

Diversity of fuzzy dark matter solitons

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According to the Schrödinger-Poisson equations, fuzzy dark matter (FDM) can form a stable equilibrium configuration, the so-called FDM soliton. In principle, given the FDM particle mass, the profile of the FDM soliton is fixed. In practice, however, there is a great diversity of structures in the Universe. Possible causes of such diversity can lie in such sources as the gravitoelectric field due to a central supermassive black hole, the gravitomagnetic field due to the system angular momentum, and an ellipsoidal baryon background. We find that the effects of the gravitomagnetic field due to the soliton's self-angular momentum are very weak while those of the other sources are considerable.

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I. INTRODUCTION

The existence of dark matter (DM) is strongly suggested from rotation curves of galaxies [1], evolution of large scale structure [2], and gravitational lensing observations [3]. Depending on the free streaming length of DM, it can be divided into cold, warm, and hot categories. Although the standard Lambda cold DM (ΛCDM) cosmological model is very successful and the latest cosmic microwave background (CMB) observations [4] suggest that CDM accounts for about 26% of today's energy density in the Universe, we still have no certainty on the existence or nature of CDM, whether sourced by particles and other objects or explained away by gravity modifications [5]. Especially, as one of the most promising candidates for the CDM particle, the weakly interacting massive particles (WIMPs) grounded on supersymmetric theories of particle physics still have not been detected [6–8]. Moreover, primordial black holes (BHs) can also serve as CDM [9], although they still have not been identified. These

null results coupled with the failures of CDM particles on subgalactic scales [10,11] imply that DM may not entirely fall under the CDM paradigm.

A promising alternative to CDM lies in an ultralight scalar field with spin-0, extraordinarily light mass ($\sim 10^{-22}$ eV/ c^2) and de Broglie wavelength comparable to a few kiloparsecs, coined fuzzy DM (FDM) [12]. Due to its large occupation numbers in galactic halos, FDM behaves as a classical field obeying the coupled Schrödinger-Poisson (SP) system of equations,

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} &= \left(-\frac{\hbar^2}{2m} \nabla^2 + m\Phi \right) \Psi, \\ \nabla^2 \Phi &= 4\pi G |\Psi|^2, \end{aligned} \quad (1)$$

where m is the mass of the FDM particle, which is described by the wave function Ψ , and the gravitational potential Φ is sourced by the FDM density $\rho = |\Psi|^2$. According to Eq. (1), FDM can condensate into a many particles ground state coined as a soliton. The SP system can also describe the evolution of FDM in an expanding universe [13–16]. Note that the SP system is just the weak field limit of the general relativistic Einstein-Klein-Gordon (EKG) system [17–19] and that the SP system should be replaced by the Gross-Pitaevskii-Poisson (GPP) system when FDM particles self-interactions exist [20,21].

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In this paper, for simplicity, we confine ourselves to the SP system in a nonexpanding universe. The evolution of this system is usually simulated numerically, including the formation, perturbation, interference/collision, and tidal disruption/deformation of FDM solitons [22–25]. Focusing on stationary solutions, one can turn to the shooting method to compute the eigenvalues of equilibrium configurations [22,26]. In this work we restrict to different situations where numerical treatment can be avoided cleverly, as illustrated below.

Since FDM solitons only depend on the mass of FDM, the SP system obeys scaling symmetry. As a result, the Universe should *a priori* present very monotonous and similar structures. Yet, there is a great diversity of structures in the Universe. Therefore, although we do not here consider the modifications of gravity or various possible FDM interactions, the real astrophysics systems must follow variants of the exact SP system. In this paper, we will consider several variants, induced by extra terms, such as the gravitoelectric field, the gravitomagnetic field, both sourced by the Weyl tensor [27], and the baryon density profile. Our motivations are as follows. First, since, as pointed out in [22,28], the SP system is the weak field limit of the EKG system, there must exist a variant system, between these two systems, for which gravitoelectromagnetism is a good approximation. While the gravitoelectric field or the gravitational potential of a supermassive BH has been discussed [24–26], the gravitomagnetic field due to the system's angular momentum has not yet been considered. Second, when the nonspherical baryon profile is located in galaxies, it results in nonspherical FDM solitons. The impact of a baryon profile with cylindrical symmetry has been quantified in [29]. In fact, the baryon profile in the Milky Way may be ellipsoidal [30,31].

This paper is organized as follows. In Sec. II, we briefly review the effects of the gravitoelectric field due to a supermassive BH on the FDM solitons. In Sec. III, we study the effects of the gravitomagnetic field due to the rotation velocity of FDM, the supermassive BH spin, and the orbital motion of the supermassive BH binary on the FDM solitons. In Sec. IV, we calculate the FDM solitons in a given ellipsoidal baryon profile. Finally, a brief summary and discussions are included in Sec. V.

II. EFFECTS OF THE GRAVITOELECTRIC FIELD

The effects of the gravitoelectric field include the tidal disruption/deformation of FDM solitons by a nearby supermassive BH [24,25] and the modified formation of FDM solitons by a central supermassive BH [26]. While the former nonspherical impacts are more complicated and have to be studied by numerical simulations, the latter spherical ones are simple and can be studied by the shooting method. In this section, we briefly review the latter cases.

First, the gravitoelectric field $\mathbf{E}_g$ derived from the gravitational potential Φ_e , sourced by a central supermassive BH, contributes to a variant of the exact SP system as

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + m(\Phi + \Phi_e) \right] \Psi, \quad (2)$$

$$\nabla^2 \Phi = 4\pi G |\Psi|^2,$$

where the supermassive BH with mass M_{bh} has a point mass potential

$$\Phi_e = -\frac{GM_{\text{bh}}}{r}. \quad (3)$$

When the system in question features spherical symmetry, the ansatz of $\Psi(r, t) = e^{-i\gamma t/\hbar} \psi(r)$, where γ can be interpreted as the ansatz energy eigenvalue, leads to the FDM soliton density $\rho(r) = |\Psi|^2 = \psi^2(r)$ and the FDM soliton mass $M = \int_0^\infty 4\pi r^2 \rho(r) dr$. After defining the system's dimensionless variables as

$$\tilde{\psi} \equiv \frac{\hbar \sqrt{4\pi G}}{mc^2} \psi, \quad (4)$$

$$\tilde{r} \equiv \frac{mc}{\hbar} r, \quad (5)$$

$$\tilde{\Phi} \equiv \frac{1}{c^2} \Phi, \quad (6)$$

$$\tilde{\gamma} \equiv \frac{1}{mc^2} \gamma, \quad (7)$$

$$\tilde{M} \equiv \frac{GMm}{\hbar c}, \quad (8)$$

$$\tilde{M}_{\text{bh}} \equiv \frac{GM_{\text{bh}}m}{\hbar c}, \quad (9)$$

the dimensionless version of Eq. (2), with the spherical ansatz, follows

$$\frac{\partial^2(\tilde{r} \tilde{\psi})}{\partial \tilde{r}^2} = 2\tilde{r} \left(\tilde{\Phi} - \tilde{\gamma} - \frac{\tilde{M}_{\text{bh}}}{\tilde{r}} \right) \tilde{\psi},$$

$$\frac{\partial^2(\tilde{r} \tilde{\Phi})}{\partial \tilde{r}^2} = \tilde{r} \tilde{\psi}^2. \quad (10)$$

Fulfilling the arbitrary normalization $\tilde{\psi}(\tilde{r}=0) = 1$ and $\tilde{\Phi}(\tilde{r}=0) = 0$,¹ the boundary conditions $\tilde{\psi}(\tilde{r}=\infty) = 0$, $\frac{\partial \tilde{\psi}}{\partial \tilde{r}}|_{\tilde{r}=0} = 0$, and $\frac{\partial \tilde{\Phi}}{\partial \tilde{r}}|_{\tilde{r}=0} = 0$, choosing the supermassive BH mass $\tilde{M}_{\text{bh}} = \{0.0, 0.5, 1.0, 1.5, 2.0, 2, 5\}$ as the input parameter and adjusting the quantized eigenvalue $\tilde{\gamma}$, we can calculate the equilibrium configurations from Eq. (10) by

¹This normalization condition for the soliton self-potential, Φ , is different from the conventional choice of normalization as for the BH potential, Φ_e , that $\Phi_e(r=\infty) = 0$. Although it produces a nonvanishing asymptotic value at infinity, $\tilde{\Phi}(\tilde{r}=\infty)$, this difference does not affect the solutions with the normalization $\tilde{\psi}(\tilde{r}=0) = 1$.

TABLE I. Input values of the dimensionless supermassive BH mass $\tilde{M}_{\text{bh}}$ and the corresponding eigenvalue $\tilde{\gamma}$ and soliton dimensionless mass $\tilde{M}$ of the ground state solutions.

$\tilde{M}_{\text{bh}}$	$\tilde{\gamma}$	$\tilde{M}$
0.0	0.6495	2.0622
0.5	0.1164	0.7916
1.0	-0.5113	0.2169
1.5	-1.0839	0.0718
2.0	-1.9767	0.0309
2.5	-3.1101	0.0159

the shooting method. Given the input parameter $\tilde{M}_{\text{bh}}$, we obtain the only stable solution from the smallest $\tilde{\gamma}$, which is the ground state. In Table I, we list the input values of $\tilde{M}_{\text{bh}}$ and the corresponding eigenvalue $\tilde{\gamma}$ and soliton mass $\tilde{M}$ of the ground state solutions. In Fig. 1, the corresponding soliton profiles are plotted. We find that the larger $\tilde{M}_{\text{bh}}$ results in a denser and more compact soliton, leading to a smaller dimensionless soliton mass $\tilde{M}$, due to the fixed normalization $\tilde{\psi}(\tilde{r} = 0) = 1$. The solutions under this normalization can be rescaled by the following scaling symmetry:

$$\tilde{\psi} \longrightarrow \lambda \tilde{\psi}, \quad (11)$$

$$\tilde{r} \longrightarrow \lambda^{-1/2} \tilde{r}, \quad (12)$$

$$\tilde{\Phi} \longrightarrow \lambda \tilde{\Phi}, \quad (13)$$

$$\tilde{\gamma} \longrightarrow \lambda \tilde{\gamma}, \quad (14)$$

$$\tilde{M} \longrightarrow \lambda^{1/2} \tilde{M}, \quad (15)$$

$$\tilde{M}_{\text{bh}} \longrightarrow \lambda^{1/2} \tilde{M}_{\text{bh}}. \quad (16)$$

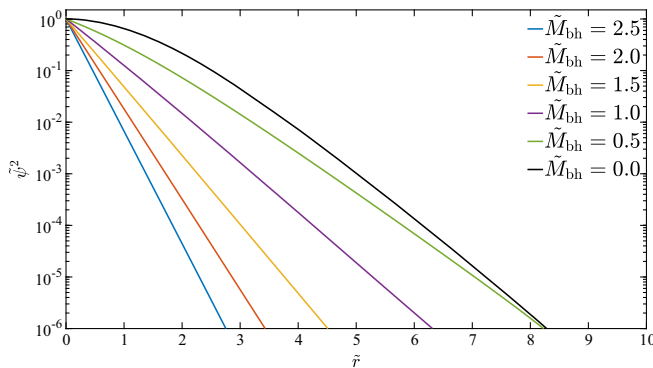


FIG. 1. Density profiles of the ground state solutions with different dimensionless supermassive BH mass $\tilde{M}_{\text{bh}}$.

III. EFFECTS OF THE GRAVITOMAGNETIC FIELD

The SP system corresponds to the weak field limit of the general relativistic EKG system [22]. A variant system can therefore account for some general relativistic corrections to the SP system, on the spectrum between the full EKG system and its weak field approximation, such as the gravitomagnetic field $\mathbf{B}_g$. Such a variant of the exact SP system reads

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + m(\Phi + \Phi_e + \Phi_m) \right] \Psi, \quad (17)$$

$$\nabla^2 \Phi = 4\pi G |\Psi|^2,$$

where the gravitational potential Φ_m is related to the gravitomagnetic field $\mathbf{B}_g$ and the angular momentum² of an FDM particle $\hat{\mathbf{L}}$. In the case of a nonuniform gravitomagnetic field, we should use the following expression, in analogy to the effect of an external magnetic field in nonrelativistic quantum mechanics:

$$\begin{aligned} \Phi_m &= \Phi_{m1} + \Phi_{m2} \\ &= \frac{i\hbar}{m} \mathbf{A}_g \cdot \nabla + \frac{1}{2} \mathbf{A}_g \cdot \mathbf{A}_g, \end{aligned} \quad (18)$$

where $\mathbf{A}_g$ is the gravitomagnetic vector potential, related to the gravitomagnetic field as $\mathbf{B}_g = \nabla \times \mathbf{A}_g$. For the vector potential component in the azimuthal angular direction ∂_ϕ , it takes the form

$$\Phi_m = -\frac{1}{mr} A_g^\phi \hat{L}_z + \frac{1}{2} A_g^\phi A_g^\phi. \quad (19)$$

Generally speaking, the linear term Φ_{m1} is much larger than the quadratic one Φ_{m2} . In this section, we will talk about some possible sources of $\mathbf{A}_g$ and their effects on the FDM solitons.

A. Gravitomagnetic field due to rotation velocity of fuzzy dark matter

The gravitational potential inside an FDM soliton sourced by its own density profile follows the spherically symmetric potential

$$\Phi = -\frac{4\pi G \int_0^r \rho(R) R^2 dR}{r} - 4\pi G \int_r^\infty \frac{\rho(R) R^2 dR}{R}, \quad (20)$$

²Note that $\Psi = \sqrt{N} \Psi_1$ [18], where Ψ_1 is the dimensional wave function of one FDM particle and N is the fixed number of particles in the soliton. As N is constant, we have $i\hbar \frac{\partial \Psi}{\partial t} = i\hbar \frac{\partial \sqrt{N} \Psi_1}{\partial t} = \sqrt{N} i\hbar \frac{\partial \Psi_1}{\partial t}$. Therefore, the Schrödinger Eq. (17) can reduce to the description of a single particle of the soliton, and the $\hat{\mathbf{L}}$ operator brings out the angular momentum for a single particle.

where we temporarily utilize the normalization $\Phi(r = \infty) = 0$ in Sec. III, as we did not solve the SP system but rather estimate and compare the effect of different potentials. If this FDM soliton is rotating around the z -axis, there should be a nonzero A_g^ϕ in the x - y plane according to gravitoelectromagnetism. To calculate the total A_g^ϕ , in the x - y plane, inside the FDM soliton, we first consider an FDM density shell with radius R rotating around the z -axis with a typical linear velocity, in the x - y plane, relative to the FDM soliton's center $v \sim 10^{-3}c$. The infinitesimal dA_g^ϕ in the x - y plane produced by this FDM spherical shell verifies

$$dA_g^\phi = \begin{cases} -\frac{4\pi G}{3c^2} \frac{v\rho(R)R^3 dR}{r^2} & (r > R), \\ -\frac{4\pi G}{3c^2} v\rho(R)r dR & (r < R). \end{cases} \quad (21)$$

Then the total A_g^ϕ produced by the whole FDM soliton sums up to

$$\begin{aligned} A_g^\phi &= -\frac{4\pi Gv}{3c^2} \left(\frac{1}{r^2} \int_0^r \rho(R)R^2 dR + r \int_r^\infty \rho(R) dR \right) \\ &= \frac{4\pi Gv}{3c^2} f(r), \end{aligned} \quad (22)$$

where the density ρ is rescaled from the dimensionless density of the black curve in Fig. 1. The scaling factor λ is determined from the comparison between the integration of the black curve and a given soliton mass M . The soliton mass M can be predicted from the halo mass M_{halo} according to the soliton-halo mass relation, whether from the version of Ref. [14]

$$M \approx 1.25 \times 10^9 \left(\frac{M_{\text{halo}}}{10^{12} M_\odot} \right)^{1/3} \left(\frac{m}{10^{-22} \text{ eV}/c^2} \right)^{-1} M_\odot, \quad (23)$$

or following the version of Ref. [32]

$$\begin{aligned} M &\approx \beta \left(\frac{m}{8 \times 10^{-23} \text{ eV}/c^2} \right)^{-3/2} \\ &\quad + \left(\frac{M_{\text{halo}}}{\gamma} \right)^\alpha \left(\frac{m}{8 \times 10^{-23} \text{ eV}/c^2} \right)^{3(\alpha-1)/2} M_\odot, \end{aligned} \quad (24)$$

where $\beta = 8.00 \times 10^6 M_\odot$, $\gamma = 10^{-5.73} M_\odot$, and $\alpha = 0.515$. The following discussion takes the Milky Way value with $M_{\text{halo}} = 1 \times 10^{12} M_\odot$ [33] as an example.

The angular momentum of each particle of the soliton being identical, and in the z direction, we compute its magnitude by dividing the FDM shells angular momentum, integrated over the soliton, by the number of FDM particles in the soliton $N = M/m$, according to

$$L_z = \frac{1}{N} \int_0^\infty \frac{8\pi}{3} \rho(R) R^3 v dR. \quad (25)$$

According to Eq. (19), the linear gravitational potential in the x - y plane yields

$$\Phi_{m1} = -\frac{1}{mr} A_g^\phi L_z, \quad (26)$$

and the quadratic gravitational potential in the x - y plane can be written as

$$\Phi_{m2} = \frac{1}{2} A_g^{\phi 2} = \frac{8\pi^2 \times 10^{-6} G^2}{9 c^2} f(r)^2. \quad (27)$$

The gravitational potentials Φ [Eq. (20)], Φ_{m1} [Eq. (26)], and Φ_{m2} [Eq. (27)] include some nontrivial integrals, which obscure their direct comparison. For clarity, we plot the gravitational potentials $|\Phi|$, $|\Phi_{m1}|$, and $|\Phi_{m2}|$ in Fig. 2. We find Φ to dominate over Φ_{m1} and Φ_{m2} in the x - y plane and thus that the latter two can be neglected.

The above introduction of angular momentum was based on a background soliton obeying Eq. (1), without rotation, that yields the density of the black curve in Fig. 1. The effect of rotation is modeled in Eq. (19) and has been shown to be negligible compared with the soliton's self-gravitating potential.

B. Gravitomagnetic field due to a supermassive black hole spin

In this section, we consider the effect not only of the gravitoelectric field but also of the angular momentum from the central supermassive BH on a soliton rotating in a similar way as Sec. III A. In this case we express potentials in terms of a dimensionless distance in units of the Schwarzschild radius $R_S = \frac{2GM_{\text{bh}}}{c^2}$. To do so we define it for an arbitrary radial distance r as $n = r/R_S$. The gravitational potential sourced by a central supermassive BH follows:

$$\Phi_e = -\frac{GM_{\text{bh}}}{r} = -\frac{c^2}{2n}. \quad (28)$$

If this supermassive BH is spinning, the magnitude of its angular momentum is $L_{\text{bh}} = \chi \frac{GM_{\text{bh}}^2}{c}$, where χ is a dimensionless spin parameter. According to gravitoelectromagnetism, the gravitomagnetic vector potential field $\mathbf{A}_g$ depends on the angular momentum $\mathbf{L}_{\text{bh}}$, which points along the z direction, as

$$\mathbf{A}_g = -\frac{G}{2c^2 r^3} \mathbf{L}_{\text{bh}} \times \mathbf{r}. \quad (29)$$

The corresponding magnitude of the gravitomagnetic vector potential field A_g^ϕ due to this spinning massive BH in the x - y plane reads

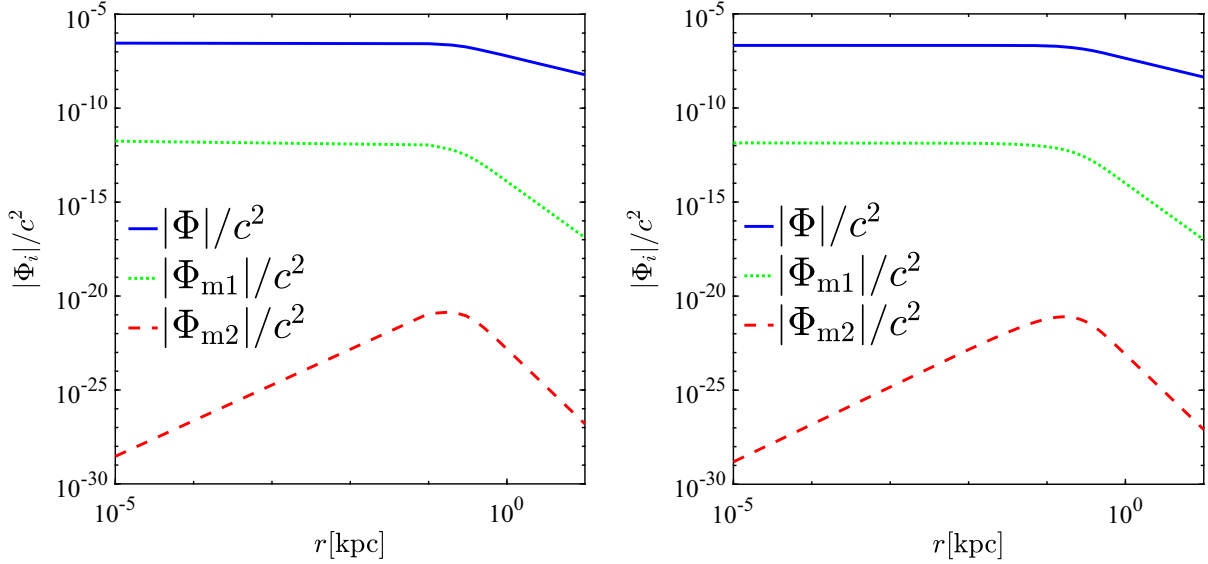


FIG. 2. Gravitational potentials $|\Phi|$ [Eq. (20)], $|\Phi_{m1}|$ [Eq. (26)], and $|\Phi_{m2}|$ [Eq. (27)] in the x - y plane, where the halo mass is similar to the Milky Way's, $M_{\text{halo}} \approx 1 \times 10^{12} M_{\odot}$ [33], and the FDM particle has the typical mass $m = 10^{-22} \text{ eV}/c^2$. In the left panel, we used the soliton-halo mass relation from Eq. (23), while the right panel used Eq. (24).

$$A_g^{\phi} = -\frac{G}{2c^2} \frac{L_{\text{bh}}}{r^2} = -\frac{\chi}{8n^2} c. \quad (30)$$

For an FDM particle, we can obtain its z direction angular momentum by Eq. (25), where $v \sim 10^{-3}c$ and ρ is rescaled from the green curve in Fig. 1. λ is determined from the comparison between the integration of the green curve and M [Eqs. (23) or (24)]. Meanwhile, M_{bh} is also rescaled from $\tilde{M}_{\text{bh}} = 0.5$ with the same λ . According to Eq. (19), the linear gravitational potential in the x - y plane follows:

$$\Phi_{m1} = -\frac{1}{mr} A_g^{\phi} L_z = \frac{1}{m} \frac{\chi}{8n^3} \frac{c}{R_S} L_z, \quad (31)$$

while the corresponding quadratic gravitational potential writes

$$\Phi_{m2} = \frac{1}{2} A_g^{\phi 2} = \frac{\chi^2}{128n^4} c^2. \quad (32)$$

Comparing the gravitational potentials from Eqs. (28), (31), and (32), we find Φ_e and Φ_{m1} to dominate over Φ_{m2} in the x - y plane and thus that the latter can be neglected for $n \geq 1$ (outside the supermassive BH horizon). For clarity, we plot the gravitational potentials $|\Phi_e|$, $|\Phi_{m1}|$, and $|\Phi_{m2}|$ in the x - y plane as a function of n in Fig. 3, where $n \geq 1$ marks that the radial distance is larger than the Schwarzschild radius of the supermassive BH. We found that Φ_{m1} is comparable to Φ_e for radii within a decade of the Schwarzschild radius. According to Sec. II, this will further modify the soliton profile obtained from Eq. (2). Then, following Eq. (25), the angular momentum of the FDM particle should be affected,

which in turn should modify Φ_{m1} through Eq. (31). This loop reveals that our method cannot yield safely an evaluation of the potential for the gravitomagnetic field induced by the spin of a supermassive BH. To do so, the spherical symmetric approach of Φ [Eq. (20)] is insufficient to account for the cylindrical symmetric gravitomagnetic potential Φ_{m1} . Solving the cylindrical symmetric system [Eq. (17)] lies beyond the scope of the present paper.

C. Gravitomagnetic field due to the orbital motion of a supermassive black hole binary

In this section, we consider the effect not only of the gravitoelectric field but also of the angular momentum from a supermassive BH binary on a soliton rotating in a similar way as Sec. III A. For simplicity, we consider an equal mass supermassive BH binary in a circular orbit in the x - y plane. These two supermassive BHs share the same mass M_{bh} and have a separation distance $S = sR_S$ between each other, where s is a dimensionless parameter. The magnitude of the angular momentum for such a binary is

$$L_{\text{bbh}} = I_{\text{bbh}} \Omega_{\text{bbh}} = \sqrt{s} \frac{GM_{\text{bh}}^2}{c}, \quad (33)$$

where the moment of inertia of this binary reduces to $I_{\text{bbh}} = \frac{1}{2} M_{\text{bh}} S^2$ and its angular frequency follows $\Omega_{\text{bbh}} = \frac{1}{\sqrt{s}} \frac{c}{S}$. According to Eq. (29), the magnitude of the gravitomagnetic field due to this binary in the x - y plane yields

$$A_g^{\phi} = -\frac{G}{2c^2} \frac{L_{\text{bbh}}}{r^2} = -\frac{\sqrt{s}}{8n^2} c. \quad (34)$$

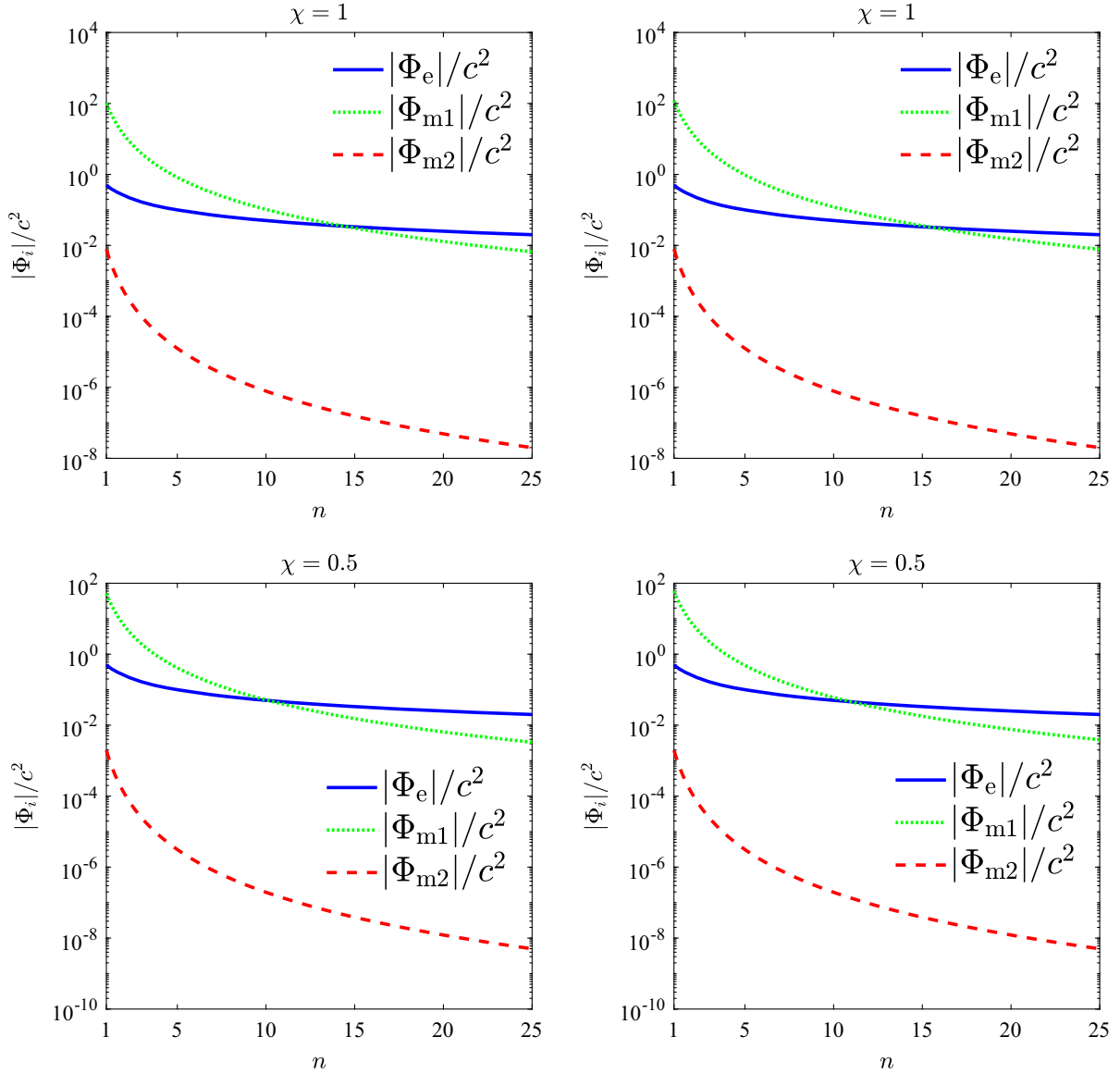


FIG. 3. Gravitational potentials $|\Phi_e|$, $|\Phi_{m1}|$, and $|\Phi_{m2}|$ in the x - y plane as a function of n from Eqs. (28), (31), and (32) respectively, where $n \geq 1$ marks that the radial distance is larger than the Schwarzschild radius of the supermassive BH. In the top subplots we set $\chi = 1$, while in the bottom subplots, $\chi = 0.5$. Concomitantly, the left panels used the soliton-halo mass relation from Eq. (23), while the right panels used Eq. (24).

In addition, for an FDM particle, we can obtain its z direction angular momentum by Eq. (25), where $v \sim 10^{-3}c$ and ρ is rescaled from the green curve in Fig. 1. λ is determined from the comparison between the integration of the green curve and M [Eqs. (23) or (24)]. Meanwhile, $2M_{\text{bh}}$ is also rescaled from $\tilde{M}_{\text{bh}} = 0.5$ with the same λ . According to Eq. (19), the linear gravitational potential in the x - y plane follows:

$$\Phi_{m1} = -\frac{1}{mr}A_g^\phi L_z = \frac{1}{m} \frac{\sqrt{s}}{8n^3} \frac{c}{R_S} L_z, \quad (35)$$

while the corresponding quadratic gravitational potential writes

$$\Phi_{m2} = \frac{1}{2}A_g^{\phi^2} = \frac{s}{128n^4}c^2. \quad (36)$$

We plot the gravitational potentials $|\Phi_e| = \frac{c^2}{n}$, $|\Phi_{m1}|$ [Eq. (35)], and $|\Phi_{m2}|$ [Eq. (36)] in the x - y plane as a function of n in Fig. 4, where $n \geq 1 + \frac{s}{2}$ marks that the radial distance is larger than the Schwarzschild radius of each supermassive BH plus half the binary separation, $R_S + \frac{s}{2}$. We found that Φ_{m1} is comparable to Φ_e for radii within a decade of the Schwarzschild radius. Similar to Sec. III B, this will further

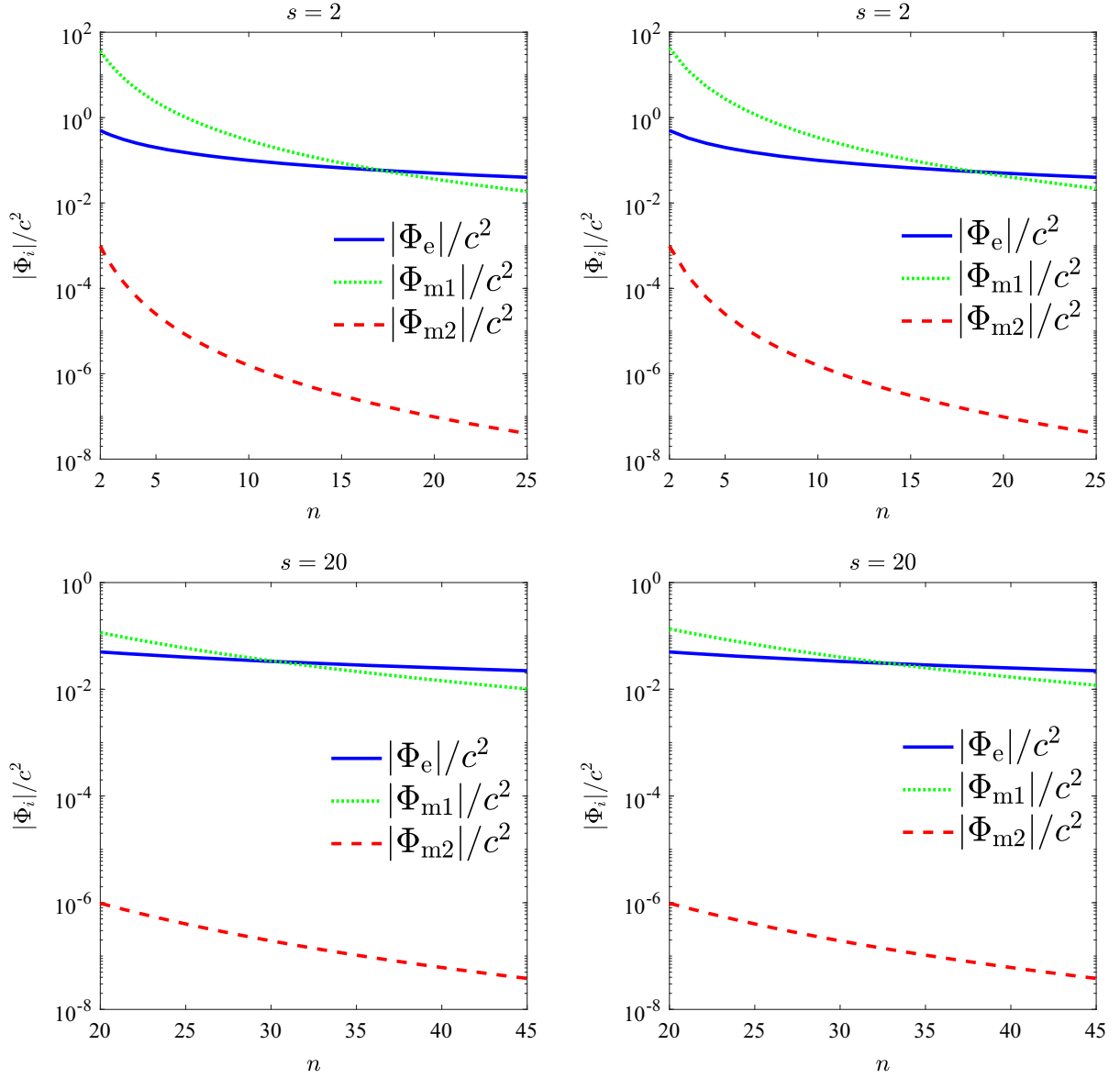


FIG. 4. Gravitational potentials $|\Phi_e| = \frac{c^2}{n} |\Phi_{m1}|$ [Eq. (35)], and $|\Phi_{m2}|$ [Eq. (36)] in the x - y plane as a function of n , where $n \geq 1 + \frac{s}{2}$ marks that the radial distance is larger than the Schwarzschild radius of each supermassive BH plus half the binary separation, $R_S + \frac{s}{2}$. In the top subplots we set $s = 2$ while in the bottom subplots, $s = 20$. Concomitantly, the left panels used the soliton-halo mass relation from Eq. (23), while the right panels used Eq. (24).

modify the soliton profile obtained from Eq. (2). Then, following Eq. (25), the angular momentum of the FDM particle should be affected, which in turn should modify Φ_{m1} through Eq. (35). Therefore, the spherical symmetric approach of Φ [Eq. (20)] is also insufficient to account for the cylindrical symmetric gravitomagnetic potential Φ_{m1} .

IV. EFFECTS OF THE ELLIPSOIDAL BARYON PROFILE

We now consider a background baryon profile in this section. We use the following variant of the exact SP system to describe the soliton with background baryons

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + m\Phi \right) \Psi, \quad \nabla^2 \Phi = 4\pi G(|\Psi|^2 + \rho_{bg}), \quad (37)$$

whose dimensionless version is

$$\frac{\partial^2(\tilde{r}\tilde{\Psi})}{\partial \tilde{r}^2} = 2\tilde{r}(\tilde{\Phi} - \tilde{\gamma})\tilde{\Psi}, \quad \frac{\partial^2(\tilde{r}\tilde{\Phi})}{\partial \tilde{r}^2} = \tilde{r}(\tilde{\Psi}^2 + \tilde{\rho}_{bg}), \quad (38)$$

where ρ_{bg} is the density of the background baryons and its dimensionless counterpart is

$$\tilde{\rho}_{\text{bg}} \equiv \left(\frac{\hbar \sqrt{4\pi G}}{mc^2} \right)^2 \rho_{\text{bg}}. \quad (39)$$

This variant SP system [Eq. (38)] also obeys the scaling symmetry when

$$\tilde{\rho}_{\text{bg}} \longrightarrow \lambda^2 \tilde{\rho}_{\text{bg}}. \quad (40)$$

More precisely, for the baryon component, we consider an ellipsoidal baryon profile fitted on the Milky Way's [30,31] three baryonic density components

$$\rho_{\text{bg}}(x, y, z) = \rho_{\text{bulge}}(x, y, z) + \rho_{\text{disk}}(x, y, z) + \rho_{\text{gas}}(x, y, z), \quad (41)$$

where ρ_{bulge} is a triaxial and bar-shaped bulge of stars at the inner few kiloparsecs of the Milky Way [34,35], ρ_{disk} represents the galactic disk density profile, and ρ_{gas} gives the diffuse gas component of the Milky Way, which takes the form of molecular, atomic, and ionized hydrogen and heavier elements. The bulge can be fitted by

TABLE II. Eigenvalue $\tilde{\gamma}$, ground state solution soliton mass $\tilde{M}$, and final values λ_0 for a set of FDM masses m , in the case of the background baryon profile $\tilde{\rho}_{\text{bg}}^x$ and using the soliton-halo mass relation Eq. (23).

m	$\tilde{\gamma}$	$\tilde{M}$	λ_0
$1 \times 10^{-22} \text{ eV}/c^2$	0.7600	1.4807	4.3088×10^{-7}
$2 \times 10^{-22} \text{ eV}/c^2$	0.7037	1.7103	3.2300×10^{-7}
$5 \times 10^{-22} \text{ eV}/c^2$	0.6644	1.9471	2.4920×10^{-7}

TABLE III. Eigenvalue $\tilde{\gamma}$, ground state solution soliton mass $\tilde{M}$, and final values λ_0 for a set of FDM masses m , in the case of the background baryon profile $\tilde{\rho}_{\text{bg}}^y$ and using the soliton-halo mass relation Eq. (23).

m	$\tilde{\gamma}$	$\tilde{M}$	λ_0
$1 \times 10^{-22} \text{ eV}/c^2$	0.7521	1.6009	3.6863×10^{-7}
$2 \times 10^{-22} \text{ eV}/c^2$	0.7022	1.7670	3.0256×10^{-7}
$5 \times 10^{-22} \text{ eV}/c^2$	0.6642	1.9483	2.4889×10^{-7}

TABLE IV. Eigenvalue $\tilde{\gamma}$, ground state solution soliton mass $\tilde{M}$, and final values λ_0 for a set of FDM masses m , in the case of the background baryon profile $\tilde{\rho}_{\text{bg}}^z$ and using the soliton-halo mass relation Eq. (23).

m	$\tilde{\gamma}$	$\tilde{M}$	λ_0
$1 \times 10^{-22} \text{ eV}/c^2$	0.7265	1.6652	3.4068×10^{-7}
$2 \times 10^{-22} \text{ eV}/c^2$	0.6859	1.8400	2.7970×10^{-7}
$5 \times 10^{-22} \text{ eV}/c^2$	0.6596	1.9922	2.3805×10^{-7}

$$\begin{aligned} \rho_{\text{bulge}}(x, y, z) &= 9.9 M_{\odot} \text{ pc}^{-3} \\ &\times \exp \left\{ - \left[x^2 + \left(\frac{y}{0.49} \right)^2 + \left(\frac{z}{0.37} \right)^2 \right]^{1/2} / 740 \text{ pc} \right\}. \end{aligned} \quad (42)$$

The disk density, because of its observed value $\rho_{\text{disk}} \approx 0.05 M_{\odot} \text{ pc}^{-3}$ [34], can be neglected compared with the bulge density. On the contrary, the diffuse gas density ρ_{gas} is a non-negligible part of the baryons in the Milky Way. Here we assume that $\rho_{\text{gas}} \approx m_{H_2} \langle n_{H_2} \rangle$, where m_{H_2} is the mass of molecular hydrogen and $\langle n_{H_2} \rangle$ is its number density in the central molecular zone [36], fitted by

$$\begin{aligned} \langle n_{H_2} \rangle &= 150 \text{ cm}^{-3} \exp \left\{ - \left[\frac{\sqrt{x^2 + (2.5y)^2} - 125 \text{ pc}}{137 \text{ pc}} \right]^4 \right. \\ &\quad \left. \times \exp \left[- \left(\frac{z}{18 \text{ pc}} \right)^2 \right] \right\}. \end{aligned} \quad (43)$$

TABLE V. Eigenvalue $\tilde{\gamma}$, ground state solution soliton mass $\tilde{M}$, and final values λ_0 for a set of FDM masses m , in the case of the background baryon profile $\tilde{\rho}_{\text{bg}}^x$ and using the soliton-halo mass relation Eq. (24).

m	$\tilde{\gamma}$	$\tilde{M}$	λ_0
$1 \times 10^{-22} \text{ eV}/c^2$	0.7773	1.4228	3.9834×10^{-7}
$2 \times 10^{-22} \text{ eV}/c^2$	0.6922	1.7784	3.7065×10^{-7}
$5 \times 10^{-22} \text{ eV}/c^2$	0.6537	2.0275	4.6674×10^{-7}

TABLE VI. Eigenvalue $\tilde{\gamma}$, ground state solution soliton mass $\tilde{M}$, and final values λ_0 for a set of FDM masses m , in the case of the background baryon profile $\tilde{\rho}_{\text{bg}}^y$ and using the soliton-halo mass relation Eq. (24).

m	$\tilde{\gamma}$	$\tilde{M}$	λ_0
$1 \times 10^{-22} \text{ eV}/c^2$	0.7682	1.5464	3.3721×10^{-7}
$2 \times 10^{-22} \text{ eV}/c^2$	0.6898	1.8245	3.5218×10^{-7}
$5 \times 10^{-22} \text{ eV}/c^2$	0.6531	2.0265	4.6720×10^{-7}

TABLE VII. Eigenvalue $\tilde{\gamma}$, ground state solution soliton mass $\tilde{M}$, and final values λ_0 for a set of FDM masses m , in the case of the background baryon profile $\tilde{\rho}_{\text{bg}}^z$ and using the soliton-halo mass relation Eq. (24).

m	$\tilde{\gamma}$	$\tilde{M}$	λ_0
$1 \times 10^{-22} \text{ eV}/c^2$	0.7395	1.6119	3.1036×10^{-7}
$2 \times 10^{-22} \text{ eV}/c^2$	0.6767	1.8899	3.2823×10^{-7}
$5 \times 10^{-22} \text{ eV}/c^2$	0.6523	2.0411	4.6054×10^{-7}

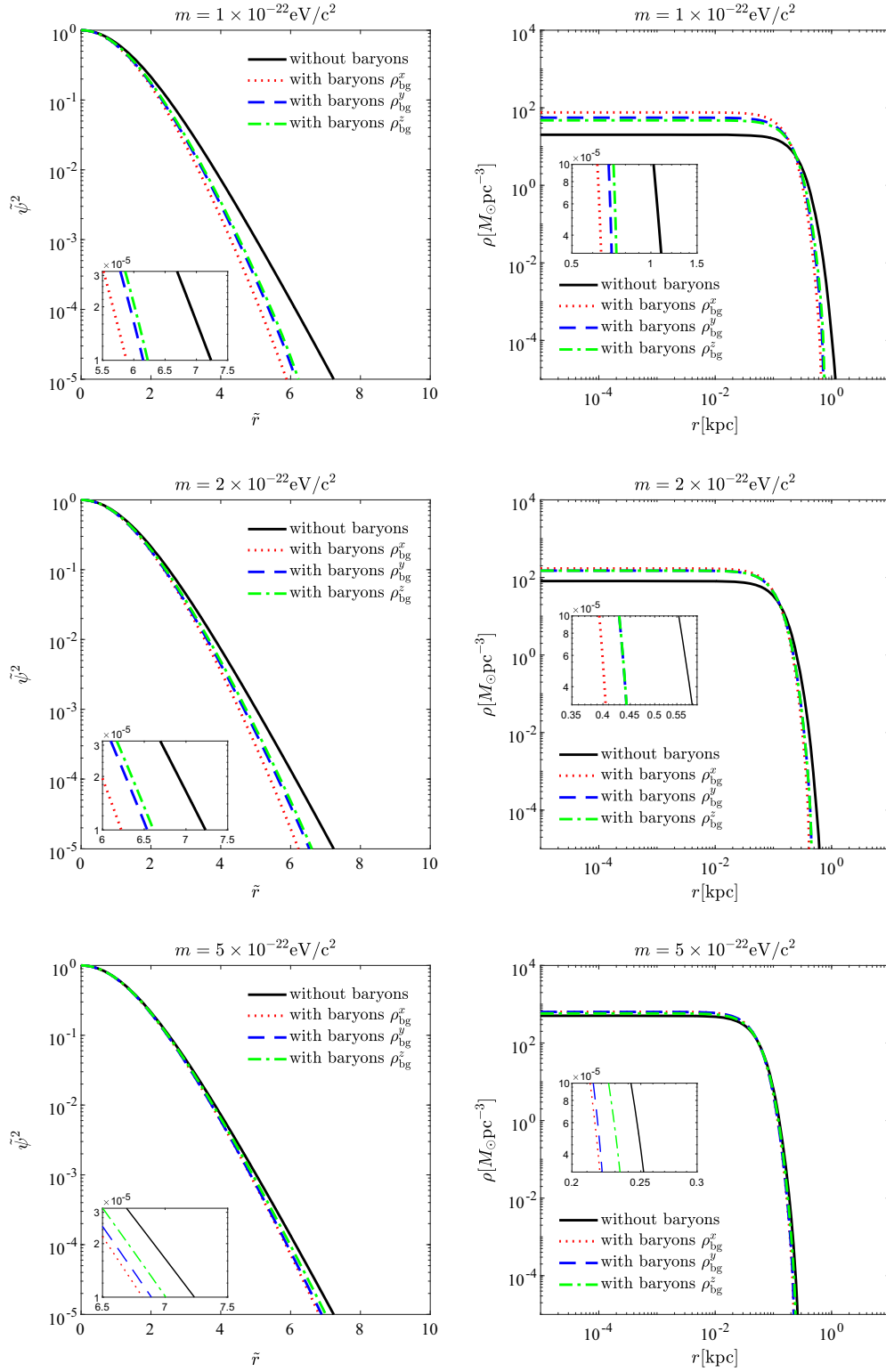


FIG. 5. Density profiles of the ground state solutions with different background baryon profile $\tilde{\rho}_{\text{bg}}^i$ for various FDM masses m and using the soliton-halo mass relation Eq. (23): first row, $m = 1 \times 10^{-22}$ eV/c²; second row, $m = 2 \times 10^{-22}$ eV/c²; and third row, $m = 5 \times 10^{-22}$ eV/c². The first column gives the normalized density profile while the second column represents the physical density profile. Black curves mark pure FDM solitons without the background baryon profile; red dotted curves show the effect of $\tilde{\rho}_{\text{bg}}^x$; blue dashed curves show that of $\tilde{\rho}_{\text{bg}}^y$; and green dash-dotted curves show that of $\tilde{\rho}_{\text{bg}}^z$.

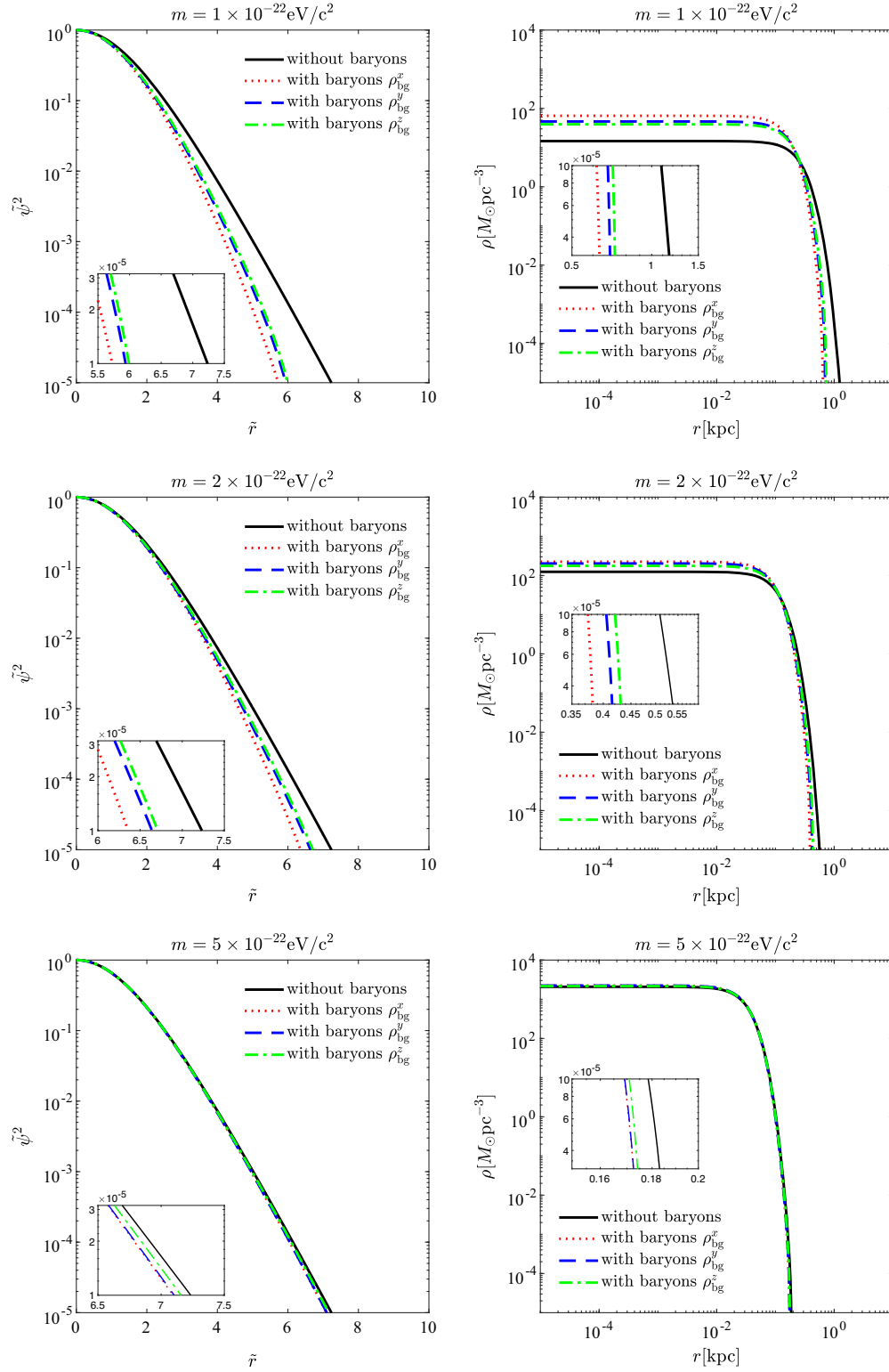


FIG. 6. Density profiles of the ground state solutions with different background baryon profile $\tilde{\rho}_{\text{bg}}^i$ for various FDM masses m and using the soliton-halo mass relation Eq. (24): first row, $m = 1 \times 10^{-22} \text{ eV}/c^2$; second row, $m = 2 \times 10^{-22} \text{ eV}/c^2$; and third row, $m = 5 \times 10^{-22} \text{ eV}/c^2$. The first column gives a normalized density profile while the second column represents the physical density profile. Black curves mark pure FDM solitons without the background baryon profile; red dotted curves show the effect of $\tilde{\rho}_{\text{bg}}^x$; blue dashed curves show that of $\tilde{\rho}_{\text{bg}}^y$; and green dash-dotted curves show that of $\tilde{\rho}_{\text{bg}}^z$.

For this nonspherical system, the shooting method fails and numerical simulations should be performed. However, we propose an approximation in which we consider three spherically averaged background baryon profiles, respectively: $\rho_{\text{bg}}^x(r) = \rho_{\text{bg}}(x = r, y = 0, z = 0)$, $\rho_{\text{bg}}^y(r) = \rho_{\text{bg}}(x = 0, y = r, z = 0)$, and $\rho_{\text{bg}}^z(r) = \rho_{\text{bg}}(x = 0, y = 0, z = r)$. The shooting method can therefore be applied to these spherical approximations $\rho_{\text{bg}}^i(r)$. Should the results by each $\rho_{\text{bg}}^i(r)$ be too markedly different, it would indicate that the original $\rho_{\text{bg}}(x, y, z)$ is exceedingly nonspherical and unable to be described by the approximation $\rho_{\text{bg}}^i(r)$.

The way we find the equilibrium configurations, following the procedure of Sec. II, uses an arbitrary normalization $\tilde{\psi}(\tilde{r} = 0) = 1$ for which the normalized soliton mass $\tilde{M} = \int_0^\infty \tilde{r}^2 \tilde{\psi}^2 d\tilde{r}$ deviates from the true dimensionless mass by a scaling factor λ . For the Milky Way, the soliton physical mass $M \approx 1.25 \times 10^9 M_\odot$ or $M \approx 1.16 \times 10^9 M_\odot$ is obtained with the soliton-halo mass relation Eqs. (23) or (24). On the other hand, taking this scaling factor into consideration, $\tilde{\rho}_{\text{bg}}^i \rightarrow \lambda^2 \tilde{\rho}_{\text{bg}}^i$ modifies the background baryon profile. To set both the soliton mass and the background baryon profile as the Milky Way's, we introduce an *a priori* scaling factor and obtain a new dimensionless variant as

$$\begin{aligned} \frac{\partial^2(\tilde{r}\tilde{\psi})}{\partial\tilde{r}^2} &= 2\tilde{r}(\tilde{\Phi} - \tilde{\gamma})\tilde{\psi}, \\ \frac{\partial^2(\tilde{r}\tilde{\Phi})}{\partial\tilde{r}^2} &= \tilde{r}(\tilde{\psi}^2 + \lambda_0^{-2}\tilde{\rho}_{\text{bg}}^i), \end{aligned} \quad (44)$$

where λ_0^{-2} is a parameter unaffected by the scaling factor λ , such that this system retains the scaling symmetry with $\lambda_0^{-2}\tilde{\rho}_{\text{bg}}^i \rightarrow \lambda^2\lambda_0^{-2}\tilde{\rho}_{\text{bg}}^i$. Given an initial λ_0 , we can obtain the normalized soliton profile $\tilde{\psi}^2$. The scaling factor λ can then be computed by requiring the soliton mass to be the Milky Way's. If $\lambda/\lambda_0 \neq 1$, we replace the initial λ_0 with the computed λ and repeat the former two steps. We stop the iteration when $\lambda/\lambda_0 - 1 < 0.0001$.

In Tables II–VII, we list the eigenvalues $\tilde{\gamma}$, ground state solution soliton mass $\tilde{M}$, and final value λ_0 for each FDM mass m , background baryon profile $\tilde{\rho}_{\text{bg}}^i$, and soliton physical mass M , respectively. We plot the normalized and the physical density profiles of the ground state solutions with different background baryon profile $\tilde{\rho}_{\text{bg}}^i$ for various FDM masses m and soliton physical masses M in Figs. 5 and 6, respectively. Comparing the soliton profiles corresponding to each $\tilde{\rho}_{\text{bg}}^i$ in every subplot, we find that the fine structure of the background baryon profile does not radically change the soliton profiles and that our treatment for the ellipsoidal baryon profile is reasonable. Contrasting the soliton profiles with different m , we find that heavier FDM particles form a denser, more compact,

and insensitive to the background baryon profile, soliton. Since, for the Milky Way, the soliton physical masses M derived from two soliton-halo mass relations Eqs. (23) and (24) are very similar, our results do not obviously depend on the choice of the soliton-halo mass relation.

V. SUMMARY AND DISCUSSION

In this paper, we present reasonable variants of the exact SP system that predicts FDM solitons in a nonrelativistic gravitational potential. Each variant aims at modeling a different effect, such as the variants due to a central supermassive BH (Sec. II), the system's own angular momentum (Sec. III), and an ellipsoidal baryon background (Sec. IV). We approximated each of them by an almost spherical system and solved them with the shooting method. We reach the same conclusion as [26], that the central supermassive BH renders the soliton denser and more compact. For the first time to the best of our knowledge, we demonstrate that the effect of the gravitomagnetic field derived from the soliton's self-angular momentum can be neglected compared to the contributions from the soliton's self-gravitating field, while the effect of the gravitomagnetic field derived from the central supermassive BH angular momentum is comparable to the contribution from the gravitoelectric field from that BH. Finally, we revisit the topic discussed in [29], showing that the baryon background is important for the soliton profile. We propose a reasonable treatment for the ellipsoidal baryon background, from which we are able to highlight the importance of the baryon background for a lighter FDM particle.

Of course, we could enlarge the parameter space of the underlying theoretical model and produce a greater diversity in the soliton solution. For example, the self-interactions between FDM particles [37] or the modified gravitational potential due to the modified gravity [38] would result in a different set of coupled equations, hence, a different soliton profile. In any case, the variants of the exact SP system presented in this paper are taking into account environmental effects, so the mapping between a basic system and its variants should be common to theoretical models.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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