

Surface growth scheme for bulk reconstruction and $T\bar{T}$ deformation

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In this paper, we study the dynamical connection between the surface growth scheme and the conformal field theory with $T\bar{T}$ deformation. By utilizing the extended one-shot entanglement distillation tensor network, we find that the iterative growth, i.e., radial evolution of homogenous and isotropic bulk minimal surfaces in asymptotically anti-de Sitter spacetime can be mapped to the $T\bar{T}$ operator flow driven by the deformation parameter. Our results show that the $T\bar{T}$ deformation can provide a dynamical mechanism for the surface growth in asymptotically anti-de Sitter spacetime, which may shed light on reconstructing bulk gravitational dynamics from the surface growth scheme.

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I. INTRODUCTION

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence establishes a profound connection between a gravitational system in an asymptotically AdS spacetime and a CFT on its boundary [1–3]. As a duality between two theories, the AdS/CFT correspondence or more general gauge-gravity duality also indicates that the boundary quantum system should encode the full information of the bulk gravity. The program to construct bulk gravity from its dual boundary quantum field theory is called bulk reconstruction [4–11]. Progresses in recent years have shown that entanglement entropy and its holographic dual, i.e., the holographic entanglement entropy [12–14], play a very important role in realizing the bulk reconstruction. However, difficulties still remain in the previous approaches of entanglement wedge reconstruction—for example, how to find a more fine description for the subregion-subregion duality, how to give a unified framework to construct both the bulk matter fields and bulk geometry, and how to construct bulk gravitational dynamics from the dual boundary quantum fields.

In recent works [15,16], a surface growth approach for bulk reconstruction in asymptotically AdS spacetime was proposed which not only can probe the fine structure of bulk entanglement, but also is able to construct both the

bulk geometry and matter fields from a boundary quantum system. In the surface growth approach, a given boundary subregion is partitioned into many smaller adjacent segments, each generating a bulk minimal surface, namely, the Ryu-Takayanagi (RT) surface; then, treating these RT surfaces as the new boundary of the bulk spacetime, a new layer of bulk minimal surfaces can continue to grow on them, which probe deeper regions of the bulk spacetime. By repeating this process layer by layer, the bulk minimal surfaces can probe arbitrary regions in the entanglement wedge, similar to the bubble growing. In addition, it has been shown that the surface growth scheme can be explicitly realized by the multiscale entanglement renormalization ansatz (MERA)-like tensor network [15,17–21], which naturally gives an emergence of bulk geometry and indicates a dynamical connection between the fine structure of entanglement and bulk spacetime.¹ For example, the entanglement of purification, which captures the fine structure in subregions of the entanglement wedge, can be naturally derived from the surface growth scheme [26] and also suggests a selection rule for surface growth in the bulk reconstruction in the AdS/BCFT (boundary CFT) correspondence [27].

¹Note that the original MERA tensor network proposal for realizing holography was only a toy model, which indirectly provided an emergent picture of discrete AdS₃ geometry (time slice) [22], while the one-shot entanglement distillation tensor network, which is a MERA-like tensor network constructed directly from the RT surfaces, can describe actual bulk excitations in the AdS/CFT correspondence [11]. To construct the continuous bulk AdS geometry from tensor networks, one can use the continuous version of the MERA, i.e., the cMERA formalism [23]. However, the problem of how to construct bulk gravitational dynamics from tensor networks has not been fully solved yet; see related progresses in [24,25].

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On the other hand, a parallel avenue of research explored the $T\bar{T}$ deformation of two-dimensional CFTs, which deforms the ultraviolet (UV) dynamics of the theory [28–30]. The $T\bar{T}$ deformation has received a lot of attention recently. For example, from the field theory side, the correlation functions [31–36] and Rényi or entanglement entropy [37–42] of $T\bar{T}$ -deformed two-dimensional field theories have been studied, and the geometric formulation of generalized root- $T\bar{T}$ deformation has been obtained [43]. From the holographic perspective, the $T\bar{T}$ deformation can alter the infrared (IR) geometry and result in a radial cutoff at the boundary of the asymptotically AdS_3 spacetime [32,44,45]; see, for example, recent reviews on $T\bar{T}$ deformation [46,47].

The renormalization group (RG) flow of the $T\bar{T}$ -deformed theory maps to bulk evolution along the radial direction, which is very similar to the process of surface growth. This similarity motivates us to ask an interesting question: Can the iterative growth of bulk minimal surfaces be linked to the operator-driven evolution induced by $T\bar{T}$ deformation? In this work, we will unify these two perspectives by utilizing the tensor network realization of the surface growth scheme and demonstrate that it can be mapped to the $T\bar{T}$ -deformed CFT. Explicitly, we generalized the one-shot entanglement distillation (OSD) tensor network to describe the general surface growth process in which bulk minimal surfaces grow layer by layer and then connected the radial evolution of bulk minimal surfaces to the $T\bar{T}$ operator flow via the generalized OSD tensor network. Our analysis revealed that the deformation parameter μ governs the iterative growth of bulk minimal surfaces, while the dynamics of the stress tensor emerges from the hierarchical entanglement structure.

The paper is organized as follows. Section II briefly reviews the original surface growth scheme; then Sec. III extends the OSed tensor network to describe the general surface growth and analyzes its continuum limit. Section IV establishes the explicit connection between the isotropic surface growth and the $T\bar{T}$ deformation. Finally, Sec. V gives the conclusions and discusses future related problems to be investigated.

II. REVIEW OF THE SURFACE GROWTH SCHEME

The surface growth scheme provides an efficient and systematic framework for reconstructing bulk spacetime through hierarchical entanglement structures [15,16]. It has been shown that this scheme can be described via generalized form of the OSED tensor network [17], which maps the iterative growth of bulk minimal surfaces to tensor network architectures.

To illustrate, consider partitioning the spatial boundary of a CFT_2 into adjacent subsystems A and B . The entanglement structure of A is encoded through an isometric tensor network, as depicted in Fig. 1. The combined

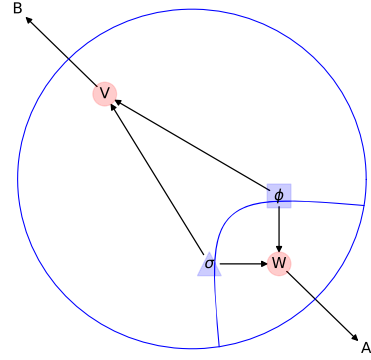


FIG. 1. The bulk minimal surface $\phi \otimes \sigma$ grows out from the subsystem A . The isometry tensor W matches to the entanglement wedge of A , and the isometry tensor V matches to the rest of bulk spacetime.

boundary state Ψ^{AB} is expressed as

$$\Psi^{AB} = V_{\alpha\beta}^B W_{\bar{\alpha}\bar{\beta}}^A \phi^{\alpha\bar{\alpha}} \sigma^{\beta\bar{\beta}}, \quad (1)$$

or, equivalently,

$$|\Psi\rangle = (V \otimes W)(|\phi\rangle \otimes |\sigma\rangle), \quad (2)$$

where W and V are isometric tensors corresponding to the entanglement wedges of A and B , respectively. The states $|\phi\rangle$ and $|\sigma\rangle$ represent maximally entangled Bell pairs in which $|\phi\rangle$ contributes to the dominant part of the entanglement entropy of A and $|\sigma\rangle$ captures the quantum fluctuations:

$$\begin{aligned}
|\phi\rangle &= \sum_{m=0}^{e^{S(A)-O(\sqrt{S(A)})}} |m\bar{m}\rangle_{\alpha\bar{\alpha}}, \\
|\sigma\rangle &= \sum_{n=0}^{e^{O(\sqrt{S(A)})}} \sqrt{\tilde{p}_{n\Delta}} |n\bar{n}\rangle_{\beta\bar{\beta}} \quad (3)
\end{aligned}$$

with Hilbert space dimensions:

$$\begin{aligned}\dim \mathcal{H}_\alpha &= e^{S(A)-O(\sqrt{S(A)})} \equiv \Delta, \\ \dim \mathcal{H}_\beta &= e^{O(\sqrt{S(A)})}, \\ \dim(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta) &= e^{S(A)},\end{aligned}\tag{4}$$

where Δ is the size of a block in eigenvalue space, $\tilde{p}_{n\Delta}$ is the average eigenvalue of the block, and $S(A)$ is the entanglement entropy of subsystem A .

Consider dividing the boundary into N identical segments and using the RT surfaces of each segment to construct the first layer; the next layer is constructed in a similar way by taking the adjacent two segments into a new one. By iterating this process, a hierarchical structure of bulk minimal surfaces emerges. Explicitly, the first layer [Fig. 2(a)] of the

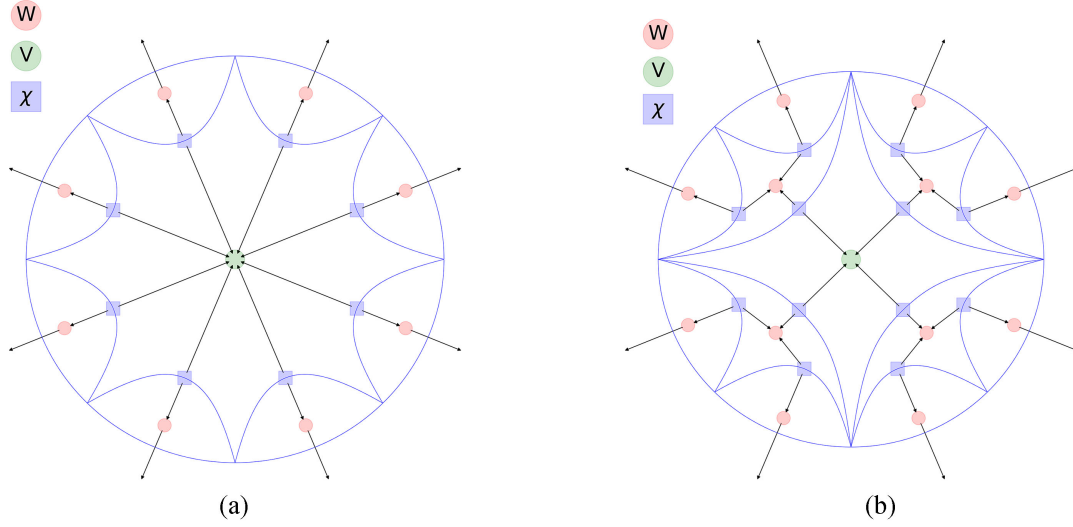


FIG. 2. The first layer (a) and the second layer (b) of the original surface growth scheme with $N = 8$, and the surfaces χ (blue squares) always anchor on the boundary in this scheme.

bulk minimal surfaces (corresponding to N segments) is described by the wave function

$$|\Psi\rangle^{(1)} = V'_N{}^{(1)} \prod_{i=1}^N W_i^{(1)} |\phi\rangle_i^{(1)} |\sigma\rangle_i^{(1)}, \quad (5)$$

the second layer which contains $N/2$ RT surfaces [Fig. 2(b)] is described by boundary state

$$|\Psi\rangle^{(2)} = V'_M{}^{(2)} \left(\prod_{j=1}^{M=N/2} W_j^{(2)} |\phi\rangle_j^{(2)} |\sigma\rangle_j^{(2)} \right) \times \left(\prod_{i=1}^N W_i^{(1)} |\phi\rangle_i^{(1)} |\sigma\rangle_i^{(1)} \right), \quad (6)$$

and subsequent layers refine the bulk geometry through progressive subdivisions; for example, the k th layer is generated by $N/2^{k-1}$ segments:

$$|\Psi\rangle^{(k)} = V'^{(k)} \prod_{j=1}^k \prod_{i=1}^{N/2^{k-1}} W_i^{(j)} |\phi\rangle_i^{(j)} |\sigma\rangle_i^{(j)}. \quad (7)$$

Each layer k corresponds to a deeper radial slice of the bulk geometry, reflecting the MERA [15–21]. However, the OSED tensor network formalism imposes a constraint in Eq. (7): The iterative subdivision process forces N to reduce by half at each layer, thereby restricting the range of k . This restriction can be removed by extending the OSED tensor network, which allows it to describe the general surface growth scheme.

III. TENSOR NETWORK DESCRIPTION OF THE GENERAL SURFACE GROWTH SCHEME

In this section, we will extend the OSED tensor network to describe the general surface growth scheme in which the next layer of bulk minimal surface is anchored on the previous layer. In addition, the original tensor network construction of surface growth scheme considered only the pure state case, in which the bulk is the pure AdS spacetime; we will also generalize the framework by changing starting points of the surface growth and extend the OSED tensor network construction to include asymptotically AdS spacetime with a black hole horizon.

A. The generalized OSED tensor network

The initial state of the boundary-horizon system is expressed as

$$\Psi^{(0)a^{(0)}b} = V_{AB}^{(0)} \Omega^{Bb} \chi_1^{(0)A_1 a_1^{(0)}} \cdots \chi_N^{(0)A_N a_N^{(0)}}, \quad (8)$$

where the indices A and $a^{(0)}$ equivalently represent the series of indices $A_1 \cdots A_N$ and $a_1^{(0)} \cdots a_N^{(0)}$, $a^{(0)}$ labels boundary segments, and b labels the horizon. The tensor $\chi^{(k)}$ combines the RT surface states $\phi^{(k)}$ and $\sigma^{(k)}$, but $\chi^{(0)}$ is the initial boundary state, regarded as the combination of the RT surface states $\phi^{(0)}$ and $\sigma^{(0)}$ for simplicity. The surface growth operation is then applied by extending bulk minimal surfaces from the turning points of previous layers. For the first layer, this yields

$$\begin{aligned}
\Psi^{(0)a^{(0)}b} &= V_{AB}^{(0)} \Omega^{Bb} \chi_1^{(0)A_1 a_1^{(0)}} \cdots \chi_N^{(0)A_N a_N^{(0)}} \\
&= V_{AB}^{(1)} \Omega^{Bb} \chi_1^{(1)A_1 a_1^{(1)}} \cdots \chi_N^{(1)A_N a_N^{(1)}} \\
&\quad \times W_{a_1^{(1)} C_1^{(0)} D_2^{(0)}}^{(1)} W_{a_2^{(1)} C_2^{(0)} D_3^{(0)}}^{(1)} \cdots W_{a_N^{(1)} C_N^{(0)} D_1^{(0)}}^{(1)} \chi_1^{(0)C_1^{(0)} D_1^{(0)} a_1^{(0)}} \cdots \chi_N^{(0)C_N^{(0)} D_N^{(0)} a_N^{(0)}} \\
&\equiv V_{AB}^{(1)} \Omega^{Bb} \chi_1^{(1)A_1 a_1^{(1)}} \cdots \chi_N^{(1)A_N a_N^{(1)}} \tilde{W}_{a^{(1)}}^{(1)(0)a^{(0)}}, \tag{9}
\end{aligned}$$

where $\tilde{W}^{(1)(0)}$ encodes the isometric transition between the first layer and the AdS boundary. The Hilbert space dimensions at the first layer are given by

$$\dim \mathcal{H}_{a^{(1)}} = e^{NS^{(1)}}, \quad \dim \mathcal{H}_b = e^{S_h}, \tag{10}$$

where $S^{(1)}$ is the entanglement entropy of a RT surface in the first layer and S_h is the entanglement entropy of the horizon. With Eq. (9), the first-layer state of the surface-horizon system can be expressed as

$$\Psi^{(1)a^{(1)}b} \equiv V_{AB}^{(1)} \Omega^{Bb} \chi_1^{(1)A_1 a_1^{(1)}} \cdots \chi_N^{(1)A_N a_N^{(1)}}, \tag{11}$$

such that Eq. (9) can be rewritten as

$$\Psi^{(0)a^{(0)}b} = \Psi^{(1)a^{(1)}b} \tilde{W}_{a^{(1)}}^{(1)(0)a^{(0)}}. \tag{12}$$

Next, repeating this process, the tensor network of the second-layer state can be expressed as

$$\begin{aligned}
\Psi^{(0)a^{(0)}b} &= V_{AB}^{(1)} \Omega^{Bb} \chi_1^{(1)A_1 a_1^{(1)}} \cdots \chi_N^{(1)A_N a_N^{(1)}} \\
&\quad \times W_{a_1^{(1)} C_1^{(0)} D_2^{(0)}}^{(1)} W_{a_2^{(1)} C_2^{(0)} D_3^{(0)}}^{(1)} \cdots W_{a_N^{(1)} C_N^{(0)} D_1^{(0)}}^{(1)} \chi_1^{(0)C_1^{(0)} D_1^{(0)} a_1^{(0)}} \cdots \chi_N^{(0)C_N^{(0)} D_N^{(0)} a_N^{(0)}} \\
&= V_{AB}^{(2)} \Omega^{Bb} \chi_1^{(2)A_1 a_1^{(2)}} \cdots \chi_N^{(2)A_N a_N^{(2)}} \\
&\quad \times W_{a_1^{(2)} C_1^{(1)} D_2^{(1)}}^{(2)} W_{a_2^{(2)} C_2^{(1)} D_3^{(1)}}^{(2)} \cdots W_{a_N^{(2)} C_N^{(1)} D_1^{(1)}}^{(2)} \chi_1^{(1)C_1^{(1)} D_1^{(1)} a_1^{(1)}} \cdots \chi_N^{(1)C_N^{(1)} D_N^{(1)} a_N^{(1)}} \\
&\quad \times W_{a_1^{(1)} C_1^{(0)} D_2^{(0)}}^{(1)} W_{a_2^{(1)} C_2^{(0)} D_3^{(0)}}^{(1)} \cdots W_{a_N^{(1)} C_N^{(0)} D_1^{(0)}}^{(1)} \chi_1^{(0)C_1^{(0)} D_1^{(0)} a_1^{(0)}} \cdots \chi_N^{(0)C_N^{(0)} D_N^{(0)} a_N^{(0)}} \\
&\equiv \Psi^{(2)a^{(2)}b} \tilde{W}_{a^{(2)}}^{(2)(1)a^{(1)}} \tilde{W}_{a^{(1)}}^{(1)(0)a^{(0)}}, \tag{13}
\end{aligned}$$

where $\tilde{W}^{(2)(1)}$ encodes the isometric transition between the second layer and the first layer and $\Psi^{(2)a^{(2)}b}$ is the second-layer state of the surface-horizon system:

$$\Psi^{(2)a^{(2)}b} = V_{AB}^{(2)} \Omega^{Bb} \chi_1^{(2)A_1 a_1^{(2)}} \cdots \chi_N^{(2)A_N a_N^{(2)}}. \tag{14}$$

The Hilbert space dimensions at the second layer are given by

$$\dim \mathcal{H}_{a^{(2)}} = e^{NS^{(2)}}. \tag{15}$$

Similarly, repeating this process generates higher-layer states. For the k th layer:

$$\Psi^{(0)a^{(0)}b} = \Psi^{(k)a^{(k)}b} \tilde{W}_{a^{(k)}}^{(k)(k-1)a^{(k-1)}} \cdots \tilde{W}_{a^{(1)}}^{(1)(0)a^{(0)}}, \tag{16}$$

where $\tilde{W}$ encodes the isometric transition between layers and consists of both the coarse grainer and disentangler [15] and $\Psi^{(k)a^{(k)}b}$ is the k th-layer state of the surface-horizon system:

$$\Psi^{(k)a^{(k)}b} = V_{AB}^{(k)} \Omega^{Bb} \chi_1^{(k)A_1 a_1^{(k)}} \cdots \chi_N^{(k)A_N a_N^{(k)}}. \tag{17}$$

The Hilbert space dimensions at the k th layer are given by

$$\dim \mathcal{H}_{a^{(k)}} = e^{NS^{(k)}}. \tag{18}$$

Crucially, the subdivision count N remains fixed across layers, eliminating the original constraint. This allows k to take any positive integer value, enabling lasting growth of the surfaces toward the horizon without dependence on N (Fig. 3).

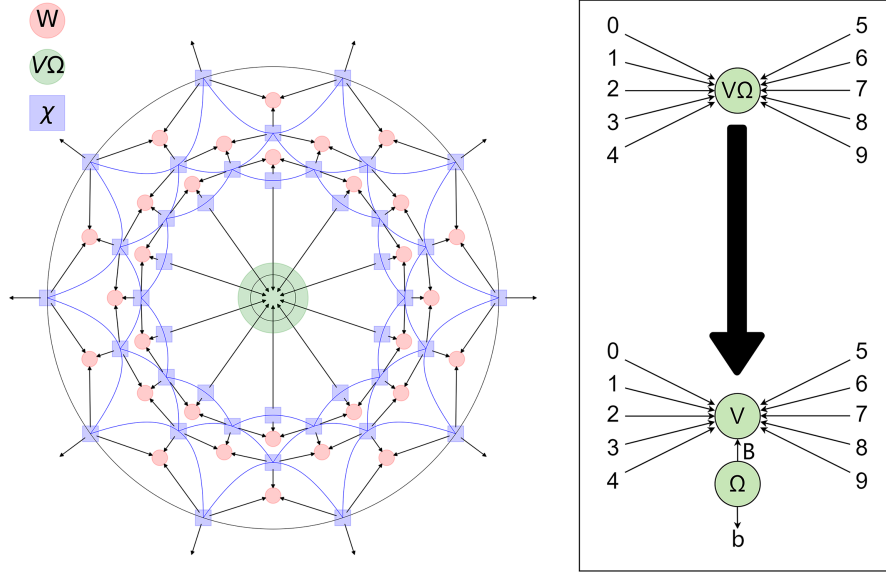


FIG. 3. The left is the tensor network of general surface growth scheme with $N = 10$. The $V\Omega$ (green circle) includes the bulk spacetime V and the horizon Ω , which have hidden labels and are shown in the right image. In this scheme, the surfaces growing out anchor on the turning points of previous surfaces.

B. The limit of the surface growth

Geometrically, the iterative surface growth process drives the constructed minimal surfaces asymptotically toward the center of the bulk spacetime. It has been shown in [16] that these surfaces will approach to the horizon after finite steps of growth. In this section, we rigorously demonstrate this behavior through a monotonicity argument and entropy constraints and show that the surface growth will reach the black hole horizon in the $k \rightarrow \infty$ limit.

Consider the i th minimal surface in the k th layer. The Hilbert space dimension associated with the index $a_i^{(k)}$ is given by

$$\dim \mathcal{H}_{a_i^{(k)}} = e^{S^{(k)}}, \quad (19)$$

where $S^{(k)}$ denotes the entanglement entropy of the k th-layer surface. Correspondingly, the Hilbert space dimension of the composite indices $C_i^{(k-1)} D_{i+1}^{(k-1)}$ from the previous layer is

$$\dim \mathcal{H}_{C_i^{(k-1)} D_{i+1}^{(k-1)}} = e^{\frac{1}{2}S^{(k-1)}} e^{\frac{1}{2}S^{(k-1)}} = e^{S^{(k-1)}}. \quad (20)$$

Because of the minimal surface growing, the isometric tensor $W_{a_i^{(k)} C_i^{(k-1)} D_{i+1}^{(k-1)}}^{(k)}$ reduces the dimension to $e^{S^{(k)}}$, implying the entropy inequality

$$S^{(k)} < S^{(k-1)}, \quad \forall k \in \mathbb{N}_+. \quad (21)$$

Setting $N = 1$ when constructing the $(k+1)$ th-layer surfaces, the new surface will coincide with the horizon and

the tensor network will be expressed as

$$\begin{aligned} \Psi^{(k)a^{(k)}b} &= V_{AB}^{(k)} \Omega^{Bb} \chi_1^{(k)A_1 a_1^{(k)}} \cdots \chi_N^{(k)A_N a_N^{(k)}} \\ &= V_{AB}^{(h)} \Omega^{Bb} \chi^{(h)Aa^{(h)}} \\ &\quad \times W_{a^{(h)} C^{(k)}}^{(h)} \chi_1^{(k)C_1^{(k)} a_1^{(k)}} \cdots \chi_N^{(k)C_N^{(k)} a_N^{(k)}}. \end{aligned} \quad (22)$$

The dimensions of Hilbert space at the $(k+1)$ th layer are given by

$$\begin{aligned} \dim \mathcal{H}_{a^{(h)}} &= e^{S_h}, \\ \dim \mathcal{H}_{C^{(k)}} &= e^{NS^{(k)}} \end{aligned} \quad (23)$$

with the entropy inequality

$$\frac{S_h}{N} \leq S^{(k)} \leq S^{(1)}. \quad (24)$$

This monotonic decrease in entropy establishes that the sequence $\{S^{(k)}\}$ is bounded and strictly decreasing. By the monotone convergence theorem, $S^{(k)}$ must converge to a finite limit as $k \rightarrow \infty$.

To identify this limit, suppose the surfaces in the $(k-1)$ th-layer surfaces coincide with the horizon. The k th-layer surfaces also coincide with the horizon, and the entanglement entropy of each segment is satisfied:

$$S^{(k)} = S^{(k-1)} = \frac{S_h}{N}. \quad (25)$$

Thus, the limit of the sequence $\{S^{(k)}\}$ converged to is

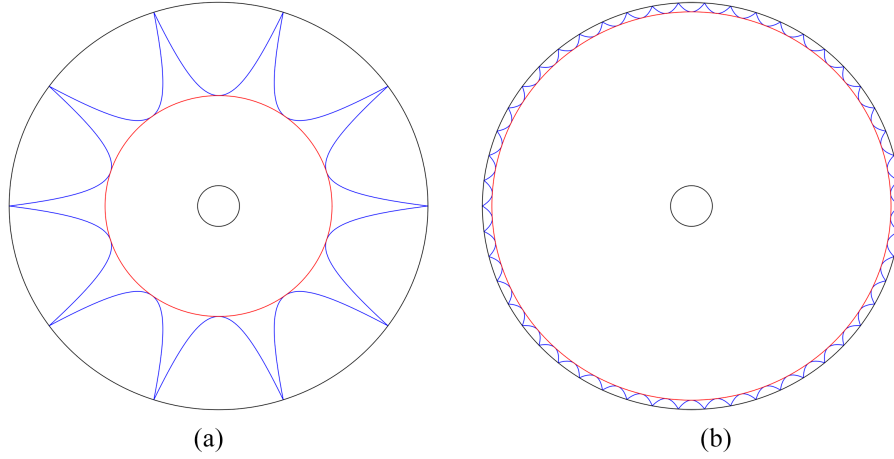


FIG. 4. The partition number N in (a) is chosen as $N = 10$ and N in (b) is chosen as $N = 50$. In the continuum limit $N \rightarrow \infty$, a layer of homogenous and isotropic discrete bulk minimal surfaces converges to a smooth radial cutoff surface (denoted in red color).

$$\lim_{k \rightarrow \infty} S^{(k)} = \frac{S_h}{N}. \quad (26)$$

Consequently, in the limit $k \rightarrow \infty$, the iteratively grown surfaces converge to the horizon, completing the geometric reconstruction of the bulk spacetime, and the surface growth process dynamically encodes the transition from boundary entanglement to horizon physics.

To establish a smooth geometric correspondence, we analyze the continuum limit of the homogenous and isotropic surface growth as the partition number $N \rightarrow \infty$, as illustrated in Fig. 4. This limit bridges the discrete tensor network construction with the continuous radial evolution of bulk geometry.

C. Geometric convergence to radial surfaces

Consider a nonrotating Banados-Teitelboim-Zanelli spacetime with metric:

$$ds^2 = -\frac{r^2 - M}{L^2} dt^2 + \frac{L^2}{r^2 - M} dr^2 + r^2 d\phi^2, \quad (27)$$

whose horizon is located at $r_h = \sqrt{M}$. The surface growth scheme generates minimal surfaces governed by the equation [16]

$$\begin{aligned} \Delta\phi &= \phi - \phi_* \\ &= -\frac{L}{r_h} \ln \left(\sqrt{1 - \frac{r_h^2}{r^2}} - \sqrt{\frac{r_h^2}{r_*^2} - \frac{r_h^2}{r^2}} \right) + \frac{L}{r_h} \ln \sqrt{1 - \frac{r_h^2}{r_*^2}} \end{aligned} \quad (28)$$

with (r_*, ϕ_*) denoting the turning point of the surface.

For surface of the $k + 1$ th layer, its end point is located at radius r_{*k} (which is the turning point of the k th layer) and its turning point is r_{*k+1} ; then Eq. (28) can be

rearranged into

$$u_{k+1} \cosh^2 \lambda = u_k, \quad (29)$$

where

$$\lambda = \frac{r_h}{L} \Delta\phi, \quad u_k = 1 - \frac{r_h^2}{r_{*k}^2}. \quad (30)$$

It is easy to derive from Eq. (28) that the sequence $\{r_{*k}\}$ is bounded and monotonically decreasing; i.e., the sequence $\{r_{*k}\}$ has a limit: $\lim_{k \rightarrow \infty} r_{*k} \equiv r_{\text{lim}}$. Then we have

$$u_{\text{lim}} (\cosh^2 \lambda - 1) = 0, \quad u_{\text{lim}} = 0, \quad r_{\text{lim}} = r_h, \quad (31)$$

for λ is a constant. Thus, we obtain the conclusion that the grown surfaces geometrically converge to the horizon in the limit $k \rightarrow \infty$ as shown in Sec. III B above.

Next, we consider the case of $N \rightarrow \infty$, or, equivalently, $\Delta\phi \rightarrow 0$.

First, the first step of surface growth constructs the first cutoff surface at radius $r_{*1} \gg r_h$ from the AdS boundary $r = r_0 \rightarrow \infty$, where r_{*1} denotes the turning point of the first layer of growing surfaces; then the change of angle $\Delta\phi$ is

$$\Delta\phi = \frac{L}{r_{*1}} \left(1 - \frac{r_{*1}^2}{2r_0^2} \right) + \mathcal{O}(r_{*1}^{-2}). \quad (32)$$

When choosing $r_{*1} \gg L$, the constructed surface has an infinitesimal angle range and the number of segments $N \rightarrow \infty$.

Second, ignoring the $\mathcal{O}(r_{*1}^{-2})$ terms, for the homogenous and isotropic surface growth, $\Delta\phi = \frac{L}{r_{*1}} \left(1 - \frac{r_{*1}^2}{2r_0^2} \right) \equiv \frac{L}{r_{*1}}$ is fixed. Denoting the radius of the k th-layer and the $(k + 1)$ th-layer growing surfaces (i.e., new cutoff surfaces)

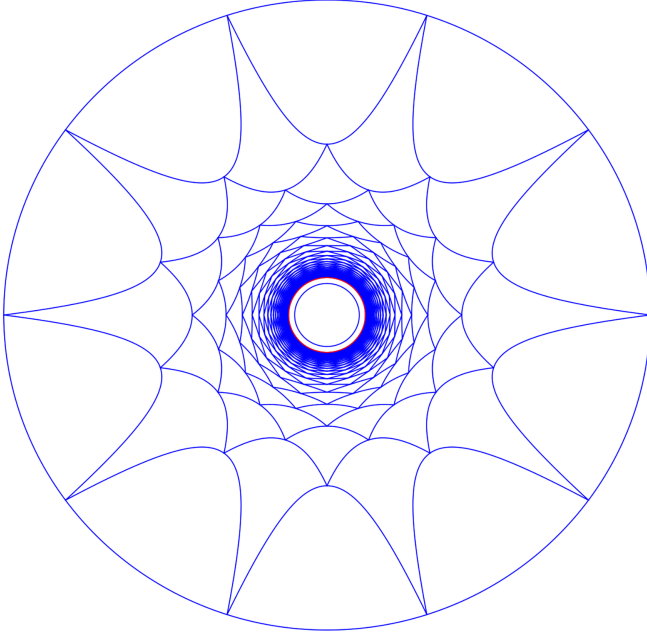


FIG. 5. Homogenous and isotropic surface growth from the AdS boundary r_0 to the horizon r_h , and the growing step number shown in the figure is 50.

as r_{*k} and r_{*k+1} , respectively, one can obtain the change of radius from Eq. (28) as

$$\Delta r_{*k} = r_{*k+1} - r_{*k} = \frac{r_{*k}(r_h^2 - r_{*k}^2)}{2\tilde{r}_{*1}^2} + \mathcal{O}(\tilde{r}_{*1}^{-4}), \quad (33)$$

where r_{*k} and r_{*k+1} correspond to r and r_* in Eq. (28), respectively. In this step, the new surface has a radial deviation

$$\left| \frac{\Delta r_{*k}}{r_{*k}} \right| \simeq \frac{(r_{*k}^2 - r_h^2)}{2\tilde{r}_{*1}^2} < \frac{1}{2}, \quad (34)$$

and when the step number k is large enough, i.e., $r_{*k} \ll \tilde{r}_{*1}$, the deviation

$$\left| \frac{\Delta r_{*k}}{r_{*k}} \right| \simeq \frac{(r_{*k}^2 - r_h^2)}{2\tilde{r}_{*1}^2} \ll 1. \quad (35)$$

Therefore, the layer of surfaces can be treated as a smooth radial surface in the $N \rightarrow \infty$ limit.

Finally, to confirm the range of surface growth, we change Eq. (33) to a differential equation with an initial condition:

$$\Delta r_{*k} \sim \frac{dr_*}{dx} = \frac{r_*(r_h^2 - r_*^2)}{2\tilde{r}_{*1}^2}, \quad \text{with } r_*|_{x=0} = r_0, \quad (36)$$

where dx is one step of surface growth. And the solution is

$$r_* = \frac{e^{\frac{r_h^2 x}{2\tilde{r}_{*1}^2}} r_h r_0}{\sqrt{e^{\frac{r_h^2 x}{2\tilde{r}_{*1}^2}} r_0^2 - r_0^2 + r_h^2}}. \quad (37)$$

Subsequently, taking the limit $x \rightarrow +\infty$, we find

$$r_*|_{x \rightarrow +\infty} = r_h. \quad (38)$$

This implies that the surface growth scheme dynamically fills the bulk spacetime from the AdS boundary r_0 to the horizon r_h , covering the radial domain (r_h, r_0) in the continuum limit $N \rightarrow \infty$. An example of $N = 10$ with 50 growing steps is shown in Fig. 5.

D. Continuum state evolution

The k th-layer state is expressed as

$$\Psi^{(0)a(0)b} = \Psi^{(k)a(k)b} \tilde{W}_{a^{(k)}}^{(k)(k-1)a^{(k-1)}} \dots \tilde{W}_{a^{(1)}}^{(1)(0)a^{(0)}} \quad (39)$$

or, equivalently,

$$|\Phi_V\rangle^{(0)} = \tilde{W}^{(1)(0)} \dots \tilde{W}^{(k)(k-1)} |\Phi_V\rangle^{(k)}. \quad (40)$$

Substituting the property of the isometric tensor $\tilde{W}$ [15]:

$$\tilde{W}^{\dagger(k)(k-1)} \tilde{W}^{(k)(k-1)} = \mathbf{1}^{(k)} \quad (41)$$

into Eq. (40), we obtain

$$|\Phi_V\rangle^{(k)} = \tilde{W}^{\dagger(k)(k-1)} \dots \tilde{W}^{\dagger(1)(0)} |\Phi_V\rangle^{(0)} \quad (42)$$

and the other property of $\tilde{W}$:

$$\langle \Phi_V |^{(k-1)} \tilde{W}^{(k)(k-1)} \tilde{W}^{\dagger(k)(k-1)} | \Phi_V \rangle^{(k-1)} = 1. \quad (43)$$

In the limit $N \rightarrow \infty$, the tensors $\tilde{W}^{\dagger}$ and the step number k are replaced by an evolution operator U and the radial parameter μ :

$$|\Phi(\mu)\rangle = U(\mu, 0) |\Phi(0)\rangle, \quad (44)$$

and the evolution operator U satisfies rules

$$\begin{aligned} U(a, b)U(b, c) &= U(a, c), \quad |a| \geq |b| \geq |c|, \\ \langle \Phi(b) | U^{\dagger}(a, b) U(a, b) | \Phi(b) \rangle &= 1, \\ U(a, b)U^{\dagger}(a, b) &= \mathbf{1}(a), \end{aligned} \quad (45)$$

where the evolution operators $U(a, b)$ with the sequence $|a| \geq |b|$ constitute a semigroup $\mathcal{G}_U$. Although there exists $U^{\dagger}(a, b)$ with $(|a| > |b|)$ such that $U(a, b)U^{\dagger}(a, b) = \mathbf{1}(a)$ —i.e., $U(a, b)$ is a coisometry—the inverse operator $U^{\dagger}(a, b)$ is not in $\mathcal{G}_U$ due to the irreversibility of surface growth. More precisely, surface growth does not diminish the information of the surface-horizon system:

$$S_{\text{surfaces}}(\mu) = S_{\text{horizon}}, \quad \forall \mu \text{ with } |\mu| \leq |\mu_h|, \quad (46)$$

but rather reduces the dimension of the Hilbert space.

Consider it from a different standpoint: If we maintain the dimension of the state unchanged during the evolution process, the evolution process can be mathematically expressed as transforming some segments of the singular density matrix into zero matrices, i.e., $\rho(a) \rightarrow \rho(a) \oplus \mathbf{0}(\bar{a})$, where $\mathbf{0}(\bar{a})$ can be removed at the end of the calculation. Then the evolution operator can be expressed as $U(\#, a) \rightarrow u(\#, a) \oplus X(\bar{a})$, where the effective operator $u(\#, a)$ is a unitary operator whose dimension is the effective dimension of $U(\#, a)$, and the other noneffective part $X(\bar{a})$ is an arbitrary square matrix which maintains the dimension of U unchanged. As the dimension reduces, the effective dimension of the evolution operator $U(a, b)$ operating on $|\Phi(b)\rangle$ diminishes, whereas the effective dimension of $U^\dagger(a, b)$ stays constant, meaning that $U^\dagger(a, b)$ is unable to operate on $|\Phi(a)\rangle$; i.e., there does not exist $U^\dagger(a, b)$ in the semigroup $\mathcal{G}_U$ such that the dimension of $|\Phi\rangle$ increases from parameter a to parameter b . Under this premise, the isometric operator $U(a, b) \equiv u(a, b) \oplus X(\bar{b})$ can be mathematically expressed as a unitary operator by setting $X(\bar{b}) = \mathbf{1}(\bar{b})$ and ignoring the operation of eliminating zero matrices in $\rho(b) \oplus \mathbf{0}(\bar{b})$ during the surface growth process.

The state of the surface-horizon system $|\Phi(\mu)\rangle$ decomposes into a radial surface $|\chi(\mu)\rangle$ and the horizon $|\Omega\rangle$:

$$|\Phi(\mu)\rangle = V(\mu)|\chi(\mu)\rangle|\Omega\rangle, \quad (47)$$

where the operator $V(\mu)$ corresponds to the isometric tensor V in the tensor network and encodes the data of the surfaces $|\chi(\mu)\rangle$ and the horizon $|\Omega\rangle$. It is obvious that the evolution operator U does not change the horizon state $|\Omega\rangle$:

$$\begin{aligned} |\Phi(\mu_1)\rangle &= V(\mu_1)|\chi(\mu_1)\rangle|\Omega\rangle \\ &= U(\mu_1, \mu_2)|\Phi(\mu_2)\rangle \\ &= U(\mu_1, \mu_2)V(\mu_2)|\chi(\mu_2)\rangle|\Omega\rangle, \end{aligned} \quad (48)$$

and the density matrix operators have the same properties:

$$\begin{aligned} \rho_V(\mu_1) &= U(\mu_1, \mu_2)\rho_V(\mu_2)U^\dagger(\mu_1, \mu_2), \\ \rho_\Omega &= U(\mu_1, \mu_2)\rho_\Omega U^\dagger(\mu_1, \mu_2), \end{aligned} \quad (49)$$

where $\rho_V(\mu) = \text{Tr}_\Omega|\Phi(\mu)\rangle\langle\Phi(\mu)|$ is the reduced density matrix operator at the surfaces arriving at parameter μ and $\rho_\Omega = \text{Tr}_\chi|\Phi(\mu)\rangle\langle\Phi(\mu)|$ is the reduced density matrix operator at the horizon. Considering a operator $\mathcal{O}$ at growing surfaces, its expectation value is

$$\text{Tr}(\rho_V(\mu)\mathcal{O}) = \text{Tr}(\rho_V(0)U^\dagger(\mu, 0)\mathcal{O}U(\mu, 0)). \quad (50)$$

Especially when the surfaces arrive at the horizon, i.e., $\mu = \mu_h$, the Hilbert spaces of ρ_V and ρ_Ω have the same dimension, and the $V(\mu_h)$ can be rewritten as a diagonal matrix by Schmidt decomposition:

$$\begin{aligned} |\Phi(\mu_h)\rangle &= V(\mu_h)|\chi(\mu_h)\rangle|\Omega\rangle \\ &= \sum_i \sqrt{p_i} |\chi(\mu_h)\rangle_i |\Omega\rangle_i, \end{aligned} \quad (51)$$

for $S(\rho_V) = S(\rho_\Omega) = S_h = -\sum_i p_i \ln p_i \sim \ln(\dim \mathcal{H}_i)$, i.e., $\dim \mathcal{H}_i \sim e^{S_h}$, where all p_i are approximately equal.

So far, we have obtained the evolution operator $U(\mu, 0)$ in the surface growth scheme, which maps to the transition of radial surfaces in asymptotic AdS_3 spacetime. Thus, it is natural to map the surface growth process to the radial evolution in semiclassical AdS_3 gravity. At the k th-layer surfaces, we can define a semiclassical code subspace:

$$\mathcal{H}_k^{\text{code}} = \text{span}\{|\gamma_k\rangle\}, \quad (52)$$

where γ_k denotes the induced metric on the k th surface in the growth process. Then considering two neighboring radial slices $\Sigma_k: \mu_k$ and $\Sigma_{k+1}: \mu_{k+1} = \mu_k + \delta\mu$, we define the thin-shell gravitational kernel by the radial Arnowitt-Deser-Minsner path integral:

$$K_g = (\gamma_{k+1}, \gamma_k) = \int_{\gamma_k}^{\gamma_{k+1}} \mathcal{D}\gamma \mathcal{D}\pi \mathcal{D}N \mathcal{D}N^a e^{iS_{\text{shell}}}, \quad (53)$$

where

$$S_{\text{shell}} = \int_{\gamma_k}^{\gamma_{k+1}} d\mu \int_{\Sigma_k} d^2x (\pi^{ab} \partial_\mu \gamma_{ab} - N\mathcal{H} - N^a \mathcal{H}_a). \quad (54)$$

Here, $\mathcal{H} = 0$ and $\mathcal{H}_a = 0$ are the radial Hamiltonian and momentum constraints of Einstein gravity, respectively. The radial gravitational transfer matrix $U_g^{(k+1, k)}$ acts as

$$|\gamma_{k+1}\rangle = U_g^{(k+1, k)} |\gamma_k\rangle = \int \mathcal{D}\gamma_k K_g(\gamma_{k+1}, \gamma_k) |\gamma_k\rangle. \quad (55)$$

Thus, we can introduce an encoding map R_k between the abstract OSED index basis and the semiclassical geometric basis:

$$R_k: |\Phi_V\rangle^{(k)} \mapsto |\gamma_k\rangle; \quad (56)$$

then the inward growth map could be defined by the radial gravitational transfer matrix

$$G_k \equiv \tilde{W}^{\dagger(k+1)(k)} = R_{k+1}^\dagger U_g^{(k+1, k)} R_k. \quad (57)$$

Subsequently, the evolution operator $U(\mu, 0)$ of surface growth scheme can be expressed as

$$U(\mu, 0) = \lim_{\delta\mu \rightarrow 0} G_{k-1} \cdots G_1 G_0 = U_g(\mu, 0), \quad (58)$$

which means that the evolution operator $U(\mu, 0)$ is equivalent to the radial gravitational transfer matrix $U_g(\mu, 0)$, and the surface growth process can map to the transition of radial surfaces in asymptotic AdS_3 spacetime precisely.

Specifically, in the large central charge limit, the Hamilton-Jacobi relation gives

$$\pi^{ab} = \frac{\sqrt{\gamma}}{2} T^{ab}, \quad (59)$$

which gives

$$\pi^{ab} \pi_{ab} - \pi^2 = \frac{\gamma}{4} (T_{ab} T^{ab} - (T_c^c)^2), \quad (60)$$

which indicates that the radial transfer matrix is precisely governed by the $T\bar{T}$ operator $\mathcal{O}_{T\bar{T}} \equiv T_{ab} T^{ab} - (T_c^c)^2$, as shown in [45], and the on-shell action can give the expression of $T\bar{T}$ flow:

$$\frac{\partial}{\partial \mu} I_\mu = -\frac{1}{2} \int d^2x \sqrt{\gamma} \mathcal{O}_{T\bar{T}}. \quad (61)$$

In summary, we have obtained the evolution operator $U(\mu, 0)$ in the generalized surface growth scheme (homogeneous and isotropic case) and find that the tensor network can realize precisely the radial gravitational transfer process in its continuum limit, i.e., $U = U_g$. Then, in the large central charge limit, the tensor network can also realize the $T\bar{T}$ flow in asymptotic AdS_3 spacetime precisely, which corresponds to the $T\bar{T}$ -deformed CFT_2 [44,45].

IV. ESTABLISH THE CONNECTION BETWEEN SURFACE GROWTH AND $T\bar{T}$ DEFORMATION

In this section, we will establish the connection between the surface growth scheme and $T\bar{T}$ deformation of CFT_2 . Let us first recall the important properties of $T\bar{T}$ deformation. From the RG perspective, the $T\bar{T}$ operator is irrelevant, inducing significant modifications to the UV dynamics of the deformed CFT. Holographically, such deformations are conjectured to alter the IR geometry of the bulk spacetime. Notably, for negative coupling constants, the $T\bar{T}$ -deformed CFT has been suggested to be dual to gravitational physics in AdS_3 with a finite radial cutoff [45].

An alternative formulation suggested that the $T\bar{T}$ -deformed CFT admits a holographic dual description as AdS_3 governed by mixed boundary conditions [44]. Crucially, in pure Einstein gravity, the mixed boundary conditions at infinity can be reinterpreted as the Dirichlet boundary conditions at a finite cutoff surface. Therefore, the holographic duality with $T\bar{T}$ deformation modifies the conventional framework of AdS/CFT correspondence, which prompts a critical inquiry into the mechanism of spacetime emergence within this generalized correspondence. By using the surface-state correspondence [48,49],

this problem has been investigated in [50], which proposed that quantum states of $T\bar{T}$ deformed CFT correspond to finite radial cutoff surface of the bulk AdS spacetime. On the other hand, note that the iterative growth of bulk minimal surfaces which describes the radial evolution in AdS spacetime may directly map to the operator-driven deformation flow governed by the $T\bar{T}$ perturbation. In other words, it is expected that the $T\bar{T}$ deformation of boundary CFT can provide a dynamical mechanism for specific configuration of the surface growth.

A. $T\bar{T}$ deformation and the surface growth in asymptotically AdS_3 spacetime

In what follows, we will establish the connection between the surface growth scheme and the $T\bar{T}$ deformation by adopting the mixed boundary conditions formalism. Considering a spacetime with a negative cosmological constant, the solution of Einstein's equations can be expressed by the Fefferman-Graham (FG) expansion [51,52]

$$ds^2 = l^2 \frac{d\rho^2}{4\rho^2} + g_{ab}(\rho, x^a) dx^a dx^b, \\ g_{ab}(\rho, x^a) = \frac{1}{\rho} (g_{ab}^{(0)}(x^a) + \rho g_{ab}^{(2)}(x^a) + \rho^2 g_{ab}^{(4)}(x^a) + \dots), \quad (62)$$

where l is the curvature radius of the AdS spacetime. Especially, the expansion truncates at the second order in three dimensions. In this case, we have

$$g_{ab}^{(2)} = 8\pi G l (T_{ab} - g_{ab}^{(0)} T_c^c) \equiv 8\pi G l \hat{T}_{ab}, \quad (63)$$

where T_{ab} is the Brown-York stress tensor on the AdS boundary and $\hat{T}_{ab} \equiv T_{ab} - g_{ab}^{(0)} T_c^c$ is the reduced stress tensor. Then we have

$$g_{ab}^{(4)} = \frac{1}{4} g_{ac}^{(2)} g^{(0)cd} g_{bd}^{(2)}. \quad (64)$$

To match the radial surface, we define the $T\bar{T}$ deformation by adding the $T\bar{T}$ operator to the Einstein-Hilbert action:

$$\frac{\partial}{\partial \mu} I_\mu = -\frac{1}{2} \int d^2x \sqrt{\gamma} \mathcal{O}_{T\bar{T}}, \quad (65)$$

in which $\mathcal{O}_{T\bar{T}} = T^{ab} T_{ab} - (T_c^c)^2$. When μ is infinitesimally varied, the change in $\sqrt{\gamma} \mathcal{O}_{T\bar{T}}$ and T_{ab} is given by the flow equation

$$\frac{\partial}{\partial \mu} (\delta I) = -\delta \left(\frac{\partial}{\partial \mu} I \right) \Leftrightarrow \frac{\partial}{\partial \mu} (\sqrt{\gamma} T_{ab} \delta \gamma^{ab}) = \delta (\sqrt{\gamma} \mathcal{O}_{T\bar{T}}), \quad (66)$$

which can be solved as [44]

$$\begin{aligned}\gamma_{ab}^{(\mu)} &= \gamma_{ab}^{(0)} - 2\mu\hat{T}_{ab}^{(0)} + \mu^2\hat{T}_{ac}^{(0)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(0)}, \\ \hat{T}_{ab}^{(\mu)} &= \hat{T}_{ab}^{(0)} - \mu\hat{T}_{ac}^{(0)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(0)}, \\ \hat{T}_{ac}^{(\mu)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(\mu)} &= \hat{T}_{ac}^{(0)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(0)},\end{aligned}\quad (67)$$

where $\hat{T}_{ab} \equiv T_{ab} - \gamma_{ab}T^c_c$. By comparing with FG expansion in asymptotically AdS₃ spacetime, the $T\bar{T}$ deformation of CFT₂ can be identified with the radial cutoff surface of AdS₃ as

$$\begin{aligned}g_{ab}^{(0)} &= \gamma_{ab}^{(0)}, \\ g_{ab}^{(2)} &= 8\pi G l \hat{T}_{ab} \equiv 2C\hat{T}_{ab}, \\ g_{ab}^{(4)} &= C^2\hat{T}_{ac}^{(0)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(0)}, \\ \rho_c &= -\frac{\mu}{C}.\end{aligned}\quad (68)$$

B. Connection of operators between $T\bar{T}$ deformation and surface growth

To establish the operator connection between surface growth dynamics and $T\bar{T}$ deformation, we construct the evolution operator formalism governing hierarchical entanglement structure propagation. Within the surface growth scheme, the metric operator $\underline{\gamma}_{ab}$ and reduced stress tensor operator $\hat{T}_{ab}$ are defined through the corresponding expectation values in $T\bar{T}$ -deformed CFT₂:

$$\begin{aligned}\text{Tr}(\rho_V(0)\hat{T}_{ab}) &= \hat{T}_{ab}^{(0)}, \quad \text{Tr}(\rho_V(0)\underline{\gamma}_{ab}) = \gamma_{ab}^{(0)}, \\ \text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}) &= \gamma_{ab}^{(\mu)} = \gamma_{ab}^{(0)} - 2\mu\hat{T}_{ab}^{(0)} + \mu^2\hat{T}_{ac}^{(0)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(0)}, \\ \text{Tr}(\rho_V(\mu)\hat{T}_{ab}) &= \hat{T}_{ab}^{(\mu)} = \hat{T}_{ab}^{(0)} - \mu\hat{T}_{ac}^{(0)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(0)},\end{aligned}\quad (69)$$

where $\hat{T}_{ab}^{(\mu)} = \langle \hat{T}_{ab} \rangle^{(\mu)}$ is the expectation value of $\hat{T}_{ab}$ in $T\bar{T}$ -deformed CFT₂ and $\gamma_{ab}^{(\mu)}$ is the deformed metric, while $\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab})$ and $\text{Tr}(\rho_V(\mu)\hat{T}_{ab})$ are the expectation values defined by Eq. (50) at the growing surfaces, and $\rho_V(\mu)$ is the density matrix operator of the surfaces that grow out. With these definitions for a pair of tensors $\text{Tr}(\rho_V(\mu)\underline{\mathcal{O}}_{ab})$ at the growing surfaces and $\mathcal{O}_{ab}^{(\mu)}$ in CFT, the expectation value of the $\underline{\mathcal{O}}_{ab}\underline{\gamma}^{bc}$ is²

$$\begin{aligned}\text{Tr}(\rho_V(\mu)\underline{\mathcal{O}}_{ab}\underline{\gamma}^{bc}) &= \text{Tr}(\rho_V(\mu)\underline{\mathcal{O}}_a^c) = \mathcal{O}_a^{c(\mu)} = \mathcal{O}_{ab}^{(\mu)}\gamma^{bc} \\ &= \text{Tr}(\rho_V(\mu)\underline{\mathcal{O}}_{ab})\text{Tr}(\rho_V(\mu)\underline{\gamma}^{bc}),\end{aligned}\quad (70)$$

²In fact, the $\mathcal{O}_{ab}$ is just a virtual symbol in CFT, and it is defined by its expectation value $\mathcal{O}_{ab}^{(\mu)}$, which can be a function: $\mathcal{O}_{ab}^{(\mu)} = (\langle \mathcal{O}_1 \rangle^{(\mu)} \cdots \langle \mathcal{O}_n \rangle^{(\mu)})_{ab}$.

and, similarly, the metric operator $\underline{\gamma}^{bc}$ or $\underline{\gamma}_{bc}$ can be factored out from the other traced operators (e.g., $\underline{\mathcal{O}}_b^a\underline{\gamma}^{bc}$ and $\underline{\mathcal{O}}^{ab}\underline{\gamma}_{bc}$) when calculating the expectation values. Especially, according to $\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ac}\underline{\gamma}^{cb}) = \text{Tr}(\rho_V(\mu)\underline{\gamma}_{ac})\text{Tr}(\rho_V(\mu)\underline{\gamma}^{cb}) = \delta_a^b$, we can also request that the metric operators are satisfied with $\underline{\gamma}_{ac}\underline{\gamma}^{cb} = \underline{1}_a^b$ and $\underline{\gamma}^{ac}\underline{\gamma}_{cb} = \underline{1}_b^a$. Furthermore, for any scalar function $\underline{F}(\underline{\gamma}_{ab}, \hat{T}_{ab})$ which has the corresponding expectation value $F^{(\mu)}$ in CFT, the metric operator $\underline{\gamma}_{ab}$ can be factored out from the operators $\underline{\gamma}_{ab}\underline{F}$ when calculating the expectation values:

$$\begin{aligned}\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}\underline{F}) &= \text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}\delta_b^a\underline{F}) \\ &= \gamma_{ae}^{(\mu)}(\delta_b^e F)^{(\mu)} = \gamma_{ab}^{(\mu)}F^{(\mu)} \\ &= \text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab})\text{Tr}(\rho_V(\mu)\underline{F}).\end{aligned}\quad (71)$$

For the $\hat{T}_{ac}^{(0)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(0)}$ term of Eq. (69), we introduce the operators $\underline{L}_{ab}$ at the growing surfaces, defined by

$$\text{Tr}(\rho_V(0)\underline{L}_{ab}) = \hat{T}_{ac}^{(0)}\gamma_{(0)}^{cd}\hat{T}_{db}^{(0)},\quad (72)$$

and with Eq. (67), we have

$$\text{Tr}(\rho_V(\mu)\underline{L}_{ab}) = \text{Tr}(\rho_V(0)\underline{L}_{ab}).\quad (73)$$

According to Eqs. (58) and (53), the evolution operator $U(\mu, 0)$ implements radial flow through path ordered exponentiation:

$$U(a, b) = \mathcal{P}e^{-i \int_b^a A(\xi) d\xi}\quad (74)$$

and

$$U^\dagger(a, b) = \mathcal{P}^*e^{i \int_b^a A(\xi) d\xi},\quad (75)$$

where ξ is the coordinate in the μ parameter space, $A^\dagger(\xi) = A(\xi)$ is Hermitian, and the symbols $\mathcal{P}$ and $\mathcal{P}^*$ mean the reverse path ordering. By moving to the ‘‘Heisenberg picture,’’ the evolution of $\underline{\gamma}_{ab}$, $\hat{T}_{ab}$, and $\underline{L}_{ab}$ is given by

$$\begin{aligned}U^\dagger(\mu, 0)\underline{\gamma}_{ab}U(\mu, 0) &= \underline{\gamma}_{ab} - 2\mu\hat{T}_{ab} + \mu^2\underline{L}_{ab}, \\ U^\dagger(\mu, 0)\hat{T}_{ab}U(\mu, 0) &= \hat{T}_{ab} - \mu\underline{L}_{ab}, \\ U^\dagger(\mu, 0)\underline{L}_{ab}U(\mu, 0) &= \underline{L}_{ab}.\end{aligned}\quad (76)$$

In the Heisenberg picture, the operators are $\mathcal{O}(\mu) = U^\dagger(\mu, 0)\mathcal{O}(0)U(\mu, 0)$, and Eq. (76) can be rewritten as

$$\begin{aligned}
 iU^\dagger(\mu, 0)[A(\mu), \underline{\gamma}_{ab}]U(\mu, 0) &= \frac{\partial}{\partial \mu} \underline{\gamma}_{ab}(\mu) = -2\hat{T}_{ab}(\mu), \\
 iU^\dagger(\mu, 0)[A(\mu), \hat{T}_{ab}]U(\mu, 0) &= \frac{\partial}{\partial \mu} \hat{T}_{ab}(\mu) = -\underline{L}_{ab}(\mu);
 \end{aligned} \tag{77}$$

then we obtain

$$\begin{aligned}
 [A(\mu), \underline{\gamma}_{ab}] &= 2i\hat{T}_{ab}, \\
 [A(\mu), \hat{T}_{ab}] &= i\underline{L}_{ab}.
 \end{aligned} \tag{78}$$

With these formulas [(77) and (78)], we obtain (please see Appendix A for more details)

$$\begin{aligned}
 [A(\mu), \underline{L}_{ab}] &= 0, \\
 [A(\mu), \underline{\gamma}^{ab}] &= -2i\underline{\gamma}^{ac}\hat{T}_{cd}\underline{\gamma}^{db}.
 \end{aligned} \tag{79}$$

Because of the fact that μ is arbitrary in the range of $0 \leq |\mu| \leq |\mu_h|$, there exists a constant operator Q separated from the operator $A(\mu)$:

$$A(\mu) = K(\mu) + Q, \tag{80}$$

such that

$$\begin{aligned}
 [K(\mu), \underline{\gamma}_{ab}] &= 0, & [Q, \underline{\gamma}_{ab}] &= 2i\hat{T}_{ab}, \\
 [K(\mu), \hat{T}_{ab}] &= 0, & [Q, \hat{T}_{ab}] &= i\underline{L}_{ab}, \\
 [K(\mu), \underline{L}_{ab}] &= 0, & [Q, \underline{L}_{ab}] &= 0, \\
 [K(\mu), \underline{\gamma}^{ab}] &= 0, & [Q, \underline{\gamma}^{ab}] &= -2i\underline{\gamma}^{ac}\hat{T}_{cd}\underline{\gamma}^{db},
 \end{aligned} \tag{81}$$

where $K(\mu)$ is the other part of $A(\mu)$ affected by the parameter μ . In the surface growth process, $K(\mu)$ can be treated as the disentangler and Q can be treated as the coarse grainer reducing the dimension of the labels. Recall that, in Eq. (16), the tensor in the surface growth process associated with $A(\mu)$ is $\tilde{W}_{a^{(k)}}^{(k)(k-1)a^{(k-1)}}$, consisting of two tensor groups, i.e., the coarse grainer and disentangler, which correspond to the operators Q and $K(\mu)$, respectively. Specifically, the coarse-grainer operator Q corresponds to dimensional reduction³ from index $a^{(k-1)}$ to $a^{(k)}$, which can be mathematically expressed as $\rho_V(k) \oplus \mathbf{0}(k) \rightarrow \rho_V(k)$, while the disentangler $K(\mu)$ accounts for the remaining components—including the surface redivision in $\chi^{(k-1)}$ and the unitary transformation on two Hilbert spaces related with each reconstructed surface in $W^{(k)}$, which can be mathematically expressed as

³In this case, the coarse-grainer operator Q is Hermitian counterintuitively, because the dimensional reduction does not diminish the information of the system, i.e., $S_V(k-1) = S_V(k)$. However, the coarse-grainer operation on the system is still irreversible due to the reduction of effective dimension.

$\rho_V(k-1) \rightarrow \rho_V(k) \oplus \mathbf{0}(k)$. Especially, there is no $K(\mu_h)$ such that $\rho_V(\mu_h) \rightarrow \rho'_V \oplus \mathbf{0}$ on the horizon and the coarse-grainer operator Q is unable to diminish the dimension, corresponding to the limit of surface growth Eq. (26).

In conclusion, the $T\bar{T}$ deformation directly maps to the radial evolution step in the surface growth scheme. The commutator structure $[Q, \underline{\mathcal{O}}]$, where Q generates the deformation of $\underline{\gamma}_{ab}$ and $\hat{T}_{ab}$ encodes the bulk diffeomorphism invariance (see Fig. 6)

$$i[Q, \underline{\mathcal{O}}] \sim \frac{\partial}{\partial \mu} \langle \mathcal{O} \rangle_\mu. \tag{82}$$

Therefore, for the homogenous, isotropic, and continuum case, the dynamical mechanism of iterative growth of bulk minimal surfaces in asymptotical AdS spacetime can be described by the radial RG flow driven by the $T\bar{T}$ deformation of boundary CFT, which provides a dynamical mechanism for the surface growth scheme and may shed light on reconstructing bulk gravitational dynamics via this scheme.

Moreover, a further evidence for the connection between the $T\bar{T}$ flow and the entanglement is from the entanglement first law [53–55]. Explicitly, the radial evolution of the boundary (the cutoff surface) is generated by the $T\bar{T}$ deformation, and the energy of a subsystem A (which can be viewed as a segment of the boundary for growing surfaces at a given layer) at the cutoff surface ρ_c with spatial interval l is

$$E_A(\mu) = \int_0^l T_{\tau\tau}^{(\mu)} d\sigma = l T_{\tau\tau}^{(\mu)}, \tag{83}$$

where τ and σ are, respectively, the timelike and spacelike coordinates on the cutoff surface [44]. Taking the partial derivative of $T_{\tau\tau}^{(\mu)}$ with respect to μ gives

$$\frac{\partial}{\partial \mu} T_{\tau\tau}^{(\mu)} = \frac{1}{2} \mathcal{O}_{T\bar{T}}, \tag{84}$$

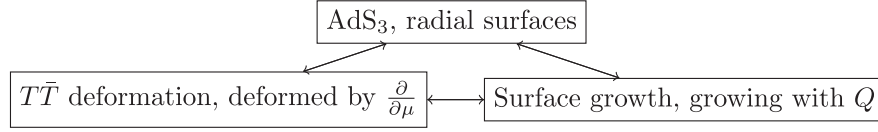
which is equivalent to

$$\frac{\partial}{\partial \mu} \frac{E_A(\mu)}{l} = \frac{1}{2} \mathcal{O}_{T\bar{T}}. \tag{85}$$

Then making small perturbations to Eq. (85) gives

$$\frac{\partial}{\partial \mu} \frac{\Delta E_A(\mu)}{l} = \frac{1}{2} \Delta \mathcal{O}_{T\bar{T}}. \tag{86}$$

Furthermore, under the low-energy excitation, analogous to the first law of thermodynamics, the variation of energy and entanglement entropy of the subsystem A can be expressed in a first-law-like relation as $\Delta E_A(\mu) = T_e \Delta S_A(\mu)$, called the entanglement first law, where T_e is termed as the

FIG. 6. The connection between the surface growth scheme and $T\bar{T}$ deformation.

entanglement temperature, which is proportional to the reciprocal of the size of the subsystem, i.e., $T_e = c/l$, with c a constant [53]. Thus, using the entanglement first law, Eq. (86) becomes

$$\frac{\partial}{\partial \mu} \left(\frac{T_e}{c} \Delta E_A(\mu) \right) = \frac{\partial}{\partial \mu} \left(\frac{T_e^2}{c} \Delta S_A(\mu) \right) = \frac{1}{2} \Delta \mathcal{O}_{T\bar{T}}. \quad (87)$$

Therefore, Eq. (87) shows that small perturbations of the $T\bar{T}$ operator will induce the radial flow of small perturbations of the holographic entanglement entropy. Equivalently, at small perturbations, the radial flow of the bulk minimal surfaces at the cutoff surface (which is described by the surface growth process) can generate the $T\bar{T}$ operator.

C. The properties of $\underline{L}_{ab}$ and the factorization formula of $T\bar{T}$ operator

In order to figure out the properties of $\underline{L}_{ab}$, we consider the expectation values $\text{Tr}(\rho_V(\mu) \hat{T}_{ab} \underline{\gamma}^{ab})$ with $\underline{\Delta}_{ab} \equiv \underline{L}_{ab} - \hat{T}_{ac} \underline{\gamma}^{cd} \hat{T}_{db}$ (the detail of calculation is in Appendix B):

$$\begin{aligned} & \text{Tr}(\rho_V(\mu) \hat{T}_{ab} \underline{\gamma}^{ab}) \\ &= \text{Tr}(\rho_V(0) U^\dagger(\mu, 0) \hat{T}_{ab} \underline{\gamma}^{ab} U(\mu, 0)) \\ &= \text{Tr}(\rho_V(0) \hat{T}_{ab} \underline{\gamma}^{ab}) + i\mu \text{Tr}(\rho_V(0) [\mathcal{Q}, \hat{T}_{ab} \underline{\gamma}^{ab}]) + \mathcal{O}(\mu^2) \\ &= \text{Tr}(\rho_V(0) \hat{T}_{ab} \underline{\gamma}^{ab}) + \mu \text{Tr}(\rho_V(0) \underline{L}_{ab} \underline{\gamma}^{ab}) \\ &\quad - 2\mu \text{Tr}(\rho_V(0) \underline{\Delta}_{ab} \underline{\gamma}^{ab}) + \mathcal{O}(\mu^2), \end{aligned} \quad (88)$$

while the expectation value $\hat{T}_{ab}^{(\mu)} \underline{\gamma}_{(\mu)}^{ab}$ has the same expansion [44]:

$$\hat{T}^{(\mu)} = \hat{T}^{(0)} + \mu \hat{T}_{ab}^{(0)} \underline{\gamma}_{(0)}^{bc} \hat{T}_{cd}^{(0)} \underline{\gamma}_{(0)}^{da} + \mathcal{O}(\mu^2). \quad (89)$$

With the definition $\text{Tr}(\rho_V(\mu) \hat{T}_{ab} \underline{\gamma}^{ab}) = \hat{T}^{(\mu)}$, we focus on the μ^1 -linear terms:

$$\text{Tr}(\rho_V(0) \underline{L}_{ab} \underline{\gamma}^{ab}) - 2\text{Tr}(\rho_V(0) \underline{\Delta}_{ab} \underline{\gamma}^{ab}) = \hat{T}_{ab}^{(0)} \underline{\gamma}_{(0)}^{bc} \hat{T}_{cd}^{(0)} \underline{\gamma}_{(0)}^{da}. \quad (90)$$

Because of the definition Eq. (72), $\underline{L}_{ab} \underline{\gamma}^{ab}$ can factor out $\underline{\gamma}^{ab}$ when calculating its expectation value:

$$\begin{aligned} \text{Tr}(\rho_V(0) \underline{L}_{ab} \underline{\gamma}^{ab}) &= \text{Tr}(\rho_V(0) \underline{L}_{ab}) \text{Tr}(\rho_V(0) \underline{\gamma}^{ab}) \\ &= \hat{T}_{ab}^{(0)} \underline{\gamma}_{(0)}^{bc} \hat{T}_{cd}^{(0)} \underline{\gamma}_{(0)}^{da}. \end{aligned} \quad (91)$$

Substituting it into Eq. (90), we find

$$\text{Tr}(\rho_V(0) \underline{\Delta}_{ab} \underline{\gamma}^{ab}) = 0. \quad (92)$$

Then, with the definition $\underline{\Delta}_{ab} = \underline{L}_{ab} - \hat{T}_{ac} \underline{\gamma}^{cd} \hat{T}_{db}$, the contractions of $\underline{L}_{ab}$ and $\hat{T}_{ac} \underline{\gamma}^{cd} \hat{T}_{db}$ have an equal expectation value:

$$\text{Tr}(\rho_V(0) \underline{L}_{ab} \underline{\gamma}^{ab}) = \text{Tr}(\rho_V(0) \hat{T}_{ab} \underline{\gamma}^{bc} \hat{T}_{cd} \underline{\gamma}^{da}), \quad (93)$$

and a factorization of the double-traced operator is obtained as

$$\begin{aligned} & \text{Tr}(\rho_V(0) \hat{T}_{ab} \underline{\gamma}^{bc} \hat{T}_{cd} \underline{\gamma}^{da}) \\ &= \text{Tr}(\rho_V(0) \hat{T}_{ab}) \text{Tr}(\rho_V(0) \underline{\gamma}^{bc}) \text{Tr}(\rho_V(0) \hat{T}_{cd}) \text{Tr}(\rho_V(0) \underline{\gamma}^{da}), \end{aligned} \quad (94)$$

which is similar as the first term in Zamolodchikov's factorization formula in CFT_2 :

$$\begin{aligned} & \langle n | (T_{ab} T^{ab} - (T_a^a)^2) | n \rangle \\ &= \langle n | T_{ab} | n \rangle \langle n | T^{ab} | n \rangle - \langle n | T_a^a | n \rangle \langle n | T_b^b | n \rangle. \end{aligned} \quad (95)$$

In order to compare with the second term in Eq. (95), let us expand the expectation values $\text{Tr}(\rho_V(\mu) \underline{\gamma}_{ab} \hat{T}_{cd} \underline{\gamma}^{cd})$ until the μ^1 -linear term (the detail of calculation is in Appendix B):

$$\begin{aligned} & \text{Tr}(\rho_V(\mu) \underline{\gamma}_{ab} \hat{T}_{cd} \underline{\gamma}^{cd}) \\ &= \text{Tr}(\rho_V(0) U^\dagger(0, \mu) \underline{\gamma}_{ab} \hat{T}_{cd} \underline{\gamma}^{cd} U(0, \mu)) \\ &= \text{Tr}(\rho_V(0) \underline{\gamma}_{ab} \hat{T}_{cd} \underline{\gamma}^{cd}) + i\mu \text{Tr}(\rho_V(0) [\mathcal{Q}, \underline{\gamma}_{ab} \hat{T}_{cd} \underline{\gamma}^{cd}]) \\ &\quad + \mathcal{O}(\mu^2) \\ &= \text{Tr}(\rho_V(0) \underline{\gamma}_{ab} \hat{T}_{cd} \underline{\gamma}^{cd}) - 2\mu \text{Tr}(\rho_V(0) \hat{T}_{ab} \hat{T}_{cd} \underline{\gamma}^{cd}) \\ &\quad + \mu \text{Tr}(\rho_V(0) \underline{\gamma}_{ab}) \text{Tr}(\rho_V(0) \underline{L}_{cd} \underline{\gamma}^{cd}) + \mathcal{O}(\mu^2). \end{aligned} \quad (96)$$

With Eq. (71), $\text{Tr}(\rho_V(\mu) \underline{\gamma}_{ab} \hat{T}_{cd} \underline{\gamma}^{cd})$ can be factorized into $\text{Tr}(\rho_V(\mu) \underline{\gamma}_{ab}) \text{Tr}(\rho_V(\mu) \hat{T}_{cd} \underline{\gamma}^{cd})$, which equals $\underline{\gamma}_{ab}^{(\mu)} \hat{T}_c^{c(\mu)}$. Then we expand $\underline{\gamma}_{ab}^{(\mu)} \hat{T}_c^{c(\mu)}$ until the μ^1 -linear term:

$$\begin{aligned}
 \gamma_{ab}^{(\mu)} \hat{T}_c^{c(\mu)} &= \gamma_{ab}^{(0)} \hat{T}_c^{c(0)} + \mu \left(\frac{\partial}{\partial \xi} (\gamma_{ab}^{(\xi)} \hat{T}_c^{c(\xi)}) \right) \Big|_{\xi=0} + \mathcal{O}(\mu^2) \\
 &= \gamma_{ab}^{(0)} \hat{T}_c^{c(0)} - 2\mu \hat{T}_{ab}^{(0)} \hat{T}_c^{c(0)} + \mu \gamma_{ab}^{(0)} \hat{T}_{cd}^{(0)} \hat{T}_{(0)}^{cd} + \mathcal{O}(\mu^2).
 \end{aligned} \tag{97}$$

Equating the μ^1 -linear terms in Eqs. (96) and (97):

$$\begin{aligned}
 &-2\mu \text{Tr}(\rho_V(0) \hat{T}_{ab} \hat{T}_{cd} \gamma_{cd}) + \mu \text{Tr}(\rho_V(0) \gamma_{ab}) \text{Tr}(\rho_V(0) \hat{T}_{cd} \gamma_{cd}) \\
 &= -2\mu \hat{T}_{ab}^{(0)} \hat{T}_c^{c(0)} + \mu \gamma_{ab}^{(0)} \hat{T}_{cd}^{(0)} \hat{T}_{(0)}^{cd},
 \end{aligned} \tag{98}$$

we can obtain

$$\text{Tr}(\rho_V(0) \hat{T}_{ab} \hat{T}_{cd} \gamma_{cd}) = \hat{T}_{ab}^{(0)} \hat{T}_c^{c(0)} \tag{99}$$

by substituting Eq. (93).

Finally, the other double-traced operator is obtained by substituting Eq. (99):

$$\begin{aligned}
 &\text{Tr}(\rho_V(0) \hat{T}_{ab} \gamma_{ab} \hat{T}_{cd} \gamma_{cd}) \\
 &= \text{Tr}(\rho_V(0) \gamma_{ab}) \text{Tr}(\rho_V(0) \hat{T}_{ab} \hat{T}_{cd} \gamma_{cd}) \\
 &= \gamma_{(0)}^{ab} \hat{T}_{ab}^{(0)} \hat{T}_{cd}^{(0)} \gamma_{(0)}^{cd} \\
 &= \text{Tr}(\rho_V(0) \hat{T}_{ab}) \text{Tr}(\rho_V(0) \gamma_{ab}) \\
 &\quad \times \text{Tr}(\rho_V(0) \hat{T}_{cd}) \text{Tr}(\rho_V(0) \gamma_{cd}),
 \end{aligned} \tag{100}$$

which corresponds to the second term in Eq. (95).

Therefore, the operator in the surface growth scheme corresponding with the $T\bar{T}$ operator in CFT_2 is given:

$$\begin{aligned}
 &\langle T_{ab} T^{ab} - (T_a^a)^2 \rangle^{(0)} \\
 &= T_{ab}^{(0)} T_{(0)}^{ab} - (T_a^{a(0)})^2 = \hat{T}_{ab}^{(0)} \hat{T}_{(0)}^{ab} - (\hat{T}_a^{a(0)})^2 \\
 &= \text{Tr}(\rho_V(0) (\hat{T}_{ab} \gamma_{bc} \hat{T}_{cd} \gamma_{da} - \hat{T}_{ab} \gamma_{ab} \hat{T}_{cd} \gamma_{cd})),
 \end{aligned} \tag{101}$$

which has the same factorization formula:

$$\begin{aligned}
 &\text{Tr}(\rho_V(0) (\hat{T}_{ab} \gamma_{bc} \hat{T}_{cd} \gamma_{da} - \hat{T}_{ab} \gamma_{ab} \hat{T}_{cd} \gamma_{cd})) \\
 &= \text{Tr}(\rho_V(0) \hat{T}_{ab}) \text{Tr}(\rho_V(0) \gamma_{bc}) \text{Tr}(\rho_V(0) \hat{T}_{cd}) \text{Tr}(\rho_V(0) \gamma_{da}) \\
 &\quad - \text{Tr}(\rho_V(0) \hat{T}_{ab}) \text{Tr}(\rho_V(0) \gamma_{ab}) \text{Tr}(\rho_V(0) \hat{T}_{cd}) \text{Tr}(\rho_V(0) \gamma_{cd}).
 \end{aligned} \tag{102}$$

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we extended the OSED tensor network to describe the more general surface growth scheme for bulk reconstruction in asymptotically AdS_3 spacetime, which provided a state-to-geometry mapping between quantum states of tensor network and bulk minimal surfaces. We then analyzed the behavior of geometric convergence and

continuum limit of the general surface growth scheme as the partition number $N \rightarrow \infty$, which bridges the tensor network construction with the continuous radial evolution of the bulk geometry. Finally, through an evolution operator formalism within the tensor network framework, we established an explicit connection between the surface growth process and $T\bar{T}$ -deformed CFT_2 . By implementing the operators γ_{ab} and $\hat{T}_{ab}$ as fundamental building blocks in this scheme, we demonstrated systematic reconstruction of bulk geometry in asymptotically AdS_3 spacetime from the operators of the boundary CFT_2 . Namely, the geometrical radial evolution in both frameworks indicates that the $T\bar{T}$ deformation of boundary CFT_2 can provide a dynamical mechanism for the surface growth in asymptotically AdS_3 spacetime. Moreover, we found further evidence that the $T\bar{T}$ deformation can be connected with the entanglement from the entanglement first law, which showed that, at small perturbations, the radial flow of the entanglement entropy (which is described by the growth of bulk minimal surfaces) can generate the $T\bar{T}$ operator, or small perturbations of the $T\bar{T}$ operator can induce the radial evolution of small perturbations of the entanglement entropy.

Note that our present framework is formulated only within the $\text{AdS}_3/\text{CFT}_2$ correspondence; there are still many interesting problems to be studied such as the extension into higher-dimensional spacetime. However, there will be many difficulties, for example, (i) resolution of nontrivial truncation in the FG expansion and (ii) the nonlocality of stress tensors, since the absence of finite FG truncation renders metric and stress tensor deformations ill defined under conventional $T\bar{T}$ operator flows in $d \geq 3$ -dimensional spacetime. In addition, the current implementation of the operators γ_{ab} and $\hat{T}_{ab}$ assumed homogenous and isotropic radial propagation, the extension into inhomogeneous and anisotropic surface growth scheme and its connection with other operator deformation requires further investigation. Furthermore, a more detailed study on the entanglement first law in the surface growth process is needed to further understand its dynamical origin; we would like to study this problem in a further work.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A

2. Eq. (79): $[A(\mu), \underline{\gamma}^{ab}]$

In this Appendix, we will derive some formulas.

We have

$$[A(\mu), \underline{\gamma}_{ab}] = 2i\hat{T}_{ab}; \quad (\text{A4})$$

1. Eq. (79): $[A(\mu), \underline{L}_{ab}]$

With Eqs. (77) and (78), we have

then $[A(\mu), \underline{\gamma}^{ab}]$ is given that

$$U^\dagger(\mu, 0)\underline{L}_{ab}(0)U(\mu, 0) = \underline{L}_{ab}(\mu). \quad (\text{A1})$$

$$\begin{aligned} [A(\mu), \underline{\gamma}^{ab}] &= [A(\mu), \underline{\gamma}^{ac}\underline{\gamma}_{cd}\underline{\gamma}^{db}] \\ &= 2[A(\mu), \underline{\gamma}^{ab}] + \underline{\gamma}^{ac}[A(\mu), \underline{\gamma}_{cd}]\underline{\gamma}^{db} \\ &= 2[A(\mu), \underline{\gamma}^{ab}] + 2i\underline{\gamma}^{ac}\hat{T}_{cd}\underline{\gamma}^{db}. \end{aligned} \quad (\text{A5})$$

In the Heisenberg picture, it can be rewritten as

$$iU^\dagger(\mu, 0)[A(\mu), \underline{L}_{ab}]U(\mu, 0) = \frac{\partial}{\partial\mu}\underline{L}_{ab}(\mu) = 0. \quad (\text{A2})$$

Finally, we obtain

$$[A(\mu), \underline{\gamma}^{ab}] = -2i\underline{\gamma}^{ac}\hat{T}_{cd}\underline{\gamma}^{db}. \quad (\text{A6})$$

Finally, we obtain

$$[A(\mu), \underline{L}_{ab}] = 0. \quad (\text{A3})$$

3. $U^\dagger(a, b)\mathcal{O}U(a, b)$ with $[K(\mu), \mathcal{O}] = 0$

The evolution operator from b to a is defined as

$$\begin{aligned} U(a, b) &= \mathcal{P}e^{-i\int_b^a A(\xi)d\xi} = 1 + (-i)\int_b^a d\xi_1 A(\xi_1) + \frac{(-i)^2}{2}\mathcal{P}\left(\int_b^a d\xi_1 \int_b^a d\xi_2 A(\xi_1)A(\xi_2)\right) + \cdots \\ &\quad + \frac{(-i)^n}{n!}\mathcal{P}\left(\int_b^a d\xi_1 \cdots \int_b^a d\xi_n A(\xi_1) \cdots A(\xi_n)\right) + \cdots, \end{aligned} \quad (\text{A7})$$

and we obtain

$$\begin{aligned} U^\dagger(a, b) &= U(b, a) = \mathcal{P}e^{-i\int_a^b A(\xi)d\xi} = 1 + (-i)\int_a^b d\xi_1 A(\xi_1) + \frac{(-i)^2}{2}\mathcal{P}\left(\int_a^b d\xi_1 \int_a^b d\xi_2 A(\xi_1)A(\xi_2)\right) + \cdots \\ &\quad + \frac{(-i)^n}{n!}\mathcal{P}\left(\int_a^b d\xi_1 \cdots \int_a^b d\xi_n A(\xi_1) \cdots A(\xi_n)\right) + \cdots \\ &= 1 + (i)\int_b^a d\xi_1 A(\xi_1) + \frac{(i)^2}{2}\mathcal{P}^*\left(\int_b^a d\xi_1 \int_b^a d\xi_2 A(\xi_1)A(\xi_2)\right) + \cdots \\ &\quad + \frac{(i)^n}{n!}\mathcal{P}^*\left(\int_b^a d\xi_1 \cdots \int_b^a d\xi_n A(\xi_1) \cdots A(\xi_n)\right) + \cdots \\ &= \mathcal{P}^*e^{i\int_b^a A(\xi)d\xi}, \end{aligned} \quad (\text{A8})$$

where $\mathcal{P}^*$ means the reverse path ordering.

Considering an operator $\mathcal{O}$ changed with the evolution operator $U(a, b)$, we can write the terms of the expansion as

$$\begin{aligned} U^\dagger(a, b)\mathcal{O}U(a, b): \\ \mu^0 \text{ term} &= \mathcal{O}, \\ \mu^1 \text{ term} &= (i)\int_b^a d\xi_1 A(\xi_1)\mathcal{O} + (-i)\mathcal{O}\int_b^a d\xi_1 A(\xi_1) \\ &= i\int_b^a d\xi_1 [A(\xi_1), \mathcal{O}] \\ &= i(a-b)[Q, \mathcal{P}], \\ &\cdots \end{aligned} \quad (\text{A9})$$

This expansion is used in Appendix B.

APPENDIX B

In this Appendix, we will derive some formulas.

1. Eq. (88): $\text{Tr}(\rho_V(\mu)\hat{T}_{ab}\underline{\gamma}^{ab})$

Expand $\text{Tr}(\rho_V(\mu)\hat{T}_{ab}\underline{\gamma}^{ab})$ until the μ^1 -linear term with (A9):

$$\begin{aligned}\text{Tr}(\rho_V(\mu)\hat{T}_{ab}\underline{\gamma}^{ab}) &= \text{Tr}(\rho_V(0)U^\dagger(\mu, 0)\hat{T}_{ab}\underline{\gamma}^{ab}U(\mu, 0)) \\ &= \text{Tr}(\rho_V(0)\hat{T}_{ab}\underline{\gamma}^{ab}) + i\mu\text{Tr}(\rho_V(0)[Q, \hat{T}_{ab}\underline{\gamma}^{ab}]) + \mathcal{O}(\mu^2),\end{aligned}\quad (\text{B1})$$

where $[Q, \hat{T}_{ab}\underline{\gamma}^{ab}]$ is given by Eq. (81):

$$[Q, \hat{T}_{ab}\underline{\gamma}^{ab}] = [Q, \hat{T}_{ab}]\underline{\gamma}^{ab} + \hat{T}_{ab}[Q, \underline{\gamma}^{ab}] = i\underline{L}_{ab}\underline{\gamma}^{ab} - 2i\hat{T}_{ab}\underline{\gamma}^{bc}\hat{T}_{cd}\underline{\gamma}^{da}.\quad (\text{B2})$$

Then the expansion $\text{Tr}(\rho_V(\mu)\hat{T}_{ab}\underline{\gamma}^{ab})$ can be expressed as

$$\text{Tr}(\rho_V(\mu)\hat{T}_{ab}\underline{\gamma}^{ab}) = \text{Tr}(\rho_V(0)\hat{T}_{ab}\underline{\gamma}^{ab}) + \mu\text{Tr}(\rho_V(0)(2\hat{T}_{ab}\underline{\gamma}^{bc}\hat{T}_{cd}\underline{\gamma}^{da} - \underline{L}_{ab}\underline{\gamma}^{ab})) + \mathcal{O}(\mu^2).\quad (\text{B3})$$

Substituting the definition $\Delta_{ab} \equiv \underline{L}_{ab} - \hat{T}_{ac}\underline{\gamma}^{cd}\hat{T}_{db}$ into the μ^1 -linear term, we obtain Eq. (88):

$$\text{Tr}(\rho_V(\mu)\hat{T}_{ab}\underline{\gamma}^{ab}) = \text{Tr}(\rho_V(0)\hat{T}_{ab}\underline{\gamma}^{ab}) + \mu\text{Tr}(\rho_V(0)\underline{L}_{ab}\underline{\gamma}^{ab}) - 2\mu\text{Tr}(\rho_V(0)\Delta_{ab}\underline{\gamma}^{ab}) + \mathcal{O}(\mu^2).\quad (\text{B4})$$

2. Eq. (96): $\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd})$

Expand $\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd})$ until the μ^1 -linear term with (A9):

$$\begin{aligned}\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}) &= \text{Tr}(\rho_V(0)U^\dagger(0, \mu)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}U(0, \mu)) \\ &= \text{Tr}(\rho_V(0)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}) + i\mu\text{Tr}(\rho_V(0)[Q, \underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}]) + \mathcal{O}(\mu^2),\end{aligned}\quad (\text{B5})$$

where $[Q, \underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}]$ is given by Eqs. (81) and (B2):

$$\begin{aligned}[Q, \underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}] &= [Q, \underline{\gamma}_{ab}]\hat{T}_{cd}\underline{\gamma}^{cd} + \underline{\gamma}_{ab}[Q, \hat{T}_{cd}\underline{\gamma}^{cd}] \\ &= 2i\hat{T}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd} + \underline{\gamma}_{ab}(i\underline{L}_{cd}\underline{\gamma}^{cd} - 2i\hat{T}_{cd}\underline{\gamma}^{de}\hat{T}_{ef}\underline{\gamma}^{fc}) \\ &= 2i\hat{T}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd} + i\underline{\gamma}_{ab}\underline{L}_{cd}\underline{\gamma}^{cd} - 2i\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{de}\hat{T}_{ef}\underline{\gamma}^{fc}.\end{aligned}\quad (\text{B6})$$

Then the expansion $\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd})$ can be rewritten as

$$\begin{aligned}\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}) &= \text{Tr}(\rho_V(0)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}) - 2\mu\text{Tr}(\rho_V(0)\hat{T}_{ab}\hat{T}_{cd}\underline{\gamma}^{cd}) \\ &\quad - \mu\text{Tr}(\rho_V(0)\underline{\gamma}_{ab}\underline{L}_{cd}\underline{\gamma}^{cd}) + 2\mu\text{Tr}(\rho_V(0)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{de}\hat{T}_{ef}\underline{\gamma}^{fc}) + \mathcal{O}(\mu^2).\end{aligned}\quad (\text{B7})$$

Because $\text{Tr}(\rho_V(0)\hat{T}_{cd}\underline{\gamma}^{de}\hat{T}_{ef}\underline{\gamma}^{fc}) = \text{Tr}(\rho_V(0)\underline{L}_{cd}\underline{\gamma}^{cd}) = \hat{T}_{cd}^{(0)}\underline{\gamma}_{(0)}^{de}\hat{T}_{ef}^{(0)}\underline{\gamma}_{(0)}^{fc}$, $\underline{\gamma}_{ab}$ can be factored out by Eq. (71):

$$\text{Tr}(\rho_V(0)\underline{\gamma}_{ab}\hat{T}_{cd}\underline{\gamma}^{de}\hat{T}_{ef}\underline{\gamma}^{fc}) = \text{Tr}(\rho_V(0)\underline{\gamma}_{ab})\text{Tr}(\rho_V(0)\hat{T}_{cd}\underline{\gamma}^{de}\hat{T}_{ef}\underline{\gamma}^{fc}) = \text{Tr}(\rho_V(0)\underline{\gamma}_{ab})\text{Tr}(\rho_V(0)\underline{L}_{cd}\underline{\gamma}^{cd}),\quad (\text{B8})$$

$$\text{Tr}(\rho_V(0)\underline{\gamma}_{ab}\underline{L}_{cd}\underline{\gamma}^{cd}) = \text{Tr}(\rho_V(0)\underline{\gamma}_{ab})\text{Tr}(\rho_V(0)\underline{L}_{cd}\underline{\gamma}^{cd}).\quad (\text{B9})$$

Substituting it into Eq. (B7), we obtain Eq. (96) finally:

$$\begin{aligned}
\text{Tr}(\rho_V(\mu)\underline{\gamma}_{ab}\hat{\underline{T}}_{cd}\underline{\gamma}^{cd}) &= \text{Tr}(\rho_V(0)\underline{\gamma}_{ab}\hat{\underline{T}}_{cd}\underline{\gamma}^{cd}) - 2\mu\text{Tr}(\rho_V(0)\hat{\underline{T}}_{ab}\hat{\underline{T}}_{cd}\underline{\gamma}^{cd}) - \mu\text{Tr}\text{Tr}(\rho_V(0)\underline{\gamma}_{ab})\text{Tr}(\rho_V(0)\underline{L}_{cd}\underline{\gamma}^{cd}) \\
&\quad + 2\mu\text{Tr}(\rho_V(0)\underline{\gamma}_{ab})\text{Tr}(\rho_V(0)\hat{\underline{T}}_{cd}\underline{\gamma}^{de}\hat{\underline{T}}_{ef}\underline{\gamma}^{fc}) + \mathcal{O}(\mu^2) \\
&= \text{Tr}(\rho_V(0)\underline{\gamma}_{ab}\hat{\underline{T}}_{cd}\underline{\gamma}^{cd}) - 2\mu\text{Tr}(\rho_V(0)\hat{\underline{T}}_{ab}\hat{\underline{T}}_{cd}\underline{\gamma}^{cd}) + \mu\text{Tr}(\rho_V(0)\underline{\gamma}_{ab})\text{Tr}(\rho_V(0)\underline{L}_{cd}\underline{\gamma}^{cd}) + \mathcal{O}(\mu^2). \quad (\text{B10})
\end{aligned}$$

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