b)]] \right\} \\ &= \frac{c}{3} \log \left\{ \sqrt{\mathcal{F}(a)\mathcal{F}(b)} [(r_*(a) - r_*(b))^2 - (t_a - t_b)^2] \right\}. \end{aligned} \quad (44)$$

Then, the generalized entropy is

$$S_{\text{gen}} = \frac{2\pi a^2}{G_N} + \frac{c}{6} \log \left\{ \mathcal{F}(a)\mathcal{F}(b) [(r_*(a) - r_*(b))^2 - (t_a - t_b)^2]^2 \right\}. \quad (45)$$

Extremizing S_{gen} with respect to t_a gives

$$\frac{\partial S_{\text{gen}}}{\partial t_a} = \frac{2c(t_a - t_b)}{3[(t_a - t_b)^2 + (r_*(a) - r_*(b))^2]} = 0, \quad (46)$$

which results in $t_a = t_b$. Then, extremizing S_{gen} with respect to the location of the island,

$$\frac{\partial S_{\text{gen}}}{\partial a} = \frac{4\pi a}{G_N} + \frac{c\mathcal{F}'(a)}{6\mathcal{F}(a)} + \frac{2cr^*(a)'}{3(r^*(a) - r^*(b))} = 0, \quad (47)$$

where $\mathcal{F}'(r)$ and $r^*(r)'$ are derivatives of $\mathcal{F}(r)$ and $r^*(r)$, respectively. By taking the approximation $a \approx r_+$, one can finally obtain the location of the quantum extremal island as

$$a \simeq r_+ + \frac{cG_N}{6\pi r_+}. \quad (48)$$

At late times, the generalized entanglement entropy with the island is given by

$$S_{\text{gen}} \simeq \frac{2\pi r_+^2}{G_N} + \frac{c}{6} \log \left(\frac{36\pi r_+^{11}(b - r_+)^2(b - r_h)}{b^3 c^2 G_N^2 (r_+ - r_h)^3} \right) + \mathcal{O}(G_N). \quad (49)$$

Note that it cannot be obtained from the continuous extremal limit of the nonextremal entanglement entropy in Eq. (36) as we discussed before.

So far, we have determined the contributions of islands to the entanglement entropy when the radiation region is either close to or far from the horizon. In both cases, the fine-grained entropy eventually reaches an upper bound. Furthermore, due to the symmetry of the Penrose diagram, the entanglement entropy of the radiation ultimately approaches twice the area entropy at the acoustic horizon.

V. PAGE CURVE AND PAGE TIME

In this section, we will discuss the Page curve. To be precise, the entanglement entropy of radiation initially increases, reaches its maximum at the Page time, and then decreases gradually. In the island description, the entanglement entropy is initially described by the result without islands; as time progresses and after reaching the Page time, it transitions to the case with islands. Thus, the Page curve can be reproduced (see Fig. 5).

In the nonextremal case, it can be expressed as

$$S_{\text{gen}} = \text{Min} \left(\frac{c}{3} \kappa_+ t, S_{\text{gen}}^{\text{with island}} \right), \quad (50)$$

where $S_{\text{gen}}^{\text{with island}}$ is the entanglement entropy with islands, given by Eqs. (27) and (36) depending on the location of the radiation region. The Page time is therefore calculated as

$$t_{\text{Page}} = \frac{6\pi r_+^2}{G_N c \kappa_+} = \frac{6\pi r_h^3 \xi (\xi + \sqrt{(\xi - 4)\xi})^2}{G_N c 4 - \xi + \sqrt{(\xi - 4)\xi}}. \quad (51)$$

In the extremal limit where $\xi \rightarrow 4$, we find that the Page time diverges, implying that the Page time cannot be well defined.

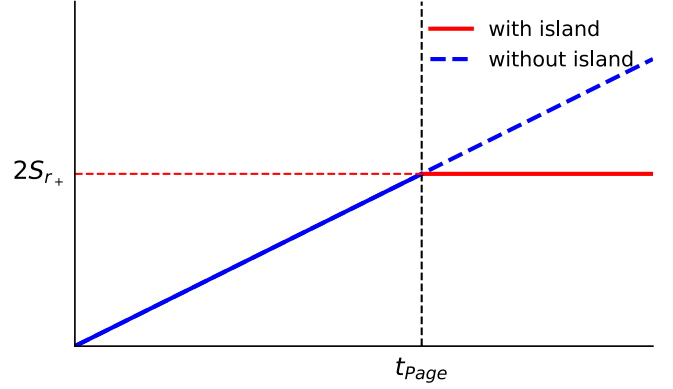


FIG. 5. Page curve of the entanglement entropy in the non-extremal case. The entanglement entropy is initially described by the result without islands and then transitions to the case with islands after reaching the Page time. The entanglement entropy remains stable and could be approximated by the area entropy of the acoustic horizon.

This is consistent with our previous discussion in Sec. IV. The Page time also diverges for the limit $\xi \rightarrow \infty$, where the acoustic horizon r_+ tends to infinity, meaning that the radiation cannot escape from the entire spacetime.

We have analyzed the Penrose diagram and calculated the entanglement entropy for the extremal case with an island, considering both close and far radiation regions. Many physical quantities diverge in this case due to the vanishing surface gravity κ_+ , suggesting close connections between entanglement entropy and causal structure of spacetime. When the island is considered, the total entanglement entropy remains stable again and is consistently equal to twice the area entropy of the acoustic horizon.

VI. CONCLUSION AND DISCUSSION

In this paper, we investigated the information paradox of the acoustic black hole in a Schwarzschild black hole background using the island formula. By adopting different methods to estimate the entanglement entropy of phonons, distinct scenarios were considered: when the radiation region is close to the acoustic horizon and when it is far from it. For the nonextremal case, we solved for the positions of islands and derived the entanglement entropy, which depends on the location of the radiation region. It was shown that the entanglement entropy can reach the steady value, and the Page curve can be found only in the presence of islands. When taking the extremal limit, the Page time will diverge, implying that the Page curve cannot be well defined in the extremal case. For the extremal case, our analysis showed that, in the absence of islands, the entanglement entropy diverges due to the vanishing surface gravity, or zero Hawking-like temperature, which implied a close connection between entanglement entropy and spacetime causal structure. This finding is distinct from previous work on RN black holes, where the divergence

was attributed to the inclusion of the black hole singularity in the island formula. Nevertheless, in the presence of islands, a finite entanglement entropy can be obtained, which to leading order is equal to twice the area entropy of the acoustic horizon.

Our results successfully reproduced the Page curve in this curved acoustic black hole system, confirming that unitarity is not violated in the analog gravity model. This could provide new insights into both analog gravity and real gravity. At present, we cannot directly observe the Hawking radiation, entanglement entropy, or the spacetime region known as the island of a real black hole. However, our results using the analog gravity model suggest a resolution to this difficulty. For instance, in the model we adopted, the position of the acoustic horizon is affected by the flow velocity of the fluid surrounding the black hole and is always larger than that of the event horizon. This will result in the island region associated with phonon entanglement being exposed outside the event horizon, which means that we can observe the island region without having to access the interior of the black hole. Second, our results indicate that the island formula may not only be applicable in the AdS/CFT correspondence, but it may also serve as a universal mechanism for restoring unitarity in any effective theory with a horizon. This is similar to the viewpoint in analog gravity research, which asserts that Hawking-like radiation arises for any test field in any Lorentzian geometry containing an event horizon, without the need to satisfy the Einstein field equations [70], and it is also similar to the study of Hawking-Unruh effect in the noninertial reference frame [71]. Third, there have also been discussions of acoustic black holes in the context of holographic duality previously. For instance, Refs. [72,73] discussed how to use analog gravity systems to simulate AdS and asymptotically AdS spacetimes. References [58,59] constructed the duality between the

phonon in the acoustic black hole and the quasinormal mode in the bulk perturbed AdS black brane through fluid-gravity duality. The present work is a generalization of applying the holographic duality to the case of acoustic black holes, and we specifically investigated the properties of entanglement entropy in analog gravity.

Several open questions remain in the current research. For instance, it is unclear how to isolate the specific contributions of photons and phonons to the entanglement entropy in such a background; it is also unclear how the resulting island structure compares to that of a standard Schwarzschild black hole. In addition, it is straightforward to generalize the calculation of entanglement entropy to other backgrounds, such as the RN black hole or AdS black holes. Moreover, although the present paper only focused on the acoustic black hole within a Schwarzschild background, the island formula can be extended to investigate general analog gravity models in flat spacetime; these systems may provide a concrete, nongeometric interpretation of the island rule from their well-defined microscopic degrees of freedom. It is also worthwhile to use analog gravity to realize the holographic principle. We leave these intriguing questions for future investigation.

ACKNOWLEDGMENTS

We would like to thank Xian-Hui Ge and Ruo-Han Wang for useful discussions. This work was supported by the National Natural Science Foundation of China (No. 12475069) and the Guangdong Basic and Applied Basic Research Foundation (No. 2025A1515011321).

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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