

Erratum: Slow and fast collective neutrino oscillations: Invariants and reciprocity [Phys. Rev. D **107**, 043024 (2023)]

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In the published version of this work, we improperly treated the singularity of the Lax vectors when they match the velocity, or frequency, of one of the polarization vectors of the system. While this has ultimately no impact on the definition of the Gaudin invariants in the two-flavor system, it does affect them in the three-flavor case. Therefore, in this Erratum, we revise Secs. III. A, III. C, and V. C of the paper. All other sections, as well as the central message and results of the paper, remain unchanged. A fully revised manuscript is available in Ref. [1], matching the corrections highlighted in this Erratum.

III. INVARIANTS AND INTEGRABILITY

A. Lax vectors

We revise here the original content of Sec. III. A of the paper. The Lax vectors, as defined in Eq. (17) of the paper

$$\mathbf{L}_u = \mathbf{B} + \sum_{w \neq u} \frac{\mathbf{P}_w}{u - w}, \quad (17)$$

obey an equation of motion that differs from Eq. (18) of the paper and should read instead

$$\dot{\mathbf{L}}_u = (u\mathbf{B} + \mathbf{P}_0) \times \mathbf{L}_u + \mathbf{B} \times \mathbf{P}_u. \quad (18)$$

For u different from any of the frequencies w of the original system of polarization vectors $\mathbf{P}_w$, the length of $\mathbf{L}_u$ is conserved. On the other hand, if u matches one of the frequencies, then its length is *not* conserved, while the Gaudin invariant $\mathbf{L}_u \cdot \mathbf{P}_u$ is still conserved.

The somewhat discontinuous behavior for the Lax vectors, which perform a pure precession only for $u \neq w$, can be cured by introducing a modified definition:

$$\tilde{\mathbf{L}}_u = \prod_w (u - w) \left[\mathbf{B} + \sum_w \frac{\mathbf{P}_w}{u - w} \right].$$

These modified Lax vectors continuously obey

$$\dot{\tilde{\mathbf{L}}}_u = (u\mathbf{B} + \mathbf{P}_0) \times \tilde{\mathbf{L}}_u.$$

On the other hand, for u equal to one of the frequencies of the original system, the modified Lax vectors coincide with the original polarization vector $\tilde{\mathbf{L}}_w = \mathbf{P}_w$, so that the invariants $\tilde{\mathbf{L}}_w \cdot \mathbf{P}_w$ are not independent constants of motion. (Still, for all other continuous values of $u \neq w$, $|\tilde{\mathbf{L}}_u|^2$ is a nontrivial invariant.)

For the fast system, the corresponding equation obeyed by $\mathbf{L}_u$ as defined in Eq. (19) of the paper now reads

$$\dot{\mathbf{L}}_u = (\mathbf{D}_0 - u\mathbf{D}_1) \times \mathbf{L}_u - u\mathbf{D}_1 \times \mathbf{D}_u. \quad (19)$$

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Once again, $\mathbf{L}_v \cdot \mathbf{P}_v$ are still the conserved Gaudin invariants as originally introduced.

C. How many invariants per mode?

We revise here the content of Sec. III. C of the paper. In Eq. (30) of the paper, which we now write in the form

$$\mathbf{L}_u^2 = \sum_{v_1, v_2} \mathbf{D}_{v_1} \cdot \mathbf{D}_{v_2} \frac{v_1 v_2}{(u - v_1)(u - v_2)}, \quad (30)$$

we separate the terms with $v_1 = v_2$ and those with $v_1 \neq v_2$. For the latter, we modify the right-hand side with the identity

$$\frac{v_1 v_2}{(u - v_1)(u - v_2)} = \frac{v_1 v_2}{v_1 - v_2} \left(\frac{1}{u - v_1} - \frac{1}{u - v_2} \right) \quad (31)$$

so that

$$\mathbf{L}_u^2 = \sum_{v_1} \frac{\mathbf{D}_{v_1}^2 v_1^2}{(u - v_1)^2} + 2 \sum_{v_1, v_2} \frac{\mathbf{D}_{v_1} \cdot \mathbf{D}_{v_2} v_1 v_2}{(v_1 - v_2)(u - v_1)} = \sum_{v_1} \frac{D_{v_1}^2 v_1^2}{(u - v_1)^2} + 2 \sum_{v_1} I_{v_1} \frac{v_1}{u - v_1}. \quad (32)$$

Therefore, the set of invariants $\mathbf{L}_u^2$ is indeed a linear combination of the Gaudin invariants I_v and the original lengths of the Bloch vectors, without introducing any additional constant of motion.

V. THREE-FLAVOR EXTENSION

Here, we present a revised version of Sec. V. C of the paper. The main changes in the results correspond to a modification of the conserved invariants K_v , which in the original version were defined as $K_v = \langle \mathbf{L}_v, \mathbf{L}_v, \mathbf{D}_v \rangle$. Below, we introduce the correct definition. The following text is a complete substitution for the original section.

C. Gaudin invariants and integrability

To identify the Gaudin invariants, it is more straightforward to express the Lax vectors in matrix form. The original EOM are $\partial_t \mathbf{D}_v = -i[\mathbf{H}_v, \mathbf{D}_v]$ with $\mathbf{H}_v = \mathbf{D}_0 - v\mathbf{D}_1$ or in vector form $\partial_t \mathbf{D}_v = \mathbf{H}_v \times \mathbf{D}_v$ with $\mathbf{H}_v = \mathbf{D}_0 - v\mathbf{D}_1$. In this case, it is convenient to start with the Lax vectors for an arbitrary complex parameter u , assumed to be different from any of the discrete values v of the original system,

$$\tilde{\mathbf{L}}_u = \sum_v \frac{\mathbf{D}_v v}{u - v} \quad \text{and} \quad \tilde{\mathbf{L}}_u = \tilde{\mathbf{L}}_u \cdot \boldsymbol{\lambda}, \quad (61)$$

obeying the original EOM $\partial_t \tilde{\mathbf{L}}_v = \mathbf{H}_v \times \tilde{\mathbf{L}}_v$ or in matrix form $\partial_t \tilde{\mathbf{L}}_v = -i[\mathbf{H}_v, \tilde{\mathbf{L}}_v]$. Since the motion is an instantaneous precession, for any u , $\text{Tr}(\tilde{\mathbf{L}}_u^n)$ is conserved for any n . As in the case of $\text{Tr}(\mathbf{D}_u^n)$, we can find independent and nontrivial invariants only for $n = 2$ or 3 . Hence, we have a continuum of invariants for any complex value of u , which clearly are not all independent. To identify the analog of the Gaudin invariants, we are going to take the limit for $u \rightarrow \bar{v}$, where $\bar{v}$ is one of the discrete velocities. This limit must be taken with care. Let us separate the regular part from the pole term

$$\tilde{\mathbf{L}}_u = \frac{\bar{v} \mathbf{D}_{\bar{v}}}{u - \bar{v}} + \mathbf{L}_u, \quad (62)$$

with

$$\mathbf{L}_u = \sum_{v \neq \bar{v}} \frac{\mathbf{D}_v v}{u - v}. \quad (63)$$

We will need only an expansion in $u - \bar{v}$ up to the first order,

$$\tilde{\mathbf{L}}_u \simeq \frac{\bar{v} \mathbf{D}_{\bar{v}}}{u - \bar{v}} + \mathbf{L}_{\bar{v}} - \mathbf{M}_{\bar{v}}(u - \bar{v}), \quad (64)$$

where

$$\mathbf{M}_{\bar{v}} = \sum_{v \neq \bar{v}} \frac{v \mathbf{D}_v}{(\bar{v} - v)^2}. \quad (65)$$

The analogous matrices are

$$\tilde{\mathbf{L}}_u \simeq \mathbf{L}_{\bar{v}} + \frac{\mathbf{D}_{\bar{v}} \bar{v}}{u - \bar{v}} - \mathbf{M}_{\bar{v}}(u - \bar{v}). \quad (66)$$

We can now deduce conservation laws for the vectors $\mathbf{L}_{\bar{v}}$ using the conservation of the pole terms of $\text{Tr}(\tilde{\mathbf{L}}_u^n)$ for $u \rightarrow \bar{v}$. For $n = 2$, the pole contributions are

$$\text{Tr}(\tilde{\mathbf{L}}_u^2) \simeq \frac{\text{Tr}(\mathbf{D}_{\bar{v}}^2) \bar{v}^2}{(u - \bar{v})^2} + \frac{2\bar{v} \text{Tr}(\mathbf{L}_{\bar{v}} \mathbf{D}_{\bar{v}})}{u - \bar{v}}; \quad (67)$$

we do not write the subdominant regular terms, since they would require more precision in the expansion of $\tilde{\mathbf{L}}_u$. Hence, we immediately find the conservation of $|\mathbf{D}_v|^2$, which we already knew about, and the conservation of $I_v = \mathbf{L}_v \cdot \mathbf{D}_v$. We now turn to $n = 3$,

$$\text{Tr}(\tilde{\mathbf{L}}_u^3) \simeq \frac{\text{Tr}(\mathbf{D}_{\bar{v}}^3) \bar{v}^3}{(u - \bar{v})^3} + \frac{3\bar{v}^2 \text{Tr}(\mathbf{L}_{\bar{v}} \mathbf{D}_{\bar{v}}^2)}{(u - \bar{v})^2} + \frac{3\bar{v}}{u - \bar{v}} [\text{Tr}(\mathbf{L}_{\bar{v}}^2 \mathbf{D}_{\bar{v}}) - \bar{v} \text{Tr}(\mathbf{D}_{\bar{v}}^2 \mathbf{M}_{\bar{v}})]. \quad (68)$$

From here, we see that there are two families of additional invariants

$$J_v = \langle \mathbf{L}_v, \mathbf{D}_v, \mathbf{D}_v \rangle \quad (69)$$

and

$$K_v = \langle \mathbf{L}_v, \mathbf{L}_v, \mathbf{D}_v \rangle - v \langle \mathbf{D}_v, \mathbf{D}_v, \mathbf{M}_v \rangle. \quad (70)$$

These are the generalizations of the two-flavor Gaudin invariants to three flavors. Notice that these invariants are all linearly independent of one another. Therefore, we have $3N$ Lagrangian degrees of freedom and $3N$ associated conservation laws for I_v , J_v , and K_v . This guarantees integrability in the mechanical sense also for the three-flavor motion.

- [1] D. F. G. Fiorillo and G. G. Raffelt, Slow and fast collective neutrino oscillations: Invariants and reciprocity, *Phys. Rev. D* **107**, 043024 (2023).