

Quantum treatment of black hole superradiance

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Rotating black holes can form dense boson clouds through superradiant instability, making Kerr black holes a powerful probe of ultralight massive bosons. Previous studies of black hole superradiance have often treated bosonic fields classically, leaving open questions about how particles are produced and how the clouds grow over time. In this work, we canonically quantize a massive scalar field around a Kerr black hole, providing a fully quantum description of black hole superradiance. We show that the evolution of the particle number in the cloud, as well as the energy and angular momentum of the scalar field, can be consistently explained within the standard framework of quantum field theory in curved spacetime. Furthermore, we prove that the growth of the cloud occurs independently of the choice of initial state. We also explore several phenomena related to a massive scalar field in a rotating black hole spacetime, including Hawking radiation, adiabatic backreaction on the black hole spin, and the direction of level transitions in the presence of self-interactions of the field. Our analysis provides a consistent quantum-mechanical perspective that includes all these phenomena.

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I. INTRODUCTION

Ultralight bosonic (ULB) particles have attracted increasing attention in recent years as one of the candidates for dark matter [1–5]. In minimal setups, they interact with ordinary matter only through gravity, making black holes (BHs) an exceptional detection channel. One remarkable gravitational phenomenon arising from the interplay between BHs and

ULBs is the BH superradiance instability, a process in which a spinning BH transfers its rotational energy to surrounding bosonic fields, generating a boson cloud. This mechanism offers a unique window into new physics [6–13].

The boson cloud and the BH can form a bound state analogous to that of a hydrogen atom, a system commonly referred to as the gravitational atom (GA). The GA displays a rich array of fascinating phenomena, one example being the emission of monochromatic gravitational waves (GWs) [14–20] with frequencies set by the mass of the ultralight boson. These frequencies can fall within the sensitivity bands of current and future GW detectors [21–26], making the GA an effective probe of ultralight bosons. When the GA is embedded in a binary system, the gravitational perturbation of the companion can deform the structure of the cloud, which, in turn, backreacts on the orbital dynamics of the companion. Such deviations in the binary orbits can potentially be detected through a variety of observational channels, thereby providing the first evidence of ULBs [27–44].

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However, most existing studies treat the bosonic field as a classical field, where the superradiant instability is described as an exponential growth in the wave amplitude over time. In this framework, the growth of the particle population in the quantum perspective remains poorly understood. The primary challenge in quantizing the system comes from the complex frequencies of the superradiant modes, which give rise to their characteristic exponential time dependence [11,45]. This makes it difficult to apply the standard procedure of field quantization based on a set of harmonic oscillators. Although a formal quantization approach has been attempted [46,47], the results were incomplete due to the lack of discrete superradiant spectra, which is yet unknown at the time. See also recent discussions on this topic [48,49] that focus exclusively on continuous modes. A closely related issue, i.e., the quantum treatment of ergoregion instabilities, has also been explored [50], offering useful conceptual guidance. In this paper, we further develop the formalism of a quantum description of superradiance along this direction.

In this work, we aim to formulate the superradiant instability in a quantum setup by applying the standard canonical quantization procedure to a massive scalar field in the Kerr metric. We present all normalizable solutions to the Klein-Gordon (KG) equation, including modes with both discrete and continuous frequencies. The discrete modes have imaginary frequencies but are normalizable differently from the quasinormal modes (QNMs). To understand how superradiance arises in the quantum framework, we introduce a pseudoparticle number corresponding to the portion of the field localized outside the BH potential barrier. Although the particle number defined in the usual way is constant, the pseudoparticle number grows exponentially with time in the quantum description as well, regardless of the choice of initial state. This growth of the pseudoparticle number can therefore be interpreted as the growth of the “boson cloud.”

This paper is organized as follows. In Sec. II, we begin with the massive KG equation in the Kerr background and build a complete set of mode functions that is orthonormalized for the mode expansion of a field. Then, we perform the canonical quantization of the massive scalar field. After quantization, in Sec. III, we analyze the behavior of the Hamiltonian and the angular momentum and calculate the evolution of the particle number with time. Introducing a pseudonumber operator, we show that the growth of the cloud is inevitable, whatever initial state is selected. Section IV then discusses related topics based on our quantum formulation, including the Hawking radiation, the adiabatic evolution of the background spacetime, and level transitions with the self-interaction turned on. Section V is devoted to the conclusion. In the Appendix, we also show the equivalence between two different expressions for the retarded Green’s function in the presence of discrete modes. We adopt the $(-, +, +, +)$ metric sign convention and set $G = \hbar = c = 1$ throughout our paper.

II. MASSIVE SCALAR FIELD AROUND A KERR BLACK HOLE

A. Field equations and Klein-Gordon norm

We consider a massive scalar field in a Kerr spacetime, whose metric is given in the Boyer-Lindquist coordinates as

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dt d\varphi + \sin^2 \theta \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\Sigma}\right) d\varphi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \quad (1)$$

where $\Sigma \equiv r^2 + a^2 \cos^2 \theta$ and $\Delta \equiv r^2 - 2Mr + a^2$, with M and a denoting the mass and spin of the black hole, respectively. The event horizon r_+ and the inner horizon r_- are the solutions of $\Delta = 0$, yielding $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. We denote the metric tensor by $g_{\mu\nu}$. The Penrose diagram of the Kerr BH is shown in Fig. 1.

We consider a noninteracting real massive scalar field Φ in the Kerr spacetime, whose action is given by

$$\mathcal{S} := \int d^4x \sqrt{-g} \mathcal{L}, \quad (2)$$

with

$$\mathcal{L} := \frac{1}{2} [-g^{\mu\nu} (\partial_\mu \Phi)(\partial_\nu \Phi) - \mu^2 \Phi^2]. \quad (3)$$

We expand the field operator Φ using a complete set of properly orthonormalized functions with respect to the (pseudo) KG inner product defined by

$$((u, v)) \equiv i \int_{\Sigma_t} d^3x \sqrt{-g} \tilde{n}_\mu j^\mu(u, v), \quad (4)$$

with $\tilde{n}^\mu$ being the normal (but nonunit) vector perpendicular to the t -constant hypersurface, defined by $\tilde{n}_\mu dx^\mu = -dt$, and

$$j^\mu(u, v) := g^{\mu\nu} [u \partial_\nu v - (\partial_\nu u) v], \quad (5)$$

so as to satisfy

$$\begin{aligned} ((u_i, \mathcal{J}u_j)) &= -((\mathcal{J}u_i, u_j)) \begin{cases} \neq 0, & (\text{for } i = j), \\ = 0, & (\text{for } i \neq j), \end{cases} \\ ((u_i, u_j)) &= ((\mathcal{J}u_i, \mathcal{J}u_j)) = 0. \end{aligned} \quad (6)$$

Usually, the conjugation operation $\mathcal{J}$ corresponds to complex conjugation, but in our analysis, we adopt a different definition which simultaneously flips the sign of ω and m in the eigenmodes, in order to apply it to the

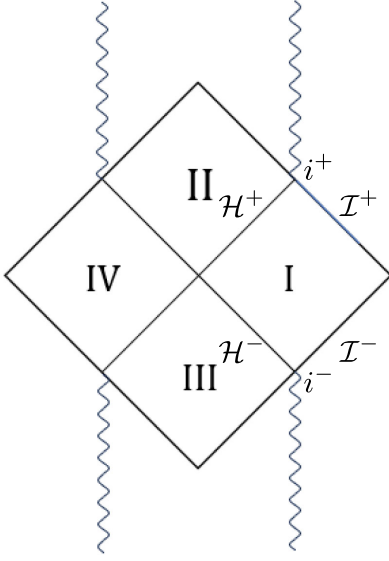


FIG. 1. The Penrose diagram of Kerr BH on the equatorial plane. Region I is the Universe we live in while region IV is the parallel universe. $\mathcal{H}^\pm$ is the future/past outer horizon, $\mathcal{I}^\pm$ is the future/past null infinity, and $i^\pm$ is the future/past timelike infinity. The wavy lines correspond to the singularities.

discrete superradiant modes. It turns out that such a sign-flipping definition of $\mathcal{J}$ is equivalent to the complex conjugation for the usual continuous modes [51]. In the usual KG inner product, v on the right-hand side in Eq. (4) is replaced with its complex conjugate, u_j^* , but we adopt this unconventional notation because the ordinary convention turns out to be not so convenient when we discuss discrete superradiant modes. Because the KG inner product is independent of the choice of the hypersurface, here we give an expression which is valid only when the hypersurface is set to be a constant-time hypersurface, for simplicity.

The equation of motion for the massive scalar field in Kerr spacetime is given by [52]

$$\square\Phi = \mu^2\Phi, \quad (7)$$

where the d'Alembertian operator $\square = \nabla^\nu \nabla_\nu$ and μ is the mass of the scalar field. As the Kerr metric is independent of t and φ , one can assume the separation of variables with respect to t and φ . We can further separate r and θ , owing to the symmetry of the Kerr spacetime [53], to obtain a complete set of modes in the form of

$$u_{\omega lm} = e^{-i\omega t} e^{im\varphi} R_{\omega lm}(r) S_{\omega lm}(\theta). \quad (8)$$

The radial part and the polar part of the equation are separated as

$$\begin{aligned} \mathcal{L}_\theta S_{\omega lm}(\theta) := & \left[\frac{d}{d\cos\theta} \sin^2\theta \frac{d}{d\cos\theta} - \frac{m^2}{\sin^2\theta} + 2am\omega \right. \\ & \left. - a^2(\omega^2 \sin^2\theta + \mu^2 \cos^2\theta) + \lambda_{lm\omega} \right] S_{\omega lm}(\theta) \\ = & 0, \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{L}_r R_{\omega lm}(r) := & \left[\frac{d}{dr} \Delta \frac{d}{dr} + \frac{(\omega(r^2 + a^2) - am)^2}{\Delta} \right. \\ & \left. - \mu^2 r^2 - \lambda_{lm\omega} \right] R_{\omega lm}(r) = 0, \end{aligned} \quad (10)$$

where $\lambda_{lm\omega}$ is a separation constant, which is determined by the regularity of the spheroidal harmonics $S(\theta)$ at both $\theta = 0$ and π . Here, we introduce an integer label $l \geq |m|$ such that $S_{lm\omega}$ reduces to the spherical harmonics Y_{lm} in the nonrotating limit $a \rightarrow 0$. We normalize the spheroidal harmonics to satisfy the orthogonality conditions

$$\int_0^\pi d\theta \sin\theta S_{\omega lm}(\theta) S_{\omega l'm}(\theta) = \frac{1}{2\pi} \delta_{ll'}, \quad (11)$$

and these conditions specify the normalization of $S_{\omega lm}(\theta)$. We should note that there is a symmetry

$$S_{\omega lm}(\theta) = S_{-\omega l-m}(\theta), \quad (12)$$

because Eq. (9) is invariant under the exchange $(\omega, m) \rightarrow (-\omega - m)$.

The radial function generally takes the asymptotic forms

$$\begin{aligned} R_{\omega lm}(r) \rightarrow & \mathcal{N}_{\omega lm} \mathcal{G}(r) \\ & \times \begin{cases} A_{\omega lm}^{\text{out}} e^{ik_H r_*} + A_{\omega lm}^{\text{in}} e^{-ik_H r_*}, & (r_* \rightarrow -\infty), \\ A_{\omega lm}^{\text{up}} e^{ik_\infty r_*} + A_{\omega lm}^{\text{down}} e^{-ik_\infty r_*}, & (r_* \rightarrow +\infty). \end{cases} \end{aligned} \quad (13)$$

Here, the tortoise coordinate r_* is defined by $dr_* := (r^2 + a^2)dr/\Delta$. The horizon $r = r_+$ and infinity $r = +\infty$ correspond to $r_* = -\infty$ and $r_* = +\infty$, respectively. The coefficients $A^{\text{in}}, A^{\text{out}}, A^{\text{up}}$, and A^{down} are, in general, complex numbers. We denote a normalization factor as $\mathcal{N}$, which we demand to be real and positive for a real ω . Determination of $\mathcal{N}$ will be done later. In the asymptotic form (13), we introduced

$$\mathcal{G}(r) := 1/\sqrt{r^2 + a^2}, \quad (14)$$

$$k_\infty(\omega) := \sqrt{\omega^2 - \mu^2}, \quad (15)$$

$$k_H(\omega, m) := \omega - m\Omega_H, \quad (16)$$

$$\Omega_H = \frac{a}{2Mr_+}. \quad (17)$$

In this paper, we take the branch cuts of k_∞ to emanate from $\omega = \pm\mu$ with an infinitesimal negative imaginary part, extending toward infinity in the lower-half complex plane.

In the present setup, the inner product $((u_{\omega lm}, u_{\omega' l' m'}))$ is nonvanishing only when $\omega' = -\omega$ and $m' = -m$. If the inner product with $\omega' + \omega \neq 0$ is nonvanishing, it should depend on time in proportion to $\exp[-i(\omega + \omega')t]$, which contradicts the fact that the KG inner product is time independent. By inserting the Kerr metric, the inner product is expressed explicitly as

$$\begin{aligned} & ((u_{\omega lm}, u_{\omega' l' m'})) \\ &= \delta_{l,l'} \delta_{m,-m'} \int dr_* \frac{f(r)}{G^2} R_{\omega lm}(r) R_{\omega' l' m'}(r), \end{aligned} \quad (18)$$

where we introduce the weight of the radial integral

$$f(r) \equiv 2\omega \left[1 - \frac{\eta a^2 \Delta}{(r^2 + a^2)^2} \right] - \frac{4aMr}{(r^2 + a^2)^2} m, \quad (19)$$

with

$$\eta \equiv 2\pi \int d(\cos \theta) \sin^2 \theta [S_{\omega lm}(\theta)]^2. \quad (20)$$

Here, we used the fact that the inner product vanishes unless $\omega = -\omega'$. The weight function behaves asymptotically as

$$f(r) \rightarrow \begin{cases} 2\omega, & (r_* \rightarrow \infty), \\ 2k_H, & (r_* \rightarrow -\infty). \end{cases} \quad (21)$$

B. Complete set of mode functions

In this section, we list all the normalizable modes with either continuous or discrete frequency spectra, in preparation for the mode expansion discussed later.

1. Continuous modes

We start with the usual mode solutions with $\omega_I = 0$, $|\omega_R| > \mu$, where ω_I and ω_R are the imaginary and real parts of ω , respectively. In this case, for each set of values of (ω, ℓ, m) , we have two independent radial solutions, which form two independent continuous spectra. In principle, discrete modes with $\omega_I = 0$ could appear, but for the system considered here, no such modes exist. It would be convenient to choose “in” mode $R_{\omega lm}^{\text{in}}$ and “up” mode $R_{\omega lm}^{\text{up}}$, which are specified by the asymptotic boundary conditions

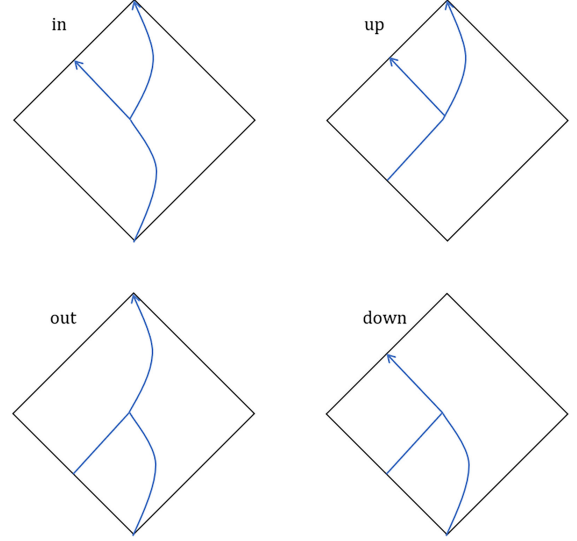


FIG. 2. All four kinds of modes in the region I of the Kerr BH Penrose diagram shown in Fig. 1. The “in” and “up” modes represent the waves ingoing and upcoming from i^- and $\mathcal{H}^-$, respectively, while the “out” and “down” modes represent the waves outgoing and down to i^+ and $\mathcal{H}^+$, respectively.

$$\begin{aligned} R_{\omega lm}^{\text{in}}(r) &\rightarrow \mathcal{N}_{\omega lm}^{\text{in}} G \\ &\times \begin{cases} e^{-ik_H r_*}, & (r_* \rightarrow -\infty), \\ A_{\omega lm}^{\text{up}} e^{+ik_\infty r_*} + A_{\omega lm}^{\text{down}} e^{-ik_\infty r_*}, & (r_* \rightarrow +\infty), \end{cases} \end{aligned} \quad (22)$$

and

$$\begin{aligned} R_{\omega lm}^{\text{up}}(r) &\rightarrow \mathcal{N}_{\omega lm}^{\text{up}} G \\ &\times \begin{cases} A_{\omega lm}^{\text{in}} e^{-ik_H r_*} + A_{\omega lm}^{\text{out}} e^{+ik_H r_*}, & (r_* \rightarrow -\infty), \\ e^{+ik_\infty r_*}, & (r_* \rightarrow +\infty). \end{cases} \end{aligned} \quad (23)$$

The “in” mode represents a scalar wave ingoing from i^- , which is partly reflected to i^+ while partly transmitted to $\mathcal{H}^+$. The “up” mode represents a scalar wave upcoming from $\mathcal{H}^-$, which is partly reflected to $\mathcal{H}^+$ while partly transmitted to i^+ . The “in” and “up” modes are depicted in the Penrose diagram shown in Fig. 2. We denote the corresponding apparently “positive” frequency functions by $u_{\omega lm}^{\text{in}}$ and $u_{\omega lm}^{\text{up}}$, respectively.

For $\omega_I = 0$ and $0 < |\omega_R| < \mu$, k_∞ becomes purely imaginary and $\text{Im } k_\infty > 0$. We chose the branch cut of k_∞ such that $R_{\omega lm}^{\text{up}}$ satisfies the decaying boundary condition at $r_* \rightarrow +\infty$. In this frequency range we only have one normalizable mode $R_{\omega lm}^{\text{up}}$, since $R_{\omega lm}^{\text{in}}$ contains the exponentially growing component at $r_* \rightarrow +\infty$.

In the region $|\omega_R| > \mu$ we can define “out” and “down” modes that satisfy the boundary conditions shown in Fig. 2,

by flipping the sign of ω and m in $R_{\omega lm}^{\text{in}}(r)$ and $R_{\omega lm}^{\text{up}}(r)$. Under this sign-flipping operation, which we denote by $\mathcal{J}$, the radial Eq. (10) remains invariant. In addition, the radial Eq. (10) is real when $\omega_I = 0$, and the solutions are real for $|\omega_R| < \mu$ except for the possibility of multiplying an irrelevant overall phase.

We assume that the normalization factors $\mathcal{N}_{\omega lm}$ are all real by appropriately choosing the irrelevant overall phases. Since $\mathcal{J}k_H = -k_H$ and $\mathcal{J}k_\infty = -k_\infty$ hold¹ in the region $|\omega_R| > \mu$, we find that we can consistently impose the condition

$$\mathcal{J}^{\mathfrak{N}} R^x = (R^x)^*, \quad \text{for } |\omega| > \mu, \quad x = \text{“in”, “up”}, \quad (24)$$

where we associate the superscript $\mathfrak{N}$ to $\mathcal{J}$ to express that ω is above the branch cut in the complex plane of ω . If we naively flip the sign of ω with $\omega_I > 0$, it should be mapped to the value below the complex plane in general. We continue to use $\mathcal{J}$ to express this transformation.

On the real axis satisfying $|\omega| < \mu$, there is no distinction between $\mathcal{J}$ and $\mathcal{J}^{\mathfrak{N}}$. For $|\omega_R| < \mu$, $\mathcal{J}k_\infty = k_\infty$ and the radial function is real, and therefore we find

$$\mathcal{J}R^{\text{up}} = R^{\text{up}} = (R^{\text{up}})^*, \quad \text{for } |\omega| < \mu, \quad (25)$$

where the first equality holds for an arbitrary ω . Hereafter, we omit the subscript ωlm unless it is ambiguous. To be consistent with the above transformation laws, we should choose the normalization constants as

$$\mathcal{J}^{\mathfrak{N}} \mathcal{N}^x = \mathcal{N}^x. \quad (26)$$

Thus, the mode functions are defined by

$$u^x := e^{-i\omega t + im\varphi} R^x(r) S(\theta). \quad (27)$$

Because $S(\theta)$ is always real for the continuous modes, combining with Eq. (12), we find

$$\mathcal{J}^{\mathfrak{N}} u^x = (u^x)^*, \quad (\text{for all continuous modes}). \quad (28)$$

Hence, we can use the operation $\mathcal{J}^{\mathfrak{N}}$, in place of the complex conjugation. Use of $\mathcal{J}^{\mathfrak{N}}$ is advantageous in the sense that we can avoid that ω^* enters in the expression that requires analytic continuation away from the real axis to ensure that the expressions are holomorphic when analytically continued away from the real axis.

From the abovementioned relation, we can read the relations among the asymptotic coefficients,

$$\mathcal{J}^{\mathfrak{N}} A^x = (A^x)^*, \quad (\text{for } |\omega| > \mu), \quad (29)$$

$$\mathcal{J}A^{\text{out}} = A^{\text{in}} = (A^{\text{out}})^*, \quad (\text{for } |\omega| < \mu). \quad (30)$$

¹Note the unusual choice of branch cuts for k_∞ in (15).

The first equality in (30) generically holds for an arbitrary ω . We introduce $\|A^x\|^2 := A^x(\mathcal{J}^{\mathfrak{N}} A^x)$. From Eqs. (29) and (30), we find $\|A^x\|^2 = |A^x|^2$ for all continuous modes.

When both $R_1(r)$ and $R_2(r)$ are solutions of the radial equation, the Wronskian

$$W[R_1(r), R_2(r)] \equiv \frac{1}{\mathcal{G}^2} \left(R_1(r) \frac{dR_2(r)}{dr_*} - \frac{dR_1(r)}{dr_*} R_2(r) \right) \quad (31)$$

becomes a constant independent of r_* . Then, from the conservation of $W[R^{\text{in}}, \mathcal{J}R^{\text{in}}]$, $W[R^{\text{up}}, \mathcal{J}R^{\text{up}}]$, and $W[R^{\text{up}}, \mathcal{J}R^{\text{in}}]$ at the infinity and the horizon, we have the identities

$$k_\infty = k_H(\|A^{\text{out}}\|^2 - \|A^{\text{in}}\|^2), \quad (32)$$

$$k_H = k_\infty(\|A^{\text{down}}\|^2 - \|A^{\text{up}}\|^2), \quad (33)$$

$$k_\infty \mathcal{J}^{\mathfrak{N}} A^{\text{up}} = -k_H A^{\text{in}} \quad (34)$$

for $|\omega| > \mu$, and from $W[R^{\text{up}}, R^{\text{in}}]$, we have

$$k_\infty A^{\text{down}} = k_H A^{\text{out}} \quad (35)$$

for an arbitrary ω .

Now, we are ready to evaluate the inner products between u^x and $\mathcal{J}u^x$. After straightforward computation, we find [54,55]

$$\begin{aligned} & ((u_{\omega lm}^{\text{up}}, \mathcal{J}^{\mathfrak{N}} u_{\omega' lm}^{\text{up}})) \\ &= 2\pi(\mathcal{N}^{\text{up}})^2(k_\infty + k_H(\|A^{\text{out}}\|^2 + \|A^{\text{in}}\|^2))\delta(\omega - \omega') \\ &= 4\pi(\mathcal{N}^{\text{up}})^2 k_H \|A^{\text{out}}\|^2 \delta(\omega - \omega'), \\ & ((u_{\omega lm}^{\text{in}}, \mathcal{J}^{\mathfrak{N}} u_{\omega' lm}^{\text{in}})) \\ &= 2\pi(\mathcal{N}^{\text{in}})^2(k_\infty(\|A^{\text{up}}\|^2 + \|A^{\text{down}}\|^2) + k_H)\delta(\omega - \omega') \\ &= 4\pi(\mathcal{N}^{\text{in}})^2 k_\infty \|A^{\text{down}}\|^2 \delta(\omega - \omega'), \end{aligned} \quad (36)$$

and the inner products of all the other pairs among u^{in} , u^{up} , $\mathcal{J}^{\mathfrak{N}} u^{\text{in}}$, and $\mathcal{J}^{\mathfrak{N}} u^{\text{up}}$ vanish. We can also see that $((u^{\text{in}}, \mathcal{J}^{\mathfrak{N}} u^{\text{up}})) = 0$ more intuitively by sending the surface to evaluate the inner product to $\mathcal{I}^- \cup i^- \cup \mathcal{H}^-$. Then, the “in” modes vanish at $\mathcal{H}^-$ and $\mathcal{I}^-$ while the “up” modes vanish at i^- and $\mathcal{I}^-$. Hence, they do not have any overlap on the surface $\mathcal{I}^- \cup i^- \cup \mathcal{H}^-$.

By choosing the normalization constants as

$$\begin{aligned} \mathcal{N}^{\text{up}} &= \frac{1}{2\sqrt{\pi}\|A^{\text{out}}\|^2}, \\ \mathcal{N}^{\text{in}} &= \frac{1}{2\sqrt{\pi}\|A^{\text{down}}\|^2}, \end{aligned} \quad (37)$$

which satisfy Eq. (26), the ortho-normalization relations among the continuous modes are summarized as

$$\begin{aligned} ((u_{\omega lm}^{\text{up}}, \mathcal{J}^{\mathfrak{N}} u_{\omega' l' m'}^{\text{up}})) &= k_H(\omega, m) \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}, \\ ((u_{\omega lm}^{\text{in}}, \mathcal{J}^{\mathfrak{N}} u_{\omega' l' m'}^{\text{in}})) &= k_\infty(\omega) \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}, \end{aligned} \quad (38)$$

with all other inner products being 0 [54,55]. Whether the mode is of positive or negative frequency is determined by the signature of the norm $k_H(\omega, m)$ and $k_\infty(\omega)$. Hence, from Eq. (38), we find that when $k_H > 0$, the mode u^{up} is the positive frequency mode, while $\mathcal{J}^{\mathfrak{N}} u^{\text{up}}$ corresponds to the negative frequency mode, as implied by the relation in Eq. (6). Conversely, when $k_H < 0$, the roles are reversed, suggesting u^{up} becomes the negative frequency mode, while $\mathcal{J}^{\mathfrak{N}} u^{\text{up}}$ becomes the positive frequency mode.

In the same manner, u^{in} is the positive frequency function for $k_\infty > 0$, i.e., for $\omega > \mu$, while $\mathcal{J}^{\mathfrak{N}} u^{\text{in}}$ is the positive frequency function in the opposite case. We did not include $1/\sqrt{|k_H|}$ and $1/\sqrt{|k_\infty|}$ in the normalization coefficients $\mathcal{N}$, to keep all the expressions holomorphic. We associate the factors k_H and k_∞ with the commutation relation between the creation and annihilation operators later. To summarize, only for the superradiant modes, u^{up} with $0 < \omega < m\Omega_H$, the roles of positive and negative frequency modes are interchanged between u^{up} and $\mathcal{J}^{\mathfrak{N}} u^{\text{up}}$, compared to the ordinary case. We denote the class of continuous modes in this range by N^- , while the class of the other continuous modes by N^+ , i.e.,

$$\begin{aligned} N^-: & \{u_{\omega lm}^{\text{up}}, \mathcal{J}^{\mathfrak{N}} u_{\omega lm}^{\text{up}}\} \quad \text{with } m > 0 \quad \text{and } 0 < \omega < m\Omega_H, \\ N^+: & \text{all the other continuous modes.} \end{aligned} \quad (39)$$

It is worth pointing out that observers floating near the horizon, whose 4-velocity u^μ should satisfy $u^\mu \partial_\mu \propto \partial_t + \Omega_H \partial_\phi$ approximately, would identify $\mathcal{J}^{\mathfrak{N}} u_{\omega lm}^{\text{up}}$ with $k_H < 0$ as a positive frequency mode without any doubt, because

$$iu^\mu \partial_\mu \log(\mathcal{J}^{\mathfrak{N}} u_{\omega lm}^{\text{up}})|_{\mathcal{H}^-} = -k_H > 0. \quad (40)$$

As u^{up} and $\mathcal{J}^{\mathfrak{N}} u^{\text{up}}$ vanish at i^- and $\mathcal{I}^-$, by considering the limit $t \rightarrow -\infty$, this argument explains why $\mathcal{J}^{\mathfrak{N}} u_{\omega lm}^{\text{up}}$ with $k_H(\omega, m) < 0$ is the positive frequency function.

2. Superradiant discrete modes

To construct a complete set of mode functions, we also need to consider the discrete spectrum, which satisfies the decaying boundary conditions on both boundaries at $r_* = \pm\infty$. We find that these boundary conditions imply $A^{\text{out}} = 0$ and $A^{\text{down}} = 0$ simultaneously at discrete frequencies with $\omega_I > 0$, which corresponds to the modes that undergo the superradiant instability. Their amplitude grows exponentially with time by extracting rotational energy from the BH [9–13].

Normalizable solutions with discrete complex frequencies with $\omega_I > 0$ can be obtained through the analytic continuation of u^{up} (or u^{in}) given in Eq. (27) with Eq. (22) [or Eq. (23)]. In the case of the analytic continuation of u^{up} , the boundary condition at the horizon is automatically satisfied as long as $\omega_I > 0$, and the decaying boundary condition toward the horizon requires that the frequencies of discrete modes $\omega = \bar{\omega}_{nlm}$ must satisfy

$$A_{\bar{\omega}_{nlm} lm}^{\text{out}} = 0. \quad (41)$$

We assign $n = l + 1$ to the mode that possesses the smallest ω_R , in analogy with hydrogen atom. We can discuss the analytic continuation of u^{in} in a similar manner, and we obtain the equivalent condition for the frequency, $A_{\bar{\omega}_{nlm} lm}^{\text{down}} = 0$, cf. Eq. (35).

Thus, the radial function labeled by nlm is expressed as

$$R_{nlm}(r) \rightarrow \mathcal{N}_{\bar{\omega}_{nlm} lm}^{\text{DSR}} \mathcal{G} \begin{cases} A_{\bar{\omega}_{nlm} lm}^{\text{in}} e^{-ik_H r_*}, & (r_* \rightarrow -\infty), \\ e^{+ik_\infty r_*}, & (r_* \rightarrow +\infty) \end{cases}. \quad (42)$$

These modes are quite different from the QNMs. The QNMs are also specified by the zeros of A^{out} . However, the QNMs are decaying, i.e., $\omega_I < 0$. For the modes not to vanish completely at a large $|r_*|$ at a late time, they exponentially grow radially toward both boundaries at $r_* \rightarrow \pm\infty$. Therefore, ω is not on the Riemann surface that we are currently concerned with, on which $\text{Im } k_\infty \geq 0$ always holds, but on another one beyond the branch cuts. Anyway, because of their exponential growth at $r_* \rightarrow \pm\infty$, QNMs are not normalizable. Nevertheless, they appear as poles in the retarded Green's function, corresponding to $A^{\text{out}} = 0$. We give an explicit form of the retarded Green's function written as a single contour integral with respect to ω in Appendix B, where we show that the retarded Green's function in the form of a single contour integral is equivalent to the one composed of the mode sum of all normalizable modes. The QNM poles are in the Riemann surface beyond the branch cut, instead of in the one that we mainly consider in this paper. When we focus on the late-time behavior, we can close the integration contour in the lower-half complex ω -plane, and those QNM poles are picked up, as usual. We do not discuss QNMs in this paper any further, because they have nothing to do with the mode expansion of the field operator, ϕ . For more details about QNMs, see [56].

The superradiant discrete mode functions are obtained as

$$u_{nlm} = e^{-i\bar{\omega}_{nlm} t + im\phi} R_{nlm}(r) S_{\bar{\omega}_{nlm} lm}(\theta). \quad (43)$$

Since the KG Eq. (7) is real and symmetric under the sign-flipping operation $\mathcal{J}$, we also have three more independent solutions,

$$u_{nlm}^*, \mathcal{J}u_{nlm}, \quad \text{and} \quad \mathcal{J}u_{nlm}^*, \quad (44)$$

where the action of $\mathcal{J}$ is the flip of the signs of $\bar{\omega}_{nlm}$ and m . In particular, the last one,

$$\mathcal{J}u_{nlm}^* = e^{-i\bar{\omega}_{nlm}^* t + im\varphi} R_{nlm}^*(r) S_{\bar{\omega}_{nlm}^* lm}^*(\theta), \quad (45)$$

belongs to the same m -modes as the original mode u , and its frequency is given by $\bar{\omega}^*$. Thus, we find that two modes, u_{nlm} with $\omega = \bar{\omega}_{nlm}$ and $\mathcal{J}u_{nlm}^*$ with $\omega = \bar{\omega}_{nlm}^*$, always appear in a pair. The frequency of the former is in the upper-half complex plane while that of the latter is in the lower-half complex plane. We should recall Eq. (12) and that we can extend the relation on the real axis,

$$\mathcal{J}R_{\omega lm}^{\text{up}} = R_{\omega lm}^{\text{up}}, \quad \text{for } \omega_I = 0 \quad \text{and} \quad |\omega_R| < \mu, \quad (46)$$

to the whole complex plane consistently, because

$$k_\infty(-\omega) = k_\infty(\omega), \quad \text{for } \omega_I \neq 0 \quad \text{or} \quad |\omega_R| < \mu. \quad (47)$$

To be consistent with Eq. (46), we should have

$$\mathcal{N}_{\bar{\omega} lm}^{\text{DSR}} = \mathcal{J} \mathcal{N}_{\bar{\omega} lm}^{\text{DSR}}. \quad (48)$$

We should stress that $\mathcal{J}R_{nlm}(r)$ does not take the form of Eq. (42). The asymptotic form of $\mathcal{J}R_{nlm}(r)$ at $r_* \rightarrow -\infty$ is $\propto A_{-\bar{\omega}_{nlm}^* l - m}^{\text{out}} e^{ik_H(-\bar{\omega}_{nlm}^*, -m)}$ and $A_{-\bar{\omega}_{nlm}^* l - m}^{\text{in}} = 0$.

We consider the KG inner product among these discrete modes. Since the KG inner product must be time independent, from the vanishing conditions of t and φ dependence, we find that the nonvanishing combination is essentially $((u_{nlm}, \mathcal{J}u_{nlm}))$ only, with others like $((u_{nlm}, u_{nlm}^*))$ and $((\mathcal{J}u_{nlm}, \mathcal{J}u_{nlm}^*))$ equal to 0. As given in Appendix A, one can calculate the inner product as

$$((u_{nlm}, \mathcal{J}u_{nlm})) = 2i(\mathcal{N}_{\bar{\omega} lm}^{\text{DSR}})^2 k_H A_{\bar{\omega} lm}^{\text{in}} \partial_{\bar{\omega}} A_{\bar{\omega} lm}^{\text{out}}. \quad (49)$$

Here, we abbreviate the subscripts nlm from $\bar{\omega}_{nlm}$, for brevity. Hence, we set the normalization coefficient to

$$\mathcal{N}_{nlm}^{\text{DSR}} = [2ik_H A_{\bar{\omega} lm}^{\text{in}} \partial_{\bar{\omega}} A_{\bar{\omega} lm}^{\text{out}}]^{-1/2}, \quad (50)$$

which is consistent with Eq. (48). With this normalization, we obtain

$$\begin{aligned} ((u_{nlm}, \mathcal{J}u_{n'l'm'})) &= ((\mathcal{J}u_{nlm}^*, u_{n'l'm'}^*)) \\ &= \delta_{nn'} \delta_{ll'} \delta_{mm'}, \end{aligned} \quad (51)$$

while all other inner products of discrete modes vanish.

We should stress that when we express the complex conjugate u_{nlm}^* , we can eliminate all complex conjugate operations except for the one acting on $\bar{\omega}_{nlm}$. Namely,

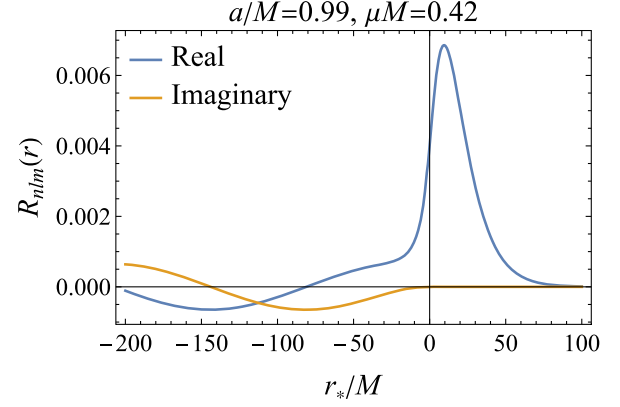


FIG. 3. Numerical demonstration of radial profile $R_{211}(r)$ of the fundamental mode. The BH and field parameters are set as $a/M = 0.99, \mu M = 0.42$. The blue line denotes the real part of R_{211} while the orange line denotes the imaginary part.

$$\begin{aligned} \hat{R}_{\bar{\omega} lm}^{\text{up}*} &= \hat{R}_{\bar{\omega}^* lm}^{\text{up}}, \\ A_{\bar{\omega} lm}^{\text{in}*} &= A_{\bar{\omega}^* lm}^{\text{out}}, \quad A_{\bar{\omega}^* lm}^{\text{in}} = A_{\bar{\omega} lm}^{\text{out}*} = 0, \\ R_{nlm}^*(r) &= \mathcal{N}_{nlm}^{\text{DSR}*} \hat{R}_{\bar{\omega} lm}^{\text{up}*}, \\ \mathcal{N}_{nlm}^{\text{DSR}*} &= [-2ik_H(\bar{\omega}^*, m) A_{\bar{\omega}^* lm}^{\text{out}} \partial_{\bar{\omega}^*} A_{\bar{\omega}^* lm}^{\text{in}}]^{-1/2}, \\ S_{\bar{\omega} lm}^*(\theta) &= S_{\bar{\omega}^* lm}(\theta), \end{aligned} \quad (52)$$

where the first equality is derived from the fact that $k_\infty^*(\bar{\omega}) = -k_\infty(\bar{\omega}^*)$, and all other equalities can be derived directly.

It would be worth mentioning how the orthogonality,

$$((u_{nlm}, u_{nlm}^*)) = 0, \quad (53)$$

is satisfied. This orthogonality relies on the orthogonality of the radial functions,

$$\int dr_* \tilde{f}(r) (r^2 + a^2) |R_{nlm}|^2 = 0, \quad (54)$$

with

$$\tilde{f}(r) := 2\omega_R \left[1 - \frac{\eta a^2 \Delta}{(r^2 + a^2)^2} \right] - \frac{4aMr}{(r^2 + a^2)^2} m. \quad (55)$$

Equation (54) is explained by the sign change in the weight function $\tilde{f}(r)$, which is related to the presence of the ergoregion. The outer region of the radial integral in Eq. (54) contributes positively, while the inner region contributes negatively.

One example of the radial function $R_{211}(r)$ is shown in Fig. 3. The bump in the real part of the radial function is on the outside of the potential barrier, which is the part that we usually identify as the axion cloud. From this plot, we find that the cancellation in the integral in Eq. (54) is caused by a long tail of $|R_{nlm}|^2$ inside the barrier, although its amplitude

is suppressed. As a result, the inner product of $((u_{nlm}, u_{nlm}^*))$ vanishes. The integral $\int dr_* (r^2 + a^2) f(r) R_{nlm}(r)^2$, which is contained in $((u_{nlm}, \mathcal{J}u_{nlm}))$, is dominated by the region outside the potential barrier, because $R_{nlm}(r)^2$ oscillates inside the barrier and gives only a negligible small contribution.

3. Mode expansion

As we have obtained a complete set of normalizable solutions with the evaluation of the KG norm in the previous subsection, we expand the field as

$$\begin{aligned} \Phi = \sum_l \left[\sum_{m=1}^l \sum_{n=l+1}^{\infty} (u_{nlm} a_{nlm} + u_{nlm}^* a_{nlm}^\dagger) \right. \\ + \mathcal{J} u_{nlm}^* b_{nlm} + \mathcal{J} u_{nlm} b_{nlm}^\dagger \\ + \sum_m \left(\int_{-\infty}^{\infty} u_{\omega lm}^{\text{up}} a_{\omega lm}^{\text{up}} d\omega \right. \\ \left. + \left(\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty} \right) u_{\omega lm}^{\text{in}} a_{\omega lm}^{\text{in}} d\omega \right) \Big]. \end{aligned} \quad (56)$$

Now, we establish the commutation relation among the expansion coefficients introduced above. Since Φ is Hermitian, together with Eq. (28) we have

$$a_{\omega lm}^{x\dagger} = \mathcal{J}^{\Re} a_{\omega lm}^x = a_{-\omega l - m}^x. \quad (57)$$

We can extract the expansion coefficients as

$$\begin{aligned} a_{nlm} &= ((\Phi, \mathcal{J}u_{nlm})), \\ a_{\omega lm}^{\text{up}} &= k_H^{-1} ((\Phi, \mathcal{J}^{\Re} u_{\omega lm}^{\text{up}})), \\ a_{\omega lm}^{\text{in}} &= k_{\infty}^{-1} ((\Phi, \mathcal{J}^{\Re} u_{\omega lm}^{\text{in}})). \end{aligned} \quad (58)$$

Using the equal-time commutation relations

$$\begin{aligned} [\Phi(t, \mathbf{x}), \Pi(t, \mathbf{x}')] &= i\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \\ [\Pi(t, \mathbf{x}), \Pi(t, \mathbf{x}')] &= 0, \\ [\Phi(t, \mathbf{x}), \Phi(t, \mathbf{x}')] &= 0, \end{aligned} \quad (59)$$

we obtain the desired commutation relations as

$$[a_{nlm}, b_{n'l'm'}^\dagger] = [b_{nlm}, a_{n'l'm'}^\dagger] = \delta_{nn'} \delta_{ll'} \delta_{mm'}, \quad (60)$$

$$[a_{\omega lm}^{\text{up}}, a_{\omega' l' m'}^{\text{up}\dagger}] = \frac{1}{k_H(\omega, m)} \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}, \quad (61)$$

$$[a_{\omega lm}^{\text{in}}, a_{\omega' l' m'}^{\text{in}\dagger}] = \frac{1}{k_{\infty}(\omega)} \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}, \quad (62)$$

and the commutators of all the other combinations vanish. We introduced the conjugate momentum $\Pi = \partial \mathcal{L} / \partial \dot{\Phi}$. Here,

we used $a_{\omega lm}^{x\dagger} = \mathcal{J}^{\Re} a_{\omega lm}^x$ for continuous modes. As anticipated before, the commutation relation of the associated expansion coefficients has an unusual frequency-dependent factor. Because of this factor, $\sqrt{|k_H|} a_{\omega lm}^{\text{up}}$ and $\sqrt{|k_{\infty}|} a_{\omega lm}^{\text{in}}$ are the annihilation operators normalized in a standard notation for $k_H > 0$ and for $k_{\infty} > 0$, respectively. They play the role of the creation operator in the opposite case. We stick to the current notation in this paper, because the expression becomes more compact and the use of $|k_H|$ hides the analyticity of expressions.

The commutation relations for the discrete modes differ from the standard case. To obtain the standard one, we introduce new operators $a_{\pm, nlm}$ by

$$\begin{aligned} a_{nlm} &= \frac{1}{\sqrt{2}} (a_{+, nlm} + a_{-, nlm}^\dagger), \\ b_{nlm} &= \frac{1}{\sqrt{2}} (a_{+, nlm} - a_{-, nlm}^\dagger), \end{aligned} \quad (63)$$

such that they satisfy the ordinary commutation relations

$$\begin{aligned} [a_{+, nlm}, a_{+, n'l'm'}^\dagger] &= \delta_{nn'} \delta_{ll'} \delta_{mm'}, \\ [a_{-, nlm}, a_{-, n'l'm'}^\dagger] &= \delta_{nn'} \delta_{ll'} \delta_{mm'}. \end{aligned} \quad (64)$$

The mode functions associated with the new operators $a_{\pm, nlm}$ are given by

$$\begin{aligned} v_{+, nlm} &= \frac{1}{\sqrt{2}} (u_{nlm} + \mathcal{J}u_{nlm}^*), \\ v_{-, nlm} &= \frac{1}{\sqrt{2}} (u_{nlm}^* - \mathcal{J}u_{nlm}). \end{aligned} \quad (65)$$

It is easy to check that they obey the following orthonormalization relations:

$$\begin{aligned} ((v_{\pm, nlm}, \mathcal{J}v_{\pm, nlm})) &= 1, \\ ((v_{\pm, nlm}, \mathcal{J}v_{\mp, nlm})) &= ((v_{\pm, nlm}, \mathcal{J}v_{\pm, nlm}^*)) = 0. \end{aligned} \quad (66)$$

The mode expansion is now reorganized as

$$\begin{aligned} \Phi = \sum_{nl} \sum_{m=1}^l (v_{-, nlm} a_{-, nlm} + v_{-, nlm}^* a_{-, nlm}^\dagger \\ + v_{+, nlm} a_{+, nlm} + v_{+, nlm}^* a_{+, nlm}^\dagger) \\ + (\text{continuum contributions with } \omega_l = 0). \end{aligned} \quad (67)$$

Since $\mathcal{J}(R_{nlm} S_{nlm}) = R_{nlm} S_{nlm}$, we find

$$v_{\pm, nlm} \approx \frac{e^{\mp i\omega_R t \pm im\varphi}}{\sqrt{2}} (R_{nlm}^* S_{nlm}^* \pm R_{nlm} S_{nlm}), \quad (68)$$

assuming $\omega_l t \ll 1$. Since ω_l is small, S_{nlm} is almost real. Hence, the real (imaginary) part of R_{nlm} corresponds to v_+

$(-v_-^*)$ at $t = 0$. As is read from Fig. 3, v_+ has a contribution also inside the potential barrier, but the contribution from the outside dominates, while v_- is concentrated in the region inside the barrier.

III. QUANTUM DYNAMICS OF SUPERRADIANCE

In this section, we discuss quantum states and their properties. In the present setup we do not have a vacuum state as the lowest energy state, which makes the procedure unconventional.

A. Hamiltonian and the reference state

The Hamiltonian is given by

$$\begin{aligned} H &\equiv \int_{\Sigma_t} d^3x \sqrt{-g} (\Pi \dot{\Phi} - \mathcal{L}) \\ &= \int_{\Sigma_t} d^3x \sqrt{-g} \left(-\frac{1}{2} g^{00} \partial_0 \Phi \partial_0 \Phi \right. \\ &\quad \left. + \frac{1}{2} g^{ij} \partial_i \Phi \partial_j \Phi + \frac{1}{2} \mu^2 \Phi^2 \right). \end{aligned} \quad (69)$$

We can show that this Hamiltonian is equivalent to

$$E := \int_{\Sigma_t} d^3x \sqrt{-g} T_{\mu\nu} \tilde{n}^\mu \xi^\nu, \quad (70)$$

where

$$T_{\mu\nu} \equiv \partial_{(\mu} \Phi \partial_{\nu)} \Phi - \frac{1}{2} g_{\mu\nu} (\partial_\rho \Phi \partial^\rho \Phi + \mu^2 \Phi^2), \quad (71)$$

and ξ^μ is the timelike Killing vector defined by $\xi^\mu \partial_\mu = \partial_t$. We notice that $\tilde{n}_\mu \xi^\mu = -1$. Parentheses on the indices mean the symmetrization. Performing the integration by parts in the spatial directions with Dirichlet boundary conditions at finite values of r_* , simplifying the expression using the KG equation, and sending the boundary to infinity ($r_* \rightarrow \pm\infty$) later, we obtain

$$\begin{aligned} H &= \int_{\Sigma_t} d^3x \sqrt{-g} \left(\xi^\nu \tilde{n}^\mu \partial_{(\mu} \Phi \partial_{\nu)} \Phi \right. \\ &\quad \left. - \frac{1}{2} \tilde{n}_\nu \xi^\nu g^{00} (\Phi \partial_0^2 \Phi + \partial_0 \Phi \partial_0 \Phi) \right). \end{aligned} \quad (72)$$

Plugging the mode expansion shown in Eq. (56), the second term in Eq. (72), and the Hamiltonian reduces to

$$H = H_D + H_C, \quad (73)$$

with

$$H_D := \sum_{nlm} \bar{\omega}_{nlm} b_{nlm}^\dagger a_{nlm} + \bar{\omega}_{nlm}^* a_{nlm}^\dagger b_{nlm}, \quad (74)$$

$$\begin{aligned} H_C &:= \sum_{lm} \left[\int_{\mu}^{\infty} d\omega \omega k_{\infty} (\mathcal{J}^{\Re} a_{\omega lm}^{\text{in}}) a_{\omega lm}^{\text{in}} \right. \\ &\quad \left. + \int_0^{\infty} d\omega \omega k_H (\mathcal{J}^{\Re} a_{\omega lm}^{\text{up}}) a_{\omega lm}^{\text{up}} \right], \end{aligned} \quad (75)$$

besides an additive constant irrelevant to the current discussion. H_D represents the contribution of discrete modes, and H_C is that of continuous modes. This is nothing but $\sum_i \omega_i n_i$ with n_i being the number operator of the i th mode.

The expression for the angular momentum along the direction of the axis of the Kerr metric, J_z , is obtained by replacing ξ^μ in Eq. (72) with the minus of the rotational Killing vector $-\xi_{(\varphi)}^\mu$ defined by $\xi_{(\varphi)}^\mu \partial_\mu = \partial_\varphi$, and hence the expression of J_z in the mode decomposition is obtained by simply replacing ω in (75) with m , i.e.,

$$J_z = J_{zD} + J_{zC}, \quad (76)$$

with

$$J_{zD} := \sum_{nlm} m (b_{nlm}^\dagger a_{nlm} + a_{nlm}^\dagger b_{nlm}), \quad (77)$$

$$\begin{aligned} J_{zC} &:= \sum_{lm} m \left[\int_{\mu}^{\infty} d\omega k_{\infty} (\mathcal{J}^{\Re} a_{\omega lm}^{\text{in}}) a_{\omega lm}^{\text{in}} \right. \\ &\quad \left. + \int_0^{\infty} d\omega k_H (\mathcal{J}^{\Re} a_{\omega lm}^{\text{up}}) a_{\omega lm}^{\text{up}} \right]. \end{aligned} \quad (78)$$

The Hamiltonian of continuous modes, H_C , is represented as two independent series of modes, i.e., “in” and “up.” “Up” modes contain oscillators belonging to N^- , which have a negative energy. Hence, H_C is unbounded from below. Recall that we have to identify $a_{\omega lm}^{\text{up}}$ not as the annihilation operator but as the creation operator for the modes in N^- . This interchange of the roles of the creation and annihilation operators is necessary to ensure the positivity of the norm of the excited states. Therefore, it is more appropriate to replace the Hamiltonian shown in Eq. (75) for the modes in N^- , excluding their zero-point energy, by

$$H_C(N^-) = \sum_{l=1}^{\infty} \sum_{m=1}^l \int_0^{m\Omega_H} d\omega (-\omega) |k_H| a_{\omega lm}^{\text{up}} (\mathcal{J}^{\Re} a_{\omega lm}^{\text{up}}), \quad (79)$$

though this is just an unessential shift of the origin of the energy.

Because the Hamiltonian is unbounded from below, we do not have the Boulware “vacuum,” the lowest-energy state. However, one can still define a Boulware-like reference state $|B\rangle_C$ by the conditions

$$\begin{aligned} a_{\omega lm}|B\rangle_C &= 0, \quad \text{for modes in } N^+, \\ (\mathcal{J}^{\Re} a_{\omega lm})|B\rangle_C &= 0, \quad \text{for modes in } N^-, \end{aligned} \quad (80)$$

and construct the Fock space $\mathcal{H}_C$ applying creation operators to it. When there is no interaction between two sectors N^+ and N^- , they can be treated as two independent quantum systems. Then, the flip of the sign of the energy is just a matter of convention and the reference state $|B\rangle_C$ is stable. However, $|B\rangle_C$ becomes unstable once the interaction between the two sectors is activated. Nonetheless, as long as free-field theory is concerned, there is no pathological behavior associated with this negativity of the excitation energy, and $|B\rangle_C$ is stable. The quantum state $|B\rangle_C$ is also singular on the horizon, as in the case of the usual Boulware vacuum state in a nonrotating BH spacetime. We will discuss later about the states that are regular on the future horizon.

Next, we discuss the Hamiltonian for the discrete modes, H_D . Together with Eqs (63) and (64), H_D can be rewritten as

$$H_D \equiv H_0 + H_S, \quad (81)$$

with

$$\begin{aligned} H_0 &:= \sum_{nlm} \bar{\omega}_R (a_+^\dagger a_+ - a_-^\dagger a_-), \\ H_S &:= \sum_{nlm} i\bar{\omega}_I (a_+^\dagger a_-^\dagger - a_+ a_-). \end{aligned} \quad (82)$$

Since the imaginary part of the superradiant discrete modes frequency is small, i.e., $\bar{\omega}_I \ll \bar{\omega}_R$, H_0 is the dominant part while H_S is a small correction to it. Then, the modes $v_{+,nlm}$ have a positive energy, while the modes $v_{-,nlm}$ have a negative energy. Because of the negative-energy modes, H_D is also unbounded from below, and hence the lowest-energy state does not exist. Note that this type of Hamiltonian appears as a nondegenerate parametric amplifier in the quantum optics [57].

Unlike the case of continuous modes we discussed above, because of the presence of H_S , the positive- and the negative-energy sectors interact with each other. Therefore, it is not only that the lowest-energy state does not exist, but that normalizable energy eigenstates themselves are absent. It is easy to prove that H_D cannot be rewritten into the diagonalized form, in which the Hamiltonian takes the form of $H_D = \sum \omega_i a_i^\dagger a_i$. If it were the case, we should have $[a_i, H_D] = \omega_i a_i$. However, as a_i should be a linear combination of a , b , $a^\dagger$, and $b^\dagger$, i.e., $a_i = \alpha a + \beta b + \gamma a^\dagger + \delta b^\dagger$, a direct computation shows that $[a_i, H_D] = -\alpha \bar{\omega}^* a - \beta \bar{\omega} b + \gamma \bar{\omega}^* a^\dagger + \delta \bar{\omega} b^\dagger$, which is incompatible with the condition $[a_i, H_D] = \omega_i a_i$ mentioned above. This completes the proof of the absence of the basis in which the positive- and negative-energy sectors decouple.

We specify a reference state for the discrete modes $|B\rangle_D$ by the conditions

$$a_{+,nlm}|B\rangle_D = a_{-,nlm}|B\rangle_D = 0, \quad (83)$$

and then we can construct the Fock space $\mathcal{H}_D$ by multiplying the creation operators. Such a reference state is not invariant under the time translation, since the annihilation operators $a_\pm$, which define the reference state, do not commute with the Hamiltonian, i.e., the operator for the time translation.

One may think that we can specify the reference state by imposing the conditions $a_{nlm}|B\rangle_D = b_{nlm}|B\rangle_D = 0$, but these conditions are equivalent to $a_+|B\rangle_D = a_-^\dagger|B\rangle_D = 0$. Then, the excited states generated by multiplying a_- (and $a_+^\dagger$) can have a negative norm. Therefore, we cannot accept this choice of the construction of the Hilbert space. The complete Hilbert space should be $\mathcal{H}_C \otimes \mathcal{H}_D$, and we choose the joint reference state as $|B\rangle \equiv |B\rangle_C \otimes |B\rangle_D$.

Since the number operators associated with discrete modes $v_{\pm,nlm}$,

$$N_{\pm,nlm} = a_{\pm,nlm}^\dagger a_{\pm,nlm}, \quad (84)$$

do not commute with the Hamiltonian, the corresponding Heisenberg operators $N_\pm(t)$ evolve in time, obeying the Heisenberg equation, $dN_\pm(t)/dt = -i[H, N_\pm(t)]$. The evolutions of $N_\pm(t)$ are easily solved to obtain

$$\begin{aligned} N_+(t) + N_-(t) &= 2 \sinh^2(\bar{\omega}_I t) - \frac{\sinh(2\bar{\omega}_I t)}{2} \\ &\times (a_+^\dagger(0)a_-^\dagger(0) + a_+(0)a_-(0)) \\ &+ \cosh(2\bar{\omega}_I t)(N_+(0) + N_-(0)), \end{aligned} \quad (85)$$

$$N_+(t) - N_-(t) = N_+(0) - N_-(0). \quad (86)$$

The exponentially growing factors in Eq. (85) roughly explain the growth of superradiant clouds, which we will discuss in more detail in the next subsection, while Eq. (86) reflects the conservation of energy.

B. Growth of superradiant clouds

The expectation value of the Hamiltonian H is time independent for any quantum state, since H is the generator of time translation and it commutes with itself. This is nothing but the energy conservation, which is the consequence of the time translation symmetry of the background Kerr spacetime. We should notice that each time slicing must be a Cauchy surface in quantum field theory. Otherwise, we cannot describe the unitary evolution of the quantum state. Hence, the escape of the energy through the boundary is strictly prohibited. On the other hand, in the situation where the superradiant condensate grows, the energy of the scalar field also looks to be growing. How can

we understand the growth of the clouds under the constraint of the energy conservation?

In order to understand the growth of superradiant condensates in the language of the quantum field theory, we first need to realize that the amplitude of the mode functions $v_{\pm}$, roughly speaking, depends on time as $|v_{\pm}| \propto \cosh(\omega_I t)$. This is because $v_{\pm}$ are composed of a superposition of exponentially growing modes, $u_{nlm}, u_{nlm}^* \propto e^{\omega_I t}$, and exponentially decaying modes, $\mathcal{J}u_{nlm}, \mathcal{J}u_{nlm}^* \propto e^{-\omega_I t}$.

The second step is to understand how the KG norm of these mode functions remains constant and the energy is conserved as well.

We explain how this cancellation occurs. For this purpose, we divide the contribution of the inner product into two pieces as $((*, *)) = ((*, *))_{\text{int}} + ((*, *))_{\text{ext}}$, where the inner product with the subscript “int” (“ext”) represents the contribution of the region inside (outside) the potential barrier. Because of the oscillatory behavior of the product of the radial mode functions R_{nlm}^2 in the internal region, we have

$$((u_{nlm}, \mathcal{J}u_{nlm}))_{\text{int}} \approx 0, \quad (87)$$

and hence we have

$$((u_{nlm}, \mathcal{J}u_{nlm}))_{\text{ext}} \approx 1, \quad (88)$$

which shows the radial mode function R_{nlm} is almost real in the external region, as is read from Fig. 3 explicitly. This means that $\mathcal{N}_{nlm}^{\text{DSR}}$ is also almost a real positive number. Using this fact, we have

$$e^{-2\bar{\omega}_I t} ((u_{nlm}, u_{nlm}^*))_{\text{ext}} \approx ((u_{nlm}, \mathcal{J}u_{nlm}))_{\text{ext}} \approx 1. \quad (89)$$

On the other hand, Eq. (53) implies

$$((u_{nlm}, u_{nlm}^*))_{\text{int}} = -((u_{nlm}, u_{nlm}^*))_{\text{ext}}. \quad (90)$$

Since $\bar{\omega}_I$ is small, we also have

$$\begin{aligned} ((u_{nlm}, \mathcal{J}u_{nlm}))_{\text{int/ext}} &\approx ((\mathcal{J}u_{nlm}^*, u_{nlm}^*))_{\text{int/ext}}, \\ e^{-2\bar{\omega}_I t} ((u_{nlm}, u_{nlm}^*))_{\text{int/ext}} &\approx e^{2\bar{\omega}_I t} ((\mathcal{J}u_{nlm}^*, \mathcal{J}u_{nlm}))_{\text{int/ext}}. \end{aligned} \quad (91)$$

Using these approximate relations (91), we find

$$\begin{aligned} ((v_{\pm}, \mathcal{J}v_{\pm}))_{\text{int/ext}} &\approx ((u_{nlm}, \mathcal{J}u_{nlm}))_{\text{int/ext}} \\ &\pm \frac{1 + e^{-4\bar{\omega}_I t}}{2} ((u_{nlm}, u_{nlm}^*))_{\text{int/ext}}. \end{aligned} \quad (92)$$

Applying the relations (87)–(90), we obtain

$$\begin{aligned} ((v_{\pm}, \mathcal{J}v_{\pm}))_{\text{int}} &\approx 0 \mp \cosh(2\bar{\omega}_I t), \\ ((v_{\pm}, \mathcal{J}v_{\pm}))_{\text{ext}} &\approx 1 \pm \cosh(2\bar{\omega}_I t). \end{aligned} \quad (93)$$

Consequently, we recover

$$((v_{\pm}, \mathcal{J}v_{\pm})) = 1. \quad (94)$$

Since the ergoregion is in the internal region, it is consistent that the energy of v_{-} modes, which are more localized in the internal region, is negative. The result is consistent with the discussion at the end of Sec. III A.

To discuss the growth of the condensate, it would be better to introduce the number operator corresponding to each axion cloud quasieigenstate. As there is a potential barrier in the radial equation for superradiant modes, we can find an approximate solution that stays localized in the external region for a short time. To describe these quasi-localized modes, we introduce a new mode function, which satisfies the initial conditions at $t = t_0$,

$$\begin{aligned} \tilde{u}_{nlm}^{(t_0)}(t_0, \mathbf{x}) &:= u_{nlm}(0, \mathbf{x})\theta(r - r_b), \\ \dot{\tilde{u}}_{nlm}^{(t_0)}(t_0, \mathbf{x}) &:= \dot{u}_{nlm}(0, \mathbf{x})\theta(r - r_b), \end{aligned} \quad (95)$$

where r_b is the point under the potential barrier at which the internal and external regions are separated. Since we mostly perform approximations in the following discussions, an accurate definition for r_b is unnecessary. We associate the annihilation and creation operators, $\tilde{a}_{nlm}^{(t_0)}$ and $\tilde{a}_{nlm}^{(t_0)\dagger}$, with the mode functions $\tilde{u}_{nlm}^{(t_0)}(t_0, \mathbf{x})$ and $\tilde{u}_{nlm}^{(t_0)*}(t_0, \mathbf{x})$, respectively. These mode functions are, of course, solutions of the KG equation, but their meaning as the modes for the superradiant condensate is limited to $t \approx t_0$. We define the pseudonumber operator

$$\tilde{N}_{nlm}^{(t_0)} := \tilde{a}_{nlm}^{(t_0)\dagger} \tilde{a}_{nlm}^{(t_0)}, \quad (96)$$

which counts the number of particles contained in the condensate labeled by nlm at $t = t_0$ in the usual sense.

Then, it is straightforward to calculate

$$\begin{aligned} \tilde{a}_{nlm}^{(t_0)} &= ((\Phi(t, \mathbf{x}), \tilde{u}_{nlm}^{(t_0)*})) = ((\Phi(t + t_0, \mathbf{x}), u_{nlm}^*))_{\text{ext}} \\ &= e^{-i\bar{\omega}_I t_0} a_{nlm} ((u_{nlm}, u_{nlm}^*))_{\text{ext}} \\ &\quad + e^{-i\bar{\omega}_I^* t_0} b_{nlm} ((\mathcal{J}u_{nlm}^*, u_{nlm}^*))_{\text{ext}} \\ &\quad + e^{i\bar{\omega}_I^* t_0} a_{nlm}^{\dagger} ((u_{nlm}^*, u_{nlm}^*))_{\text{ext}} \\ &\quad + e^{i\bar{\omega}_I t_0} b_{nlm}^{\dagger} ((\mathcal{J}u_{nlm}, u_{nlm}^*))_{\text{ext}} \\ &\approx e^{-i\bar{\omega}_I t_0} a_{nlm} + e^{-i\bar{\omega}_I^* t_0} b_{nlm} \\ &= \sqrt{2} e^{-i\bar{\omega}_R t_0} (a_+ \cosh(\bar{\omega}_I t_0) + a_-^{\dagger} \sinh(\bar{\omega}_I t_0)), \end{aligned} \quad (97)$$

where $((\dots))_{\text{ext}}$ is assumed to be evaluated at $t = 0$, as before. Here, we neglect the contributions from the

continuous modes in the third equality, because their contribution to the expectation value of $\langle B|\tilde{N}_{nlm}^{(t_0)}|B\rangle$ is t_0 independent. Then, the expectation value of $\tilde{N}_{nlm}^{(t_0)}$ is easily evaluated as

$$\langle B|\tilde{N}_{nlm}^{(t_0)}|B\rangle = 2 \sinh^2(\bar{\omega}_I t_0). \quad (98)$$

From the expression (98), we find that the particle number $\langle B|\tilde{N}_{nlm}|B\rangle$ increases for the reference state as the absolute value of time $|t|$ increases. We also find that $\langle B|\tilde{N}_{nlm}^{(0)}|B\rangle = 0$ for the reference state, besides the possible time-independent contribution from the continuous modes. Since the choice of the origin of time is arbitrary, we find that there exists a quantum state such that $\langle \tilde{N}_{nlm}^{(t_0)} \rangle = 0$ for each value of an arbitrarily chosen t_0 .

To prove the increase of the expectation value of the number of particles contained in a cloud, it would be enough to observe that, for an arbitrary state $|\psi\rangle$, we have

$$\begin{aligned} & \langle \psi | \tilde{N}_{nlm}^{(\delta t)} - 2\tilde{N}_{nlm}^{(0)} + \tilde{N}_{nlm}^{(-\delta t)} | \psi \rangle \\ &= 4 \sinh^2(\bar{\omega}_I \delta t) \langle \psi | a_+^\dagger a_+ + a_-^\dagger a_- + 1 | \psi \rangle \\ &\geq 4 \sinh^2(\bar{\omega}_I \delta t), \end{aligned} \quad (99)$$

where we neglect the additional contribution from the continuous modes, again. The above inequality implies

$$\begin{aligned} & \left\langle \psi \left| \frac{d^2 \tilde{N}_{nlm}^{(t_0)}}{dt_0^2} \right| \psi \right\rangle \Big|_{t_0=0} \\ &= \lim_{\delta t \rightarrow 0} \frac{\langle \psi | \tilde{N}_{nlm}^{(\delta t)} - 2\tilde{N}_{nlm}^{(0)} + \tilde{N}_{nlm}^{(-\delta t)} | \psi \rangle}{\delta t^2} \geq 4\bar{\omega}_I^2, \end{aligned} \quad (100)$$

which can be generalized to any t_0 . Hence, whatever initial state one may choose, the growth of the superradiant condensates is inevitable.

Finally, we briefly discuss how the discrete superradiant modes extract energy and angular momentum from the central black hole. For this purpose, we evaluate the radial component of the² energy flux at a finite value of r_* , which is given by

$$T_{\mu\nu} \hat{r}^\mu \xi^\nu = \partial_{(\mu} \Phi \partial_{\nu)} \Phi \hat{r}^\mu \xi^\nu, \quad (101)$$

where $\hat{r}_\mu$ is the unit normal $\propto \partial r / \partial x^\mu$. Taking the expectation value in the region near the horizon, we can evaluate the energy flux passing through $r = \text{constant}$ surface in a

²To compute the expectation value of the energy flux, one needs to renormalize the expression. Here, we only show how the exponential growth and the positivity of the energy flux can be seen and leave the renormalization procedure to the future work. For a similar reason, we also omit the renormalization procedure for the energy density below.

unit coordinate time interval of t in the increasing direction of r as

$$\begin{aligned} & \langle B | T_{\mu\nu} | B \rangle \hat{r}^\mu \xi^\nu \\ & \rightarrow -\frac{1}{2} \sum_{nl} \sum_{m>1}^l (\bar{\omega} k_H^* + \bar{\omega}^* k_H) (|u_{nlm}|^2 - |\mathcal{J}u_{nlm}|^2) \\ &= \sum_{nl} \sum_{m>1}^l (m\bar{\omega}_R \Omega_H - |\bar{\omega}|^2) (\mathcal{N}_{\bar{\omega}lm}^{\text{DSR}})^2 |A_{\bar{\omega}lm}^{\text{in}}|^2 \\ & \times (e^{2\text{Im} \bar{\omega}_{nlm}(t+r_*)} - e^{2\text{Im} \bar{\omega}_{nlm}(-t+r_*)}). \end{aligned} \quad (102)$$

The energy flux near H^+ is basically positive, as long as

$$m\bar{\omega}_R \Omega_H - |\bar{\omega}|^2 > 0, \quad (103)$$

which is the extension of the superradiance condition to a complex value of ω [12].

In the near horizon limit the wave vector k_μ , which is determined by the normal vector of the phase of the horizon ingoing wave, is given by

$$k_\mu dx^\mu \rightarrow -\omega dt - k_H dr + m d\varphi. \quad (104)$$

Raising the subscript of k_μ as

$$k^\mu \partial_\mu \rightarrow \frac{(2Mr_+)^2 k_H}{\Sigma \Delta} [\partial_t - \partial_{r_*} + \Omega_H \partial_\varphi], \quad (105)$$

we can confirm that the wave vector is almost parallel to the ingoing null vector, independently of the value of ω . Hence, the propagation direction of the superradiant discrete modes is also ingoing for all modes. The energy flux in (102), pointing toward the increasing direction of r , is the consequence of the flow of the negative energy toward the horizon.

We can also understand the growth from the viewpoint of the energy density, without relying on the particle number. The energy density is given by

$$\mathcal{E} = \langle B | T_{\mu\nu} \xi^\mu \tilde{n}^\nu | B \rangle. \quad (106)$$

Using the fact that $\xi^\mu \partial_\mu u_{nlm} = -i\bar{\omega} u_{nlm}$, we have

$$\begin{aligned} \mathcal{E} &= \bar{\omega} \tilde{n}_\mu j^\mu(u, \mathcal{J}u) + \bar{\omega}^* \tilde{n}_\mu j^\mu(u^*, \mathcal{J}u^*) \\ &+ \bar{\omega}_R (\tilde{n}_\mu j^\mu(u, u^*) + \tilde{n}_\mu j^\mu(\mathcal{J}u, \mathcal{J}u^*)) \\ &+ \bar{\omega}_I (\tilde{n}_\mu j^\mu(u, \mathcal{J}u^*) + \tilde{n}_\mu j^\mu(\mathcal{J}u, u^*)) \\ &+ (\text{total derivative terms}), \end{aligned} \quad (107)$$

for the energy density of the superradiant discrete mode. Regarding the part other than the total derivative term, we have a simple expression, which reads

$$\mathcal{E} \supset \sum_{nl} \sum_{m>1}^l \bar{\omega}_R (-g^{00} \bar{\omega}_R + g^{0\varphi} m) (|u_{nlm}|^2 + |\mathcal{J}u_{nlm}|^2). \quad (108)$$

After integrating over the angular dependence, $(-g^{00} \bar{\omega}_R + g^{0\varphi} m)$ reduces to a factor proportional to $\tilde{f}(r)$, which means this term is positive definite in the outer region. What is more, $(|u_{nlm}|^2 + |\mathcal{J}u_{nlm}|^2)$ indicates an exponential growth of $e^{2\bar{\omega}_{l,nlm}t}$, showing that the energy density should be positive and exponentially growing outside. The total derivative terms, which read $1/2 \sqrt{-g^{00}} h^{ij} \times \nabla_i (1/\sqrt{-g^{00}} \Phi \nabla_j \Phi)$, can also have the same exponential growth, but when one integrates over the outer region, they only give the boundary term, and we do not expect them to be comparable with the contribution inside the outer region since the amplitude of the field on the boundary is very small.

The exponential growth does not contradict the conservation of the Hamiltonian, as we have discussed before. The energy densities in the outer and inner regions have the opposite sign and when we integrate on Σ_t , they cancel each other. Therefore, the total energy remains conserved, while the energy density in the cloud region outside the potential barrier grows exponentially in time.

IV. DISCUSSIONS

A. Hawking radiation

So far, we consider the Boulware vacuum state as the reference state, but it is known to be singular at the horizon and physically irrelevant [58]. As a substitute, the Hartle-Hawking vacuum, which represents the thermal equilibrium state, is often considered. However, a more physically relevant state will be the Unruh state [59], in which the Hartle-Hawking vacuum [60] is assigned only to the “up” modes, which are nonvanishing at H^- . The “in” modes are kept unchanged to respect the natural vacuum choice at $\mathcal{I}^-$ for the massless case, or at i^- for the massive case. We can do the same in the present setup for the “up” modes, but the situation for the discrete modes is different. In Sec. III B we have observed the unbounded growth of the quasiparticle number regardless of the choice of quantum state, which proves that the discrete modes do not have any stationary state. Namely, there is no thermal equilibrium state, i.e., no Hartle-Hawking-like state, for discrete modes. Therefore, it does not make much sense to search for the natural state analogous to the usual Unruh vacuum for discrete modes. Furthermore, since the discrete modes decay exponentially at large distances, they do not contribute to the radiation observed by a distant observer at any finite time. Consequently, the standard calculation of Hawking radiation can be extended to massive fields without any essential modification.

To identify the initial quantum state, including the discrete modes, after the formation of a rotating black hole via a gravitational collapse, it would be necessary to evolve the initial vacuum state in the dynamical spacetime, which is beyond the scope of this paper. However, we can imagine that the state at a time $t = t_0$ immediately after the formation of a rotating BH will not have a very large occupation number of quasiparticles, $\tilde{N}_{nlm}^{(t_0)}$, and the state of the continuous modes will be approximated by the Unruh state.

Since we have found that the discrete modes do not contribute to Hawking radiation, we focus on the continuous modes in the following discussion. The singular behavior at the horizon is caused by the “up” modes, but the treatment of these modes is well understood, see [45,55]. Here, we would recapitulate the nontrivial point different from the non-rotating case [61,62]. When considering only observables outside the horizon, the Unruh state is obtained by replacing the quantum state of the “up” modes to the thermal state with respect to $|k_H|$, which is the energy observed in the very vicinity of the horizon.

The Hawking radiation is purely described by the “up” modes, and the number spectrum observed by the observers at a large distance is simply given by the product of the gray-body factor $1/\|A^{\text{out}}\|^2$ and the expectation value of the number of the outgoing particles near the horizon. Hence, we have

$$I_\omega := \frac{1}{\|A^{\text{out}}\|^2} \langle H | a_{\omega lm}^{\text{up}\dagger} a_{\omega lm}^{\text{up}} | H \rangle = \frac{\|A^{\text{out}}\|^{-2}}{k_H (e^{\beta k_H} - 1)}, \quad (109)$$

where the state $|H\rangle$ is the Hartle-Hawking state. The number spectrum vanishes for $|\omega| < \mu$, as $u^{\text{up}}(r)$ decays at a large r exponentially in this case. Namely, the above formula applies only for $|\omega| > \mu$.

The derivation of the detected spectrum for the superradiant modes ($k_H < 0$) is slightly different from the standard case. Since the roles of the creation and annihilation operators are interchanged for the superradiant modes, the expectation value of the number operator for the observers at $r_* \rightarrow \infty$ is computed as

$$\begin{aligned} \langle a_{\omega lm}^{\text{up}\dagger} a_{\omega lm}^{\text{up}} \rangle &= \frac{1}{-k_H} \sum_{q=1}^{\infty} (q+1) e^{q\beta k_H} \\ &= \frac{1}{-k_H} \frac{e^{-\beta k_H}}{e^{-\beta k_H} - 1} \\ &= \frac{1}{k_H (e^{\beta k_H} - 1)}. \end{aligned} \quad (110)$$

Although the path is different, the final result is exactly the same as in the usual case with $k_H > 0$. For both positive and negative k_H , I_ω given in (109) remains positive. At the first sight, the spectrum of the Hawking radiation I_ω apparently diverges at $k_H = 0$. However, from Eq. (35), we find

$$\|A^{\text{out}}\|^{-2} = \frac{-k_H^2 \|A^{\text{down}}\|^{-2}}{k_\infty (\mathcal{J}k_\infty)} = \frac{k_H^2}{|k_\infty|^2} \|A^{\text{down}}\|^{-2}. \quad (111)$$

Since $\|A^{\text{down}}\|^2 \neq 0$ at $k_H = 0$ and $|k_\infty|^2 \neq 0$ for $|\omega| > \mu$, Eq. (111) implies that the spectrum of the Hawking radiation I_ω remains finite at $k_H = 0$. In general, even for $k_\infty = 0$, the “up” mode radial function simply becomes $\hat{R}^{\text{up}} \rightarrow 1$ for $r_* \rightarrow \infty$, and is not singular. Therefore, $\hat{R}^{\text{up}}$ is a non-vanishing finite real function, and A^{out} should not vanish. Hence, we expect $\|A^{\text{out}}\|^{-2}$ remains to have the factor k_H^2 even for $k_\infty = 0$.

B. Problems with backreaction

So far, we consider a scalar field on a fixed background Kerr spacetime. However, in a realistic situation the central black hole will spin down because of the extraction of angular momentum due to superradiance. There are several attempts to formulate the quantum adiabatic evolution [63], but their quantization procedure is incomplete with limited results. In this section, we discuss how to describe the evolution of the system in such a time-dependent background, particularly focusing on the superradiant discrete modes.

1. Justification of the adiabatic evolution prescription

Let us consider the superradiant discrete mode in a slowly varying background. For simplicity, we assume that only the spin parameter of the black hole changes adiabatically in time. In principle, our discussion can be extended to more general situations in which the black hole mass also evolves, or in which the effective mass of the field varies due to self-interactions.

When the background evolution is adiabatic, i.e., $\epsilon \equiv |\nabla_\mu \log a|/|\bar{\omega}| \ll 1$, we usually assume that the fastest-growing superradiant discrete mode with the complex frequency $\bar{\omega}$ continuously grows, even though the exact shape of the mode function and the frequency change in time. When the background becomes time dependent, all that we need to do is to replace the mode functions with the ones that solve the KG equation on the given dynamical background, which is not so easy to perform analytically. However, when the background value of a changes as a function of a time $\tau(t, r) := t - \Delta t(r)$, we can show that the mode function

$$u_{nlm}^{(0)}(x) := e^{-i \int^\tau d\tau' \bar{\omega}(\tau') + im\varphi} \hat{R}_{nlm}^{(\tau)}(r) S_{\bar{\omega}(\tau)lm}(\theta) \quad (112)$$

gives a sufficiently good approximation, where $\bar{\omega}(\tau) := \bar{\omega}(a(\tau))$ and

$$\hat{R}_{nlm}^{(\tau)}(r) := e^{-i\bar{\omega}(\tau)\Delta t(r)} R_{nlm}^{(a(\tau))}(r), \quad (113)$$

with $R_{nlm}^{(a)}(r)$ being the discrete radial mode function for a given spin parameter a . In $\hat{R}_{nlm}^{(\tau)}(r)$ the violent $\bar{\omega}$ dependence

is assumed to be eliminated by the multiplicative phase factor $e^{-i\bar{\omega}(\tau)\Delta t(r)}$ for a large negative r_* , which will be discussed in more detail below. When we vary the background parameters, the label n is assumed to be continuous, even though the order of growth rates may change. Namely, $\bar{\omega}_{nlm}$ is supposed to have no discrete jump.

To show that $u_{nlm}^{(0)}$ gives a good approximation, we consider an improvement of the solution by perturbatively adding corrections to $u_{nlm}^{(0)}$ around each reference time slice τ_* as

$$u_{nlm} = u_{nlm}^{(0)} + u_{nlm}^{(1,\tau_*)} + \dots \quad (114)$$

We expand the KG equation

$$(\square^{(a(\tau))} - \mu^2)u_{nlm} = 0, \quad (115)$$

where $\square^{(a(\tau))}$ is the d'Alembertian operator on the background with the variable $a(\tau(x))$. To first order in ϵ at around $\tau = \tau_*$, using the retarded Green's function $G_R^{(a)}(x, x')$ for a fixed value of a (see Appendix B), we find

$$u_{nlm}^{(1,\tau_*)}(x) = \int d^4x' \sqrt{-g(x')} G_R^{(a_*)}(x, x') \mathcal{S}(x'), \quad (116)$$

with $a_* := a(\tau_*)$, and the source term is given by

$$\mathcal{S}(x) := (-\square^{(a(\tau))} + \mu^2)u_{nlm}^{(0)}(x), \quad (117)$$

which should be of $\mathcal{O}(\epsilon)$ because it vanishes when a is constant. Therefore, we find that the other modes are simply suppressed to be $\mathcal{O}(\epsilon)$.

However, the contribution of the superradiant mode to the Green's function may secularly modify the evolution of the amplitude of the dominant mode, $u_{nlm}^{(0)}$. However, as we shall see below, this variation is expected to be, at most, $\mathcal{O}(\epsilon^2)$, and hence it remains small at $\mathcal{O}(\epsilon)$ even after the reaction timescale $\mathcal{O}(\epsilon^{-1})$, as long as the gradient of the background is sufficiently small.

First, we should notice that the source contains only the frequencies close enough to $\bar{\omega}_{nlm}^{(a_*)}$, as the frequency of the source $\mathcal{S}$ is limited to the modulation of the seed frequency $\bar{\omega}_{nlm}^{(a_*)}$ caused by low-frequency perturbations. In general, we can expand u_{nlm} in terms of a complete set of mode functions, $\{u_{nlm}^{(a_*)}, u_{\Lambda}^{(a_*)}\}$ for $a = a_*$, where Λ represents the labels of all modes with the frequency close to $\bar{\omega}_{nlm}$, excluding the seed mode labeled by nlm itself. Here, we associate the superscript “(a)” to specify the modes on the background with $a = a_*$.

Formally, we expand u_{nlm} as $u_{nlm} \approx c_{nlm}(\tau)u_{nlm}^{(a_*)} + \sum_{\Lambda} c_{\Lambda}(\tau)u_{\Lambda}^{(a_*)}$ on the time slice $\tau = \tau_*$ with slowly changing coefficients depending on τ . By construction, we can

choose $c_{nlm}(\tau_*) = 1$ as the initial state. We can also construct $\mathcal{J}u_{nlm}$ in the same way. Then, from the conservation of the KG inner product $((u_{nlm}, \mathcal{J}u_{nlm}))$, we find $\|c_{nlm}\|^2 = 1 - \sum_{\Lambda} |c_{\Lambda}|^2$. As c_{Λ} is sourced by $\mathcal{S} = \mathcal{O}(\epsilon)$, we have $c_{\Lambda} \sim \mathcal{O}(\epsilon)$. This implies $(c_{nlm} - 1) + (\mathcal{J}c_{nlm} - 1) = \mathcal{O}(\epsilon^2)$. However, we may worry if each of c_{nlm} and $(\mathcal{J}c_{nlm})$ may change. Namely, the question is whether $dc_{nlm}(\tau)/d\tau|_{\tau=\tau_*} = \mathcal{O}(\epsilon^2)$ or not. As the amplitude of u_{nlm} is increasing in time, one can always tune the origin of the time coordinate such that $c_{nlm}(\tau_*) = 1$, as mentioned above.

To examine the change rate of $c_{nlm}(\tau)$, we focus on the “particle number” defined by the integral of $n_{\mu}^{(\tau)} j^{\mu}(u_{nlm}, u_{nlm}^*)$ in some finite region between $r_{\min}$ and $r_{\max}$, where $n_{\mu}^{(\tau)}$ is the unit vector normal to the constant- τ time slice and j^{μ} is defined in Eq. (5). In the adiabatic evolution, the evolution of the overall amplitude corresponding to c_{nlm} is determined by the balance between the change rate of this particle number and the number fluxes.

We can estimate the number flux at the boundaries at $r_{\min}$ and $r_{\max}$. The particle number flux differs from the case with fixed a primarily because of two effects. One is the flux due to the extra modes $\{u_{\Lambda}\}$ and the other is the correction to the flux carried by $u_{nlm}^{(a_*)}$ caused by the adiabatic change of a . The former is manifestly $\mathcal{O}(\epsilon^2)$ because the flux is quadratic in the amplitude of modes, which is $\mathcal{O}(\epsilon)$. The latter is also $\mathcal{O}(\epsilon^2)$ because this correction is $\mathcal{O}(\epsilon)$ higher-order correction to the original flux. However, the original flux carried by $u_{nlm}^{(a_*)}$ is already $\mathcal{O}(\epsilon)$, since the smallness of this flux is the origin of the adiabatically slow evolution of a . Thus, we find that the correction is at most $\mathcal{O}(\epsilon^2)$. Therefore, we can conclude $d|c_{nlm}(\tau)|^2/d\tau|_{\tau=\tau_*} = \mathcal{O}(\epsilon^2)$, or equivalently, $dc_{nlm}(\tau)/d\tau|_{\tau=\tau_*} = \mathcal{O}(\epsilon^2)$, and the correction to the adiabatic evolution of the number of particles contained in the cloud is suppressed by ϵ , which justifies the use of the adiabatic approximation.

In the above discussions, we assumed that the variation of $u_{nlm}^{(0)}(x)$ through $\bar{\omega}(\tau)$, i.e., $\frac{du_{nlm}^{(0)}(x)}{d\bar{\omega}(\tau)} \frac{d\bar{\omega}(\tau)}{d\tau}$ is small $\propto \epsilon$. This condition might be violated at a large negative r_* . Extracting the τ dependence in this region, we find

$$u_{nlm}^{(0)} \propto e^{-i \int^{\tau} d\tau' \bar{\omega}(\tau') - i\bar{\omega}(\tau)(\Delta t(r) + r_*)}.$$

Taking its τ derivative through $\bar{\omega}(\tau)$, we have

$$\frac{d \log u_{nlm}^{(0)}}{d\bar{\omega}(\tau)} \frac{d\bar{\omega}(\tau)}{d\tau} = -i \frac{d\bar{\omega}(\tau)}{d\tau} (\Delta t(r) + r_*). \quad (118)$$

For both terms to be small, we should choose $\Delta t(r) = -r_*$ in this region, i.e., the time slicing should be along the outgoing null coordinate in the near horizon region. This is not so demanding, because all the fluxes are pointing toward

the horizon, and the propagation speed gets close to the speed of light there.

2. Vanishing of a superradiant mode

As we change the parameters of the system like a and μ , the superradiance frequency $\bar{\omega}_{nlm}$ varies. There will be a boundary of the parameter space at which a sequence of superradiant modes stops existing. Here, we discuss what happens at this boundary.

The boundary will be characterized by $\bar{\omega}_I := \text{Im } \bar{\omega}_{nlm} = 0$. As we have shown in Sec. III B, superradiant poles specified by $\bar{\omega}_{nlm}$ can exist only in the range (103). Therefore, when $\bar{\omega}_I$ gets close to 0, we should have

$$\bar{\omega}_R(m\Omega_H - \bar{\omega}_R) = -\bar{\omega}_R k_H(\bar{\omega}_R, m) > 0, \quad (119)$$

where $\bar{\omega}_R := \text{Re } \bar{\omega}$. In fact, we can prove that at the boundary the above condition is saturated, i.e., $k_H(\bar{\omega}_R, m) = 0$, if $\bar{\omega}_R < \mu$ at the boundary. Recall that the superradiant mode is given by an analytic continuation of u_{nlm}^{up} off the real axis. As long as the region of ω does not cross the branch cut, the exponential decaying condition is guaranteed, except on the real axis. If $\bar{\omega}_R < \mu$ in the $\bar{\omega}_I \rightarrow 0$ limit, the un-normalized radial function

$$\hat{R}_{nlm}^{\text{up}} := R_{nlm}^{\text{up}} / \mathcal{N}_{nlm}^{\text{up}} \quad (120)$$

becomes real in this limit, i.e., $A^{\text{in}} = A^{\text{out}*}$. Since $A_{\bar{\omega}_{nlm}}^{\text{out}} = 0$ according to the definition of a discrete mode, discrete modes cannot exist, as long as we can define A^{in} and A^{out} . However, the distinction between “in”-going waves and “out”-going waves becomes unclear for $k_H = 0$. Therefore, we conclude that $k_H = 0$ should be satisfied when a discrete mode crosses the real axis. The same conclusion follows from the implication that the contribution of the mode of our interest to the energy flux evaluated in Eq. (102) should vanish for $\bar{\omega}_I \rightarrow 0$.

On the other hand, if $\bar{\omega}$ hits the boundary of $\bar{\omega}_I = 0$ with $\bar{\omega}_R > \mu$, the un-normalized “up”-type radial function can have an unsuppressed flux to infinity, and hence $\omega_I = 0$ can be realized because of the balance between the fluxes from the horizon and to infinity. As a result, we cannot specify the value of $\bar{\omega}_R$ at $\omega_I = 0$ from the formal discussion. In this case, the boundary is on the branch cut. From the continuity, it is expected that the sequence continues to the next Riemann surface, in which $\text{Im } k_{\infty} < 0$. We are not completely sure, but this case may not happen. The superradiant modes with a lower frequency ω_R and a faster growth rate, which are usually of our interest, always cross $\omega_I = 0$ with $\omega_R < \mu$. In both cases, after continuously changing the parameters further beyond the boundary values, the mode should become a QNM.

In the latter case, the flux evaluated near the horizon (102) does not vanish in the limit $\omega_I \rightarrow 0$. This flux is to be compensated by the energy flux at a large positive r_* . We can evaluate the flux in the same way as before, to obtain

$$\begin{aligned}
& \langle B | T_{\mu\nu} | B \rangle \hat{r}^\mu \hat{\xi}^\nu \\
& \rightarrow \sum_{nl} \sum_{m>1}^l (\mathcal{N}_{\bar{\omega}lm}^{\text{DSR}})^2 (\bar{\omega} k_\infty^*(\bar{\omega}) + \bar{\omega}^* k_\infty(\bar{\omega})) \\
& \times e^{-2(\text{Im } k_\infty) r_*} \cosh(2\bar{\omega}_I t), \tag{121}
\end{aligned}$$

at a large r_* , and we can confirm that the propagation speed of the energy flux, which we define by the slope of the constant energy flux surface, is given by

$$\begin{aligned}
\pm \frac{dr_*}{dt} &= v_{nlm} := \frac{\bar{\omega}_I}{\text{Im } k_\infty(\bar{\omega})} \\
&= \frac{\text{Re } k_\infty(\bar{\omega})}{\bar{\omega}_R} \approx \left. \frac{d\omega}{dk_\infty} \right|_{\omega=\bar{\omega}_R}, \tag{122}
\end{aligned}$$

which approximately agrees with the group velocity for $\bar{\omega}_I \ll \bar{\omega}_R$. In the last approximate equality, we assumed $\bar{\omega}_I \ll \bar{\omega}_R$. We find that the flux given in Eq. (121) cancels the flux evaluated in Eq. (102) at $\omega_I = 0$, using the relation $(A^{\text{in}})^2 = -k_\infty/k_H$ obtained from Eq. (32) with the aid of Eq. (30).

C. The direction of level transitions of an interacting field

We consider an interacting field in this section. For concreteness, we consider the $\lambda\phi^4$ interaction, in which a 2-2 scattering process is possible. We consider two processes as examples. The classical treatment of scattering induced by the self-interaction has been studied in various literature, see, for example, [64–66].

1. Decay of the secondary cloud

In the first example, we assume that the discrete mode with $(n, l, m) = (2, 1, 1)$ is the fastest-growing mode and forms a dominant component of the cloud. We consider the process in which two particles in this dominant mode make a transition to a particle labeled as $(n, 1, 1)$ with $n \geq 3$ and an “up” mode particle with $\omega = \omega_c := \text{Re}(2\omega_{211} - \omega_{n11})$ and $m = 1$. Note that $\omega_c < \mu$, and thus no flux to infinity, i.e., “in” modes are not involved. In this case, the actual direction of the reaction turns out to be opposite, and as a result, the growth of the secondary cloud is suppressed by the interaction.

It is convenient to use the mode function introduced in Eq. (95) and the associated creation and annihilation operators, $\tilde{a}_{nlm}^\dagger$ and $\tilde{a}_{nlm}$. Picking up the term related to this process at around $t = t_0$, the interaction Hamiltonian is schematically written as

$$H_{\text{int}} = \mathcal{O} + \mathcal{O}^\dagger + \dots, \tag{123}$$

with

$$\mathcal{O} = \sum_l \mathcal{A}_l \tilde{a}_{n11}^{(t_0)} (\tilde{a}_{211}^{(t_0)})^\dagger{}^2 a_{\omega_c l 1}^{(\text{up})\dagger}, \tag{124}$$

where $\mathcal{A}_l$ represents the amplitude for the process concerning the mode specified by the label $(\omega_c, l, 1)$, which we treat as a discrete mode for brevity. In the following, we consider a single l -mode, for simplicity. Here, the point is that the continuous “up” mode particle involved in the above reaction is superradiant, and hence the creation and annihilation operators are interchanged, as discussed in Sec. III A. Then, the probability of the forward process described by the operator $\mathcal{O}$ composed of one annihilation and three creation operators, which can be interpreted as the decay of the particle labeled with $(n, 1, 1)$ into three particles, two $(2, 1, 1)$ and one $(\omega_c, l, 1)$, is proportional to

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \propto \tilde{N}_{n11}^{(t_0)} (\tilde{N}_{211}^{(t_0)} + 1) (\tilde{N}_{211}^{(t_0)} + 2) \langle a_{\omega_c l m}^{(\text{up})\dagger} a_{\omega_c l m}^{(\text{up})\dagger} \rangle. \tag{125}$$

On the other hand, the probability of the inverse process is given by

$$\langle \mathcal{O} \mathcal{O}^\dagger \rangle \propto (\tilde{N}_{n11}^{(t_0)} + 1) \tilde{N}_{211}^{(t_0)} (\tilde{N}_{211}^{(t_0)} - 1) \langle a_{\omega_c l m}^{(\text{up})\dagger} a_{\omega_c l m}^{(\text{up})} \rangle. \tag{126}$$

For low-energy “up” modes with $\omega < \mu$, it would be appropriate to take the thermal state, and the occupation number of the dominant mode should be large. Therefore, we have

$$\begin{aligned}
\langle \mathcal{O}^\dagger \mathcal{O} \rangle &\propto \tilde{N}_{n11}^{(t_0)} (\tilde{N}_{211}^{(t_0)})^2 \frac{e^{\beta|k_H(\omega_c, 1)|}}{e^{\beta|k_H(\omega_c, 1)|} - 1}, \\
\langle \mathcal{O} \mathcal{O}^\dagger \rangle &\propto (\tilde{N}_{n11}^{(t_0)} + 1) (\tilde{N}_{211}^{(t_0)})^2 \frac{1}{e^{\beta|k_H(\omega_c, 1)|} - 1}. \tag{127}
\end{aligned}$$

Unless $\tilde{N}_{n11}^{(t_0)}$ is very close to 0, the forward process, i.e., the decay of a particle in the secondary cloud in the state $(n, 1, 1)$, dominates the inverse process. We also find that the two processes balance at

$$\tilde{N}_{n11}^{(t_0)} = \frac{1}{e^{\beta|k_H(\omega_c, m)|} - 1}, \tag{128}$$

which is not a large number, compared to the particle number of the macroscopic axion cloud one considered in the literature.

2. Decay of the primary cloud

The situation is different when we consider higher m -mode as the secondary cloud. In this case, an example of the most relevant interaction would be

$$\mathcal{O}' = (\tilde{a}_{211}^{(t_0)})^2 \tilde{a}_{322}^{(t_0)\dagger} a_{\omega'_c 00}^{(\text{up})\dagger}, \tag{129}$$

with $\omega'_c := \text{Re}(2\omega_{211} - \omega_{322})$. The operator $\mathcal{O}'$ represents the process of the decay of two particles in the dominant

cloud $[(2,1,1)$ state] into a particle in the secondary cloud $[(3,2,2)$ state] and that in the continuous nonsuperradiant “up” mode with $m = 0$. In this case, the probabilities of the forward process and the reverse process are, respectively, given by

$$\begin{aligned}\langle \mathcal{O}^\dagger \mathcal{O} \rangle &\propto (\tilde{N}_{322}^{(t_0)} + 1)(\tilde{N}_{211}^{(t_0)})^2 \frac{e^{\beta|k_H(\omega'_c, 0)|}}{e^{\beta|k_H(\omega'_c, 0)|} - 1}, \\ \langle \mathcal{O} \mathcal{O}^\dagger \rangle &\propto \tilde{N}_{322}^{(t_0)}(\tilde{N}_{211}^{(t_0)})^2 \frac{1}{e^{\beta|k_H(\omega'_c, 0)|} - 1}.\end{aligned}\quad (130)$$

The net transition rate is determined by the difference between the forward process and the reverse process, i.e., $\propto ((\mathcal{O}^\dagger \mathcal{O}) - (\mathcal{O} \mathcal{O}^\dagger))$. Therefore, when the occupation number of the secondary cloud is large enough, the factor depending on the Hawking temperature drops off in the net transition rate, which justifies the computation of the transition rate based on the classical energy flux toward the horizon.

V. CONCLUSION

Superradiant instability has been extensively studied as a classical phenomenon, but a clear quantum interpretation has long been lacking. In this paper, we developed a quantum framework of superradiance by performing the canonical quantization of a massive real scalar field around a Kerr black hole, including both continuous and discrete modes, which are normalizable.

For the discrete modes corresponding to those in the superradiant clouds, the standard normalization fails because the inner product $((u_{nlm}, u_{nlm}^*))$ vanishes (otherwise the inner product necessarily becomes time dependent). This orthogonality is due to the cancellation between the contributions from the inside and outside of the potential barrier. Instead of the ordinary complex conjugation, the conjugate mode having a nonvanishing inner product with u_{nlm} is obtained via the operation $\mathcal{J}$, which flips the sign of (ω, m) to $(-\omega, -m)$. This choice of a complete orthonormal set leads to a commutation relation differing from the standard one after canonical quantization [see Eq. (60)]. To construct a pair of positive- and negative-frequency functions that are complex conjugate to each other, we relax the requirement that the mode function be the eigenfunction of time translation, and introduce new modes $v_\pm$ as linear combinations of u_{nlm} and $\mathcal{J}u_{nlm}$ among the discrete modes. In this construction, v_+ is the one that has more weight on the outside of the potential barrier, while v_- is opposite.

After performing the mode expansion, we derive explicit expressions for the Hamiltonian and angular momentum. We consider a “Boulware-like” stationary reference state, albeit finding that no truly stationary state exists for the discrete modes. We then study the time evolution of these discrete modes, which we expect to represent superradiant

clouds. The growth is attributed to the time dependence of the mode function itself. The norm of the functions $v_\pm$ is conserved, although they grow in time. This conservation of the norm may look contradictory, but it can be explained by the negative contribution of the conserved number density inside the potential barrier. The constancy of the norm simply results from a cancellation between the increase in the positive particle number outside the barrier and the negative particle number inside. To capture the growth of the superradiant cloud, we introduced a quasiparticle number that counts only the portion of the number density outside the potential barrier. Using this definition, we have shown that this quasiparticle number grows exponentially in time, regardless of the choice of initial state.

We also explored several related topics. For the continuous modes, we considered the Unruh vacuum state, and calculated the spectrum of Hawking radiation. If we naively apply the Bose-Einstein statistical factor, the flux to infinity seems to become negative for modes whose energy measured by a local observer near the horizon is negative. However, such a pathology is evaded by the flip of the sign in the gray-body factor, explaining why the energy flux remains positive. Furthermore, we discuss how the back-reaction on the black hole spin influences the evolution of superradiant clouds. We show that it is appropriate to evolve the cloud adiabatically, based on the flux balance with respect to the particle number. We also clarify what occurs when the spin decreases to the saturation value, at which point the superradiant mode transitions into a QNM. Finally, we give a quantum mechanical explanation to the direction of level transitions that arises in the presence of self-interactions.

Our paper raised a number of interesting follow-up questions, for example, as mentioned earlier, how to choose a physically realistic initial state corresponding to the state after the gravitational collapse. A complete quantum mechanical analysis should be formulated in a dynamical spacetime describing BH formation through collapse, which has not been studied before. Moreover, the quantization procedure can be extended to other types of fields, such as free fermions and vectors, where new issues or phenomena may arise.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: NORMALIZATION OF DISCRETE MODES

Here, we discuss the normalization of the discrete modes. We evaluate Eq. (18). For this purpose, we evaluate η at $\omega = \bar{\omega}_{nlm}$, using the equation of motion for $S_{nlm}(\theta)$. Taking the derivative of Eq. (9) with respect to ω , we have

$$\int d(\cos \theta) S(\theta) \left[\left(2am - 2a^2 \omega \sin^2 \theta + \frac{d\lambda}{d\omega} \right) S(\theta) + \mathcal{L}_\theta \frac{dS(\theta)}{d\omega} \right] = 0. \quad (\text{A1})$$

The term with $dS(\theta)/d\omega$ is removed by performing integration by parts, owing to the self-adjoint nature of $\mathcal{L}_\theta$ and the regularity of $S(\theta)$ on both boundaries. From this identity, we obtain

$$\eta = \frac{m}{a\omega} + \frac{1}{2a^2\omega} \frac{d\lambda}{d\omega}. \quad (\text{A2})$$

Substituting this identity into $f(r)$ given in Eq. (19), we obtain

$$f(r) = 2\omega - \frac{2am}{r^2 + a^2} - \frac{\Delta}{(r^2 + a^2)^2} \frac{d\lambda}{d\omega}. \quad (\text{A3})$$

Denote $R_{nlm}(r)$ by using the un-normalized mode function $R_{\omega lm}^{\text{up}}(r)$ as $R_{nlm}(r) = \mathcal{N}_{nlm}^{\text{DSR}} \hat{R}_{\bar{\omega}_{nlm} lm}^{\text{up}}(r)$. Then, we take the derivative of Eq. (10) in the same way, to obtain

$$\begin{aligned} & \int dr_* \frac{f(r)}{\mathcal{G}^2} R_{nlm}^2 \\ &= -(\mathcal{N}_{nlm}^{\text{DSR}})^2 \int dr \hat{R}_{nlm}^{\text{up}} \mathcal{L}_r \frac{d\hat{R}_{\omega lm}^{\text{up}}}{d\omega} \Big|_{\omega=\bar{\omega}_{nlm}} \\ &= \left[\frac{(\mathcal{N}_{nlm}^{\text{DSR}})^2}{\mathcal{G}^2} \left(\frac{d\hat{R}_{\omega lm}^{\text{up}}}{dr_*} \frac{d\hat{R}_{\omega lm}^{\text{up}}}{d\omega} - \hat{R}_{\omega lm}^{\text{up}} \frac{d^2 \hat{R}_{\omega lm}^{\text{up}}}{dr_* d\omega} \right) \right]_{\omega=\bar{\omega}_{nlm}} \Big|_{r_*=-\infty}^{r_*=\infty}. \end{aligned} \quad (\text{A4})$$

Substituting Eq. (23) and

$$\begin{aligned} & \frac{dR_{\omega lm}(r)}{d\omega} \Big|_{\omega=\bar{\omega}_{nlm}} \\ & \rightarrow \mathcal{N}_{nlm}^{\text{DSR}} \mathcal{G} \begin{cases} \frac{dA_{\omega lm}^{\text{out}}}{d\omega} e^{ik_H r_*} + (\dots) e^{-ik_H r_*}, & (r_* \rightarrow -\infty), \\ e^{+ik_\infty r_*}, & (r_* \rightarrow +\infty), \end{cases} \end{aligned} \quad (\text{A5})$$

we find

$$\int dr_* \frac{f(r)}{\mathcal{G}^2} R_{nlm}^2 = 2i(\mathcal{N}_{nlm}^{\text{DSR}})^2 k_H A_{\omega lm}^{\text{in}} \frac{dA_{\omega lm}^{\text{out}}}{d\omega} \Big|_{\omega=\bar{\omega}_{nlm}}. \quad (\text{A6})$$

Using the first equality in Eq. (30), we arrive at Eq. (51).

APPENDIX B: RETARDED GREEN'S FUNCTION

In quantum theory, we often use the retarded Green's function given by

$$G_R(x, x') = i\theta(t - t') [\Phi(x), \Phi(x')], \quad (\text{B1})$$

which is manifestly 0 when $t < t'$. It is easy to show that this Green's function satisfies

$$(\square - \mu^2) G_R(x, x') = -\frac{\delta^4(x - x')}{\sqrt{-g(x)}}. \quad (\text{B2})$$

By using the unit vector n^μ normal to a constant- t time slice Σ_t ,

$$\square \Phi = \frac{1}{\sqrt{-g}} \partial_\mu \{ \sqrt{-g} (-n^\mu n^\nu + h^{\mu\nu}) \partial_\nu \Phi \}. \quad (\text{B3})$$

Since $h^{\mu\nu} n_\nu = 0$, $\partial_\mu \sqrt{-g} h^{\mu\nu} \partial_\nu$ does not contain a time derivative. The time derivative is only in $\partial_\mu \sqrt{-g} n^\mu n^\nu \partial_\nu$. One time derivative should act on $\theta(t - t')$, but the contribution with the first derivative acting on $\theta(t - t')$ vanishes because of the equal time commutation relation, $[\Phi(t, \mathbf{x}), \Phi(t, \mathbf{x}')] = 0$. So, the nontrivial contribution remains only when the second derivative acting on G_R operates on $\theta(t - t')$. Recalling that the conjugate momentum is defined by $\Pi(x) := n^\mu \partial_\mu \Phi(x)$, we have

$$(\square - \mu^2) G_R(x, x') = -i\delta(t - t') [\Pi(t, \mathbf{x}), \Phi(t, \mathbf{x}')]. \quad (\text{B4})$$

Using the commutation relation at the same time, $[\Pi(t, \mathbf{x}), \Phi(t, \mathbf{x}')] = -i\delta^3(\mathbf{x} - \mathbf{x}')/\sqrt{-g}$, we obtain Eq. (B2).

More explicitly, substituting the mode decomposition of $\Phi(x)$, we find

$$\begin{aligned}
 G_R(x, x') &= i\theta(t-t') \sum_{lm} \left\{ \int_{-\infty}^{\infty} \frac{d\omega}{k_H(\omega, m)} u_{\omega lm}^{\text{up}}(x) (\mathcal{J}^{\Re} u_{\omega lm}^{\text{up}}(x')) + \left(\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty} \right) \frac{d\omega}{k_{\infty}(\omega)} u_{\omega lm}^{\text{in}}(x) (\mathcal{J}^{\Re} u_{\omega lm}^{\text{in}}(x')) \right. \\
 &\quad \left. + \sum_n u_{nlm}(x) (\mathcal{J} u_{nlm}(x')) + (\mathcal{J} u_{nlm}^*(x)) u_{nlm}^*(x') \right\} \\
 &= i\theta(t-t') \sum_{lm} \left[\left(\int_{-\infty}^{\infty} d\omega \frac{\hat{R}_{\omega lm}^{\text{up}}(r) (\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{up}}(r'))}{4\pi k_H(\omega, m) \|A_{\omega lm}^{\text{out}}\|^2} + \left(\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty} \right) d\omega \frac{k_{\infty}(\omega) \hat{R}_{\omega lm}^{\text{in}}(r) (\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{in}}(r'))}{4\pi k_H^2(\omega, m) \|A_{\omega lm}^{\text{out}}\|^2} \right) \right. \\
 &\quad \times e^{-i\omega(t-t') + im(\varphi-\varphi')} S_{\omega lm}(\theta) S_{\omega lm}(\theta') + \sum_n (R_{nlm}(r) R_{nlm}(r') S_{\bar{\omega}_{nlm} lm}(\theta) S_{\bar{\omega}_{nlm} lm}(\theta')) \\
 &\quad \left. \times e^{-i\bar{\omega}_{nlm}(t-t')} + R_{nlm}^*(r) R_{nlm}^*(r') S_{\bar{\omega}_{nlm}^* lm}(\theta) S_{\bar{\omega}_{nlm}^* lm}(\theta') e^{-i\bar{\omega}_{nlm}^*(t-t')} e^{im(\varphi-\varphi')} \right]. \quad (\text{B5})
 \end{aligned}$$

On the other hand, we can construct the retarded Green's function assuming the decomposition into the Fourier modes and harmonics in the three-dimensional timelike surface spanned by (t, θ, φ) as

$$\begin{aligned}
 \tilde{G}_R(x, x') &= \sum_{lm} \int_C \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G_{\omega lm}(r, r') \\
 &\quad \times e^{im(\varphi-\varphi')} S_{\omega lm}(\theta) S_{\omega lm}(\theta'), \quad (\text{B6})
 \end{aligned}$$

where the integration contour C is chosen to run from $-\infty$ to $+\infty$ in the complex plane, lying entirely above all $\bar{\omega}_{nlm}$, i.e., the zeros of $A_{\omega lm}^{\text{out}}$. If $G_{\omega lm}(r, r')$ satisfies

$$\mathcal{L}_r G_{\omega lm}(r, r') = -\delta(r-r'), \quad (\text{B7})$$

since $\square = (\sin \theta / \sqrt{-g}) \mathcal{L}_r + \dots$, we find

$$\begin{aligned}
 (\square - \mu^2) \tilde{G}_R(x, x') &= -\frac{\sin \theta}{\sqrt{-g}} \delta(r-r') \sum_{lm} \int_C \frac{d\omega}{2\pi} \\
 &\quad \times e^{-i\omega(t-t') + im(\varphi-\varphi')} S_{\omega lm}(\theta) S_{\omega lm}(\theta') \\
 &= -\frac{\delta^4(x-x')}{\sqrt{-g(x)}}. \quad (\text{B8})
 \end{aligned}$$

We can construct $G_{\omega lm}(r, r')$ that satisfies Eq. (B7) by choosing

$$\begin{aligned}
 G_{\omega lm}(r, r') &= \frac{i}{2k_H A_{\omega lm}^{\text{out}}} (\hat{R}_{\omega lm}^{\text{up}}(r) \hat{R}_{\omega lm}^{\text{in}}(r') \theta(r-r') \\
 &\quad + \hat{R}_{\omega lm}^{\text{up}}(r') \hat{R}_{\omega lm}^{\text{in}}(r) \theta(r'-r)), \quad (\text{B9})
 \end{aligned}$$

where we use $W[\hat{R}^{\text{up}}, \hat{R}^{\text{in}}] = -2ik_H A_{\omega lm}^{\text{out}}$. From the choice of the contour C , we can easily show that $\tilde{G}(x, x') = 0$ for $t < t'$ by closing the integration in the upper-half complex plane.

Now, the problem is to show the equivalence between the expressions given in (B5) and (B6). For this purpose, we consider the case in which $t > t'$ and $r > r'$. The equivalence in the case with $r < r'$ can be shown in the same way. We start with rewriting $G_R(x, x')$ given in Eq. (B5). The term with $\hat{R}_{\omega lm}^{\text{in}}(r) (\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{in}}(r'))$ is rewritten by using $\int_{-\infty}^{-\mu} d\omega (\dots) = \int_{\mu}^{\infty} d\omega \mathcal{J}^{\Re}(\dots)$ as

$$-i \sum_{lm} \left(\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty} \right) d\omega \frac{k_{\infty}(\omega) (\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{in}}(r)) \hat{R}_{\omega lm}^{\text{in}}(r')}{4\pi k_H^2(\omega, m) \|A_{\omega lm}^{\text{out}}\|^2}. \quad (\text{B10})$$

Using the following identities:

$$\begin{aligned}
 \mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{up}}(r) &= (\mathcal{J}^{\Re} A_{\omega lm}^{\text{in}}) (\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{in}}(r)) \\
 &\quad + (\mathcal{J}^{\Re} A_{\omega lm}^{\text{out}}) (\hat{R}_{\omega lm}^{\text{in}}(r)), \quad (\text{B11})
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{in}}(r) &= (\mathcal{J}^{\Re} A_{\omega lm}^{\text{up}}) (\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{up}}(r)) \\
 &\quad + (\mathcal{J}^{\Re} A_{\omega lm}^{\text{down}}) (\hat{R}_{\omega lm}^{\text{up}}(r)) \\
 &= \frac{k_H(\omega, m)}{k_{\infty}(\omega)} [-A_{\omega lm}^{\text{in}} (\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{up}}(r)) \\
 &\quad + (\mathcal{J}^{\Re} A_{\omega lm}^{\text{out}}) (\hat{R}_{\omega lm}^{\text{up}}(r))], \quad (\text{B12})
 \end{aligned}$$

with

$$\mathcal{J}^{\Re} A_{\omega lm}^{\text{in}} = \begin{cases} A_{\omega lm}^{\text{out}}, & (|\omega| < \mu), \\ \frac{k_{\infty}(\omega)}{k_H(\omega, m)} A_{\omega lm}^{\text{up}}, & (|\omega| > \mu), \end{cases} \quad (\text{B13})$$

the whole contribution of the continuous modes for $t > t'$ becomes

$$\begin{aligned}
G_R^{\text{conti}}(x, x') = & \frac{i}{4\pi} \sum_{lm} \left[\int_{-\infty}^{\infty} \hat{R}_{\omega lm}^{\text{up}}(r) (\mathcal{J}^{\Re} A^{\text{out}}) \hat{R}_{\omega lm}^{\text{in}}(r') + \int_{-\mu}^{\mu} A^{\text{out}} \hat{R}_{\omega lm}^{\text{up}}(r) (\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{in}}(r')) \right. \\
& \left. - \left(\int_{-\infty}^{-\mu} + \int_{\mu}^{\infty} \right) (\mathcal{J}^{\Re} A^{\text{out}}) \hat{R}_{\omega lm}^{\text{up}}(r) \hat{R}_{\omega lm}^{\text{in}}(r') \right] \frac{d\omega}{k_H \|A^{\text{out}}\|^2} \\
& \times e^{-i\omega(t-t') + im(\varphi - \varphi')} S_{\omega lm}(\theta) S_{\omega lm}(\theta').
\end{aligned} \tag{B14}$$

In this calculation, the contributions coming from the first term in (B11) and the first term in (B12) mutually cancel in the regions $(-\infty, -\mu)$ and (μ, ∞) .

Here, one can prove

$$\begin{aligned}
A^{\text{out}} &= \mathcal{J}^{\Re} A^{\text{in}}, \quad \|A^{\text{out}}\|^2 = \|A^{\text{in}}\|^2, \\
\mathcal{J}^{\Re} \hat{R}_{\omega lm}^{\text{in}} &= \mathcal{J} \hat{R}_{\omega lm}^{\text{in}}, \quad (\text{for } |\omega| < \mu),
\end{aligned} \tag{B15}$$

which we can apply to the second term in the square brackets of (B14). We flip the sign of the integration variable ω in the third term, by applying $\int_{-\infty}^{-\mu} d\omega(\dots) = \int_{\mu-i\epsilon}^{\infty-i\epsilon} d\omega \mathcal{J}(\dots)$, and using

$$\begin{aligned}
\mathcal{J} \mathcal{J}^{\Re} A^{\text{out}} &= \mathcal{J}^{\Re} A^{\text{in}}, \quad \mathcal{J} \|A^{\text{out}}\|^2 = \|A^{\text{in}}\|^2, \\
\mathcal{J} \hat{R}_{\omega lm}^{\text{up}} &= \hat{R}_{\omega lm}^{\text{up}},
\end{aligned} \tag{B16}$$

we find

“The second and the third terms

in the square brackets of Eq.(B14)”

$$\begin{aligned}
&= \frac{i}{4\pi} \sum_{lm} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\omega \frac{\hat{R}_{\omega lm}^{\text{up}}(r) (\mathcal{J} \hat{R}_{\omega lm}^{\text{in}}(r'))}{k_H A^{\text{in}}} \\
&\times e^{-i\omega(t-t') + im(\varphi - \varphi')} S_{\omega lm}(\theta) S_{\omega lm}(\theta').
\end{aligned} \tag{B17}$$

The integration path is now combined to be continuous from $-\infty - i\epsilon$ to $\infty - i\epsilon$, and the asymptotic behavior of the integrand at a large $|\omega|$ is $\propto \exp(-i\omega(t-t') - i\omega(r_* - r'_*))$. Hence, we can close the integration contour in the lower-half complex plane. The pole of $1/A^{\text{in}}$ is corresponding to the pole of $1/A^{\text{out}}$, i.e.,

$$A_{\bar{\omega}^* lm}^{\text{in}} = 0. \tag{B18}$$

Notice that

$$R_{nlm}^*(r) R_{nlm}^*(r') = \frac{\hat{R}_{\bar{\omega}^* lm}^{\text{up}}(r) \hat{R}_{\bar{\omega}^* lm}^{\text{up}}(r')}{-2ik_H(\bar{\omega}^*, m) A_{\bar{\omega}^* lm}^{\text{out}} \partial_{\bar{\omega}^*} A_{\bar{\omega}^* lm}^{\text{in}}}, \tag{B19}$$

and

$$\mathcal{J} \hat{R}_{\bar{\omega}^* lm}^{\text{in}} = \frac{\hat{R}_{\bar{\omega}^* lm}^{\text{up}}}{\mathcal{J} A_{\bar{\omega}^* lm}^{\text{in}}} = \frac{R_{\bar{\omega}^* lm}^{\text{up}}}{A_{\bar{\omega}^* lm}^{\text{out}}}. \tag{B20}$$

The integral (B17) completely cancels the contribution from the term that contains the factor $R_{nlm}^*(r) R_{nlm}^*(r')$ in (B5).

Then, the remaining part of $G_R(x, x')$ with $t > t'$ and $r > r'$ is

$$\begin{aligned}
G_R(x, x') = & i \left[\sum_{lm} \int_{-\infty}^{\infty} d\omega \frac{\hat{R}_{\omega lm}^{\text{up}}(r) \hat{R}_{\omega lm}^{\text{in}}(r')}{4\pi k_H A^{\text{out}}} e^{-i\omega(t-t') + im(\varphi - \varphi')} S_{\omega lm}(\theta) S_{\omega lm}(\theta') \right. \\
& \left. + \sum_n \frac{\hat{R}_{\bar{\omega}^* lm}^{\text{up}}(r) \hat{R}_{\bar{\omega}^* lm}^{\text{up}}(r')}{2ik_H(\bar{\omega}^* lm, m) A_{\bar{\omega}^* lm}^{\text{in}} \partial_{\bar{\omega}^*} A_{\bar{\omega}^* lm}^{\text{out}}} e^{-i\bar{\omega}^* lm(t-t')} S_{\bar{\omega}^* lm}(\theta) S_{\bar{\omega}^* lm}(\theta') e^{im(\varphi - \varphi')} \right].
\end{aligned} \tag{B21}$$

In a similar manner, the integration contour in the first term can be changed to cancel the second term to arrive at the expression of the first term in Eq. (B6). Therefore, we complete the proof of the equivalence of the two forms of retarded Green's function. This equivalence shows that the complete set of solutions selected based on the normalizability, which leads to the retarded Green's function given in Eq. (B5), is fully consistent with the alternative definition of the retarded Green's function given in Eq. (B6).

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