

Impact of primordial magnetic fields on the first-order electroweak phase transition

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We numerically study how the primordial magnetic field affects the first-order electroweak phase transition in the early Universe. We observe the following: 1) the phase transition process would be slowed down by the magnetic field; 2) the phenomenon of vortex structure of the Higgs condensation appears when the homogenesis hypermagnetic field $g'B_Y^x/m_W^2 \gtrsim 3.63$; and, 3) the helical hypermagnetic field can dramatically enhance the sphaleron rate and validate the generation of the baryon asymmetry through the chiral anomaly.

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I. INTRODUCTION

Recently, the evidence of the stochastic gravitational wave background (SGWB) has been reported by pulsar timing array experiments around the world: PPTA [1], EPTA and InPTA [2], NanoGrav [3], and CPTA [4]. The first-order phase transition (PT) at QCD scale has been considered as one compelling possibility to address the signatures [5–14]. The SGWB produced by the strong first-order electroweak PT [15–17] might be able to detect by future space-based interferometer, such as the Laser Interferometer Space Antenna (LISA) [18], Taiji [19], and TianQin [20]. Aside from SGWB, during the first-order electroweak PT, it was found that the bubble collision dynamics can drive the production of the primordial magnetic fields (MFs) [21–27]. The MFs generated in the early Universe might influence the Higgs vacuum and yield formation of a typical vortex structure known as the Ambjørn-Olesen condensation [28–33]. MFs might affect the electroweak sphaleron rate during the electroweak crossover [34,35] and the first-order PT [36]. After considering amplification mechanisms [37–42], these MFs can explain the MFs in the Milky Way [43], galaxy clusters [44–46], and the voids of large-scale structures

[47–50], and be probed by γ -ray observations of blazars [51–53].

The first-order electroweak PT, as a generic prediction of new physics beyond the Standard Model (SM) of particle physics [54–56], provides one of the three Sakharov conditions [57], i.e., the out-of-equilibrium conditions, and makes it possible to explain the baryon asymmetry of the Universe through electroweak baryogenesis [58,59]. Different from the microphysics for baryogenesis, Refs. [60–63] build the relationship between the baryon number and the helicity of primordial MF through the Abelian anomaly. Refs. [64,65] further found that source terms arising from the chiral anomaly in the background of a helical hypermagnetic field can drive the baryon asymmetry generation after considering the decay of MF helicity around the SM crossover.

In this work, we intend to study the first-order electroweak PT in a MF background. We conduct lattice field simulations to include the nonlinear dynamics during the PT. We analyze the impact of the primordial MFs on the PT speed and investigate the Ambjørn-Olesen condensation during the PT. We also evaluate how the helical magnetic field affects the Chern-Simons number evolution, altering the sphaleron rate. Finally, we observe that the helical MF can alter the baryon number and the lepton number during first-order PT, which provides a chance to address the baryon asymmetry of the Universe.

II. THE PHYSICAL PICTURE

We consider an external $U(1)_Y$ gauge field Y_{ex}^μ that does not vary over time to achieve an external MF, and the

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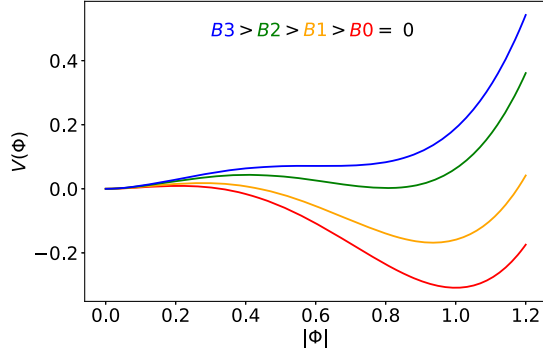


FIG. 1. Schematic diagram of the additional contribution to the Higgs potential function (3) from the term proportional to $(B_Y^{\text{ex}})^2 \Phi^\dagger \Phi$ obtained from covariant derivatives (2).

complete electroweak Lagrangian is

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{1}{4} W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4} Y_{\mu\nu} Y^{\mu\nu} - \frac{1}{2} Y_{\mu\nu}^{\text{ex}} Y^{\mu\nu} - \mathcal{V}(\Phi). \quad (1)$$

Φ is the Higgs doublet, and $W_{\mu\nu}^a$ and $Y_{\mu\nu}$ represent the field strengths of the $\text{SU}(2)_L$ and $\text{U}(1)_Y$ gauge fields, respectively. Here we omit $Y_{\mu\nu}^{\text{ex}} Y^{\mu\nu}$ since it contributes only a constant. D_μ is the covariant derivative

$$D_\mu = \partial_\mu - ig \frac{\sigma^a}{2} W_\mu^a - ig' \frac{1}{2} (Y_\mu + Y_\mu^{\text{ex}}), \quad (2)$$

where σ^a , $a = 1, 2, 3$ are Pauli matrices, and the coupling constants are $g = 0.65$ and $g' = 0.53g$.

There are three classes of first-order electroweak PT models motivated by multiple particle physics models beyond the SM [66], i.e., thermally driven, tree-level driven, and loop driven, depending on the contributions of tree-level, quantum (loop), and thermal contributions in the thermal effective potential, that give raise to a pair of minima separated by a potential barrier at finite temperature in the early Universe [56,67]. The potential $\mathcal{V}(\Phi)$ under study belongs to the tree-level driven class, and is expressed as

$$\mathcal{V}(\Phi) = -\mu^2 \Phi^\dagger \Phi + A(\Phi^\dagger \Phi)^{3/2} + \lambda(\Phi^\dagger \Phi)^2, \quad (3)$$

that can be tested by colliders through the cubic Higgs coupling detection [55,68,69]. Its potential shape is indicated by the red line in Fig. 1, which arises from particle physics beyond the SM, such as new scalars that couple with the SM Higgs. Specially, during the PT, we have an extra contribution to the potential from the time-independent external MF through

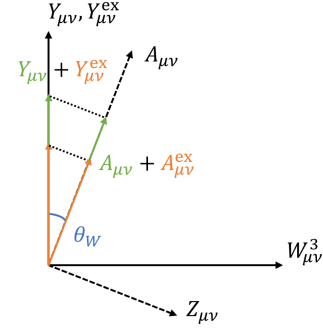


FIG. 2. A graphic representation of the conversion from hyper-MF $Y_{\mu\nu}$ into electromagnetic field $A_{\mu\nu}$ during the electroweak first order PT.

$$\Delta V \supset \frac{g'^4}{4} Y_\mu^{\text{ex}} Y_\mu^{\text{ex}} \Phi^\dagger \Phi. \quad (4)$$

Before the electroweak PT, the electromagnetic field $\mathbf{B}^{\text{ex}} = \nabla \times \mathbf{A}^{\text{ex}}$ had not been well defined and was replaced by a hyper-MF $\mathbf{B}_Y^{\text{ex}} = \nabla \times \mathbf{Y}^{\text{ex}}$. The relationship between them is $g' \mathbf{B}_Y^{\text{ex}} = e \mathbf{B}^{\text{ex}}$; here $e = g \sin \theta_W$ is the electric charge with θ_W being the Weinberg angle (weak mixing angle). The term would cancel the quartic term in Eq. (3) to some extent and change the PT process accordingly for different MF distributions at different spacetimes. For illustration, the influence of the external MF strength on the Higgs potential is demonstrated in Fig. 1. As the external MF increases, the value of $|\Phi|$ at the true vacuum continuously decreases, and the potential energy increases. If the external MF continues to increase, the true and false vacua degenerate, and ultimately, only one vacuum with $|\Phi| = 0$ remains, with the electroweak symmetry being restored.

After the breaking of electroweak symmetry, the relationship between the electromagnetic field and the original gauge field is [70]

$$\begin{aligned} A_{\mu\nu} &= W_{\mu\nu}^3 \sin \theta_W + (Y_{\mu\nu} + Y_{\mu\nu}^{\text{ex}}) \cos \theta_W \\ &\quad - i \frac{2}{g v^2} [(D_\mu \Phi)^\dagger (D_\nu \Phi) - (D_\nu \Phi)^\dagger (D_\mu \Phi)] \sin \theta_W \\ &= W_{\mu\nu}^3 \sin \theta_W + (Y_{\mu\nu} + Y_{\mu\nu}^{\text{ex}}) \cos \theta_W - S_{\mu\nu} \sin \theta_W, \end{aligned} \quad (5)$$

where the last term is the contribution of scalar current, which we denote as $S_{\mu\nu}$. Figure 2 shows the transition of a hyper-MF to an electromagnetic field during the first-order PT. First-order PTs can generate MFs through bubble collisions, serving as the seeds of the present-day cosmic MF [22]. When the external MF is stronger, it enters the scalar flow $S_{\mu\nu}$ [see Eq. (5)] through covariant derivatives [see Eq. (2)], further enhancing the MF generated by the PT.

In the following, we intend to explore how the MF affects the baryon asymmetry generation during the

first-order PT. For such a purpose, we first derive the definition of the Chern-Simons number in the background of an external MF. The chiral anomaly in electroweak theory gives the relationship between fermions and gauge fields [71–74]

$$\partial_\mu j_B^\mu = N_g \left[\frac{g^2}{16\pi^2} \text{Tr}(W_{\mu\nu} \tilde{W}^{\mu\nu}) - \frac{g^2}{32\pi^2} Y_{\mu\nu} \tilde{Y}^{\mu\nu} \right], \quad (6)$$

where $N_g = 3$ is the number of fermion generations, and the Hodge dual is $\tilde{X}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} X_{\rho\sigma}$. Integrating the above equation yields the change in baryon number density

$$n_B = N_g \frac{\Delta N_{\text{CS}}(t)}{V} = N_g \frac{N_{\text{CS}}(t) - N_{\text{CS}}(0)}{V}, \quad (7)$$

where $N_{\text{CS}}(t)$ is the Chern-Simons number [75]

$$\begin{aligned} \frac{N_{\text{CS}}(t)}{V} = \frac{1}{V} \frac{1}{32\pi^2} \epsilon^{ijk} \int d^3\mathbf{x} \left[-g^2 (Y_i + Y_i^{\text{ex}}) (Y_{jk} + Y_{jk}^{\text{ex}}) \right. \\ \left. + g^2 \left(W_i^a W_{jk}^a - \frac{g}{3} \epsilon^{abc} W_i^a W_j^b W_k^c \right) \right], \end{aligned} \quad (8)$$

and $V = L^3$ is the lattice volume. The U(1) component of N_{CS} corresponds to the hypermagnetic helicity

$$\begin{aligned} h_Y = \frac{H_Y}{V} = \frac{1}{V} \epsilon^{ijk} \int d^3\mathbf{x} (Y_i + Y_i^{\text{ex}}) (Y_{jk} + Y_{jk}^{\text{ex}}) \\ = \frac{1}{V} \int d^3\mathbf{x} (\mathbf{Y} + \mathbf{Y}^{\text{ex}}) \cdot (\mathbf{B}_Y + \mathbf{B}_Y^{\text{ex}}), \end{aligned} \quad (9)$$

up to a negative proportionality factor. During first-order PT, bubble collisions trigger abrupt changes in the Higgs field Φ distributions in different spacetime. This seeds the variation of the $g' \text{Im}[\Phi^\dagger D_i \Phi]$ term in the Y_i equation of motion, see Eqs. (14), (15) and also Appendix A, driving rapid Y_i field evolution that generates nonzero contributions in the Chern-Simons number and also the helicity from $\mathbf{Y} \cdot \mathbf{B}_Y$, $\mathbf{Y} \cdot \mathbf{B}_Y^{\text{ex}}$ and $\mathbf{Y}^{\text{ex}} \cdot \mathbf{B}_Y$. Meanwhile, during the PT, the U(1)_Y field strength $Y_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu$ additionally contributes to electromagnetism through $A_{\mu\nu} \supset Y_{\mu\nu} \cos \theta_W$ [22].

Bubble dynamics during the PT triggers abrupt changes in the Higgs field Φ distributions in different spacetimes, governed by the first equation in the following Eq. (14). The distributions of the nonvanishing Higgs field value when the PT starts to yield the Higgs winding number N_H [76]

$$N_H = -\frac{1}{24\pi^2} \epsilon^{ijk} \int d^3\mathbf{x} \text{tr}(\partial_i \Phi^\dagger \partial_j \Phi \partial_k \Phi^\dagger \Phi), \quad (10)$$

with [77]

$$\Phi = \frac{1}{|\Phi|} (i\sigma^2 \Phi^*, \Phi) \quad (11)$$

being a unitary matrix, N_H would grow along with the changing of the Chern-Simons number and stabilize when the PT is complete; see the following section.

During the PT, the baryon (and lepton) asymmetry of the Universe can be calculated through the relationship between the change in baryon (and lepton) number and the variation of the Chern-Simons number

$$\eta_B(t) = \frac{n_B}{s} = \eta_L(t) = \frac{n_L}{s} = \frac{45N_g}{2\pi^2 g_{*S} T^3} \frac{\Delta N_{\text{CS}}(t)}{V}. \quad (12)$$

Here, $g_{*S} = 106.75$ is the effective number of degrees of freedom in entropy before the PT. The time-varying hypermagnetic helicity in the early Universe enters the anomalous process, leading to baryon and lepton number asymmetry while preserving B – L conservation [62,63,78–83]

$$\frac{d\eta_B^Y}{dt} = \frac{d\eta_L^Y}{dt} = -\frac{45N_g}{2\pi^2 g_{*S} T^3} \left(\frac{g^2}{16\pi^2} \frac{dh_Y}{dt} \right). \quad (13)$$

Temporal evolution of the Chern-Simons number and hypermagnetic helicity thus originates from interactions between the preexisting external helical hypermagnetic field and PT-generated magnetic fields. The net baryon number generation is subsequently quenched after the PT completes.

III. THE SIMULATION SETUP

To study the PT dynamics and the accompanied Chern-Simons number variation under the background of an external MF, we evolve the equations of motion (EOM) of the system derived from Eq. (1). Under the temporal gauge $W_0 = Y_0 = Y_0^{\text{ex}} = 0$, the EOM is given by

$$\begin{aligned} \partial_0^2 \Phi &= D_i D_i \Phi - \frac{\partial \mathcal{V}}{\partial \Phi^\dagger}, \\ \partial_0^2 W_i^a &= -\partial_k W_{ik}^a - g\epsilon^{abc} W_k^b W_{ik}^c + g \text{Im}[\Phi^\dagger \sigma^a (D_i \Phi)], \\ \partial_0^2 Y_i &= -\partial_k Y_{ik} + g' \text{Im}[\Phi^\dagger (D_i \Phi)], \end{aligned} \quad (14)$$

along with two Gauss constraints,

$$\begin{aligned} \partial_0 \partial_j Y_j - g' \text{Im}[\Phi^\dagger \partial_0 \Phi] &= 0, \\ \partial_0 \partial_j W_j^a + g\epsilon^{abc} W_j^b \partial_0 W_j^c - g \text{Im}[\Phi^\dagger \sigma^a \partial_0 \Phi] &= 0. \end{aligned} \quad (15)$$

We here employ natural units where $\hbar = c = k_B = 1$. The background MF effects are included in the EOM with its initial condition to be explored later. We neglect the expansion of the Universe and the accompanied temperature variation, and also the evolution of background MFs during the first-order PT, since it is a rapid process. Please refer to *Equations of Motion on Lattice* of Appendix A for the discretized equations of motion and Gaussian constraints on the lattice.

To evolve the EOM, the primary initial conditions are given by the thermal fluctuation spectra under thermal equilibrium describing the Higgs field Φ and its conjugate momentum field Π before PT

$$\mathcal{P}_{\Phi_i}(k) = \frac{1}{\omega_k} \frac{1}{e^{\omega_k/T} - 1}, \quad \mathcal{P}_{\Pi_i}(k) = \frac{\omega_k}{e^{\omega_k/T} - 1}, \quad (16)$$

where $\omega_k = \sqrt{k^2 + m_{\text{eff}}^2}$. k and m_{eff} are the physical momentum and effective mass of the Higgs field.

In the continuum, we have

$$\langle \Phi_i(\mathbf{k}) \Phi_j(\mathbf{k}') \rangle = (2\pi)^3 \mathcal{P}_{\Phi_i}(k) \delta(\mathbf{k} - \mathbf{k}') \delta_{ij}, \quad (17)$$

$$\langle \Pi_i(\mathbf{k}) \Pi_j(\mathbf{k}') \rangle = (2\pi)^3 \mathcal{P}_{\Pi_i}(k) \delta(\mathbf{k} - \mathbf{k}') \delta_{ij}, \quad (18)$$

$$\langle \Phi_i(\mathbf{k}) \Pi_j(\mathbf{k}') \rangle = 0. \quad (19)$$

Converting it into a discrete form on the lattice, we get

$$\langle |\Phi_i(\mathbf{k})|^2 \rangle = \left(\frac{N}{\Delta x} \right)^3 \mathcal{P}_{\Phi_i}(k), \quad \langle \Phi_i(\mathbf{k}) \rangle = 0, \quad (20)$$

$$\langle |\Pi_i(\mathbf{k})|^2 \rangle = \left(\frac{N}{\Delta x} \right)^3 \mathcal{P}_{\Pi_i}(k), \quad \langle \Pi_i(\mathbf{k}) \rangle = 0, \quad (21)$$

where N^3 denotes the total number of lattice points and Δx is the physical lattice spacing. $\Phi_i(\mathbf{k})$ and $\Pi_i(\mathbf{k})$ satisfy the point-independent Gaussian distribution in the momentum space, which contains all modes from the infrared truncation $k_{\text{IR}} = 2\pi/(N\Delta x)$ to the ultraviolet cutoff k_{CutOff} . For the gauge fields, we set their initial values to 0. However, to satisfy the Gaussian constraint [Eq. (15)], we must assign an initial value to the conjugate momentum field of the gauge field according to certain rules. For specific details on *Initialization of Gauge Fields*, see Appendix B.

When PT occurs, the Higgs field quantum tunnels from the false vacuum $|\Phi| \simeq 0$ to the true vacuum $|\Phi| = v$, nucleating a bubble at $t = t_0$. Therefore, to study the first-order PT dynamics, another initial condition, i.e., the implementation of the vacuum bubbles on lattice, is necessary. In the simulation, each lattice site near the false vacuum at each time step has a probability p_{bubble} to nucleate a vacuum bubble, which has the profile

$$\Phi(t_0, r) = \frac{v}{2} \left[1 - \tanh \left(\frac{r - R_0}{l_w} \right) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (22)$$

$$\dot{\Phi}(t_0, r) = 0, \quad (23)$$

where r denotes the distance from bubble center, R_0 denotes the radius of bubble, and l_w denotes the bubble wall thickness. Then, the nucleated bubbles expand and collide with each other to propel the PT process.

TABLE I. Model parameters and setup in the simulation.

v	Δx	Δt	μ^2	A	λ	R_0	l_w	p_{bubble}
1	0.2	0.04	0.68	2.59	1.60	$13\Delta x$	$6\Delta x$	5×10^{-8}

The MFs' strength, correlation length, and helicity serve as the third initial condition for the evolution of the EOM given by Eq. (14). Considering the correlation length of the primordial MFs, generated through inflation [84–88], the first-order PT [21–27,89], we consider two types of external hyper-MFs with distinct correlation lengths, defined as $\lambda_{BY} = \int dk k^{-1} E_{BY}(k) / \int dk E_{BY}(k)$, where $\int dk E_{BY}(k) = \int d^3x \frac{1}{2} [B_Y^{\text{ex}}(\mathbf{x})]^2 / V$. The infinite correlation length hyper-MF with helicity is set by $Y^{\text{ex}\mu} = (0, 0, x B_Y^{\text{ex}}, h_{\text{factor}} L B_Y^{\text{ex}})$, where $h_{\text{factor}} \in [-1, 1]$, and $L = N\Delta x$ represents the edge length of the lattice box. For hyper-MFs with finite correlation lengths, we adopt the following form in Fourier space [90]: $\tilde{B}_{Yi}^{\text{ex}}(\mathbf{k}) = B_{\text{ini}} \Theta(k - k_{\text{UV}}) (\delta_{ij} - \hat{k}_i \hat{k}_j - i \sigma_M \epsilon_{ijl} \hat{k}_l) g_j(\mathbf{k}) k^n$. Here $\Theta(k - k_{\text{UV}})$ is the Heaviside function, ensuring the hyper-MF has a finite correlation length comparable to the mean bubble separation $R_* = (V/N_{\text{bubble}})^{1/3}$ during the percolation process of vacuum bubbles in the PT. $\mathbf{g}(\mathbf{k})$ is the Fourier transformation of a Gaussian-distributed random vector field that is δ -correlated in all three dimensions. The helicity degree is controlled by $\sigma_M = 0, 1, -1$, corresponding to nonhelical, maximally positive, and maximally negative helicities, respectively. The specific implementation of hyper-MFs on the lattice is detailed in the *Hypermagnetic Field on Lattice* section of Appendix C.

To investigate the effects of the MFs' correlation length, we conduct numerical simulations on two different sizes of three-dimensional lattices, with lattice spacing of Δx and periodic boundary conditions:

- (i) $N^3 = 128^3$: we simulate a homogeneous MF and a spectral distribution MF with $\lambda_B \sim R_*/4 \sim R_0$, where R_0 is the initial vacuum bubble radius. We conduct 20 simulations with $n_{\text{timestep}} = 10000$ for each MF strength and helicity setting.
- (ii) $N^3 = 512^3$: we simulate the spectral distribution MF with $\lambda_B \sim R_*$. Due to computational constraints, we limit the simulations to 4 runs with $n_{\text{timestep}} = 3600$ for each combination of $\sigma = \pm 1$ and hyper-MF strength.

The model parameters and lattice setup in the simulation are presented in Table I.

IV. NUMERICAL RESULTS

Through evolving the EOM [see Eq. (14)] on lattice, in this section, we numerically study the following ingredients in the background of the hyper-MF: the PT speed, the phenomena of the Ambjørn-Olesen condensation, the

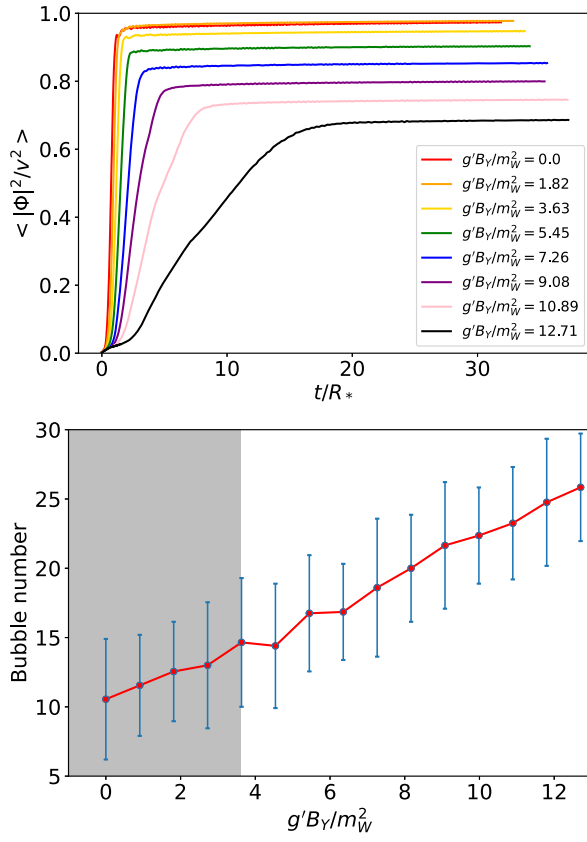


FIG. 3. Top: evolution of Φ^2 over time under nonhelical homogeneous hyper-MF. Bottom: the number of bubbles nucleated under nonhelical homogeneous hyper-MF. The gray area indicates the hyper-MF strength where Higgs condensation has not yet occurred.

electroweak sphaleron rate, and the baryon and lepton number generation through the chiral anomaly.

A. Phase transition speed

We first present the variations of the Higgs field and the number of bubbles under homogeneous nonhelical hyper-MF in Fig. 3. It can be seen that when the MF strength increases, the PT speed becomes slower and more bubbles are nucleated, which reflects that the strong MF affects the electroweak vacuum. When a bubble is nucleated, the strong MF acts rapidly on the true vacuum area inside the bubble. This not only reduces the vacuum expectation value of the Higgs field which further confirms the behavior demonstrated in Fig. 1, but also slows down the expansion speed of the bubble, causing the false vacuum to occupy the space of the Universe for a longer period, which further results in the creation of more vacuum bubbles.

The effects of homogeneous helical hyper-MF on the PT speed and the number of bubbles are similar to those without helicity. As shown in the Fig. 4, the helical homogeneous hyper-MF makes the PT speed slower and also makes the final stable field value smaller. This shows

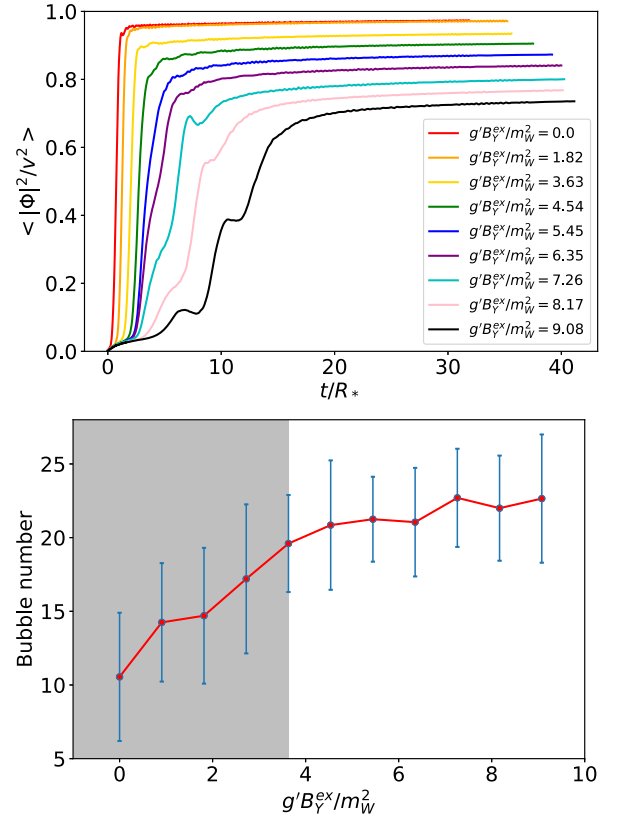


FIG. 4. Top: evolution of Φ^2 over time under different helical homogeneous hyper-MF strength with $h_{\text{factor}} = 1$. Bottom: the number of bubbles nucleated under different MF strengths with $h_{\text{factor}} = 1$. The gray area indicates the magnetic field strength where Higgs condensation has not yet occurred.

that the speed of bubble expansion is slowed down in the hyper-MF with helicity, and the bubble expansion speed is further reduced compared with the hyper-MF without helicity. The helicity will also slightly affect the PT process. As shown in Fig. 5, as the initial helicity increases, the PT speed will decrease slightly, and the final value of $\langle |\Phi|^2 \rangle$ will also become smaller. This is just as we stated before, the speed at which the bubble expands is further slowed down by the helicity.

From Figs. 6 and 7, it can be seen that the correlation length has an impact on the PT speed. When the correlation length is small, the “transition” region between regions with high MF strength and regions with low MF strength is small, and the PT will quickly complete in the regions with low MF strength, while in the regions with high MF strength, the PT will not occur at all. This leads to a PT speed in this situation that is not significantly different from the time difference without a MF, but $\langle |\Phi|^2 \rangle$ will decrease. On the contrary, MFs with larger correlation lengths have larger transition regions, where the MF strength is moderate and PT can occur here, but at a slower speed. The extreme case is a homogeneous MF with an infinite correlation length.

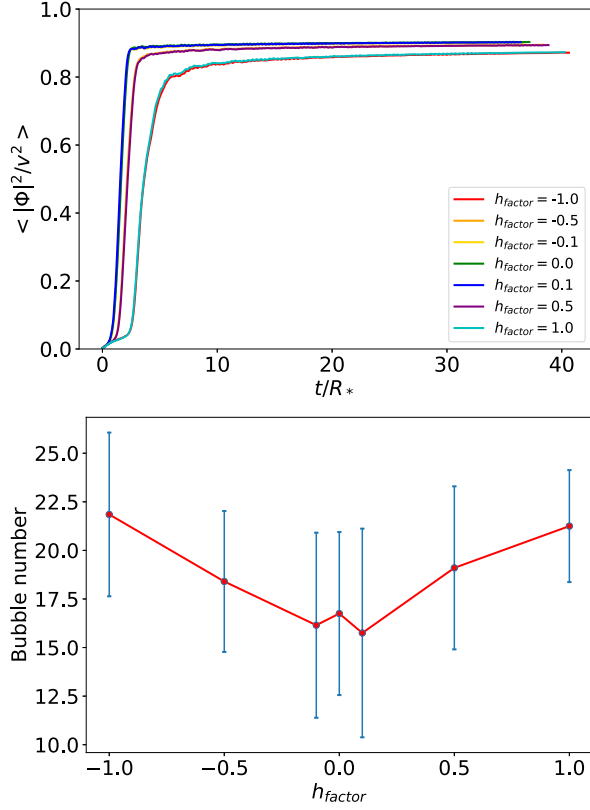


FIG. 5. Top: evolution of Φ^2 over time under the same homogeneous hyper-MF strength ($g'B_{Y,rms}^{ex}/m_W^2 = 5.45$) and different h_{factor} . Bottom: the number of bubbles nucleated under different h_{factor} with homogeneous hyper-MF strength being fixed at $g'B_{Y,rms}^{ex}/m_W^2 = 5.45$.

We observe that, regardless of the helicity or correlation scale of the magnetic field, the vacuum expectation value of the Higgs field consistently decreases with increasing hyper-MF strength following the PT. This phenomenon is analogous to the inverse magnetic catalysis observed in QCD [91,92]. It is posited that this behavior is related to the Ambjørn-Olesen condensation, which will be discussed in the next subsection. In the case of a crossover, a similar inverse magnetic catalysis effect on the Higgs field under an external magnetic field has also been reported by [33,34].

B. Ambjørn-Olesen condensation

When the process of electroweak symmetry breaking occurs in a MF background, the configuration of the broken phase field will be closely related to the MF strength. There exist two critical MFs as follows:

$$g'B_{Yc1} = eB_{c1} = m_W^2, \quad (24)$$

$$g'B_{Yc2} = eB_{c2} = m_H^2. \quad (25)$$

As the external MF increases, the Higgs and gauge fields exhibit different behaviors [28–32,93–96]:

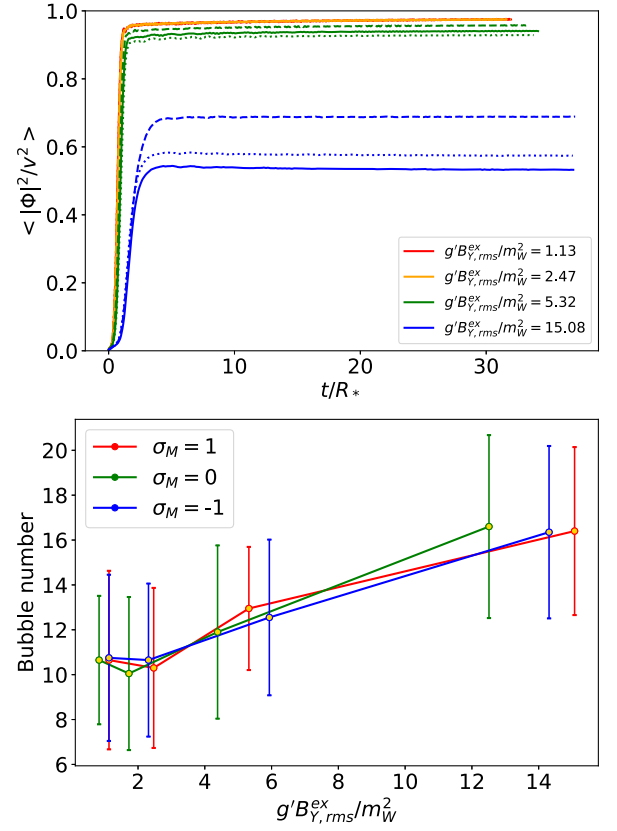


FIG. 6. The evolution of Φ^2 (top) and the number of bubbles nucleated (bottom) under different hyper-MF strength with spectral distribution in case of $N = 128$. In the top panel, the solid line indicates $\sigma_M = 1$, the dotted line indicates $\sigma_M = -1$, and the dashed line indicates $\sigma_M = 0$, respectively. The root mean square hyper-MF strength $B_{Y,rms}^{ex} = \sqrt{2\rho_{BY}^{ex}}$, where ρ_{BY}^{ex} is the volume-averaged hyper-MF energy density.

- (i) When $B < B_{c1}$, the MF has no effect on the broken phase. As symmetry breaking progresses, the Higgs field will gradually fall into the broken phase and stabilize there.
- (ii) When $B_{c1} < B < B_{c2}$, the original electroweak vacuum becomes unstable due to the emergence of an imaginary part in the vacuum energy. To compensate for this instability, the $SU(2)_L$ gauge field forms a hexagonal vortex line structure (with the direction of the line following the direction of the external MF). This phenomenon is known as the Ambjørn-Olesen condensation. Since the $SU(2)_L$ gauge field couples with the Higgs field through covariant derivatives (2), the Higgs field also produces a similar hexagonal vortex line structure, resulting in a decrease in the vacuum expectation value.
- (iii) When $B > B_{c2}$, the electroweak symmetry will be restored. This means that the vortex line structure of gauge fields and the Higgs field will disappear, and the vacuum expectation value of the Higgs field will

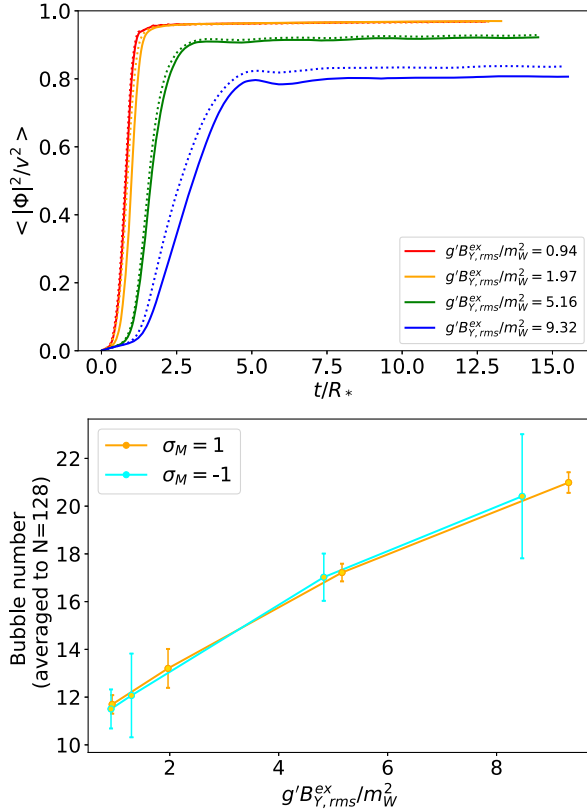


FIG. 7. The evolution of Φ^2 (top) and the number of bubbles nucleated (bottom) under different hyper-MF strength with spectral distribution in case of $N = 512$. In the top panel, the solid line indicates $\sigma_M = 1$, and the dotted line indicates $\sigma_M = -1$. For ease of comparison, the number of bubbles shown in the bottom panel has been scaled to the equivalent number of bubbles for the $N = 128$ case.

return to 0. At this point, the external MF will be replaced by the $U(1)_Y$ gauge field.

In the case of crossover, the Ambjørn-Olesen condensation phenomenon has been verified through lattice simulation methods [33]. Reference [97] studied the electroweak PT dynamics under external MF and did not observe the Ambjørn-Olesen phase, which might be due to the quantum fluctuations there. We will demonstrate that Ambjørn-Olesen condensation indeed can occur in the context of first-order PT, and it bears little relation to the helicity of the MF.

Figure 8 shows the changes of $\overline{\Phi^2}$ and $\overline{B^2} = \frac{1}{NN_t} \sum_{t,z} B^2(t, x, y, z)$ under different homogeneous hyper-MFs. When $g'B_Y^{\text{ex}}/m_W^2 > 3.63$, the Higgs condensate phenomenon appears. It can be seen that $\overline{\Phi^2}$ is relatively small at the center of the vortex, which is exactly the opposite of $\overline{B^2}$. It is worth mentioning that every time step of the Higgs field is involved in the averaging, while MF is only averaged every 100 steps, so the picture of $\overline{B^2}$ is

rougher. We also find that the stronger the hyperMF, the denser the vortex.

When the background MF is entirely provided by the $U(1)$ hyper-MF, after the electroweak symmetry breaking, the electromagnetic field part will also undergo a phenomenon similar to the W and Z condensation. For analytical calculations and lattice simulations related to this phenomenon, see Refs. [33,98,99].

Slices of $|\Phi|^2$ and hypermagnetic field energy at the end of the simulation are shown in Fig. 9. The hyper-MF with a spectral distribution is no longer homogeneous in space, but with some places having high energy and some places being almost zero. At the end of the simulation, because the background MF does not change with time, the distribution of hyper-MF energy is similar to that at the beginning. However, it should be noted that $|\Phi|^2$ at the end tells us that the PT is only successfully and quickly completed in areas with small hyper-MFs, while in places with large hyper-MFs, Φ is still in the symmetric phase. Due to the random hyper-MF distribution, unlike the homogeneous hyper-MF with a fixed direction, the Higgs condensation phenomenon cannot be observed.

C. Electroweak sphaleron

With the PT dynamics affected by the helicity of the hyper-MF as studied in the last subsection, we expect the behavior of Chern-Simons number variation during the first-order electroweak PT should also be dramatically different from the SM *crossover* case. We present the relationship between the Chern-Simons number (and the Higgs winding number) and the helicity of the MFs in Figs. 10 and 11, which reflect the increase of the N_{CS} and N_H as the helicity of the MFs grows. According to our previous research [22], in a first-order PT, a MF is generated due to the collision of bubbles. But when this process occurs in the context of a hyper-MF, the situation will undergo some changes. When a PT occurs, there would be a slight increase in MF strength as the bubble nucleates. Subsequently, the bubble expands, and the MF will stabilize for a short period of time. When bubbles collide with each other, which is a very intense dynamic process, the N_{CS} rapidly increases or decreases. Moreover, the stronger the external MF, the larger the N_{CS} generated. The magnitude of the Higgs winding number N_H increases with the proceeding of PT when the Higgs field value increases. After the PT is completed, the Chern-Simons number and the Higgs winding number tend to stabilize. As depicted for the two external MF scenarios, the N_{CS} grows as the strength and helicity of the hyper-MF increase. Meanwhile, we observe the difference between the change of Chern-Simons number ΔN_{CS} and the change of the Higgs winding number ΔN_H during the PT. The relationship between the Chern-Simons number and the MF helicity here does not support the existence of the electroweak strings reported in Refs. [100–103], since we did not assume any *Ansätze* for gauge fields.

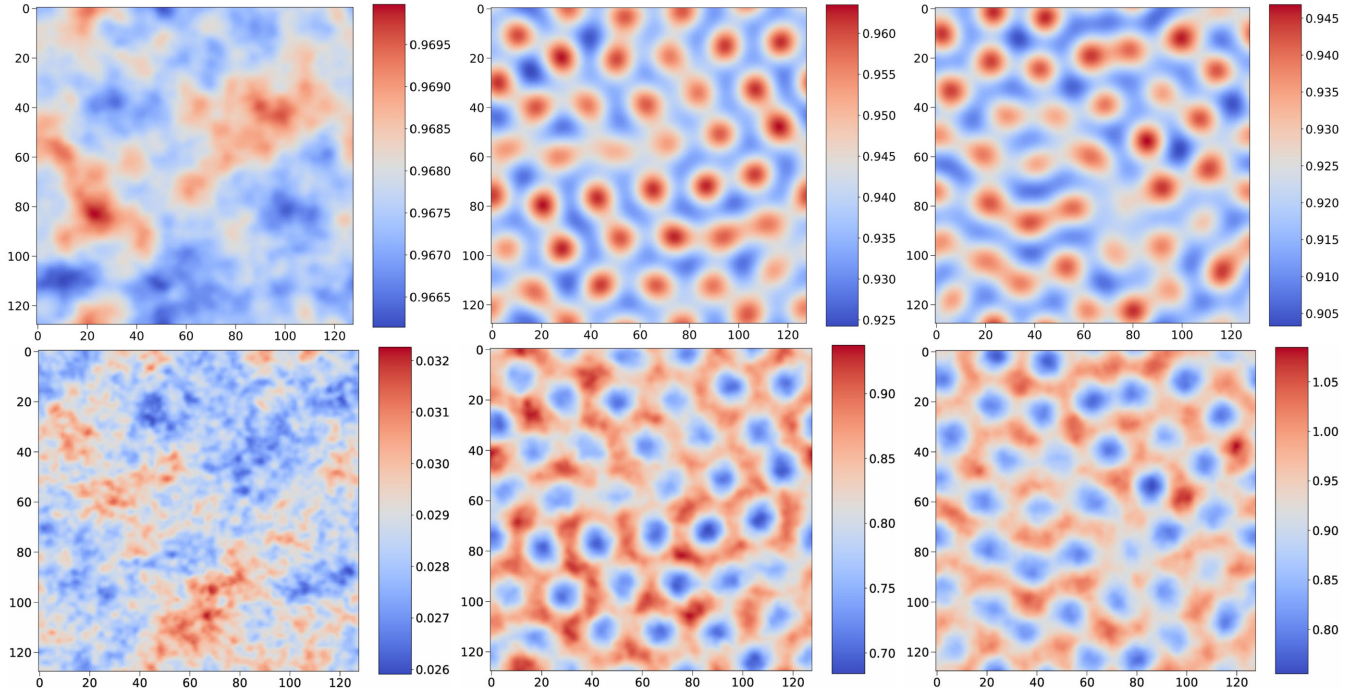


FIG. 8. $\overline{\Phi^2}$ (top) and $\overline{B^2}$ (bottom) under different homogeneous hyper-MFs. The left two plots indicate the case where $g'B_Y^{\text{ex}}/m_W^2 = 0$, and the middle and right two plots are the scenarios with nonhelical and helical homogeneous hyper-MFs ($h_{\text{factor}} = 1$) with $g'B_Y^{\text{ex}}/m_W^2 = 3.63$ where the Higgs condensations appear. The red and blue colors correspond to higher and lower field values, respectively.

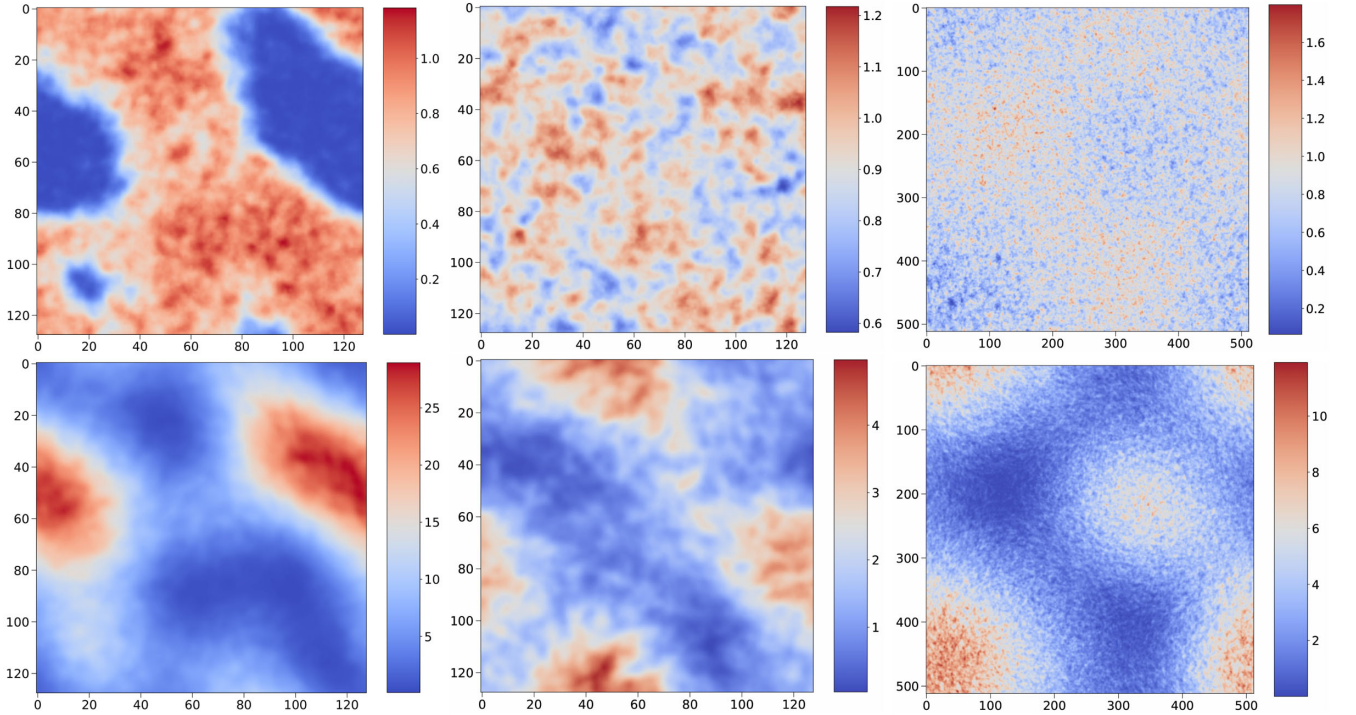


FIG. 9. Slice of $|\Phi|^2$ (top) and hypermagnetic energy (bottom) at the end of the simulation. The hyper-MF has a spectral distribution with correlation length of $\lambda_{BY} \sim R_0$ and with spectral indices of 0 (left) and 1 (middle), and with $\lambda_{BY} \sim R_*$, $n = 0$ (right), respectively. The color code is the same as in Fig. 8.

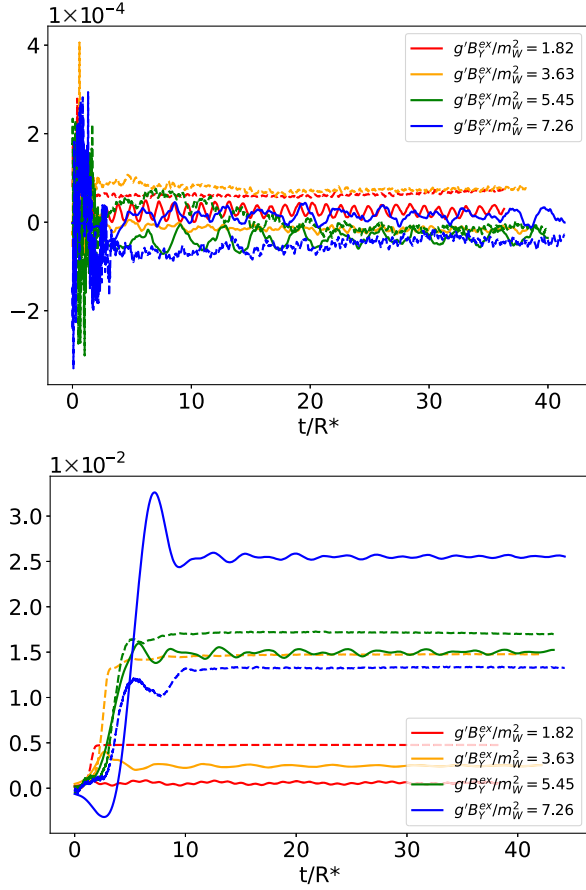


FIG. 10. The variation of Chern-Simons number N_{CS} (solid line) and Higgs winding number N_H (dashed line) over time under homogeneous external MFs with $h_{\text{factor}} = 0$ (top) and $h_{\text{factor}} = 1$ (bottom).

We first use the diffusion rate usually adopted to describe the evolution of the Chern-Simons number close to thermal equilibrium, which is also known as the sphaleron rate

$$\Gamma(t) = \frac{\langle (\Delta N_{CS}(t))^2 \rangle}{Vt} \quad (26)$$

to measure the intensity of the change of the volume averaged Chern-Simons number with the definition of N_{CS} being given by Eq. (8). Angle brackets indicate ensemble averaging, which is the average of multiple runs. It approaches a constant value, known as the diffusion constant, in the limit of infinite time and volume [34,104].

We first study the electroweak (EW) sphaleron rate behavior under the impact of the homogeneous helical MF. We find that a stronger helical hyper-MF background will bring about a more drastic change in N_{CS} , which is reflected by the behavior of $\ln \Gamma(t)/T^4$. As shown in Fig. 12, the increase of hyper-MF strength will significantly elevate $\Gamma(t)$, that is, the rate of baryon production is faster. If the hyper-MF is fixed and the helicity is increased, the rate of baryon production will also increase, but the effect is not

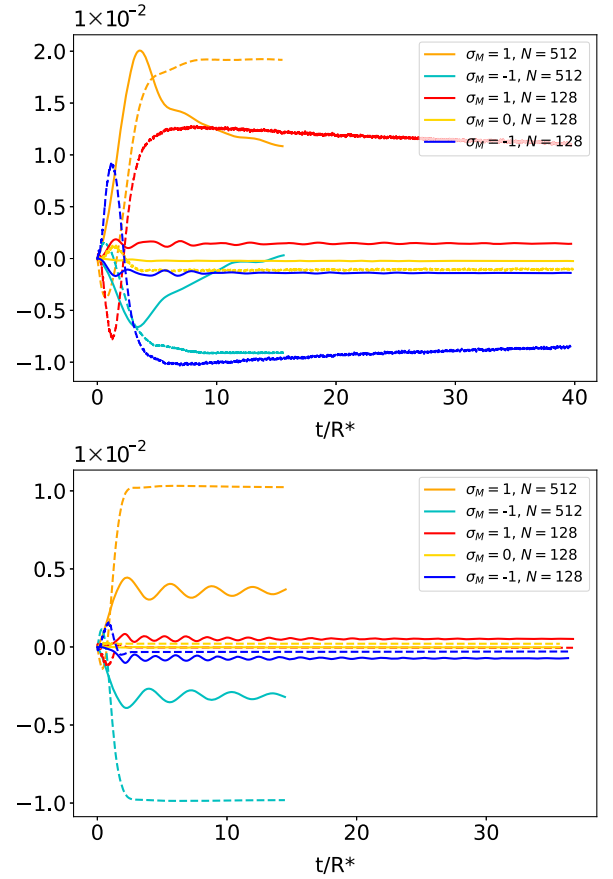


FIG. 11. The variation of Chern-Simons number N_{CS} (solid line) and winding number N_H (dashed line) over time under spectral-distributed external MFs with $n = 0$ (top) and $n = 1$ (bottom).

as obvious as that brought by increasing the MF strength. The change of sphaleron rate $\Gamma(t)$ with time under different spectral-distributed hyper-MF is shown in Fig. 13. The lines with the same color represent the same spectral index and similar hyper-MF strength. It can be seen that as the MF energy increases, the sphaleron rate also increases.

We here calculate the time average of the sphaleron rate for different hyper-MF when it is stable after $t/R_* > t_s/R_* = 2.67$, denoted as

$$\bar{\Gamma} = \frac{1}{t_f - t_s} \int_{t_s}^{t_f} dt \Gamma(t), \quad (27)$$

where t_f is the end of the simulation. As shown in Fig. 14, as the MF and helicity increase, $\ln \bar{\Gamma}_\infty/T^4$ also increases. The subscript ∞ represents that the external hyper-MF under study has an infinite correlation length. We can fit the dependence of $\ln \bar{\Gamma}_\infty/T^4$ on MF strength and h_{factor}

$$\ln \frac{\bar{\Gamma}_\infty}{T^4} = (0.78|h_{\text{factor}}|^{-0.5} + 0.58) \left(\frac{B_Y^{\text{ex}}}{T^2} \right) + \ln \frac{\Gamma_{\text{sym}}}{T^4}, \quad (28)$$

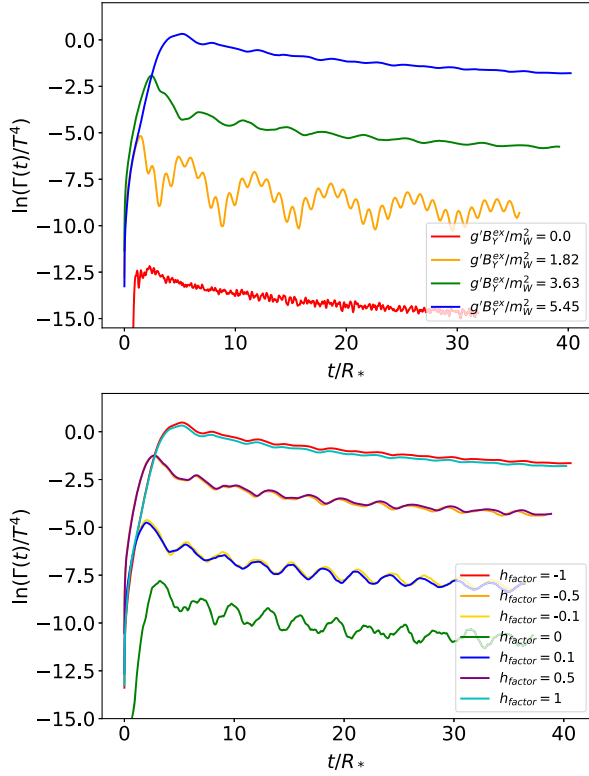


FIG. 12. Top: changes of $\ln \Gamma(t)/T^4$ over time under different homogeneous helical MFs with $h_{\text{factor}} = 1$. Bottom: changes of $\ln \Gamma(t)/T^4$ over time at different initial helicity with homogeneous hyper-MF fixed at $g'B_Y^{\text{ex}}/m_W^2 = 5.45$.

which is shown in Fig. 14 by the green dashed line. $\Gamma_{\text{sym}}/T^4 = 6.23 \times 10^{-7}$ represents the value of the sphaleron rate in the symmetric phase [105], which is showed by the gray line in Fig. 14. Here, T indicates the PT temperature. We can see that the sphaleron rate can grow by many orders with the increase of the strength and helicity of the external hyper-MF.

Similarly, we extract the electroweak sphaleron rate under the spectrum distribution hyper-MF background and plot the variation of $\ln \bar{\Gamma}/T^4$ versus hyper-MF strength in Fig. 15 and use the green dashed line to show the following fitting results:

$$\ln \frac{\bar{\Gamma}_{R_0}}{T^4} = 3.34|\sigma_M| \left(\frac{B_{Y,\text{rms}}}{T^2} \right)^{0.32} + \ln \frac{\Gamma_{\text{sym}}}{T^4}, \quad (29)$$

$$\ln \frac{\bar{\Gamma}_{R_*}}{T^4} = 6.93|\sigma_M| \left(\frac{B_{Y,\text{rms}}}{T^2} \right)^{0.40} + \ln \frac{\Gamma_{\text{sym}}}{T^4}. \quad (30)$$

As indicated by the figure and the above two formulas, the electroweak sphaleron rate increases by several orders when the correlation length increases from R_0 to R_* .

Additionally, there is an alternative definition of the sphaleron rate [77,106],

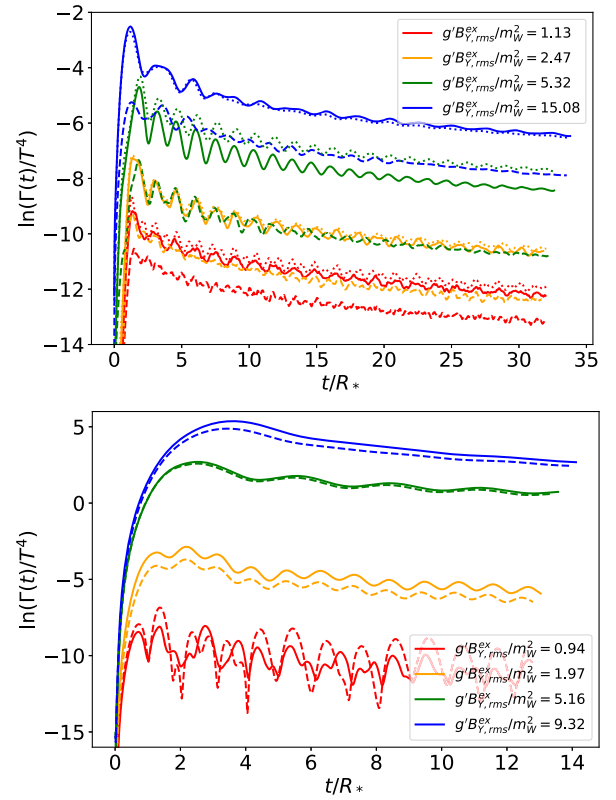


FIG. 13. Evolution of $\ln \Gamma(t)/T^4$ over time under different index n of the spectrum distributed hyper-MF and correlation length ($\lambda_B \sim R_0$ on the top and $\lambda_B \sim R_*$ on the bottom). The solid line indicates $\sigma_M = 1$, the dashed line indicates $\sigma_M = -1$, and the dotted line indicates $\sigma_M = 0$, respectively.

$$\Gamma(t) = \frac{1}{V} \frac{d}{dt} [\langle (\Delta N_{\text{CS}}(t))^2 \rangle - \langle \Delta N_{\text{CS}}(t) \rangle^2], \quad (31)$$

whose time integral differs from Eq. (26) only by a term of $\langle \Delta N_{\text{CS}}(t) \rangle^2$. When this term is negligibly small, the time integral of Eq. (31) becomes identical to Eq. (26), which is precisely the case in our simulations.

D. Baryon and lepton asymmetry

With the property of the Chern-Simons number under the hyper-MF at hand, we now investigate the time variation of the total baryon asymmetry (η_B) through Eq. (7) and the one induced by the helical hyper-MF effect (η_B^Y) through the chiral anomaly [83], i.e., Eq. (13). The changes of η_B and η_B^Y over time are shown in Figs. 16 and 17. When PT occurs, both η_B and η_B^Y increase or decrease simultaneously, depending on the sign of their helicity, and tend to stabilize upon completion of the PT. In the case of a spectral distribution hyper-MF, we present η_B and η_B^Y for different helical hyper-MF spectra. It can be seen that $\eta_B^Y \gtrsim \eta_B$, and η_B^Y occupies the dominant part of η_B . For both the homogeneous and spectral distribution of the helical hyper-MF scenarios, we have that the generation of the

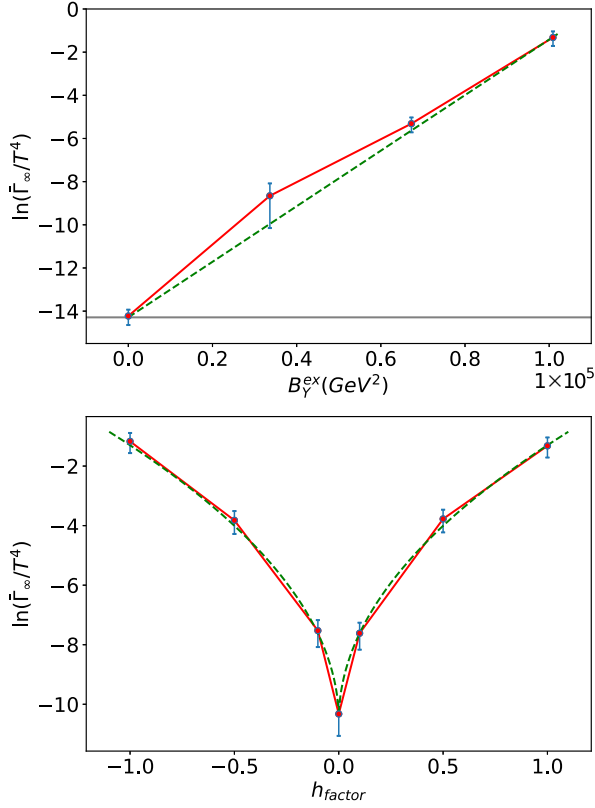


FIG. 14. Top: changes of $\ln \bar{\Gamma}_\infty/T^4$ over different homogeneous, helical MFs. The gray line represents the value of the sphaleron rate in the symmetric phase $\Gamma_{\text{sym}}/T^4 = 6.23 \times 10^{-7}$ [105]. Bottom: changes of $\ln \bar{\Gamma}_\infty/T^4$ over different initial helicity with homogeneous hyper-MF fixed at $g'B_Y/m_W^2 = 5.45$. The fitting result of Eq. (28) is represented by the green dashed line.

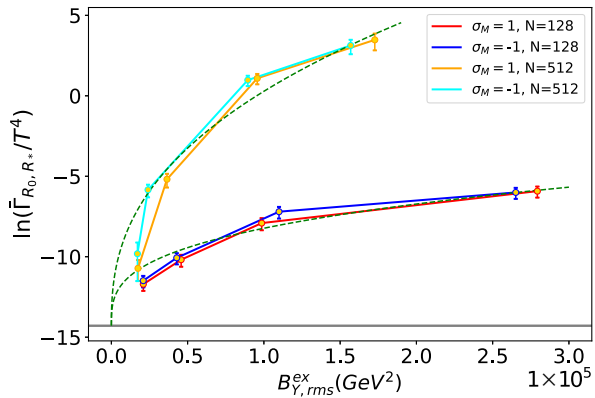


FIG. 15. Changes of $\ln \bar{\Gamma}_{R_0, R_*}/T^4$ over different hypermagnetic energy with $\sigma_M = \pm 1$ and correlation length of $\lambda_B \sim R_0$ and $\lambda_B \gtrsim R_*$. The green dashed line represents the fitting result of Eqs. (29) and (30). The gray line represents the value of the sphaleron rate in the symmetric phase $\Gamma_{\text{sym}}/T^4 = 6.23 \times 10^{-7}$ [105].

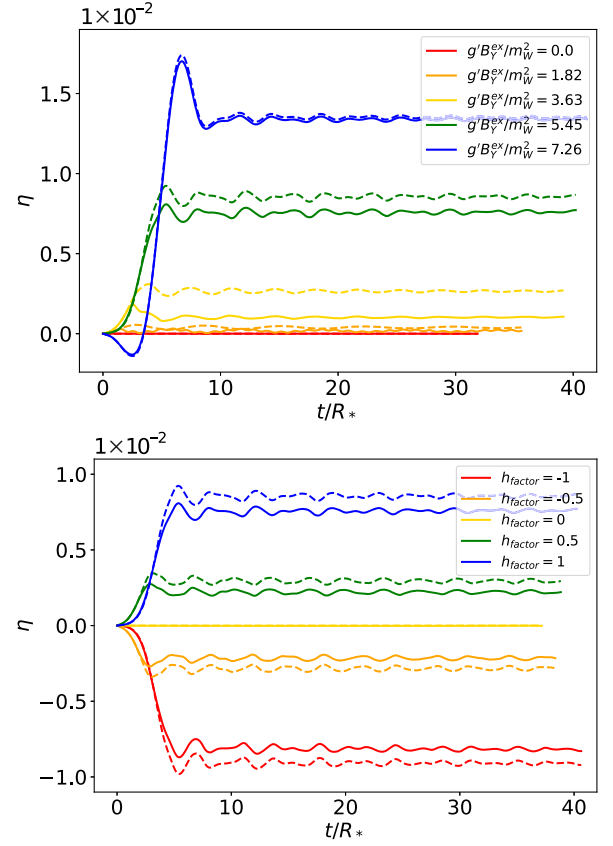


FIG. 16. Top: the variation of η_B (solid line) and η_B^Y (dashed line) over time under different homogeneous helical hyper-MF strengths with $h_{\text{factor}} = 1$. Bottom: the variation of η_B (solid line) and η_B^Y (dashed line) over time under different h_{factor} with $g'B_Y^{\text{ex}}/m_W^2 = 5.45$.

baryon asymmetry increases as the helicity and magnitude of the hyper-MF increase.

E. Remark on the MF

Consider MFs with finite correlation lengths and under fully helical conditions, by using the magnetohydrodynamic equation, we can obtain the relationship between the MF during the electroweak PT epoch and the magnetic field at present. The (comoving) correlation scale evolves as $\lambda_{\text{rec}}/\lambda_{\text{EW}} = (a_{\text{rec}}/a_{\text{EW}})(\tau_{\text{rec}}/\tau_{\text{EW}})^{2/3}$, and the MF strength evolves as $B_{\text{rec}}/B_{\text{EW}} = (a_{\text{rec}}/a_{\text{EW}})^{-2}(\tau_{\text{rec}}/\tau_{\text{EW}})^{-1/3}$ [107–109], where τ denotes conformal time, and the subscripts “rec” and “EW” refer to the recombination epoch and the electroweak PT epoch, respectively. If the MF is nonhelical, then the MF will follow the following relationship: $\lambda_{\text{rec}}/\lambda_{\text{EW}} = (a_{\text{rec}}/a_{\text{EW}})(\tau_{\text{rec}}/\tau_{\text{EW}})^{1/2}$, and the MF strength evolves as $B_{\text{rec}}/B_{\text{EW}} = (a_{\text{rec}}/a_{\text{EW}})^{-2}(\tau_{\text{rec}}/\tau_{\text{EW}})^{-1}$ [107]. After the recombination epoch, the density of charged particles in the Universe sharply decreased, and the MF was considered to no longer evolve, but only passively “diluted” with the expansion of the universe as

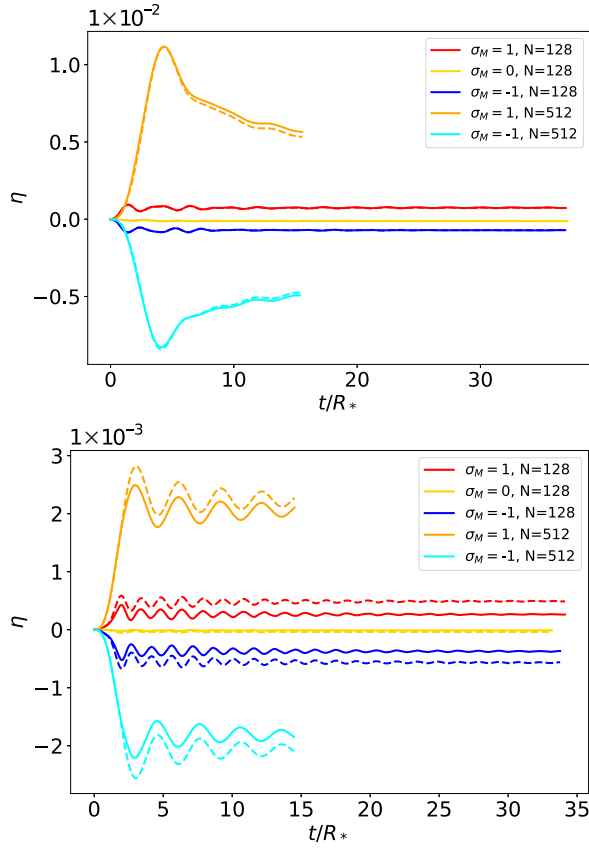


FIG. 17. The variation of η_B (solid line) and η_B^Y (dashed line) over time under different spectrum distributed hyper-MF's σ_M with the spectral index of $n = 0$ (top) and $n = 1$ (bottom).

$B \propto a^{-2}$. These are the two types of magnetic field strength conversion relationships shown in Fig. 18. Figure 18 also shows the range of MF strength (depending on its correlation length) required to achieve the correct matter-antimatter asymmetry obtained in Ref. [110], as well as the MF required for the Higgs condensate in this work. According to current astronomical observations, the upper limit of MF strength in the Universe today is determined by cosmic microwave background (CMB), which is approximately 10^{-9} Gauss on CMB scale $\lambda_B \gtrsim 1$ Mpc [111], and the lower limit is set by observations of blazars as [52]

$$B \gtrsim 10^{-17} \text{ Gauss} \begin{cases} (\lambda_B/1 \text{ Mpc})^{-1/2}, & \lambda_B < 1 \text{ Mpc}, \\ 1, & \lambda_B > 1 \text{ Mpc}. \end{cases} \quad (32)$$

During the electroweak PT, the Universe is dominated by radiation, and the MF energy density should not exceed that of the radiation energy density

$$\rho_r = \frac{\pi^2}{30} g_{*S} T^4, \quad (33)$$

where $g_{*S} = 106.75$ is the effective number of degrees of freedom in entropy before the PT. If we take the phase transition temperature as $T_{EW} = 125$ GeV, the upper limit of (hyper-)MF strength is $g'B_Y/m_W^2 = 7.07$. In our simulation, for the MF with spectral distribution, $n < 1$ can cause the MF strength to exceed this limit. The above limitations are shown in Fig. 18.

V. CONCLUSIONS

In this work, we numerically investigated the first-order electroweak PT in the background of primordial MF through three-dimensional lattice simulations. We investigated various effects of the primordial MF on the first-order electroweak PT, encompassing PT speed, Ambjørn-Olesen condensation, sphaleron rate, and baryon and lepton asymmetry generations. Firstly, we found that an external MF slows down the speed of PT. This is because the expansion speed of vacuum bubbles in a MF slows down, which also leads to the generation of more bubbles in a stronger MF. If the external MF is helical, the PT speed will slow down slightly further. Secondly, when the MF exceeds the first critical value, the Ambjørn-Olesen condensation phenomenon occurs, where the Higgs field is arranged into a regular hexagonal structure due to the influence of the $SU(2)_L$ gauge field. As the external MF increases, this hexagonal structure becomes increasingly dense. This phenomenon applies to both helical and non-helical external MFs. In the case of a spectrum distribution MF, since the MF does not have a definite direction, we cannot observe the Ambjørn-Olesen condensation phenomenon. In regions where the MF is very strong, the electroweak PT does not occur at all, and the electroweak symmetry remains (or is restored). After symmetry

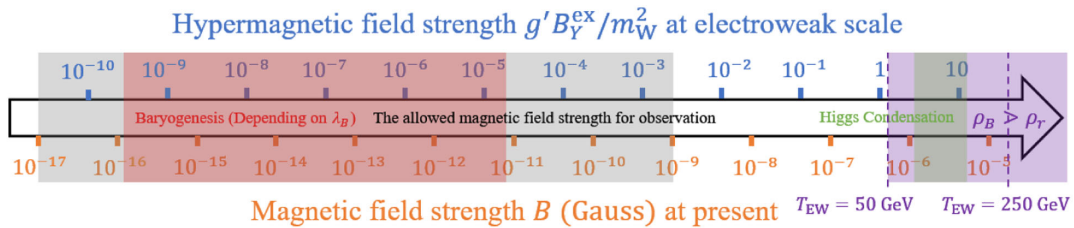


FIG. 18. Schematic diagram of the range of the cosmic MF. The observations of CMB and blazar, respectively, limit the upper and lower limits of MF strength [42,63]. The radiation energy density in the early Universe limited the upper limit of the MF strength at that time. The range of MF strength that can achieve correct matter-antimatter asymmetry obtained in Ref. [110] is also shown in the figure.

breaking, the $SU(2)_L$ gauge field breaks down into the $U(1)_{\text{em}}$ electromagnetic field, so the MF also exhibits a similar hexagonal structure.

Sphaleron rate Γ is an important physical quantity that characterizes the rate of baryon number production. We studied the changes in Γ under different preexisting MF strengths, helicities, and correlation lengths. We found that as the MF increases and the helicity increases, Γ will also increase. The correlation scale mainly affects the power-law dependence of Γ on external MF. We detailed here the mechanism to generate the baryon asymmetry through the hyper-MF, where the MF produced during the PT ensures the time variation of hyper-MF helicity, which drives baryon asymmetry through the chiral anomaly. For the relationship between baryon asymmetry and the properties of the hyper-MF, and its detectability at present, we refer to Ref. [110].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: EQUATIONS OF MOTION ON LATTICE

The Higgs field Φ is assigned to lattice sites, while gauge fields reside on links between neighboring sites via link variables

$$U_i(t, x) = \exp \left[-ig\Delta x \frac{\sigma^a}{2} W_i^a(t, x) \right], \quad (\text{A1})$$

$$V_i(t, x) = \exp \left[-ig'\Delta x \frac{1}{2} Y_i(t, x) \right], \quad (\text{A2})$$

$$V_i^{\text{ex}}(t, x) = \exp \left[-ig'\Delta x \frac{1}{2} Y_i^{\text{ex}}(t, x) \right]. \quad (\text{A3})$$

Fields Φ , U_i , V_i , and V_i^{ex} are evaluated at integer time steps $t, t + \Delta t, \dots$, with their conjugate momenta Π , E , F defined at midpoints $t + \Delta t/2, t + 3\Delta t/2, \dots$. Here $x + i$ ($x - i$) indicates displacement forward (backward) along the i -axis. The link variables $U_i(t, x)$, $V_i(t, x)$, and $V_i^{\text{ex}}(t, x)$ parallelly transport Φ from site $x + i$ to x , while their Hermitian conjugates map Φ from x to $x + i$. The covariant derivatives of $\Phi(t, x)$ are then

$$D_i\Phi = \frac{1}{\Delta x} [U_i(t, x)V_i(t, x)V_i^{\text{ex}}(t, x)\Phi(t, x + i) - \Phi(t, x)], \quad (\text{A4})$$

$$D_0\Phi = \frac{1}{\Delta t} [\Phi(t + \Delta t, x) - \Phi(t, x)]. \quad (\text{A5})$$

Plaquette operators are given by

$$U_{ij}(t, x) = U_j(t, x)U_i(t, x + j)U_j^\dagger(t, x + i)U_i^\dagger(t, x), \quad (\text{A6})$$

$$V_{ij}(t, x) = V_j(t, x)V_i(t, x + j)V_j^\dagger(t, x + i)V_i^\dagger(t, x). \quad (\text{A7})$$

$$U_{0i}(t, x) = U_i(t, x)U_i^\dagger(t + \Delta t, x), \quad (\text{A8})$$

$$V_{0i}(t, x) = V_i(t, x)V_i^\dagger(t + \Delta t, x). \quad (\text{A9})$$

V_{ij}^{ex} and V_{0j}^{ex} share the same form of V_{ij} and V_{0j} , respectively.

The conjugate momenta relate to field updates via

$$\Phi(t + \Delta t, x) = \Phi(t, x) + \Delta t \Pi(t + \Delta t/2, x), \quad (\text{A10})$$

$$V_i(t + \Delta t, x) = \frac{g'}{2} \Delta x \Delta t E_i(t + \Delta t/2, x) V_i(t, x), \quad (\text{A11})$$

$$U_i(t + \Delta t, x) = g\Delta x \Delta t F_i(t + \Delta t/2, x) U_i(t, x). \quad (\text{A12})$$

Note that $V_{\text{ex}i}(t, x)$ is static and thus lacks a conjugate momentum.

The lattice action reads

$$\begin{aligned}
S = \sum_{x,t} \Delta t \Delta x^3 \Big\{ & (D_0 \Phi)^\dagger (D_0 \Phi) - \sum_i (D_i \Phi)^\dagger (D_i \Phi) - \mathcal{V}(\Phi) + \left(\frac{2}{g \Delta t \Delta x} \right)^2 \sum_i \left(1 - \frac{1}{2} \text{Tr } U_{0i} \right) \\
& + \left(\frac{2}{g' \Delta t \Delta x} \right)^2 \sum_i (1 - \text{Re } V_{0i}) - \frac{2}{g^2 \Delta x^4} \sum_{i,j} \left(1 - \frac{1}{2} \text{Tr } U_{ij} \right) - \frac{2}{g'^2 \Delta x^4} \sum_{i,j} (1 - \text{Re } V_{ij}) \Big\} \\
& + \left(\frac{2}{g' \Delta t \Delta x} \right)^2 \sum_i (\text{Im } V_{0i}) (\text{Im } V_{\text{ex}0i}) - \frac{2}{g'^2 \Delta x^4} \sum_{i,j} (\text{Im } V_{ij}) (\text{Im } V_{\text{ex}ij}).
\end{aligned} \tag{A13}$$

Setting the functional derivative of S to zero yields the following equations of motion:

$$\begin{aligned}
\Pi(t + \Delta t/2, x) = \Pi(t - \Delta t/2, x) + \Delta t \Big\{ & \frac{1}{\Delta x^2} \sum_i [U_i(t, x) V_i(t, x) V_{\text{ex}i}(t, x) \Phi(t, x + i), \\
& - 2\Phi(t, x) + U_i^\dagger(t, x - i) V_i^\dagger(t, x - i) V_{\text{ex}i}^\dagger(t, x - i) \Phi(t, x - i)] - \frac{\partial \mathcal{V}}{\partial \Phi^\dagger} \Big\},
\end{aligned} \tag{A14}$$

$$\begin{aligned}
\text{Im}[E_k(t + \Delta t/2, x)] = \text{Im}[E_k(t - \Delta t/2, x)] + \Delta t \Big\{ & \frac{g'}{\Delta x} \text{Im}[\Phi^\dagger(t, x + k) U_k^\dagger(t, x) V_k^\dagger(t, x) V_{\text{ex}k}^\dagger(t, x) \Phi(t, x)] \\
& - \frac{2}{g' \Delta x^3} \sum_i \text{Im}[V_k(t, x) V_i(t, x + k) V_k^\dagger(t, x + i) V_i^\dagger(t, x) \\
& + V_i(t, x - i) V_k(t, x) V_i^\dagger(t, x + k - i) V_k^\dagger(t, x - i)] \Big\},
\end{aligned} \tag{A15}$$

$$\begin{aligned}
\text{Tr}[i\sigma^m F_k(t + \Delta t/2, x)] = \text{Tr}[i\sigma^m F_k(t - \Delta t/2, x)] + \Delta t \Big\{ & \frac{g}{\Delta x} \text{Re}[\Phi^\dagger(t, x + k) U_k^\dagger(t, x) V_k^\dagger(t, x) V_{\text{ex}k}^\dagger(t, x) i\sigma^m \Phi(t, x)] \\
& - \frac{1}{g \Delta x^3} \sum_i \text{Tr}[i\sigma^m U_k(t, x) U_i(t, x + k) U_k^\dagger(t, x + i) U_i^\dagger(t, x) \\
& + i\sigma^m U_k(t, x) U_i^\dagger(t, x + k - i) U_k^\dagger(t, x - i) U_i(t, x - i)] \Big\}.
\end{aligned} \tag{A16}$$

Gauss constraints derived similarly are

$$\frac{1}{\Delta x} \sum_i \text{Im}[E_i(t, x) - E_i(t, x - i)] - g' \text{Im}[\Pi^\dagger(t, x) \Phi(t, x)] = 0, \tag{A17}$$

$$\frac{1}{\Delta x} \text{Tr} \sum_i i\sigma^m [F_i(t, x) - U_i^\dagger(t, x - i) F_i(t, x - i) U_i(t, x - i)] - g \text{Re}[\Pi^\dagger(t, x) i\sigma^m \Phi(t, x)] = 0. \tag{A18}$$

APPENDIX B: INITIALIZATION OF GAUGE FIELDS

The gauge fields are initialized to zero values, resulting in link variables of 1 for U(1) and $1_{2 \times 2}$ for SU(2). To satisfy the Gauss constraint [Eq. (15)], the conjugate momentum fields of the gauge fields require initialization through specific procedures.

We first define

$$J_0^Y(t, x) = \text{Im}[\Pi^\dagger(t, x) \Phi(t, x)], \tag{B1}$$

$$J_0^a(t, x) = \text{Re}[\Pi^\dagger(t, x) i\sigma^a \Phi(t, x)]. \tag{B2}$$

Given $V_i(t = 0, x) = 1$ and $U_i(t = 0, x) = 1_{2 \times 2}$, the Gauss constraints in Eqs. (A17) and (A18) simplify to

$$\frac{1}{\Delta x} \sum_i \text{Im}[E_i(x) - E_i(x-i)] = g' J_0^Y(x), \quad (\text{B3})$$

$$\frac{1}{\Delta x} \text{Tr} \sum_i i \sigma^m [F_i(x) - F_i(x-i)] = g J_0^a(x). \quad (\text{B4})$$

The $t = 0$ specification is omitted here for conciseness. We introduce $\dot{\mathcal{Y}}_i(k)$ and $\dot{\mathcal{W}}_i^a(k)$ as the Fourier transforms of $\text{Im } E_i(x)$ and $\text{Tr}[i\sigma^m F_i(x)]$, respectively

$$\text{Im}[E_i(x)] = \frac{1}{N^3} \sum_k \dot{\mathcal{Y}}_i(k) e^{-i2\pi k(x+i/2)/N}, \quad (\text{B5})$$

$$F_i^m(x) = \text{Tr}[i\sigma^m F_i(x)] = \frac{1}{N^3} \sum_x \dot{\mathcal{W}}_i^m(k) e^{-i2\pi k(x+i/2)/N}. \quad (\text{B6})$$

These definitions utilize the connections between gauge momentum fields and link variables

$$E_i(x) = \frac{2}{g' \Delta x \Delta t} \exp \left[-ig' \Delta x \frac{1}{2} \dot{\mathcal{Y}}_i(x) \right], \quad (\text{B7})$$

$$F_i(x) = \frac{1}{g \Delta x \Delta t} \exp \left[-ig \Delta x \frac{\sigma^a}{2} \dot{\mathcal{W}}_i^a(x) \right]. \quad (\text{B8})$$

Fourier transforming both sides of Eqs. (B3) and (B4), while accounting for link fields spanning lattice sites, yields

$$\begin{aligned} & \frac{1}{\Delta x N^3} \sum_x \sum_i \sum_k [\dot{\mathcal{Y}}_i(k) e^{-i2\pi k(x+i/2)/N} - \dot{\mathcal{Y}}_i(k) e^{-i2\pi k(x-i/2)/N}] e^{i2\pi k'x/N} \\ &= \frac{1}{\Delta x N^3} \sum_k \sum_i \sum_x \dot{\mathcal{Y}}_i(k) e^{-i2\pi(k-k')x/N} [e^{-i\pi k_i/N} - e^{i\pi k_i/N}] \\ &= \frac{-2i}{\Delta x} \sum_i \dot{\mathcal{Y}}_i(k') \sin \frac{\pi k'_i}{N} \\ &= g' \sum_x J_0^Y(x) e^{i2\pi k'x/N} \\ &= g' J_0^Y(k'). \end{aligned} \quad (\text{B9})$$

Solving for $\dot{\mathcal{Y}}_i(k)$ gives

$$\dot{\mathcal{Y}}_i(k) = \sin \frac{\pi k_i}{N} \left(\sum_i \sin^2 \frac{\pi k_i}{N} \right)^{-1} \frac{i}{2} g' \Delta x J_0^Y(k). \quad (\text{B10})$$

Analogous derivation for the SU(2) sector produces

$$\dot{\mathcal{W}}_i^m(k) = \sin \frac{\pi k_i}{N} \left(\sum_i \sin^2 \frac{\pi k_i}{N} \right)^{-1} \frac{i}{2} g \Delta x J_0^m(k). \quad (\text{B11})$$

The coordinate-space conjugate momenta follow via inverse Fourier transforms using Eqs. (B5) and (B6). Complete link variables are then initialized through

$$E_i(x) = \sqrt{\left(\frac{2}{g' \Delta x \Delta t} \right)^2 - [\text{Im } E_i(x)]^2} + i \text{Im } E_i(x), \quad (\text{B12})$$

$$F_i(x) = \sqrt{\left(\frac{1}{g \Delta x \Delta t} \right)^2 - \sum_m \left(\frac{F_i^m(x)}{2} \right)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \frac{\sigma^m}{2} F_i^m(x). \quad (\text{B13})$$

APPENDIX C: HYPERMAGNETIC FIELD ON LATTICE

The infinite correlation length hypermagnetic field configuration with helicity takes the form $Y^{\text{ex}\mu} = (0, 0, xB_Y^{\text{ex}}, h_{\text{factor}}LB_Y^{\text{ex}})$, satisfying

$$\begin{aligned} \nabla \times Y^{\text{ex}} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & xB_Y^{\text{ex}} & h_{\text{factor}}LB_Y^{\text{ex}} \end{vmatrix} \\ &= \partial_x(xB_Y^{\text{ex}})\hat{z} = B_Y^{\text{ex}}\hat{z}, \end{aligned} \quad (\text{C1})$$

where B_Y^{ex} denotes constant field strength, $h_{\text{factor}} \in [-1, 1]$ adjusts helicity, and $L = N\Delta x$ is the lattice size.

Discretization replaces continuum coordinates (x, y, z) with lattice indices (n_1, n_2, n_3)

$$\begin{aligned} \partial_x(xB_Y^{\text{ex}})\hat{z} &= \frac{1}{\Delta x} [(n_1 + 1)\Delta x - n_1\Delta x] B_Y^{\text{ex}}\hat{z} \\ &= B_Y^{\text{ex}}\hat{z}, \quad n_1 \neq N-1. \end{aligned} \quad (\text{C2})$$

Boundary behavior differs due to periodicity. At $n_1 = N-1$, $Y^{\text{ex}}(t, N, n_2, n_3) = Y^{\text{ex}}(t, 0, y, z) = 0$ yields

$$\begin{aligned}\partial_x(xB_Y^{\text{ex}})\hat{z}|_{x=(N-1)\Delta x} &= \frac{1}{\Delta x}[0 - (N-1)\Delta x]B_Y^{\text{ex}}\hat{z} \\ &= -(N-1)B_Y^{\text{ex}}\hat{z} \neq B_Y^{\text{ex}}\hat{z}.\end{aligned}\quad (\text{C3})$$

This discrepancy is resolved by setting $Y_1^{\text{ex}}(t, N-1, n_2, n_3) = -Nn_2\Delta x B_Y^{\text{ex}}$, resulting in

$$\begin{aligned}\nabla \times \mathbf{Y}^{\text{ex}}|_{x=(N-1)\Delta x} &= (\partial_x Y_2^{\text{ex}} - \partial_y Y_1^{\text{ex}})|_{x=(N-1)\Delta x}\hat{z} \\ &= \left\{ -(N-1)B_Y^{\text{ex}} + \frac{N}{\Delta x}[(n_2+1)\Delta x - n_2\Delta x]B_Y^{\text{ex}} \right\}\hat{z} \\ &= B_Y^{\text{ex}}\hat{z}.\end{aligned}\quad (\text{C4})$$

ensuring boundary-internal consistency. The complete external U(1) field specification is

$$Y_1^{\text{ex}}(t, N-1, n_2, n_3) = -Nn_2\Delta x B_Y^{\text{ex}}, \quad (\text{C5})$$

$$Y_1^{\text{ex}}(t, n_1 \neq N-1, n_2, n_3) = 0, \quad (\text{C6})$$

$$Y_2^{\text{ex}}(t, n_1, n_2, n_3) = n_1\Delta x B_Y^{\text{ex}}, \quad (\text{C7})$$

$$Y_3^{\text{ex}}(t, n_1, n_2, n_3) = h_{\text{factor}}LB_Y^{\text{ex}}, \quad (\text{C8})$$

$$Y_0^{\text{ex}}(t, n_1, n_2, n_3) = 0. \quad (\text{C9})$$

The quantization condition [91] $B_Y^{\text{ex}}(N\Delta x)^2 = \frac{2}{g}2\pi N_B$, $N_B \in \mathbb{Z}$ is applied.

Finite correlation length hyper-MFs utilize the spectral representation [90]

$$\tilde{B}_{Yi}^{\text{ex}}(\mathbf{k}) = B_{\text{ini}}\Theta(k - k_{\text{UV}})(\delta_{ij} - \hat{k}_i\hat{k}_j - i\sigma_M\epsilon_{ijl}\hat{k}_l)g_j(\mathbf{k})k^n. \quad (\text{C10})$$

The corresponding vector potential in Fourier space is

$$\tilde{Y}_l^{\text{ex}}(\mathbf{k}) = iB_{\text{ini}}(\epsilon_{lmn}\hat{k}_m - i\sigma_M\delta_{ln})g_n(\mathbf{k})k^{n-1}. \quad (\text{C11})$$

Consistency with $\mathbf{B}^{\text{ex}}(\mathbf{x}) = \nabla \times \mathbf{Y}^{\text{ex}}(\mathbf{x})$ follows from

$$\begin{aligned}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{\mathbf{B}}^{\text{ex}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} &= \nabla \times \int \frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{\mathbf{Y}}^{\text{ex}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}i\mathbf{k} \times \tilde{\mathbf{Y}}^{\text{ex}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}.\end{aligned}\quad (\text{C12})$$

The relation $\tilde{B}_i^{\text{ex}}(\mathbf{k}) = i\epsilon_{ijl}k_j\tilde{Y}_l^{\text{ex}}(\mathbf{k})$ combined with (C11) yields

$$\begin{aligned}i\epsilon_{ijl}k_j\tilde{Y}_l^{\text{ex}}(\mathbf{k}) &= -\epsilon_{ijl}k_jB_{\text{ini}}(\epsilon_{lmn}\hat{k}_m - i\sigma_M\delta_{ln})g_n(\mathbf{k})k^{n-1} \\ &= -B_{\text{ini}}[(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\hat{k}_m\hat{k}_j - i\sigma_M\epsilon_{ijn}\hat{k}_j]g_n(\mathbf{k})k^n \\ &= B_{\text{ini}}(-\hat{k}_i\hat{k}_n + \delta_{in} + i\sigma_M\epsilon_{ijn}\hat{k}_j)g_n(\mathbf{k})k^n \\ &= B_{\text{ini}}(\delta_{ij} - \hat{k}_i\hat{k}_j - i\sigma_M\epsilon_{ijl}\hat{k}_l)g_j(\mathbf{k})k^n,\end{aligned}\quad (\text{C13})$$

matching (C10).

Implementation of (C10) proceeds as follows:

- (i) Generate coordinate-space Gaussian random field $\mathbf{g}(\mathbf{x})$ with δ -correlation.
- (ii) Fourier transform to obtain $\mathbf{g}(\mathbf{k}) = \int d^3\mathbf{x}\mathbf{g}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$.
- (iii) Compute $\tilde{\mathbf{Y}}^{\text{ex}}(\mathbf{k})$ via (C11).
- (iv) Apply inverse Fourier transform: $\mathbf{Y}^{\text{ex}}(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{\mathbf{Y}}^{\text{ex}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$.
- (v) Construct link variables V_l^{ex} using (A3).

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