

Radiative decays of $J/\psi, \psi(2S) \rightarrow \gamma\eta^{(\prime)}$ in perturbative QCD with relativistic corrections

Jun-Kang He[✉], Chao-Jie Fan^{✉,*}, and Cong Wang^{✉,†}

College of Physics and Electronic Science, Hubei Normal University, Huangshi 435002, China



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We present the first calculation of the radiative decays $J/\psi, \psi(2S) \rightarrow \gamma\eta^{(\prime)}$ in perturbative QCD that includes the order- q^2 relativistic corrections in all three short-distance contributions, namely the quark-antiquark, two-gluon, and QED contributions. The amplitudes are found to be remarkably insensitive to the light-cone distribution amplitude and to the light-quark mass, a robustness that persists through order q^2 and makes the predictions correspondingly reliable. The relativistic correction enhances the J/ψ branching ratios by roughly a factor of 2, narrowing their shortfall from experiment, whereas for the $\psi(2S)$ it is about twice as large as for the J/ψ and the low-order expansion converges poorly. In two representative η - η' mixing schemes, the ratio $\mathcal{R}_{1S} = \mathcal{B}(\gamma\eta')/\mathcal{B}(\gamma\eta)$ proves sharply sensitive to the mixing angle and favors the smaller of the two. The predicted $\psi(2S)$ rates lie well above the data in both channels, already at leading order, and most severely for the anomalously small $\gamma\eta$ channel. Such a discrepancy suggests that a mechanism beyond the hard perturbative process is at work. As a physically motivated attempt, we explore the η_c -mixing contribution, which adds coherently to the perturbative one and is comparable to it in the $\gamma\eta$ channel, and find that the interference can bring the $\psi(2S)$ rates into agreement with the data, although its extraction is limited by a strong sensitivity to the mixing parameters.

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I. INTRODUCTION

The discovery of the J/ψ [1,2] opened the field of charmonium physics and provided a uniquely clean window on the strong interaction. As the lowest $c\bar{c}$ vector state, the J/ψ , together with its radial excitation $\psi(2S)$, lies in the transition region between the perturbative and nonperturbative regimes of QCD, where the charm-quark mass sets a semihard scale at which the strong coupling is neither small nor large and perturbative and nonperturbative effects are closely intertwined [3–6]. The Okubo-Zweig-Iizuka (OZI) forbidden radiative decays $J/\psi, \psi(2S) \rightarrow \gamma\eta^{(\prime)}$ are a prime manifestation of this interplay. In the parton picture they proceed through the annihilation of the $c\bar{c}$ pair into a photon and at least two gluons, which subsequently convert into the $\eta^{(\prime)}$ [5,6], so that they probe at once the conversion of gluons into light hadrons, the gluonic content of the $\eta^{(\prime)}$, and the η - η' mixing that organizes the pseudoscalar nonet,

the ratio of the two rates being one of the classic observables for extracting the mixing angle.

Reflecting this perturbative-nonperturbative interplay, these decays have been described by two distinct theoretical approaches. The first is nonperturbative, and several variants have been developed. In the formulation of Novikov *et al.* [7] the transition is assumed to be dominated by the $U_A(1)$ anomaly and is controlled by the gluonic matrix elements $\langle 0 | G_{\mu\nu}^a \tilde{G}^{a,\mu\nu} | \eta^{(\prime)} \rangle$, subsequently estimated in the large- N_c expansion and the QCD multipole formalism [8]. The Feldmann-Kroll-Stech (FKS) scheme [9] reexpresses these matrix elements through the phenomenological constants f_q , f_s and the mixing angle ϕ . In this picture the anomaly form of the width carries a factor $(M_{J/\psi}/m_c^2)^4$ [7], whose steep dependence on the charm mass renders the absolute normalization strongly uncertain, while the mixing angle extracted from the radiative ratio under the assumption of anomaly dominance, $\phi \simeq 39^\circ$, is tied to that assumption. The same decays have also been analyzed with effective-Lagrangian and vector-meson-dominance methods that describe the J/ψ and $\psi(2S)$ transitions on a common footing [10,11]. A third, closely related description regards the decay as proceeding through a small η_c component admixed into the $\eta^{(\prime)}$. This $\eta^{(\prime)}$ - η_c mixing picture was introduced by Chao [12,13] and adopted in the FKS analysis [9] and is in fact the same physics as the anomaly mechanism, since the gluonic $U_A(1)$ anomaly that couples

* Contact author: fancj@hbnu.edu.cn

† Contact author: wangj@hbnu.edu.cn

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the $\eta^{(\prime)}$ to two gluons is what mixes a $c\bar{c}$ component into it. We note, however, that these η_c -mixing estimates commonly omit the radial overlap form factor associated with probing the η_c far off its mass shell. When it is neglected, the large phase-space factor associated with the highly energetic photon can lead to a substantial overestimate of the η_c -mixing contribution to these OZI-forbidden radiative decays, an enhancement that is likely unphysical.

The second approach is perturbative. The amplitude for these decays is generated at leading twist by the subprocess $c\bar{c} \rightarrow \gamma g^* g^*$, with the virtual gluons hadronizing into the $\eta^{(\prime)}$ through its light-cone distribution amplitudes (DAs). This perturbative QCD approach was pioneered by Körner *et al.* [14] and by Kühn [15] and revisited by several groups using light-cone DAs [16–20]. In a recent study [21] we computed $J/\psi \rightarrow \gamma\eta^{(\prime)}$ in this framework keeping the light-quark masses in the loop integrals and found that the sum of the one-loop integrals is infrared finite and remarkably insensitive both to the light-quark masses and to the shape of the $\eta^{(\prime)}$ DAs. As a consequence the hard mechanism alone reproduces the measured ratio $\mathcal{R}_{1S} = \mathcal{B}(J/\psi \rightarrow \gamma\eta')/\mathcal{B}(J/\psi \rightarrow \gamma\eta)$, while the individual branching ratios come out somewhat below the data. The mixing angle extracted from $\mathcal{R}_{1S}$ in the hard picture, $\phi \simeq 33.5^\circ$, is appreciably smaller than the FKS value obtained under the assumption of anomaly dominance. That analysis established that the hard mechanism is capable of describing $J/\psi \rightarrow \gamma\eta^{(\prime)}$ at the level of present data.

Common to all of these treatments is that the initial charmonium is described in the nonrelativistic, weak-binding approximation, in which the relative motion of the c and $\bar{c}$ is neglected. Charmonium is, however, only moderately nonrelativistic, with a mean squared heavy-quark velocity $\langle v^2 \rangle \simeq 0.3$, so corrections of relative order v^2 need not be small, and their importance is well established across charmonium production and decay. A prominent example on the production side is the exclusive process $e^+e^- \rightarrow J/\psi + \eta_c$, whose cross section measured at the B factories exceeds the leading-order nonrelativistic QCD (NRQCD) prediction by nearly an order of magnitude, a discrepancy that the relativistic corrections of order v^2 , together with the QCD radiative corrections, are essential to resolve [22,23]. On the decay side, relativistic corrections have likewise been found to be substantial in exclusive charmonium decays, both in the three-gluon decay $J/\psi \rightarrow ggg$ [24] and in exclusive hadronic channels such as $J/\psi \rightarrow p\bar{p}$ and $\psi(nS) \rightarrow \rho\pi$ [25,26]. The underlying reason is that the expansion parameter v^2 is not small, so the corrections of order v^2 can reach order unity, in particular whenever the leading-order amplitude is itself suppressed. The radially excited $\psi(2S)$ is even more sensitive, being a more loosely bound excited state whose mean squared velocity is substantially larger than that of the $1S$ states [26,27]. The v^2 expansion parameter is

correspondingly larger, so the relativistic corrections are enhanced and the expansion converges more slowly, an effect already seen in $\psi(2S)$ production [28]. Our recent study of the three-gluon decays $\psi(2S), \Upsilon(2S) \rightarrow ggg$ reaches a similar conclusion [27]. In addition, the related P -wave radiative decays $h_c \rightarrow \gamma\eta^{(\prime)}$ were also found to receive significant relativistic corrections [29].

A reliable description of $J/\psi, \psi(2S) \rightarrow \gamma\eta^{(\prime)}$, and especially of the $\psi(2S)$ modes, therefore requires the relativistic corrections to the initial bound state to be included systematically. We do so within the Bethe-Salpeter formalism. The present work provides, to the best of our knowledge, the first calculation of these decays in which the q^2 relativistic corrections to the charmonium are included consistently in all three short-distance contributions, namely the quark-antiquark, two-gluon, and QED contributions, using the same covariant Salpeter projector and q^2 expansion as in our three-gluon analyses [24,27], now convolved with the $\eta^{(\prime)}$ light-cone DAs.

A further motivation comes from the $2S$ states. Experimentally the $\psi(2S) \rightarrow \gamma\eta^{(\prime)}$ rates are strongly suppressed: $\mathcal{B}(\psi(2S) \rightarrow \gamma\eta') = (1.24 \pm 0.04) \times 10^{-4}$ and $\mathcal{B}(\psi(2S) \rightarrow \gamma\eta) = (9.2 \pm 1.8) \times 10^{-7}$ [30,31], to be compared with $\mathcal{B}(J/\psi \rightarrow \gamma\eta') = (5.28 \pm 0.06) \times 10^{-3}$ and $\mathcal{B}(J/\psi \rightarrow \gamma\eta) = (1.090 \pm 0.013) \times 10^{-3}$ [31]. The naive “12% rule” follows from the measured leptonic widths and predicts that exclusive $\psi(2S)$ rates be about 12% of the corresponding J/ψ ones. Relative to this expectation, the $\gamma\eta'$ mode is suppressed by roughly a further factor of 5 and the $\gamma\eta$ mode by more than 2 orders of magnitude. This pattern is reminiscent of the long-standing $\rho\pi$ puzzle in $\psi(2S)$ decays [4,26] and may share its dynamical origin. Because the $2S$ radial wave function carries a node, a relativistic treatment able to resolve the short-distance overlap is the natural tool to probe the $2S$ rates, which makes $J/\psi, \psi(2S) \rightarrow \gamma\eta^{(\prime)}$ a particularly instructive testing ground for the relativistic-correction framework.

The remainder of this paper is organized as follows. In Sec. II we set up the calculation and derive the quark, gluon, and QED contributions in turn, including the relativistic corrections. The numerical analysis is presented in Sec. III, and we summarize in Sec. IV.

II. THEORETICAL FRAMEWORK

A. Quark-content contribution

For the quark content of the $\eta^{(\prime)}$, the decay proceeds in the parton picture through the annihilation of the $c\bar{c}$ pair into a photon and two virtual gluons, which subsequently convert into the $q\bar{q}$ content of the $\eta^{(\prime)}$. At leading order this is a one-loop process, a representative diagram of which is shown in Fig. 1.

We work in the rest frame of the decaying charmonium ψ_n , where $n = 1, 2$ labels J/ψ and $\psi(2S)$, and denote by K

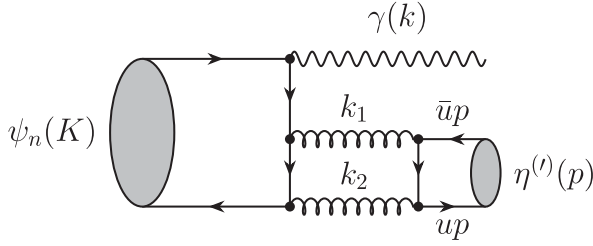


FIG. 1. A representative Feynman diagram for the quark-content contribution to $\psi_n \rightarrow \gamma \eta^{(\prime)}$, in which the $c\bar{c}$ pair annihilates into a photon and two virtual gluons that couple to the $q\bar{q}$ content of the $\eta^{(\prime)}$. Five further diagrams are obtained by permuting the photon and gluon attachments on the charm line.

the four-momentum of the charmonium of mass M , by k that of the real photon, and by $p = K - k$ that of the outgoing pseudoscalar of mass $m \equiv m_{\eta^{(\prime)}}$. Following the treatment of Körner *et al.* [14], which we also adopted in our previous nonrelativistic study of $J/\psi \rightarrow \gamma \eta^{(\prime)}$ [21], we evaluate the amplitude in two steps: the $\psi_n \rightarrow \gamma g^* g^*$ and $g^* g^* \rightarrow \eta^{(\prime)}$ amplitudes are computed separately and then joined through the two gluon propagators and the loop integration over the gluon momentum.

We first consider the $\psi_n \rightarrow \gamma g^* g^*$ amplitude. In the Bethe-Salpeter framework it is the four-dimensional convolution of the Bethe-Salpeter wave function $\Psi(K, q)$ with the hard kernel, the c and $\bar{c}$ carrying momenta $K/2 \pm q$,

$$A = \sqrt{3} \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[\Psi(K, q) \mathcal{O}(q)], \quad (1)$$

where $\sqrt{3}$ is the color factor of the $c\bar{c}$ pair. The relative momentum of the pair is decomposed into a component $q_{\parallel}$ longitudinal to K and a transverse component $\hat{q}$ with $\hat{q} \cdot K = 0$,

$$q^\mu = q_{\parallel}^\mu + \hat{q}^\mu, \quad q_{\parallel}^\mu = \frac{q \cdot K}{M^2} K^\mu. \quad (2)$$

In the rest frame of ψ_n , $K = (M, \mathbf{0})$ and the transverse component is purely spatial, $\hat{q} = (0, \hat{\mathbf{q}})$, with $\hat{\mathbf{q}}$ the relative three-momentum. For heavy quarkonium, whose constituents move nonrelativistically, the $c\bar{c}$ interaction can be treated as instantaneous, so the Bethe-Salpeter kernel does not depend on the relative-energy (longitudinal) component $q_{\parallel}$ [32,33]. In the same nonrelativistic regime the relative energy is small compared with the heavy-quark mass, so the constituent momenta reduce to $K/2 \pm \hat{q}$ and the hard kernel $\mathcal{O}$ likewise depends only on $\hat{q}$. The relative-energy dependence of $\Psi(K, q)$ is then integrated out over $q_{\parallel}$, which defines the equal-time Salpeter wave function as the longitudinal projection of the Bethe-Salpeter amplitude,

$$\psi(\hat{q}) = \frac{i}{2\pi} \int dq_{\parallel} \Psi(K, q), \quad (3)$$

and reduces the amplitude to the three-dimensional convolution [24,27]

$$A = -i\sqrt{3} \int \frac{d^3 \hat{q}}{(2\pi)^3} \text{Tr}[\psi(\hat{q}) \mathcal{O}(\hat{q})]. \quad (4)$$

The Salpeter wave function of the 1^{--} state factorizes into a scalar radial function $\chi(\hat{q})$ and a Dirac projector $\mathcal{P}(\hat{q})$ carrying the spin structure

$$\psi(\hat{q}) = \chi(\hat{q}) \mathcal{P}(\hat{q}). \quad (5)$$

Being a scalar, $\chi(\hat{q})$ can be pulled out of the Dirac trace and carries the entire normalization. The Dirac structure of the 1^{--} wave function is organized by the power-counting rule of Ref. [32], in which the covariants are ordered in powers of $1/M$. Retaining the leading and next-to-leading covariants, that is up to first order in $\hat{q}$, all Dirac structures reduce to a form sharing the single scalar function $\chi(\hat{q})$, so that the spin structure is carried entirely by the projector $\mathcal{P}(\hat{q})$ while $\chi(\hat{q})$ controls the convolution. The explicit reduction for the 1^{--} state is given in Ref. [33]. The projector $\mathcal{P}(\hat{q})$ then reads

$$\mathcal{P}(\hat{q}) = M \not{\epsilon} - \not{\epsilon} \not{K} + \frac{M}{m_c} (\hat{q} \cdot \epsilon) - \frac{1}{m_c} \not{K} (\hat{q} \cdot \epsilon) + \frac{1}{m_c} \not{K} \not{\hat{q}}, \quad (6)$$

where $\epsilon \equiv \epsilon(K)$ is the charmonium polarization vector with $\epsilon \cdot K = 0$. The first two terms, $M \not{\epsilon} - \not{\epsilon} \not{K}$, give the leading 3S_1 component in the nonrelativistic limit, while the terms linear in $\hat{q}$ are the subleading Dirac covariants induced by the relative motion of the $c\bar{c}$ pair. The next covariants, of order $\hat{q}^2$, are suppressed by a further power of $1/M$ in this counting [32] and are dropped. This truncation is sufficient for the q^2 relativistic corrections retained here. The dynamical correction is absorbed entirely into the soft scalar function $\chi(\hat{q})$, which is kept in full. The kinematical correction originates from the internal-momentum dependence of the hard-kernel denominators and becomes significant only when the internal momentum drives a propagator toward its threshold [24]. The $\hat{q}$ dependence carried by the Dirac projector enters only the numerator, where it is immaterial to the kinematical correction. For the $\psi(2S)$ the larger $\langle v^2 \rangle$ enhances these corrections, and their convergence is examined in Sec. III.

The hard kernel $\mathcal{O}(\hat{q})$ is the amplitude for $c\bar{c} \rightarrow \gamma g^* g^*$, in which the photon and the two gluons are attached to the charm line in all six orderings. With the charm momenta $K/2 \pm \hat{q}$, the hard kernel reads

$$\begin{aligned} \mathcal{O}(\hat{q}) = & i Q_c e g_s^2 \frac{\delta^{ab}}{6} \not{\epsilon}^*(k_2) S_F(l_2) \not{\epsilon}^*(k_1) S_F(l_1) \not{\epsilon}^*(k) \\ & + (5 \text{ permutations}), \end{aligned} \quad (7)$$

where $\varepsilon^*(k)$ is the photon polarization vector, $\varepsilon^*(k_1)$ and $\varepsilon^*(k_2)$ the gluon polarization vectors, a and b the gluon color indices, and $S_F(l) = (\not{l} + m_c)/(l^2 - m_c^2)$ the two charm propagators, whose momenta along the upper and lower charm lines are $l_1 = K/2 + \hat{q} - k$ and $l_2 = K/2 + \hat{q} - k - k_1$.

Since $\chi(\hat{q})$ is sharply peaked at $\hat{q} \simeq 0$, the trace $\text{Tr}[\mathcal{P}(\hat{q})\mathcal{O}(\hat{q})]$ is Taylor expanded to second order in $\hat{q}$,

$$\text{Tr}[\mathcal{P}(\hat{q})\mathcal{O}(\hat{q})] = \mathcal{T}_0 + \hat{q}^\mu \mathcal{T}_\mu + \hat{q}^\mu \hat{q}^\nu \mathcal{T}_{\mu\nu} + \mathcal{O}(\hat{q}^3), \quad (8)$$

where

$$\begin{aligned} \mathcal{T}_0 &= \text{Tr}[\mathcal{P}(0)\mathcal{O}(0)], & \mathcal{T}_\mu &= \left. \frac{\partial \text{Tr}[\mathcal{P}(\hat{q})\mathcal{O}(\hat{q})]}{\partial \hat{q}^\mu} \right|_{\hat{q}=0}, \\ \mathcal{T}_{\mu\nu} &= \left. \frac{1}{2} \frac{\partial^2 \text{Tr}[\mathcal{P}(\hat{q})\mathcal{O}(\hat{q})]}{\partial \hat{q}^\mu \partial \hat{q}^\nu} \right|_{\hat{q}=0}. \end{aligned} \quad (9)$$

The linear term vanishes by parity. Inside the convolution $\int d^3\hat{q} \chi(\hat{q})(\dots)$ the scalar function $\chi(\hat{q})$ depends only on $|\hat{q}|$ and is isotropic, so only the angular average of $\hat{q}^\mu \hat{q}^\nu$ over the directions of $\hat{q}$ survives. This average is

$$\langle \hat{q}^\mu \hat{q}^\nu \rangle_{\text{dir}} = |\hat{q}|^2 P_T^{\mu\nu}, \quad P_T^{\mu\nu} = \frac{1}{3} \left(-g^{\mu\nu} + \frac{K^\mu K^\nu}{M^2} \right), \quad (10)$$

where the transverse projector $P_T^{\mu\nu}$ projects onto the three-dimensional subspace orthogonal to K , the factor $1/3$ being the isotropic average over the three spatial directions in the rest frame. This replacement is valid only under the $\hat{q}$ integration and is a consequence of the isotropy of $\chi(\hat{q})$. The directionally averaged trace is then

$$\langle \text{Tr}[\mathcal{P}(\hat{q})\mathcal{O}(\hat{q})] \rangle_{\text{dir}} = \mathcal{T}_0 + |\hat{q}|^2 P_T^{\mu\nu} \mathcal{T}_{\mu\nu} + \mathcal{O}(\hat{q}^4). \quad (11)$$

Inserting this into the three-dimensional convolution and carrying out the $\hat{q}$ integration, the bound state enters only through its two moments at the origin,

$$\begin{aligned} \int \frac{d^3\hat{q}}{(2\pi)^3} \chi(\hat{q}) &= \sqrt{\frac{1}{4M}} \sqrt{\frac{1}{4\pi}} R_{\psi_n}(0), \\ \int \frac{d^3\hat{q}}{(2\pi)^3} |\hat{q}|^2 \chi(\hat{q}) &= -\sqrt{\frac{1}{4M}} \sqrt{\frac{1}{4\pi}} \nabla^2 R_{\psi_n}(0), \end{aligned} \quad (12)$$

where $\sqrt{1/(4M)}$ arises from the relativistic normalization of the scalar function $\chi(\hat{q})$ and $\sqrt{1/(4\pi)}$ is the S -wave spherical harmonic. Retaining the relative-momentum

expansion through the q^2 relativistic correction, the $\psi_n \rightarrow \gamma g^* g^*$ amplitude takes the compact form

$$A = -i\sqrt{3} \sqrt{\frac{1}{4M}} \sqrt{\frac{1}{4\pi}} \left[R_{\psi_n}(0) \mathcal{T}_0 - \nabla^2 R_{\psi_n}(0) P_T^{\mu\nu} \mathcal{T}_{\mu\nu} \right]. \quad (13)$$

The first term, governed by the wave function at the origin $R_{\psi_n}(0)$, is the leading-order amplitude, and the second, governed by its Laplacian $\nabla^2 R_{\psi_n}(0)$, is the q^2 relativistic correction. The neglected $\mathcal{O}(q^4)$ terms are set by the higher moments of the wave function.

The two virtual gluons are then coupled to the $\eta^{(\prime)}$ through its quark content. The quark-antiquark content of the $\eta^{(\prime)}$ is described by the twist-2 light-cone matrix element [34]

$$\begin{aligned} \langle \eta^{(\prime)}(p) | \bar{q}_\alpha(x) q_\beta(y) | 0 \rangle \\ = \frac{i}{4} f_{\eta^{(\prime)}}^q (\not{p} \gamma_5)_{\beta\alpha} \int_0^1 du e^{i(\bar{u}p \cdot y + up \cdot x)} \phi^q(u) + \dots, \end{aligned} \quad (14)$$

where the higher-twist contributions are omitted and $f_{\eta^{(\prime)}}^q$ is the flavor decay constant of the $\eta^{(\prime)}$. The flavor decay constants of the η and η' are not independent. In the quark-flavor basis of the η - η' system with a single mixing angle [9] they are parametrized by the two decay constants f_q and f_s and the mixing angle ϕ , whose values we specify in Sec. III A. The twist-2 DA is

$$\phi^q(u) = 6u\bar{u} \left[1 + \sum_{n=2,4,\dots} c_n^q(\mu) C_n^{3/2}(2u-1) \right], \quad (15)$$

with $\bar{u} = 1 - u$ and u the momentum fraction carried by the quark and $c_n^q(\mu)$ the scale-dependent Gegenbauer coefficients. Because the $\eta^{(\prime)}$ carries a flavor-singlet component, the quark DA does not evolve autonomously. Under QCD evolution its coefficients $c_n^q(\mu)$ mix with those of the two-gluon DA introduced in Sec. II B, and the coupled Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution equations and the associated anomalous dimensions are given in Ref. [35]. With the quark content of the $\eta^{(\prime)}$ thus specified, the two virtual gluons couple to it by attaching to the light $q\bar{q}$ pair. Replacing that pair by the matrix element of Eq. (14) and convolving the hard $g^* g^* \rightarrow q\bar{q}$ subprocess with the DA over the momentum fraction u gives the effective $g^* g^* \rightarrow \eta^{(\prime)}$ vertex

$$M^{\mu\nu} = -i(4\pi\alpha_s) \delta_{ab} \epsilon^{\mu\nu\rho\sigma} k_{1\rho} k_{2\sigma} \sum_{q=u,d,s} \frac{f_{\eta^{(\prime)}}^q}{6} \int_0^1 du \left(\frac{\phi^q(u)}{\bar{u}k_1^2 + uk_2^2 - u\bar{u}m^2 - m_q^2} + (u \leftrightarrow \bar{u}) \right), \quad (16)$$

with k_1, k_2 the gluon momenta, m_q the light-quark q mass, and m the $\eta^{(\prime)}$ mass.

The two subamplitudes are now joined through the gluon lines. Since the gluons are internal, their polarization vectors are replaced by the gluon propagators, and the upper $\psi_n \rightarrow \gamma g^* g^*$ amplitude of Eq. (13), stripped of those polarization vectors, enters as the tensor $A^{\mu\nu}$. Contracting it with the vertex $M_{\mu\nu}$ through the two propagators and integrating over the gluon loop momentum yields the decay amplitude

$$M_T = \frac{1}{2} \int \frac{d^4 k_1}{(2\pi)^4} A^{\mu\nu} M_{\mu\nu} \frac{i}{k_1^2} \frac{i}{k_2^2}, \quad (17)$$

with $k_2 = p - k_1$ and the factor $1/2$ compensating the interchange of the two identical gluons already summed in $A^{\mu\nu}$ and $M^{\mu\nu}$. Let $T^{\alpha\beta}$ denote M_T with the polarization vectors of the ψ_n and the photon removed. Lorentz covariance, parity, and gauge invariance fix it to the single parity-odd structure $T^{\alpha\beta} \propto \epsilon^{\alpha\beta\mu\nu} K_\mu k_\nu$, so the decay $\psi_n(1^{--}) \rightarrow \gamma(1^-) + \eta^{(\prime)}(0^{-+})$ has one independent helicity amplitude, extracted with the projector [14]

$$h^{\alpha\beta} = -\frac{i}{2(K \cdot k)} \epsilon^{\alpha\beta\mu\nu} K_\mu k_\nu, \quad (18)$$

normalized so that its contraction with $T^{\alpha\beta}$ returns the physical helicity amplitude,

$$H^q = h^{\alpha\beta} T_{\alpha\beta}. \quad (19)$$

Taking the Dirac trace, coupling the two gluons to the $\eta^{(\prime)}$ through Eq. (16), assembling the loop as in Eq. (17), and summing the six diagrams together with the flavor sum and the loop integration, the quark helicity amplitude takes the explicit form

$$H^q = \frac{2Q_c}{2\sqrt{3}} \sqrt{4\pi\alpha_e(4\pi\alpha_s)^2} \sqrt{\frac{1}{16\pi M}} \sum_{q=u,d,s} f_{\eta^{(\prime)}}^q \int_0^1 du \phi^q(u) \times \left[R_{\psi_n}(0) I_0^q(u) - \nabla^2 R_{\psi_n}(0) I_2^q(u) \right], \quad (20)$$

with $Q_c = 2/3$ the charm electric charge. The leading-order loop kernel is

$$I_0^q(u) = \int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{G_1 G_2 L} \left[\frac{N_A}{P_0 P_1} + \frac{N_B}{P_0 P_2} + \frac{N_C}{P_1 P_2} \right] + (u \leftrightarrow \bar{u}). \quad (21)$$

The five propagators are, with k_1 the loop momentum and $k_2 = p - k_1$, $G_1 = k_1^2 + i\epsilon$ and $G_2 = k_2^2 + i\epsilon$ for the two gluon lines, $L = (k_1 - up)^2 - m_q^2 + i\epsilon$ for the light-quark line at momentum fraction u , $P_0 = (k - K/2)^2 - m_c^2 + i\epsilon$ for the loop-momentum-independent charm propagator on the photon-emitting end of the charm line, and $P_1 = (k_1 - K/2)^2 - m_c^2 + i\epsilon$, $P_2 = (k_1 + k - K/2)^2 - m_c^2 + i\epsilon$ for the two loop-momentum-dependent charm propagators between consecutive gluon-charm vertices. The six Feynman

diagrams generated by permuting the photon and the two gluon attachments on the charm line are pairwise equal under the interchange of the identical gluons, collapsing to three independent topologies $X \in \{A, B, C\}$, namely two four-point box configurations ($X = A, B$, with the photon attached at either end of the charm line), in which P_0 factors out of the loop, and a five-point pentagon ($X = C$, with the photon attached between the two gluons), in which both k_1 -dependent charm lines remain off shell. The $(u \leftrightarrow \bar{u})$ term in Eq. (21) accounts for the second orientation of the light-quark loop, obtained through $u \rightarrow \bar{u} = 1 - u$ in L alone, since L is the only propagator that depends on u . Throughout the hard kernel we adopt the weak-binding approximation $m_c = M/2$. It is consistent with the treatment of the Salpeter wave function above, whose spinor structure is retained only to first order in $\hat{q}/M$. Our analyses of the three-gluon decays [24,27], built on the same projector and q^2 expansion, adopt it likewise. The three numerators are then polynomials in the scalar products

$$N_A = 32iM \left[(k \cdot k_1)^2 - (k \cdot k_1)(k_1 \cdot K) + k_1^2 k \cdot K \right]. \quad (22)$$

$$N_B = 32iM \left[(k \cdot k_1)^2 - (k \cdot k_1)(k_1 \cdot K) - (k_1 \cdot K)(k \cdot K) - (k \cdot k_1)(k \cdot K) + k_1^2 k \cdot K + M^2 k \cdot k_1 \right]. \quad (23)$$

$$N_C = 32iM \left[2(k \cdot k_1)(k_1 \cdot K) - \frac{M^2(k \cdot k_1)^2}{k \cdot K} - k_1^2 k \cdot K \right]. \quad (24)$$

The forms quoted here are the leading term of the relative-momentum expansion, equal to the hard kernel evaluated at $\hat{q} = 0$, where the c and $\bar{c}$ each carry half of the ψ_n momentum. The split into topologies A, B , and C corresponds to the three positions of the photon vertex along the charm line relative to the two gluon vertices. Individually they are not gauge invariant, the photon Ward identity being realized only by the telescoping sum over insertion points whose boundary terms vanish for the on-shell charm pair of the weak-binding projection. Only the sum over $X \in \{A, B, C\}$ is physical, and it reproduces the compact loop function of Ref. [21]. In addition, Bose symmetry under the interchange of the two gluons, $k_1 \rightarrow p - k_1$, relates the two boxes through $N_B = N_A(k_1 \rightarrow p - k_1)$ and maps N_C onto itself.

Beyond this leading order, the q^2 -correction kernel $I_2^q(u)$ is generated by restoring the relative-momentum dependence in the charm-line momenta as $K/2 \pm \hat{q}$ and taking the spherical average of the second derivative at $\hat{q} = 0$ according to Eq. (11). Because $\hat{q}$ enters the loop integrand only through the charm-line momenta, the second-derivative operation acts exclusively on the three charm propagators $P_0(\hat{q})$, $P_1(\hat{q})$, and $P_2(\hat{q})$. The two gluon propagators $G_{1,2}$ and the light-quark propagator L are inert under $\partial_{\hat{q}}$ and

remain in $I_2^q(u)$ in the very same form as in $I_0^q(u)$. Each application of the Laplacian on a charm propagator either raises its multiplicity by one or shifts a P_0 into a P_1 or P_2 , so the charm-line denominator structure of $I_2^q(u)$ is considerably richer than at leading order. The result is organized by the powers (a, b) of the charm denominators, with $D_{a,b}^A = P_0^a P_1^b$ and $D_{a,b}^B = P_0^a P_2^b$ for the two boxes and $D_{a,b}^C = P_1^a P_2^b$ for the pentagon, the powers running over $a, b \geq 1$ with $a + b \leq 4$ to give six distinct denominators per topology and 18 in all. Keeping these denominators in their derivative form, rather than placing them over a common denominator, makes the Passarino-Veltman type of each term manifest. With the same $(G_1 G_2 L)^{-1}$ skeleton inherited from leading order,

$$I_2^q(u) = \int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{G_1 G_2 L} \sum_{X \in \{A, B, C\}} \sum_{(a,b)} \frac{N_{a,b}^X(k_1)}{D_{a,b}^X} + (u \leftrightarrow \bar{u}). \quad (25)$$

For the three lowest, undifferentiated $(a, b) = (1, 1)$ terms the numerators take the compact factorized forms

$$N_{1,1}^A = \frac{32i}{3M} \left[2k_1 \cdot (k + K)(k \cdot K) + M^2 k_1 \cdot p - \frac{M^4 k \cdot k_1}{k \cdot K} \right], \quad (26)$$

$$N_{1,1}^C = \frac{64i}{3M} \left[4(k \cdot k_1)(k_1 \cdot K) + \frac{2M^2(k \cdot k_1)(k_1 \cdot K)}{k \cdot K} - 2(k_1 \cdot K)^2 - \frac{3M^2(k \cdot k_1)^2}{k \cdot K} - k_1 \cdot (k + K)(k \cdot K) - 3k_1^2 k \cdot K + M^2 k_1 \cdot K + 2M^2 k \cdot k_1 - \frac{M^4 k \cdot k_1}{k \cdot K} \right], \quad (27)$$

with $N_{1,1}^B = N_{1,1}^A(k_1 \rightarrow p - k_1)$. The remaining 15 numerators with $a + b > 2$ are polynomials of total degree up to 3 in the same loop scalar products. Their explicit forms, generated symbolically from the $\hat{q}^2$ derivative and verified by reconstruction against the original integrand, are collected in the Appendix. The subsequent numerical evaluation proceeds in two stages. The five propagators of the pentagon topology are linearly dependent in the loop momentum k_1 , so the pentagon is first reduced to four- and lower-point integrals by partial fractioning over k_1 with the ApartFF routine of FeynCalc [36–38]. The resulting four-, three- and two-point tensor integrals are then passed to Package-X [39,40], which performs the tensor reduction of the numerators and expresses the amplitude through the standard scalar Passarino-Veltman functions.

The leading-order analysis of Ref. [21] already noted a striking empirical feature of the loop kernel. $I_0^q(u)$ is, to very good accuracy, independent of both the momentum fraction u and the light-quark mass m_q across their

physically relevant ranges. That observation was reported there but only briefly commented upon, and its true origin was not identified. The present analysis brings out the same near constancy in the q^2 -correction kernel $I_2^q(u)$, a result by no means obvious in advance given the much richer charm-line denominator structure of $I_2^q(u)$ relative to $I_0^q(u)$. The numerical study of Sec. III confirms that under independent variations of the DA $\phi^q(u)$ and of m_q across the full physical interval the two kernels move by at most a few percent. The recurrence of this property at the relativistic level is too systematic to be coincidental and traces back to a sharp interplay between the analytic form of the integrands in Eqs. (21) and (25) and the hard-mechanism character of the underlying decay.

The analytic content of this interplay is most clearly seen at the integrand level. By construction u and m_q appear in the integrands of Eqs. (21) and (25) solely through the single light-quark propagator $L = (k_1 - up)^2 - m_q^2$. The gluon propagators $G_{1,2}$, the charm propagators P_0, P_1, P_2 and every numerator N_X and $N_{a,b}^X$ depend only on the hard scales m_c, M and on the fixed kinematic invariant $k \cdot K = (M^2 - m^2)/2$. Because $\hat{q}$ enters exclusively through the charm-line momenta, this single light-quark propagator is inert under the spherical $\hat{q}^2$ derivative and the identical factor L appears in $I_0^q(u)$ and in $I_2^q(u)$. The relativistic correction redistributes powers among the charm propagators but introduces no new u - or m_q -dependent denominator. Whatever insensitivity to u and m_q is exhibited by $I_0^q(u)$ is therefore inherited by $I_2^q(u)$ at the structural level, independent of the more elaborate charm sector of the latter, a fact that explains, at one stroke, both why the two kernels share the same flatness pattern and why the property is not a leading-order accident.

This structural fact is turned into a quantitative insensitivity by the strong hierarchy of scales that characterizes the decay,

$$m_q \lesssim 0.1 \text{ GeV} \ll m \lesssim 1 \text{ GeV} < m_c \approx M/2. \quad (28)$$

The loop integral is controlled by the charm propagators, whose denominators place the dominant support of the integrand at hard momenta $k_1^2 \sim m_c^2$. Writing $L = k_1^2 - 2u(k_1 \cdot p) + u^2 m^2 - m_q^2$, the light-quark mass enters only as m_q^2 and is turned by the loop integration into a chiral logarithm $(m_q^2/m_c^2) \ln(m_c^2/m_q^2)$, which vanishes as $m_q \rightarrow 0$ and stays below 2% for the physical value. The momentum fraction enters through the other two terms. The linear term $2u(k_1 \cdot p)$ is built from the hard product $k_1 \cdot p$ and is not small in itself, but it is odd about $u = 1/2$ and cancels against its $u \leftrightarrow \bar{u}$ partner in the symmetrized integrand of Eqs. (21) and (25), which enforces $I_{0,2}^q(u) = I_{0,2}^q(\bar{u})$. The quadratic term $u^2 m^2$ carries only the soft scale $p^2 = m^2$, so together with the symmetrization

it leaves in L a residual $u\bar{u}m^2 = [\frac{1}{4} - (u - 1/2)^2]m^2$. Its u -independent part $\frac{1}{4}m^2$ merely shifts the denominator uniformly and produces no modulation, so the only genuine u dependence resides in the $(u - 1/2)^2m^2$ piece. Set against the hard scale $m_c^2 \sim k_1^2$ that controls L , this soft shift changes the kernel only fractionally, by an amount even about $u = 1/2$ and of order $(u - 1/2)^2m^2/m_c^2$, a few percent at most across the physical range, and a direct evaluation of the scalar integrals confirms both this estimate and the chiral-logarithmic dependence on m_q .

One might worry that this hard-region argument is undermined by the soft region of the loop, where one gluon approaches its mass shell, $k_1 \rightarrow p$, and the two gluon denominators and the light-quark denominator vanish together. There an individual diagram is strongly enhanced and does become sensitive to m_q and to the shape of ϕ^q , precisely the region in which the soft scales could reassert themselves. This enhancement does not survive the sum over the six diagrams. Their numerators combine into a single gauge-invariant structure that vanishes fast enough as the soft configuration is approached to overcome the singularity of the propagators, so that the soft region contributes negligibly and the summed kernel is infrared finite. The same numerator zero cancels the light-quark pole when that line goes on shell, so that the chiral limit $m_q \rightarrow 0$ is reached smoothly. Gauge invariance thus switches off the one region that could have reinstated a strong dependence on u and m_q , and the kernel is left under the control of the hard region alone.

This flatness is not a generic feature of hard exclusive amplitudes but depends on which line the momentum fraction sits on. Here it sits on the soft, subdominant light-quark line, whose only hard imprint is the linear term that the symmetrization removes. In the electromagnetic transition form factor of the $\eta^{(\prime)}$, by contrast, the momentum fraction sits on the hard active-quark propagator, whose virtuality is set by u together with the large scale Q^2 . Such a hard propagator enters the amplitude as a pole $1/(uQ^2)$, and the $u \leftrightarrow \bar{u}$ symmetrization adds the crossed term $1/(\bar{u}Q^2)$. The hard kernel is then the power $1/(u\bar{u}Q^2)$, in which u multiplies the hard scale rather than a soft one, so that no scale hierarchy suppresses its variation and it grows without bound toward the end points $u \rightarrow 0, 1$. The convolution accordingly weights the inverse moments of the DA and remains genuinely sensitive to the shape of the DA.

Physically, the near constancy of both kernels reflects the dominance of the hard charm scale over every other scale in the process. The $c\bar{c}$ pair annihilates into a photon and two virtual gluons whose typical virtualities are themselves of order m_c^2 , much larger than m^2 and m_q^2 , so that the

$g^*g^* \rightarrow \eta^{(\prime)}$ subprocess is probed at a momentum scale well above any soft scale of the $\eta^{(\prime)}$. To the hard subprocess the $\eta^{(\prime)}$ accordingly appears as a quasipointlike composite object, much as a probe of large virtuality resolves only the integrated structure of a hadron in the deep-inelastic regime. The flatness of the loop kernel is the direct expression of this scale separation. Convolved with such a flat kernel, the DA then contributes essentially through its normalization alone, its zeroth Mellin moment, while its detailed shape and the value of m_q enter only through the small $(u - 1/2)^2m^2/m_c^2$ modulation.

A further and independent sign that the effect is set by the hard charm scale, and not by the particular kinematics of the radiative channel, comes from the closely related Dalitz process $J/\psi \rightarrow \eta^{(\prime)} e^+ e^-$ that we have studied in Ref. [41]. There the real photon is replaced by a virtual one of invariant mass $q^2 = M_{e^+e^-}^2$, which introduces an additional external scale into the hard kernel. The same near independence of the loop kernel on u and m_q is found to persist throughout the accessible dilepton region $q^2 \lesssim 1 \text{ GeV}^2$, for the simple reason that m_c^2 remains far larger than q^2 and continues to fix the scale of the loop. The flatness is therefore a genuine property of the hard charm loop, stable against a change of the external photon virtuality, and not an artifact of the on-shell point $q^2 = 0$.

The flatness of the quark-content kernels carries a direct phenomenological consequence. Because $I_0^q(u)$ and $I_2^q(u)$ vary so little across the integration region, the convolution that builds the decay amplitude becomes insensitive to the detailed shape of the DA and to the value of m_q so that the amplitude is fixed almost entirely by the normalization of the DA, namely the decay constants $f_{\eta^{(\prime)}}^q$. The branching ratios predicted for $\psi_n \rightarrow \gamma \eta^{(\prime)}$ inherit this insensitivity. Section III demonstrates it directly with the numerical helicity amplitudes, which show how weakly the prediction varies with the DA shape and with m_q .

B. Gluon-content contribution

The $\eta^{(\prime)}$ also carries an intrinsic two-gluon Fock component. The gluon-content contribution proceeds through the same hard subprocess $\psi_n \rightarrow \gamma g^* g^*$, but the two gluons now couple directly to this two-gluon component of the $\eta^{(\prime)}$ through its leading-twist gluonic DA, without the intermediate light-quark loop of the quark contribution. The two gluons carry momentum fractions u and $\bar{u}$ of the $\eta^{(\prime)}$ momentum p , and since these are fixed by the DA the process is tree level, with no loop integration. A representative diagram is shown in Fig. 2.

The leading-twist two-gluon matrix element is [35,42,43]

$$\langle \eta^{(\prime)}(p) | A_a^a(x) A_b^b(y) | 0 \rangle = \frac{1}{4} \epsilon_{a\beta\rho\sigma} \frac{k^\rho p^\sigma}{p \cdot k} \frac{C_F}{\sqrt{3}} \frac{\delta^{ab}}{8} f_{\eta^{(\prime)}}^g \int_0^1 du e^{i(u p \cdot x + \bar{u} p \cdot y)} \frac{\phi^g(u)}{u\bar{u}}, \quad (29)$$

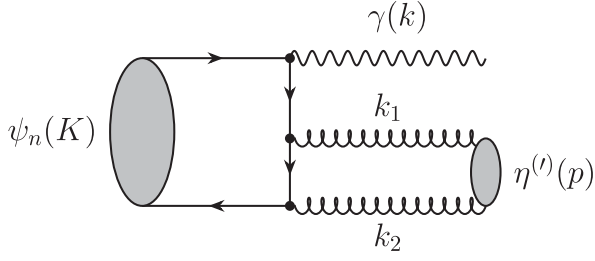


FIG. 2. A representative Feynman diagram for the gluon-content contribution to $\psi_n \rightarrow \gamma \eta^{(r)}$, in which the $c\bar{c}$ pair annihilates into a photon and two collinear gluons that directly form the two-gluon content of the $\eta^{(r)}$. Two further diagrams are obtained by permuting the photon and gluon attachments on the charm line.

with $f_{\eta^{(r)}}^g = \frac{1}{\sqrt{3}}(f_{\eta^{(r)}}^u + f_{\eta^{(r)}}^d + f_{\eta^{(r)}}^s)$ and the gluonic twist-2 DA

$$\phi^g(u) = 30u^2\bar{u}^2 \sum_{n=2,4,\dots} c_n^g(\mu) C_{n-1}^{5/2}(2u-1). \quad (30)$$

Contracting the $\psi_n \rightarrow \gamma g^* g^*$ amplitude of Eq. (17) with this gluonic vertex, inserting the two gluon propagators at the fixed momenta up and $\bar{u}p$, and projecting onto the helicity amplitude with the projector of Eq. (18) yields, with the relativistic correction in the form of Eq. (13),

$$H^g = \frac{2Q_c}{9} \sqrt{4\pi\alpha_e(4\pi\alpha_s)} \frac{1}{\sqrt{\pi M}} \frac{f_{\eta^{(r)}}^g}{M} \int_0^1 du \frac{\phi^g(u)}{u\bar{u}} \times [R_{\psi_n}(0)I_0^g(u) - \nabla^2 R_{\psi_n}(0)I_2^g(u)]. \quad (31)$$

The two kernels I_0^g and I_2^g are, as in the quark content, the value of the hard amplitude at $\hat{q} = 0$ and its q^2 correction, and the three diagrams are labeled $X \in \{A, B, C\}$ by the position of the photon vertex on the charm line, as in the quark case. In the weak-binding approximation $m_c = M/2$ the leading-order kernel reduces to the two terms of topologies A and B,

$$I_0^g(u) = \frac{N_g^A}{P_0 P_u} + \frac{N_g^B}{P_0 P_{\bar{u}}}, \quad (32)$$

topology C, in which the photon is emitted between the two gluon vertices, being absent at this order. Each diagram of that topology is proportional to the squared component of its gluon momentum orthogonal to the plane spanned by the photon and quarkonium momenta. The leading-twist projection assigns the gluons the collinear momenta up and $\bar{u}p$ with vanishing transverse components, and the contribution drops out. The three charm propagators on the line are

$$\begin{aligned} P_0 &= -k \cdot K + i\epsilon, \\ P_u &= -u[M^2\bar{u} + (k \cdot K)(u - \bar{u})] + i\epsilon, \\ P_{\bar{u}} &= -\bar{u}[M^2u + (k \cdot K)(\bar{u} - u)] + i\epsilon, \end{aligned} \quad (33)$$

left in factored form rather than combined over the common denominator $P_0 P_u P_{\bar{u}}$. The propagator P_0 stands apart, since it carries no gluon momentum and is fixed by the external photon and quarkonium momenta alone so that it is independent of u and collapses to the constant $-k \cdot K$, while P_u and $P_{\bar{u}}$ carry the gluon momenta up and $\bar{u}p$. Across the physical interval the constant P_0 and the denominators P_u , $P_{\bar{u}}$ stay nonzero, the latter reaching zero only at the end points $u = 0, 1$, where the gluonic DA suppresses the integrand. No charm line is therefore brought on shell as u is integrated, the hard kernel remains real, and the gluon-content amplitude develops no absorptive part. This conclusion extends unchanged to the q^2 correction, whose denominators are only higher powers $P_0^a P_u^b$, $P_0^a P_{\bar{u}}^b$, and $P_u^a P_{\bar{u}}^b$ of the same factors and open no new threshold. The numerators are

$$N_g^A = \frac{1}{2}(k \cdot K)M^2u, \quad (34)$$

with $N_g^B = -N_g^A(u \leftrightarrow \bar{u})$. The relative minus sign reflects Bose symmetry. Interchanging the two identical gluons swaps u and $\bar{u}$ and maps topology A into B, while the antisymmetric $\epsilon_{\alpha\beta\rho\sigma}$ structure of the matrix element of Eq. (29) changes sign, so the kernel must be odd under $u \leftrightarrow \bar{u}$, just as ϕ^g is. Summing the two terms reproduces the compact form of the leading-order gluon kernel obtained in Ref. [21],

$$I_0^g(u) = \frac{2x(2u-1)}{1-x^2(1-2u)^2}, \quad (35)$$

where $x = m^2/M^2$ denotes the squared ratio of the $\eta^{(r)}$ to the charmonium mass.

The q^2 correction is obtained by restoring the charm-line momenta as $K/2 \pm \hat{q}$ and taking the spherical average of the second derivative at $\hat{q} = 0$, equivalent to the transverse projection $\frac{1}{2}P_T^{\mu\nu}\partial^2/\partial\hat{q}^\mu\partial\hat{q}^\nu$. Being tree level it involves no loop integration. The leading-order vanishing of topology C rests on the weak-binding configuration itself, with both charm quarks carrying the momentum $K/2$. The relative momentum shifts the charm momenta to $K/2 \pm \hat{q}$, the cancellation no longer operates away from this symmetric point, and the second $\hat{q}$ derivative revives this topology, although only through its lowest term. Collecting the result by the same powers (a, b) as in Eq. (25) gives

$$I_2^g(u) = \sum_{X \in \{A, B, C\}} \sum_{(a, b)} \frac{N_{g, a, b}^X}{D_{a, b}^X}, \quad (36)$$

with $D_{a, b}^A = P_0^a P_u^b$, $D_{a, b}^B = P_0^a P_{\bar{u}}^b$, and $D_{a, b}^C = P_u^a P_{\bar{u}}^b$, the sums for $X = A, B$ running over $a + b \leq 4$ and topology C contributing only $(a, b) = (1, 1)$, every other term vanishing. The nonzero numerators of topology A are

$$\begin{aligned}
N_{g,1,1}^A &= \frac{1}{6}(2(k \cdot K) + M^2), & N_{g,1,2}^A &= \frac{1}{6}(k \cdot K)u(4(k \cdot K)u + 3M^2), \\
N_{g,2,1}^A &= \frac{1}{6}(k \cdot K)u(4(k \cdot K) + 3M^2), & N_{g,1,3}^A &= \frac{2}{3}(k \cdot K)^3 u^3, \\
N_{g,2,2}^A &= \frac{2}{3}(k \cdot K)^3 u^2, & N_{g,3,1}^A &= \frac{2}{3}(k \cdot K)^3 u,
\end{aligned} \tag{37}$$

the topology- C numerator $N_{g,1,1}^C = \frac{1}{3}(u - \bar{u})((k \cdot K) - M^2)$, and those of topology B fixed by the symmetry $N_{g,a,b}^B = -N_{g,a,b}^A (u \leftrightarrow \bar{u})$.

As the compact kernel of Eq. (35) shows, at leading order the gluon contribution is suppressed by the overall factor $x = m^2/M^2$, reflecting the near-on-shell gluons and the Ore-Powell form of their coupling to the $\eta^{(\prime)}$ [44,45]. The q^2 correction is a relativistic effect smaller still, so the gluon contribution is subleading to the quark one. Its dependence on the DA, on the other hand, differs in character from the quark case. In the leading-order kernel the denominator departs from unity only at order x^2 and varies little across the momentum-fraction range, much as in the quark case, while the numerator carries an explicit and unsuppressed dependence on u . The gluon hard kernel is therefore genuinely u dependent rather than flat, and the convolution probes the shape of the gluonic DA rather than its normalization alone. The relativistic correction is built from the same denominators raised to higher powers and from numerators that likewise carry the momentum-fraction dependence, so the leading-order and q^2 gluon kernels probe the DA in the same way. Both stand in contrast to the flatness established in Sec. II A, where the hard charm scale dominating the loop integration strongly suppresses the momentum-fraction dependence. Moreover, the gluon process involves no light-quark propagator, so unlike the quark contribution it carries no dependence on the light-quark mass m_q .

C. QED contribution

Finally, the $\eta^{(\prime)}$ can be produced electromagnetically. The $c\bar{c}$ pair annihilates into a single virtual photon, $\psi_n \rightarrow \gamma^*$, which produces the light-quark-antiquark pair of the $\eta^{(\prime)}$. The real photon is radiated from this light-quark line, leaving one internal light-quark propagator, and the pair is

projected onto the meson by the quark DA. This contribution shares the helicity structure and the bound-state projection of the quark one. The two-gluon exchange between the charm and light-quark lines is replaced by a single virtual-photon exchange, so this part of the amplitude is of order α_e instead of α_s^2 and carries the squared light-quark charges Q_q^2 . The process is tree level and is shown in Fig. 3.

A distinctive feature of this contribution is that the $c\bar{c}$ pair annihilates at a single γ^* vertex, so the charm line carries no internal propagator. The hard amplitude is then independent of the relative momentum $\hat{q}$, and the q^2 relativistic correction vanishes identically, leaving the QED contribution purely at leading order,

$$\begin{aligned}
H^{\text{QED}} &= -Q_c (4\pi\alpha_e)^{3/2} \sqrt{\frac{3}{\pi M}} R_{\psi_n}(0) \\
&\times \sum_{q=u,d,s} \frac{Q_q^2 f_{\eta^{(\prime)}}^q}{M} \int_0^1 du \phi^q(u) I^{\text{QED}}(u), \tag{38}
\end{aligned}$$

with the $\hat{q}$ -independent QED kernel

$$I^{\text{QED}}(u) = \frac{1-x}{2} \left(\frac{1}{u - u\bar{u}x - m_q^2/M^2 + i\epsilon} + (u \leftrightarrow \bar{u}) \right). \tag{39}$$

This kernel fixes how the QED amplitude depends on the light-quark mass and on the DA. The light-quark mass appears only through the strongly power-suppressed ratio m_q^2/M^2 in the propagator, so that the kernel is almost independent of it. With the $i\epsilon$ prescription the propagator acquires an imaginary part only where its denominator vanishes, at $u \simeq m_q^2/M^2$. This point lies very close to the

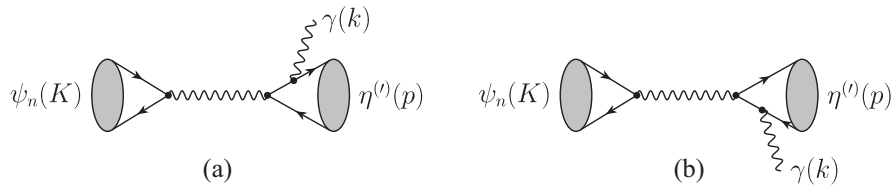


FIG. 3. Feynman diagrams for the QED contribution $\psi_n \rightarrow \gamma^* \rightarrow \gamma \eta^{(\prime)}$, in which the $c\bar{c}$ pair annihilates into a single virtual photon that produces the light-quark-antiquark pair of the $\eta^{(\prime)}$. In contrast to the quark-antiquark and two-gluon contributions, the real photon is radiated from a final-state light-quark line, the quark in (a) and the antiquark in (b).

end point $u = 0$, where the DA nearly vanishes, so the absorptive part is very small and scales as m_q^2/M^2 .

The two terms of Eq. (39) form a $u \leftrightarrow \bar{u}$ pair, so the kernel is even under this exchange. It is peaked near the end points rather than flat, but its $1/(u\bar{u})$ growth is offset by the end point suppression $u\bar{u}$ of the leading-twist DA, leaving a regular convolution. This evenness leaves the convolution dominated by the DA normalization, and the shape enters only through the low Gegenbauer moments. The QED amplitude is therefore set mainly by the decay constants and is weakly sensitive to the shape. This is opposite of the gluon kernel of Eq. (35), which is odd in $u \leftrightarrow \bar{u}$ and so weights the shape of the DA rather than its normalization, and it is distinct from the quark contribution, where the flatness produced by the loop integration removes the shape dependence directly.

III. NUMERICAL ANALYSIS

A. Decay width and input parameters

The partial width for $\psi_n \rightarrow \gamma \eta^{(\prime)}$ is fixed by the modulus squared of the single helicity amplitude, to which the quark, gluon, and QED contributions add coherently,

$$\Gamma(\psi_n \rightarrow \gamma \eta^{(\prime)}) = \frac{M^2 - m^2}{16\pi M^3} \frac{2}{3} |H^q + H^g + H^{\text{QED}}|^2. \quad (40)$$

The first factor is the two-body phase space for a massless photon recoiling against the $\eta^{(\prime)}$. With the quantization axis taken along the $\eta^{(\prime)}$ flight direction,¹ conservation of the angular momentum along the axis, $\lambda_V = \lambda_\eta - \lambda_\gamma = -\lambda_\gamma$, permits only the two helicity configurations $(\lambda_V, \lambda_\gamma) = (+1, -1)$ and $(-1, +1)$, which are equal by parity and define the single independent amplitude H . The factor 2 is then the sum over these two nonzero terms, and the factor $1/3$ the average over the three spin states of the initial ψ_n .

Of the three contributions the quark term, in which the $c\bar{c}$ pair annihilates into $\gamma g^* g^*$ and the two gluons materialize the $\eta^{(\prime)}$ through its leading quark-antiquark Fock component, dominates both channels. The gluon contribution, in which the two collinear gluons couple to the two-gluon component of the $\eta^{(\prime)}$, enters only as a correction. As shown in Sec. II B, its hard kernel scales with the ratio m^2/M^2 of the light meson to the charmonium mass and vanishes in the chiral limit, so the gluon contribution is strongly suppressed for the lighter η and grows for the heavier and more singletlike η' . The QED contribution $c\bar{c} \rightarrow \gamma^* \rightarrow \gamma \eta^{(\prime)}$, in which the light quarks of the $\eta^{(\prime)}$ are produced electromagnetically rather than through the two gluons, is suppressed relative to the quark amplitude by

$\sim \alpha_e/\alpha_s^2$ and by the light-quark electric charges, and is the smallest. The quark and gluon amplitudes are complex while the QED amplitude is real, the absorptive parts arising from internal propagators that reach their mass shell.

The evaluation of Eq. (40) requires three groups of inputs. These are the charmonium and meson kinematics together with the nonrelativistic wave-function parameters, the DAs of the quark and gluon Fock components of the $\eta^{(\prime)}$, and the η - η' mixing parameters that fix the flavor decay constants. We specify them in turn. The charmonium masses $M_{J/\psi} = 3.097$ and $M_{\psi(2S)} = 3.686$ GeV, the meson masses $m_\eta = 0.548$ and $m_{\eta'} = 0.958$ GeV, the fine-structure constant $\alpha_e = 1/137$, and the strong coupling $\alpha_s(M_Z) = 0.1180$ are taken from the Particle Data Group [31]. The strong coupling is run at one loop to the hard scale $\mu = M_{\psi_n}/2$ of each decay, giving $\alpha_s(M_{J/\psi}/2) = 0.294$ and $\alpha_s(M_{\psi(2S)}/2) = 0.276$.

The first nonperturbative ingredient is the charmonium wave function at the origin. The leading-order amplitude is governed by the wave function itself, $R_{\psi_n}(0)$, and the relativistic correction by its Laplacian $\nabla^2 R_{\psi_n}(0)$, which is controlled by the internal motion of the charm quarks. Both are fixed from the Cornell potential of Eichten and Quigg in its frozen- α_s form, a Coulomb-plus-linear potential whose coupling runs at short distance and saturates at long distance [46,47]. The wave functions at the origin, $|R_{J/\psi}(0)|^2 = 1.0952$ and $|R_{\psi(2S)}(0)|^2 = 0.6966$ GeV³, are taken from the tabulation of Eichten and Quigg [48], and solving the radial Schrödinger equation in the same potential gives the ratios

$$\frac{\nabla^2 R_{J/\psi}(0)}{R_{J/\psi}(0)} = -0.53 \text{ GeV}^2, \quad \frac{\nabla^2 R_{\psi(2S)}(0)}{R_{\psi(2S)}(0)} = -1.59 \text{ GeV}^2. \quad (41)$$

The $\psi(2S)$ value is 3 times the J/ψ one, the radially excited state being spatially more extended and less tightly bound, so that the q^2 corrections are intrinsically far more important there. For the J/ψ the ratio in Eq. (41) agrees with the Cornell-potential evaluations of the corresponding order- v^2 NRQCD matrix element in Refs. [49,50], and for the $\psi(2S)$ it is somewhat larger than the value adopted in Ref. [26], which follows from the Gremm-Kapustin relation with the binding energy taken as $M_{\psi(2S)} - 2m_c$. The wave function at the origin itself can be checked independently, since the leptonic width $\Gamma(\psi_n \rightarrow e^+ e^-)$ is fixed by $R_{\psi_n}(0)$. Evaluated from the leptonic-width formula with its first-order QCD correction [51], using the α_s values above, the adopted $R_{\psi_n}(0)$ reproduce the measured $\Gamma(J/\psi \rightarrow e^+ e^-)$ and $\Gamma(\psi(2S) \rightarrow e^+ e^-)$ [31] to about 2% and 10%, respectively. This check is what controls the potential-model dependence of the predictions, which

¹We choose $+\hat{z}$ along the $\eta^{(\prime)}$ momentum, so that the photon moves along $-\hat{z}$. This convention fixes the nonzero helicity configurations to coincide with those of Ref. [14] and underlies the helicity-projector construction of Sec. II.

TABLE I. The three DA models of Ref. [21]. The table lists the Gegenbauer moments a_2^q, a_4^q of the quark DA and a_2^g of the two-gluon DA at $\mu_0 = 1$ GeV. The spread among the models quantifies the DA systematic.

Model	a_2^q	a_4^q	a_2^g
I	0.10	0.10	-0.26
II	0.20	0.00	-0.31
III	0.25	-0.10	-0.25

would otherwise be substantial. Among the four standard potentials tabulated earlier by the same authors [52], $|R_{1S}(0)|^2$ ranges from 0.810 to 1.454 GeV³ and $|R_{2S}(0)|^2$ from 0.418 to 0.927 GeV³, ranges that cover the values adopted here, and their constituent charm masses lie between 1.48 and 1.84 GeV, a range that likewise covers the weak-binding value $m_c = M/2$ used in the hard kernel. Because the radiative and the leptonic widths carry the same factor $|R_{\psi_n}(0)|^2$, it cancels in their ratio, so that the branching ratios given below are in effect predictions of $\Gamma(\psi_n \rightarrow \gamma\eta^{(\prime)})/\Gamma(\psi_n \rightarrow e^+e^-)$, with the measured leptonic width fixing their overall size. What remains of the model dependence is then the accuracy with which the leptonic widths are reproduced, 2% and 10%. Beyond that common factor, the relativistic correction involves the wave function only through the ratio $\nabla^2 R_{\psi_n}(0)/R_{\psi_n}(0)$, which was compared with other determinations above.

The DAs are the second nonperturbative ingredient. With the conventions of Sec. II, the leading-twist quark-anti-quark DA is fixed by its first two Gegenbauer moments a_2^q and a_4^q and the two-gluon DA by its lowest moment a_2^g . Because the $\eta^{(\prime)}$ amplitudes are not precisely known, we use the three representative models of Table I, taken from the $\eta^{(\prime)}$ transition-form-factor analysis of Ref. [43]. At the reference scale $\mu_0 = 1$ GeV the three profiles differ markedly. Model I has small moments and is single humped, close to the asymptotic amplitude $6u\bar{u}$ of perturbative QCD [53,54], and a single broad peak centered at $u = 1/2$. Model II, with $a_2^q = 0.20$ and $a_4^q = 0$, is broader and flat topped, its central value lowered below the asymptotic one, of the moderate type favored by light-cone sum rules [42]. Model III, with the largest $a_2^q = 0.25$ and a negative $a_4^q = -0.10$, suppresses the midpoint into a double-humped profile, with a dip at $u = 1/2$ and two maxima displaced toward the end points, in the manner of the Chernyak-Zhitnitsky amplitude [34,55]. The a_2^g values of the three models span those of the more recent lattice-QCD [56,57] and QCD sum-rule [58] determinations, so the set still brackets the present spread of $\eta^{(\prime)}$ DAs. The moments are quoted at $\mu_0 = 1$ GeV and evolved to the hard scale $\mu = M/2$ by the coupled quark-gluon ERBL equations [35], which damp the differences as the moments shrink toward the asymptotic limit.

TABLE II. The two η - η' mixing schemes in the quark-flavor basis [9], given by the decay constants f_q, f_s and the mixing angle ϕ . “Lattice” denotes the determination of the Extended Twisted Mass Collaboration [59] and “ η' TFF” the transition-form-factor analysis [60], the latter evaluated with $f_\pi = 0.1302$ GeV.

Scheme	f_q (GeV)	f_s (GeV)	ϕ (°)
Lattice	0.1386(44)	0.1707(33)	39.3(2.0)
η' TFF	0.1419(26)	0.1250(52)	33.5(0.9)

The third ingredient is the η - η' mixing. In the quark-flavor basis the physical η and η' are superpositions of the unphysical states $\eta_q = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $\eta_s = s\bar{s}$, rotated into the mass eigenstates by a single angle ϕ ,

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \eta_q \\ \eta_s \end{pmatrix}. \quad (42)$$

In the FKS scheme [9], where the decay constants f_q and f_s of the η_q and η_s components are assumed to be common to the η and the η' , their flavor decay constants are fixed by these two constants and the single mixing angle ϕ ,

$$\begin{aligned} f_{\eta}^{u(d)} &= \frac{f_q}{\sqrt{2}} \cos \phi, & f_{\eta}^s &= -f_s \sin \phi, \\ f_{\eta'}^{u(d)} &= \frac{f_q}{\sqrt{2}} \sin \phi, & f_{\eta'}^s &= f_s \cos \phi. \end{aligned} \quad (43)$$

These three parameters are phenomenological, fixed not from first principles but from physical processes so that different determinations return somewhat different values, the mixing angle in particular varying by several degrees. We therefore take two representative determinations as inputs, collected in Table II. They agree on f_q but differ appreciably in the mixing angle. The lattice calculation of Ref. [59] gives the larger angle, $\phi \simeq 39^\circ$, which sits close to the long-standing value of the FKS analysis of low-energy η - η' phenomenology [9]. The transition-form-factor (TFF) analysis of Ref. [60] gives a markedly smaller angle, $\phi \simeq 33.5^\circ$, of the kind preferred by hard, short-distance amplitudes, as we also found at leading order from $J/\psi \rightarrow \gamma\eta^{(\prime)}$ itself [21]. We evaluate the J/ψ observables in both, adopting the lattice scheme as the default and retaining the η' TFF scheme for the comparison of total widths.

B. Results for the J/ψ

We now present the numerical results, taking $J/\psi \rightarrow \gamma\eta^{(\prime)}$ as the example and working in the lattice mixing scheme, the η' TFF scheme yielding the same pattern. The purpose of this subsection is to exhibit how the helicity amplitudes depend on the two soft inputs of the hard mechanism, the DA and the

TABLE III. Quark-component helicity amplitude for $J/\psi \rightarrow \gamma\eta^{(\prime)}$ in the lattice mixing scheme, in units of 10^{-3} GeV, for the three DA models and $m_q = 0.01, 0.05, 0.10$ GeV. H_{LO}^q is the leading order and $H_{q^2}^q$ the q^2 correction, with $H^q = H_{\text{LO}}^q + H_{q^2}^q$.

Model	m_q (GeV)	$J/\psi \rightarrow \gamma\eta$		$J/\psi \rightarrow \gamma\eta'$	
		H_{LO}^q	$H_{q^2}^q$	H_{LO}^q	$H_{q^2}^q$
I	0.01	$-1.01 - 0.31i$	$-0.45 - 0.07i$	$-5.31 - 3.58i$	$-1.86 - 1.68i$
I	0.05	$-1.02 - 0.30i$	$-0.45 - 0.06i$	$-5.32 - 3.57i$	$-1.87 - 1.66i$
I	0.10	$-1.02 - 0.29i$	$-0.44 - 0.04i$	$-5.36 - 3.53i$	$-1.90 - 1.62i$
II	0.01	$-1.02 - 0.31i$	$-0.46 - 0.08i$	$-5.31 - 3.61i$	$-1.87 - 1.71i$
II	0.05	$-1.02 - 0.31i$	$-0.45 - 0.06i$	$-5.33 - 3.60i$	$-1.88 - 1.69i$
II	0.10	$-1.02 - 0.29i$	$-0.44 - 0.04i$	$-5.37 - 3.56i$	$-1.91 - 1.65i$
III	0.01	$-1.02 - 0.31i$	$-0.46 - 0.07i$	$-5.32 - 3.63i$	$-1.88 - 1.72i$
III	0.05	$-1.02 - 0.31i$	$-0.46 - 0.06i$	$-5.33 - 3.62i$	$-1.89 - 1.70i$
III	0.10	$-1.03 - 0.29i$	$-0.44 - 0.04i$	$-5.38 - 3.58i$	$-1.91 - 1.66i$

light-quark mass m_q . This near insensitivity was already noted at leading order in our earlier study [21]. Here we confirm it and examine how the q^2 correction depends on the same two inputs. We scan the three DA models of Table I, evolved to the hard scale $\mu = M_{\psi_n}/2$, and three light-quark masses $m_q = 0.01, 0.05, 0.10$ GeV whose range brackets the physical u , d , and s masses.

The quark, gluon, and QED amplitudes are collected in Tables III–V. The quark and QED tables list nine entries each, for the three DA models and three light-quark masses, and the gluon table three, one per DA model. For the quark and gluon components we separate the helicity amplitude into a leading-order part H_{LO}^X and a q^2 relativistic correction $H_{q^2}^X$, with $H^X = H_{\text{LO}}^X + H_{q^2}^X$ ($X = q, g$) so that the size of the correction can be read off directly. The QED component receives no q^2 relativistic correction, with its helicity amplitude denoted $H_{\eta^{(\prime)}}^{\text{QED}}$. All amplitudes are in units of 10^{-3} GeV. Of the three contributions the quark amplitude is by far the largest. The gluon amplitude is about 4% of it for $\gamma\eta$ and 11% for $\gamma\eta'$, and the QED amplitude about 24% for $\gamma\eta$ and 7% for $\gamma\eta'$. The three add constructively but differ in analytic structure. The quark amplitude is a genuine one-loop object and is complex, its imaginary, absorptive part generated wherever the internal light-quark and gluon lines of the $g^*g^* \rightarrow \eta^{(\prime)}$ subgraph reach their mass shell over the bulk of the momentum-fraction integration. The QED and gluon amplitudes are real. In the QED term the single light-quark propagator reaches its mass shell only near the end point of the momentum fraction, where the quark DA vanishes, so its absorptive part is negligible.² The gluon amplitude comes from $c\bar{c}$

annihilation into two collinear gluons, and in the weak-binding kinematics $m_c = M/2$ its two charm propagators never reach their mass shell, so it develops no absorptive part either. The total is thus quark dominated, the small real gluon and QED terms adding to its real part while its imaginary part is carried by the quark loop alone.

As Table III shows, the dominant quark amplitude is almost completely insensitive to the two soft inputs of the hard mechanism, the DA and the light-quark mass. Across

TABLE IV. Gluon-component helicity amplitude for $J/\psi \rightarrow \gamma\eta^{(\prime)}$ in the lattice mixing scheme, in units of 10^{-3} GeV, with H_{LO}^g the leading order and $H_{q^2}^g$ the q^2 correction. The component is independent of m_q .

Model	$J/\psi \rightarrow \gamma\eta$		$J/\psi \rightarrow \gamma\eta'$	
	H_{LO}^g	$H_{q^2}^g$	H_{LO}^g	$H_{q^2}^g$
I	-0.03	-0.03	-0.51	-0.50
II	-0.03	-0.03	-0.61	-0.60
III	-0.03	-0.03	-0.49	-0.48

TABLE V. QED-component helicity amplitude for $J/\psi \rightarrow \gamma\eta^{(\prime)}$ in the lattice mixing scheme, in units of 10^{-3} GeV, for the three DA models and $m_q = 0.01, 0.05, 0.10$ GeV. The tabulated H^{QED} is real and receives no q^2 correction.

Model	m_q (GeV)	H_{η}^{QED}	$H_{\eta'}^{\text{QED}}$
I	0.01	-0.36	-0.58
I	0.05	-0.36	-0.58
I	0.10	-0.37	-0.59
II	0.01	-0.36	-0.58
II	0.05	-0.36	-0.58
II	0.10	-0.37	-0.59
III	0.01	-0.35	-0.56
III	0.05	-0.35	-0.56
III	0.10	-0.35	-0.57

²The QED amplitude carries a negligible imaginary part, about 10^{-9} GeV at the representative $m_q = 0.01$ GeV against a real part of order 10^{-4} GeV; it comes from the light-quark pole of the kernel near the end point $u \simeq m_q^2/M^2$, is suppressed by m_q^2/M^2 , and is discarded.

$m_q = 0.01\text{--}0.10$ GeV it moves by less than 1%, and across the three DA models it varies by less than 2%, so the leading contribution is fixed to high accuracy independent of these poorly constrained quantities. This robustness was already established at leading order in our earlier work [21] and is now seen to survive the q^2 correction. The QED amplitude in Table V is equally insensitive, changing by about 2% across the m_q range and by about 3% across the DA models. The gluon amplitude carries no light-quark propagator and is accordingly independent of m_q , but it is the one component with a non-negligible dependence on the DA, as Table IV shows. Notably, the two gluon amplitudes depend on the DA by a similar amount, but the $\gamma\eta$ one is so small that its variation falls below 10^{-5} GeV and it appears unchanged in the table. The quark and QED kernels are smooth functions of the momentum fraction whose convolution washes out the higher Gegenbauer moments, so these two barely respond to the DA shape. The gluon instead couples directly to the two-gluon DA and its kernel preserves that shape, so the model-to-model change of the gluon moment passes almost undiluted into the amplitude, which varies by up to about 20%. This is the largest variation among the three component amplitudes, yet the gluon amplitude is itself the smallest of the three, so in the coherent sum the variation amounts to only about 1%, leaving the full amplitude essentially independent of the DA. We note in passing that these entries carry no mixing-parameter error, since the amplitudes barely depend on the light-quark mass and are thus blind to its flavor, so the mixing factors out as a single overall weight, independent of the DA and m_q , that leaves the dependence shown here untouched. We therefore defer its uncertainty to the branching ratios below rather than repeat it on each amplitude.

The relativistic correction acts differently on the three contributions. The quark term, which dominates, is enhanced substantially, its modulus $|H^q|$ growing by about 40% in both channels while its phase shifts by only a couple of degrees, so the correction scales the dominant amplitude up rather than rotating it. The absorptive part of H^q comes from the internal light-quark and gluon lines of the loop going on shell, and it shrinks with the meson mass, vanishing as $x = m^2/M^2 \rightarrow 0$. At leading order this reproduces the behavior found in our earlier study [21], and the q^2 correction, built from higher powers of these propagators, has an absorptive part that vanishes faster still. The gluon term, though small, is enhanced most strongly in relative terms, its q^2 piece comparable to the leading order and roughly doubling the real gluon amplitude. The QED term receives no correction, its single-point $c\bar{c}$ annihilation leaving no charmonium relative momentum in the kernel. Summed over the three, the correction enlarges the total $|H|$ by about a third to 40%. Its size, comparable to the charmonium mean squared velocity $\langle v^2 \rangle \simeq 0.3$, is what one expects of the leading relativistic correction in the

velocity expansion of charmonium [3,4,49,61]. In an analysis carried to the same order the q^2 correction to the three-gluon width of the J/ψ is 77% [24], comparable in size to the 70%–80% by which the widths grow here.

The radiative widths follow from the quark, gluon, and QED amplitudes through Eq. (40) as the modulus squared of their coherent sum weighted by the two-body phase space. This modulus squared is expanded and truncated at order q^2 , so that the relativistic correction enters through its interference with the leading order. The near insensitivity to the soft inputs carries over to the widths, which move by only about 5% across the three DA models and the m_q range, so we adopt the representative model I at $m_q = 0.01$ GeV and let the mixing parameters carry the theoretical uncertainty. A branching ratio is what an experiment measures most directly, so we present our results in that form and compare them with the measured values. Table VI gives the leading-order branching ratio $\mathcal{B}_{\text{LO}}$ and the q^2 -corrected one $\mathcal{B}_{q^2}$ in the two mixing schemes, together with the measured $\mathcal{B}_{\text{exp}}$ from the Particle Data Group [31], the error quoted for each scheme combining that scheme's mixing parameters with the measured total width. The leading order typically falls short of the data by a factor of 2–3, and the relativistic correction is essential in closing the gap, increasing each branching ratio by roughly a factor of 1.8. With it the η' TFF scheme reaches about 55% of the measured $\gamma\eta'$ branching ratio and about 60% of the $\gamma\eta$ one, while the lattice scheme reproduces the $\gamma\eta'$ branching ratio to within 25% but undershoots $\gamma\eta$ by a factor of about 7, an imbalance that originates in the mixing angle and is taken up below.

The remaining deficit may be attributed to the uncertainties in $R_{\psi_n}(0)$ and $\alpha_s(\mu)$, since the dominant quark contribution scales as $|R_{\psi_n}(0)|^2 \alpha_s^4$. Neither is included in the quoted errors. The value of $|R_{J/\psi}(0)|^2$ varies by nearly a factor of 2 among the potential models quoted in Sec. III A, which would carry over directly into the branching ratios, but the one adopted here reproduces $\Gamma(J/\psi \rightarrow e^+e^-)$ to

TABLE VI. Branching ratios $\mathcal{B}(J/\psi \rightarrow \gamma\eta^{(\prime)})$ in units of 10^{-3} , for the representative model I DA at $m_q = 0.01$ GeV, in the lattice and η' TFF mixing schemes. Here $\mathcal{B}_{\text{LO}}$ is the leading-order branching ratio, $\mathcal{B}_{q^2}$ the one with the q^2 relativistic correction included, and $\mathcal{B}_{\text{exp}}$ the measured value, while the last row gives the ratio $\mathcal{R}_{1S}$ of the $\gamma\eta'$ to the $\gamma\eta$ branching ratio.

Process	Lattice		η' TFF		$\mathcal{B}_{\text{exp}}$ [31]
	$\mathcal{B}_{\text{LO}}$	$\mathcal{B}_{q^2}$	$\mathcal{B}_{\text{LO}}$	$\mathcal{B}_{q^2}$	
$J/\psi \rightarrow \gamma\eta$	0.09(4)	0.15(6)	0.38(4)	0.66(7)	1.090(13)
$J/\psi \rightarrow \gamma\eta'$	2.25(10)	4.01(17)	1.59(8)	2.83(14)	5.28(6)
$\mathcal{R}_{1S}$	24(10)	26(11)	4.2(5)	4.3(6)	4.84(8)

2%, and it is that agreement, not that variation, that measures how well the leading-order amplitude is fixed. What survives enters through the relativistic correction, a variation of $\nabla^2 R_{J/\psi}(0)/R_{J/\psi}(0)$ by $\pm 30\%$ moving the branching ratios by $\pm 13\%$. The strong coupling is particularly delicate at the charm scale, where its slowly converging running renders the one-loop value $\alpha_s(M_{J/\psi}/2) = 0.294$ adopted here appreciably lower than the two-loop value $\alpha_s(M_{J/\psi}/2) = 0.34$ used in Ref. [21]. Replacing the one-loop value by the two-loop one raises the branching ratios by 60%–70%, while moving the scale from $\mu = M/2$ to $\mu = M$ at one loop lowers them to between 40% and 50% of their central values, so that they are determined only up to a factor of about 2 either way. This sensitivity is intrinsic to a hard kernel of leading order in α_s , where the μ dependence carried by $\alpha_s^4(\mu)$ is left uncompensated. It is compensated only against the logarithms of the next order, so that a next-to-leading-order calculation, a two-loop computation here, would be expected to narrow the range. It is worth noting that with the two-loop coupling the η' TFF results rise to $\mathcal{B}_{q^2}(\gamma\eta) = 1.10(11) \times 10^{-3}$ and $\mathcal{B}_{q^2}(\gamma\eta') = 4.82(22) \times 10^{-3}$, in very good agreement with the measured values, and that the leading-order branching ratios are consistent with those of Ref. [21].

The two channels pull in opposite directions on the individual branching ratios, so the sharper test is their ratio $\mathcal{R}_{1S} = \mathcal{B}(J/\psi \rightarrow \gamma\eta')/\mathcal{B}(J/\psi \rightarrow \gamma\eta)$. In it the dependence on the wave function at the origin $R_{J/\psi}(0)$ and the coupling α_s largely cancels. As these inputs are the main source of model and scale dependence, $\mathcal{R}_{1S}$ is a far more reliable prediction than either branching ratio alone. Both channels are built from the same mixing parameters f_q, f_s , and ϕ , so in propagating their uncertainty to $\mathcal{R}_{1S}$ we keep numerator and denominator correlated. The last row of Table VI compares the two schemes with the measured $\mathcal{R}_{1S} = 4.84(8)$. The η' TFF scheme is consistent with it, $\mathcal{R}_{1S} = 4.2(5)$ at leading order and 4.3(6) with the q^2 correction, close to, though slightly below, the measured value, whereas the lattice scheme gives $\mathcal{R}_{1S} = 24(10)$ and 26(11), far too large even allowing for its inflated uncertainty. The difference is the mixing angle. The larger lattice angle drives the η flavor combination $\sqrt{2}f_q \cos\phi - f_s \sin\phi$ toward its zero, which suppresses the $\gamma\eta$ rate and inflates the ratio. Because the combination is small there, the $\gamma\eta$ channel responds sharply to ϕ , so the mixing-angle uncertainty dominates the error and swells that of the lattice $\mathcal{R}_{1S}$ beyond 40%. That $\mathcal{R}_{1S}$ is fixed already at leading order and barely shifted by the relativistic correction makes it a robust discriminator, and it selects the smaller mixing angle, in agreement with the leading-order conclusion of Ref. [21]. It is also consistent with the mixing angle extracted from our analyses of the radiative transitions $h_c \rightarrow \gamma\eta^{(\prime)}$ [29,62].

C. Results for the $\psi(2S)$

The $\psi(2S)$ amplitudes share most of the structure found for the J/ψ . They show a similar dependence on m_q and the DA, the dominant quark amplitude being nearly insensitive to both, and almost the same hierarchy of the three contributions. What changes markedly from the J/ψ is the size of the relativistic correction, which is substantially larger. This is intrinsic to a radial excitation, whose broader, nodal wave function gives a ratio $\nabla^2 R(0)/R(0)$ about 3 times the J/ψ value in magnitude, as shown in Eq. (41). The q^2 corrections are correspondingly larger, as detailed below.

As for the J/ψ , we present the $\psi(2S)$ helicity amplitudes in the lattice scheme to show the size of the relativistic correction. Table VII gives the quark, gluon, and QED amplitudes H^q, H^g , and H^{QED} in the $\gamma\eta$ and $\gamma\eta'$ channels, in units of 10^{-3} GeV. Each amplitude is separated into a leading-order part LO and its q^2 correction. A single entry is representative, model I for the DA and $m_q = 0.01$ GeV. The quark amplitude is again by far the largest. The gluon amplitude is about 3% of it for $\gamma\eta$ and 9% for $\gamma\eta'$, and the QED amplitude is about 20% and 6%.

The relativistic correction follows the same pattern as for the J/ψ but is considerably larger, as Table VII shows. The quark term again dominates, and its q^2 piece is now nearly as large as the leading order itself, so that $|H^q|$ almost doubles in both channels, compared with the growth of about 40% found for the J/ψ . The gluon term, the smallest of the three, is enhanced most strongly in relative terms, its q^2 piece reaching about twice the leading order. The QED term again receives no correction, its single-point $c\bar{c}$ annihilation leaving no charmonium relative momentum in the kernel. Summed over the three, the correction enlarges the total $|H|$ by some 70%–85%, roughly twice the effect in the J/ψ , in step with the larger ratio $\nabla^2 R_{\psi(2S)}(0)/R_{\psi(2S)}(0)$ of Eq. (41). Notably, the q^2 quark amplitude in the $\gamma\eta$ channel appears in Table VII with no imaginary part, although the leading order carries one. The imaginary part of H^q originates from the internal light-quark and gluon lines going on shell and diminishes as $x = m_P^2/M^2$ decreases, and the absorptive part of the q^2

TABLE VII. Helicity amplitudes of the three contributions to $\psi(2S) \rightarrow \gamma\eta^{(\prime)}$ in the lattice mixing scheme, for model I at $m_q = 0.01$ GeV, in units of 10^{-3} GeV. For each contribution LO is the leading-order helicity amplitude and q^2 its q^2 relativistic correction, the full amplitude being their sum.

Contribution	$\psi(2S) \rightarrow \gamma\eta$		$\psi(2S) \rightarrow \gamma\eta'$	
	LO	q^2	LO	q^2
Quark H^q	-0.55 - 0.13i	-0.53	-3.04 - 1.59i	-2.53 - 1.52i
Gluon H^g	-0.01	-0.02	-0.19	-0.40
QED H^{QED}	-0.22	0	-0.35	0

TABLE VIII. Branching ratios $\mathcal{B}(\psi(2S) \rightarrow \gamma\eta^{(\prime)})$ in units of 10^{-4} , for model I at $m_q = 0.01$ GeV, in the lattice and η' TFF mixing schemes. Here $\mathcal{B}_{\text{LO}}$ is the leading-order branching ratio, $\mathcal{B}_{q^2}$ the one with the q^2 relativistic correction included, and $\mathcal{B}_{\text{exp}}$ the measured value [30,31], while the last row gives the ratio $\mathcal{R}_{2S}$ of the $\gamma\eta'$ to the $\gamma\eta$ branching ratio.

Process	Lattice		η' TFF		$\mathcal{B}_{\text{exp}}$ [31]
	$\mathcal{B}_{\text{LO}}$	$\mathcal{B}_{q^2}$	$\mathcal{B}_{\text{LO}}$	$\mathcal{B}_{q^2}$	
$\psi(2S) \rightarrow \gamma\eta$	0.07(3)	0.18(7)	0.30(3)	0.76(8)	0.0092(18)
$\psi(2S) \rightarrow \gamma\eta'$	1.76(9)	4.71(23)	1.24(7)	3.33(18)	1.24(4)
$\mathcal{R}_{2S}$	23(9)	26(11)	4.2(5)	4.4(6)	135(27)

correction vanishes faster still, as it is built from higher powers of these propagators. Both $\psi(2S)$ channels sit at small x , and the $\gamma\eta$ one has the smallest x of all the channels considered in this work, so its absorptive parts are the most strongly suppressed. Numerically, that of the q^2 term is at the 10^{-6} GeV level and is dropped from the table.

The branching ratios and their ratio $\mathcal{R}_{2S} = \mathcal{B}(\psi(2S) \rightarrow \gamma\eta')/\mathcal{B}(\psi(2S) \rightarrow \gamma\eta)$ in Table VIII show a picture qualitatively unlike that for the J/ψ . The theory overshoots the data in both channels already at leading order, and the relativistic correction, enlarging each branching ratio by a further factor of about 2.5, widens the gap. In the lattice scheme the $\gamma\eta'$ rate is comparable to experiment at leading order but reaches almost 4 times the measured value once the correction is included, while the $\gamma\eta$ rate exceeds its measurement by about a factor of 8 at leading order and by some 20 after the correction. In the η' TFF scheme the $\gamma\eta'$ rate agrees with experiment at leading order and overshoots it by nearly a factor of 3 with the correction, while the $\gamma\eta$ rate lies more than a factor of 30 above the data at leading order and close to 2 orders of magnitude above with the correction. The ratio makes the tension sharpest. The theoretical $\mathcal{R}_{2S}$, 26(11) in the lattice scheme and 4.4(6) in the η' TFF one, falls far below the measured 135(27), a discrepancy neither representative mixing scheme can accommodate. The largeness of the measured ratio stems mainly from the anomalously small $\gamma\eta$ rate.

The overshoot in both channels reflects the difficulty of the relativistic expansion for the $\psi(2S)$. The q^2 correction is as large as the leading order, so the low-order relative-momentum expansion converges poorly, and the node of the $2S$ radial wave function, whose short-distance structure a low-order Taylor expansion cannot capture, compounds the problem. The same poor convergence was found in our recent study of the three-gluon decay of the $\psi(2S)$ [27], where the relative correction reaches 2.70 and the q^2 -truncated width turns negative, and it is the general behavior of the radial excitation. The corrections to $\psi(nS) \rightarrow \rho\pi$ and to $\psi(2S) \rightarrow p\bar{p}$ are of order unity, the corresponding analyses assigning $\langle v^2 \rangle \simeq 0.64$ to the $\psi(2S)$ against $\simeq 0.3$ for the J/ψ [25,26], and the v^2 corrections to

$\psi(2S)$ gluon fragmentation raise the cross section by factors of 3.3–6.6 [63]. Measured against these, the enhancement found here, which tracks the ratio of the two $\langle v^2 \rangle$, is what the velocity expansion of a radially excited charmonium leads one to expect. Beyond the expansion itself, the overshoot of the individual branching ratios might in part be laid to the inputs. The uncertainties in $R_{\psi(2S)}(0)$ and in $\alpha_s(\mu)$ are of much the same size here as for the J/ψ in Sec. III B, but they act on the two channels alike, so that they largely cancel in $\mathcal{R}_{2S}$. What survives is far too small to bring $\mathcal{R}_{2S}$ near the measured value. The charm mass, fixed in the hard kernel by the weak-binding approximation of Sec. II, is a further input to be tested and one to which the two charmonia respond very differently. Lowering it to the pole-type value 1.4 GeV used for the charm quark in NRQCD analyses, at the lower end of the constituent range quoted in Sec. III A, and keeping the Cornell wave-function inputs fixed, leaves the J/ψ leading-order widths within 7% and raises its q^2 -corrected branching ratios by 40% in $\gamma\eta$ and 27% in $\gamma\eta'$. For the $\psi(2S)$ the same change raises the branching ratios by factors between 2.7 and 4.3, the wider spread of internal momentum that enhances the q^2 term also sharpening the response of the charm propagator that lies close to its mass shell. The ratios are far less affected, $\mathcal{R}_{1S}$ moving from 26 to 24 and $\mathcal{R}_{2S}$ from 26 to 18 in the lattice scheme, and from 4.3 to 3.8 and from 4.4 to 2.8 in the η' TFF one, neither of the $\psi(2S)$ values approaching the measured ratio. The severe deficit of $\mathcal{R}_{2S}$ can therefore hardly be explained by the hard mechanism alone, which keeps the two channels within an order of magnitude of each other and cannot produce so strong a suppression of the $\gamma\eta$ rate. This suggests that a mechanism beyond the hard perturbative process may also play a significant role in these decays [10]. In the following we explore the η_c -mixing contribution as one such candidate, although a complete understanding of why the measured $\mathcal{R}_{2S}$ lies so far above the hard-mechanism prediction remains an open question.

D. Reassessment of the η_c -mixing contribution

The mechanism analyzed above proceeds by annihilating the $c\bar{c}$ pair at short distance into the photon and the partons that build up the light meson. The same decay also receives a contribution from a second, physically distinct mechanism, in which the $c\bar{c}$ pair is not annihilated but survives and enters the final state through the η_c component of the meson. In the picture introduced by Chao [12,13], the physical η and η' each carry a small admixture of the charmonium η_c , generated by the $U_A(1)$ anomaly, which couples both the light-quark content and the η_c to two gluons and lets the two oscillate into each other. The decay $\psi_n \rightarrow \gamma\eta^{(\prime)}$ can then proceed as the magnetic-dipole transition $\psi_n \rightarrow \gamma\eta_c$ on this component. While the annihilation contribution is fixed by the charmonium wave function at the origin, this transition is set by the radial overlap of the

ψ_n and η_c wave functions, which depends on the photon energy. In the $\gamma\eta^{(\prime)}$ channels that energy is large and the η_c is probed far off its mass shell. This large energy confines the overlap to short distances, a small part of the extended wave functions, so the transition is suppressed, an effect encoded in a radial overlap form factor. The off-shell η_c then converts into the $\eta^{(\prime)}$ through the two soft gluons of the anomaly that realize the mixing. In one mechanism the $c\bar{c}$ pair annihilates and in the other it survives. Yet both reach the same $\gamma\eta^{(\prime)}$ final state through different components of the $\eta^{(\prime)}$, so their amplitudes add coherently.

To gauge the size of the η_c -mixing contribution we now construct its amplitude, whose structure is fixed completely by the quantum numbers. Both $\psi_n \rightarrow \gamma\eta^{(\prime)}$ and $\psi_n \rightarrow \gamma\eta_c$ are $1^{--} \rightarrow \gamma 0^{++}$ transitions and carry the single Lorentz structure $\epsilon_{\mu\nu\lambda\sigma} K^\lambda k^\sigma$, so the η_c -mixing amplitude reads

$$\mathcal{M}^{\eta_c}(\psi_n \rightarrow \gamma\eta^{(\prime)}) = c_{\eta_c}^{\eta^{(\prime)}} g_{\psi_n\gamma\eta_c} F_n(k_{\eta^{(\prime)}}) \epsilon_{\mu\nu\lambda\sigma} K^\lambda k^\sigma \epsilon^\mu(K) \epsilon^{*\nu}(k), \quad (44)$$

where the form factor is evaluated at the photon energy $k_{\eta^{(\prime)}} = (M^2 - m_{\eta^{(\prime)}}^2)/(2M)$ of the channel in the ψ_n rest frame. The amplitude is built from three quantities, the η_c admixture $c_{\eta_c}^{\eta^{(\prime)}}$, the radial overlap form factor F_n , and the effective coupling $g_{\psi_n\gamma\eta_c}$, which we now fix in turn.

The admixture is taken from the charm-content analysis of Ref. [64]. There the $c\bar{c}$ pair created by the axial-vector current converts into two gluons through the charm-quark loop. For the heavy charm this loop is a short-distance object and is integrated out, and the matrix element of the resulting effective interaction in the $\eta^{(\prime)}$ defines the charm decay constant of the light pseudoscalar $f_{\eta^{(\prime)}}^c$.³ The admixture follows as $c_{\eta_c}^{\eta^{(\prime)}} = f_{\eta_c}^c / f_{\eta_c}$, where the η_c decay constant $f_{\eta_c} = 0.387(7)$ GeV is taken from the lattice determination of Ref. [65]. In the lattice mixing scheme this gives $c_{\eta_c}^{\eta} = -0.00310(15)$ and $c_{\eta_c}^{\eta'} = -0.00820(47)$, in line with the values of Ref. [66] and about half those extracted phenomenologically in the FKS analysis [9].

The form factor is the radial overlap of the initial charmonium with the η_c at photon energy k [67],

$$F_n(k) = \int_0^\infty dr r^2 R_{10}(r) R_{n0}(r) j_0(kr/2), \quad (45)$$

with R_{10} and R_{n0} the radial wave functions of the η_c and the initial charmonium, $n = 1$ the allowed $1S \rightarrow 1S$ transition of the J/ψ , and $n = 2$ the hindered $2S \rightarrow 1S$ one of the

$\psi(2S)$, for which $F_2(0) = 0$ by orthogonality. For these radial wave functions we adopt a simple but analytic approximation, the harmonic-oscillator form, which gives

$$F_1(k) = e^{-k^2 a^2/16}, \quad F_2(k) = \frac{\sqrt{6}}{48} (ka)^2 e^{-k^2 a^2/16}. \quad (46)$$

The oscillator length a , the essential nonperturbative input, is related to the charmonium size by the harmonic-oscillator identity $\langle r^2 \rangle_{1S} = 3a^2/2$. We fix it from the $1S$ root-mean-square radius of the same Cornell potential [46,47] that determines the wave functions at the origin in Sec. III A. Solving the radial Schrödinger equation with this potential gives $\langle r^2 \rangle_{1S}^{1/2} = 0.39$ fm, and hence $a = 1.62$ GeV⁻¹. This radius agrees with recent potential-model determinations of the charmonium size [68]. Against the numerical form factors of the Cornell potential the oscillator forms deviate by no more than a few percent, and at the large photon momenta of the $\gamma\eta^{(\prime)}$ channels the two are nearly identical. Their energy dependence, $k^3|F_1|^2$ for the allowed and k^7 for the hindered transition, is moreover consistent with the photon-energy dependence of the η_c line shapes observed by CLEO-c and BESIII in $\psi_n \rightarrow \gamma\eta_c$ away from the resonance peak [69,70].

The effective coupling is anchored to the measured rates.

The same vertex with $c_{\eta_c}^{\eta^{(\prime)}} \rightarrow 1$ describes the on-shell transition $\psi_n \rightarrow \gamma\eta_c$, whose width $\Gamma = |g_{\psi_n\gamma\eta_c}|^2 |F_n(k_{\eta_c})|^2 k_{\eta_c}^3 / (12\pi)$, with k_{η_c} the photon energy of this on-shell transition, inverts to

$$|g_{\psi_n\gamma\eta_c}| = \frac{1}{|F_n(k_{\eta_c})|} \sqrt{\frac{12\pi\Gamma^{\text{exp}}(\psi_n \rightarrow \gamma\eta_c)}{k_{\eta_c}^3}}, \quad (47)$$

where $\Gamma^{\text{exp}}(\psi_n \rightarrow \gamma\eta_c)$ is the measured width [31]. Determining the coupling from the measured width is necessary because the magnetic-dipole transitions are not yet under full theoretical control. For $J/\psi \rightarrow \gamma\eta_c$ the nonrelativistic prediction overshoots the measured rate by a factor of 2–3, and the relativistic corrections of relative order v^2 restore agreement with experiment within sizable uncertainties [67]. For the hindered $2S \rightarrow 1S$ transition, dominated by relativistic corrections and complicated further by the proximity of the $\psi(2S)$ to the open-charm threshold, a reliable theoretical prediction is more difficult still [67,71–73]. Anchoring to data absorbs this difficulty but fixes only the magnitude of the coupling, leaving its phase undetermined.

Applying the helicity projector to Eq. (44) gives the η_c -mixing helicity amplitude H^{η_c} . Let $H^{\text{ann}} \equiv H^q + H^g + H^{\text{QED}}$ denote the summed helicity amplitude of the quark, gluon, and QED contributions analyzed before. As argued at the beginning of this section, H^{η_c} adds coherently to

³Explicitly $f_{\eta^{(\prime)}}^c = -\Delta i_5(m_{\eta^{(\prime)}}^2/m_c^2)f_{\eta^{(\prime)}}^u$, with $\Delta i_5(z) = -1 + [\pi - 2 \arctan \sqrt{4/z - 1}]^2/z \approx z/12$ at small z [64], so the charm content is suppressed by $m_{\eta^{(\prime)}}^2/m_c^2$. The charm mass here is taken to be the current mass, $m_c(m_c) = 1.273(5)$ GeV [31].

TABLE IX. Branching ratios of $\psi_n \rightarrow \gamma\eta^{(\prime)}$ from the coherent amplitude at the best-fit phases $\delta_{1S} = 192^\circ$ and $\delta_{2S} = 18^\circ$, in the lattice mixing scheme, compared with the measured values [31]. The J/ψ entries are in units of 10^{-3} and the $\psi(2S)$ entries in 10^{-4} .

Process	$\mathcal{B}_{\text{fit}}$	$\mathcal{B}_{\text{exp}}$
$J/\psi \rightarrow \gamma\eta$	0.65(14)	1.090(13)
$J/\psi \rightarrow \gamma\eta'$	9.34(52)	5.28(6)
$\psi(2S) \rightarrow \gamma\eta$	0.0092(47)	0.0092(18)
$\psi(2S) \rightarrow \gamma\eta'$	1.97(29)	1.24(4)

H^{ann} , giving the total amplitude

$$H = H^{\text{ann}} + H^{\eta_c}. \quad (48)$$

Here $H^{\eta_c} = |H^{\eta_c}|e^{i\delta_{\eta_c}}$, with δ_{1S} and δ_{2S} treated as two free parameters, each common to the $\gamma\eta$ and $\gamma\eta'$ channels of its charmonium, and fixed by a best fit to experiment.

The fit is anchored to the two measured $\gamma\eta$ branching ratios, which we carry out in the lattice scheme as a representative case, and returns $\delta_{1S} \simeq 192^\circ$ and $\delta_{2S} \simeq 18^\circ$. Table IX collects the four resulting branching ratios. The interference brings the $\psi(2S)$ into agreement with experiment in both channels and substantially improves the J/ψ , whose $\gamma\eta$ and $\gamma\eta'$ rates come to about half and twice the measured values. It is worth noting that the two fitted phases differ by nearly 180° , which means the η_c -mixing amplitudes of the two charmonia are almost opposite in sign. This relative sign finds a natural interpretation in the magnetic-dipole dynamics, where the relativistic correction is opposite in sign to the leading order [67,71,72]. The leading order dominates the J/ψ amplitude but vanishes for the $\psi(2S)$ by the $1S$ – $2S$ orthogonality, leaving the $\psi(2S)$ amplitude as the correction alone, opposite in sign to that of the J/ψ .

Encouraging as this is, the coherent results obtained above should be regarded as tentative. We have deliberately fixed the essential nonperturbative inputs of both contributions, the wave function at the origin of H^{ann} and the radial overlap of the η_c -mixing one, from the same Cornell potential, so as to minimize the model dependence of their coherence. A residual dependence nonetheless remains. Both contributions depend on the mixing parameters, and in the $\gamma\eta$ channel H^{ann} is especially sensitive to them through the near cancellation in $\sqrt{2}f_q \cos\phi - f_s \sin\phi$. The relativistic expansion, moreover, converges poorly for the $\psi(2S)$, which degrades the reliability of H^{ann} there. These limitations make a precise extraction of the interference difficult, and a more reliable determination of the relative phases is left to a dedicated future study.

IV. SUMMARY

We have computed the OZI-forbidden radiative decays $J/\psi, \psi(2S) \rightarrow \gamma\eta^{(\prime)}$ in perturbative QCD, resolving the

amplitude into its quark-antiquark, two-gluon, and QED contributions and including the leading q^2 relativistic correction to each. A central result is the remarkable stability of these amplitudes against the soft inputs of the hard mechanism. The dominant quark-antiquark contribution is almost completely insensitive to the DA and to the light-quark mass, varying by less than 2% across three DA models and by less than 1% over the light-quark-mass range, while the smaller two-gluon and QED contributions carry only a mild DA dependence and are likewise nearly independent of the light-quark mass. This robustness holds at leading order and survives the q^2 correction, confirming the leading-order finding of our earlier work [21,41] and extending it to relative order q^2 . Recent lattice studies have raised as an open question the light-quark-mass dependence of $J/\psi \rightarrow \gamma\eta^{(\prime)}$ [74,75], while our perturbative analysis, borne out numerically, points to a weak dependence. This insensitivity to the soft inputs makes our predictions for the process correspondingly reliable.

The q^2 correction enhances the amplitude in every channel, but by amounts that differ markedly between the two charmonia. For the J/ψ this enhancement is controlled, about 40% at the amplitude level and in line with the charmonium mean squared velocity $\langle v^2 \rangle \simeq 0.3$, and it roughly doubles the branching ratios and so narrows their sizable shortfall from the measured values. For the $\psi(2S)$ the enhancement is about twice as large as for the J/ψ . There the q^2 term is comparable to the leading order, a poor convergence that undermines the relative-momentum expansion and makes a quantitative account of the $\psi(2S)$ relativistic correction difficult.

Comparing with experiment in two representative mixing schemes, the lattice and the η' TFF, we find the relativistic correction essential in approaching the measured J/ψ branching ratios, while their ratio $\mathcal{R}_{1S}$ favors the smaller mixing angle. Both this ratio and the $\gamma\eta$ branching ratio are highly sensitive to the mixing angle, so they discriminate sharply between the schemes. For the $\psi(2S)$ the annihilation contribution overshoots both channels, the $\gamma\eta'$ rate moderately and the anomalously small $\gamma\eta$ rate by far more. As a candidate resolution we examined the interference with the η_c -mixing contribution, whose helicity amplitude, normalized to the measured $\psi_n \rightarrow \gamma\eta_c$ rates and suppressed by a radial overlap form factor, is comparable in size to the annihilation one in the $\gamma\eta$ channel. Added coherently, it can bring both $\psi(2S)$ channels back to their measured values. The strong mixing-parameter sensitivity of the $\gamma\eta$ rates and of the ratios, however, ties this interference closely to the choice of mixing scheme and makes a clean extraction of the coherence between the two contributions difficult. Sharpening this extraction will require better-constrained mixing parameters and a firmer handle on the phase of the η_c -mixing amplitude. The poor convergence of the $\psi(2S)$ relativistic expansion, in turn, calls for retaining the full relative-momentum dependence

of the hard kernel rather than its q^2 truncation. We leave both to future work.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: EXPLICIT FORMS OF THE HIGHER DENOMINATOR-POWER NUMERATORS

$N_{a,b}^X$ IN EQ. (25)

The higher denominator-power numerators $N_{a,b}^X$ ($a + b > 2$) of the q^2 -correction kernel $I_2^q(u)$ are listed below in the weak-binding approximation $m_c = M/2$, with $p \equiv K - k$ the $\eta^{(l)}$ momentum,

$$N_{1,2}^A = \frac{32i}{3M} \left[-\frac{2M^2(k \cdot k_1)^2(k_1 \cdot K)}{k \cdot K} + \frac{2M^2(k \cdot k_1)^3}{k \cdot K} - 2(k \cdot k_1)(k_1 \cdot K)(k \cdot K) \right. \\ \left. - 2k_1^2(k_1 \cdot K)(k \cdot K) - 2k_1^2(k \cdot k_1)(k \cdot K) - M^2(k \cdot k_1)(k_1 \cdot K) \right. \\ \left. + 4M^2(k \cdot k_1)^2 + 4M^2k_1^2k \cdot k_1 - \frac{M^4(k \cdot k_1)^2}{k \cdot K} + k_1^2(k \cdot K)^2 + 2M^2k_1^2k \cdot K \right]. \quad (A1)$$

$$N_{1,3}^A = \frac{128i}{3M} \left[-(k \cdot k_1)(k_1 \cdot K)^3 + (k \cdot k_1)^2(k_1 \cdot K)^2 + k_1^2(k_1 \cdot K)^2(k \cdot K) \right. \\ \left. + M^2k_1^2(k \cdot k_1)(k_1 \cdot K) - M^2k_1^2(k \cdot k_1)^2 - M^2(k_1^2)^2k \cdot K \right]. \quad (A2)$$

$$N_{2,1}^A = \frac{32i}{3M} \left[-4(k \cdot k_1)(k_1 \cdot K)(k \cdot K) + 4(k \cdot k_1)^2(k \cdot K) - 3M^2(k \cdot k_1)(k_1 \cdot K) \right. \\ \left. + 3M^2(k \cdot k_1)^2 + 4k_1^2(k \cdot K)^2 + 3M^2k_1^2k \cdot K \right]. \quad (A3)$$

$$N_{2,2}^A = \frac{128i}{3M} \left[(k \cdot k_1)(k_1 \cdot K)^2(k \cdot K) - (k \cdot k_1)^2(k_1 \cdot K)(k \cdot K) - M^2(k \cdot k_1)^2(k_1 \cdot K) \right. \\ \left. + M^2(k \cdot k_1)^3 - k_1^2(k_1 \cdot K)(k \cdot K)^2 + M^2k_1^2(k \cdot k_1)(k \cdot K) \right]. \quad (A4)$$

$$N_{3,1}^A = \frac{128i}{3M} \left[-(k \cdot k_1)(k_1 \cdot K)(k \cdot K)^2 + (k \cdot k_1)^2(k \cdot K)^2 + k_1^2(k \cdot K)^3 \right]. \quad (A5)$$

$$N_{1,2}^C = \frac{32i}{3M} \left[8(k \cdot k_1)(k_1 \cdot K)^2 - 8(k \cdot k_1)^2(k_1 \cdot K) - 4k_1^2(k \cdot k_1)(k_1 \cdot K) \right. \\ \left. - \frac{4M^2(k \cdot k_1)^2(k_1 \cdot K)}{k \cdot K} + \frac{4M^2(k \cdot k_1)^3}{k \cdot K} + \frac{2M^2k_1^2(k \cdot k_1)^2}{k \cdot K} \right. \\ \left. + 6(k \cdot k_1)(k_1 \cdot K)(k \cdot K) - 4k_1^2(k_1 \cdot K)(k \cdot K) + 4k_1^2(k \cdot k_1)(k \cdot K) \right. \\ \left. + 2(k_1^2)^2k \cdot K + 4M^2(k \cdot k_1)(k_1 \cdot K) - 3M^2(k \cdot k_1)^2 \right. \\ \left. - \frac{2M^4(k \cdot k_1)^2}{k \cdot K} - 3k_1^2(k \cdot K)^2 - 2M^2k_1^2k \cdot K \right]. \quad (A6)$$

$$\begin{aligned}
N_{1,3}^C = \frac{128i}{3M} & \left[2(k \cdot k_1)(k_1 \cdot K)^3 - \frac{M^2(k \cdot k_1)^2(k_1 \cdot K)^2}{k \cdot K} + 4(k \cdot k_1)(k_1 \cdot K)^2(k \cdot K) \right. \\
& - k_1^2(k_1 \cdot K)^2(k \cdot K) - 6M^2(k \cdot k_1)^2(k_1 \cdot K) - 2M^2k_1^2(k \cdot k_1)(k_1 \cdot K) \\
& + \frac{2M^4(k \cdot k_1)^3}{k \cdot K} + \frac{M^4k_1^2(k \cdot k_1)^2}{k \cdot K} + 2(k \cdot k_1)(k_1 \cdot K)(k \cdot K)^2 \\
& - 2k_1^2(k_1 \cdot K)(k \cdot K)^2 - M^2(k \cdot k_1)^2(k \cdot K) + 2M^2k_1^2(k \cdot k_1)(k \cdot K) \\
& \left. + M^2(k_1^2)^2k \cdot K - k_1^2(k \cdot K)^3 \right]. \tag{A7}
\end{aligned}$$

$$\begin{aligned}
N_{2,2}^C = \frac{128i}{3M} & \left[2(k \cdot k_1)(k_1 \cdot K)^3 - \frac{M^2(k \cdot k_1)^2(k_1 \cdot K)^2}{k \cdot K} + 2(k \cdot k_1)(k_1 \cdot K)^2(k \cdot K) \right. \\
& - k_1^2(k_1 \cdot K)^2(k \cdot K) - 3M^2(k \cdot k_1)^2(k_1 \cdot K) - 2M^2k_1^2(k \cdot k_1)(k_1 \cdot K) \\
& + \frac{M^4(k \cdot k_1)^3}{k \cdot K} + \frac{M^4k_1^2(k \cdot k_1)^2}{k \cdot K} - k_1^2(k_1 \cdot K)(k \cdot K)^2 \\
& \left. + M^2k_1^2(k \cdot k_1)(k \cdot K) + M^2(k_1^2)^2k \cdot K \right]. \tag{A8}
\end{aligned}$$

The remaining numerators follow from the Bose symmetry under the gluon interchange $k_1 \rightarrow p - k_1$, which gives $N_{a,b}^B = N_{a,b}^A(k_1 \rightarrow p - k_1)$ and $N_{a,b}^C = N_{b,a}^C(k_1 \rightarrow p - k_1)$.

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