

Minimal-doubling and single-Weyl fermion Hamiltonians

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We develop a systematic Hamiltonian formulation of minimally doubled lattice fermions in $(3 + 1)$ dimensions, derive their nodal structures (structures of zeros), and classify their symmetry patterns for both four-component Dirac and two-component Weyl constructions. Motivated by recent single-Weyl proposals based on Bogoliubov-de Gennes representation, we argue that the corresponding single-Weyl Hamiltonians are obtained from the minimal-doubling Hamiltonians supplemented by an appropriate species-splitting mass term, and we reexamine the non-on-site symmetry protecting the physical Weyl node in terms of a Ginsparg-Wilson-type relation. We then construct a one-parameter family of deformations that preserves all the symmetries and demonstrate that, once the parameter exceeds a critical value, additional Weyl nodes emerge and the system exits the single-node regime. This indicates that in interacting theories radiative corrections can generate symmetry-allowed counterterms, so maintaining the desired single-Weyl phase generically requires “moderate” parameter tuning.

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I. INTRODUCTION

Formulating chiral fermions on a lattice is a long-standing problem at the intersection of quantum field theory, condensed-matter band theory, and nonperturbative gauge dynamics [1,2]. The Nielsen-Ninomiya no-go theorem is most generally formulated in terms of conserved on site charges with discrete spectra, together with locality, Hermiticity, and translation invariance. Under these standard assumptions, left- and right-handed gapless fermions must appear in matching numbers within each charge sector, thereby forbidding an isolated Weyl mode [3–6]. Well-established lattice regularizations of vectorlike gauge theories, such as Wilson fermions and staggered fermions, achieve practical control of doublers at the cost of either explicitly breaking chiral symmetry (breaking a conserved on site charge) or introducing additional tastes [7–14]. Domain-wall and overlap fermions realize single-flavor setups at the cost of breaking locality or on site chiral symmetry [15–20]. The improved and combined approaches of Wilson and staggered fermions have also been proposed, including the generalized Wilson fermions [21–28], the staggered-Wilson fermions [25,29–40], and the central-branch Wilson fermion [25,33,41–44]. By contrast, minimally doubled fermions realize the smallest

allowed number of species within the no-go theorem while preserving an exact on site chiral symmetry, typically by sacrificing full hypercubic invariance and other discrete symmetries [45–66].

While minimally doubled fermions have primarily been discussed in Euclidean Lagrangian formulations, a Hamiltonian viewpoint is independently motivated. First, it is the natural language for real-time dynamics and for the classification of gapless topological band structures such as Weyl semimetals [67]. Second, it is foundational for Hamiltonian lattice gauge theory [8,9] and for emerging quantum-simulation approaches, where one often implements the dynamics directly in the operator formalism rather than through a path integral. Despite this situation, the Hamiltonian formulation of minimal-doubling fermions has not yet been investigated.

Another motivation for the present work arises from recent proposals aiming to isolate a single Weyl node by rewriting the Hamiltonian in a doubled Bogoliubov-de Gennes (BdG) or Nambu particle/hole representation and adding a momentum-dependent pairing term in the Hamiltonian formalism [68]. Such models can possess exact, momentum-dependent conserved charges that commute with the lattice Hamiltonian while acting as a non-on-site generator in the position space. From the viewpoint of lattice chiral symmetry, these conserved non-on-site charges are naturally interpreted as Hamiltonian analogues of Ginsparg–Wilson-type chiral symmetries [69–74]: they are conserved, but their spectra need not be quantized, and the associated “chiral” transformations are generically momentum or position dependent. Classification of the similar single-Weyl Hamiltonians are found in Ref. [75].

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For the attempt to define chiral fermions in staggered formulations and the related works, see Refs. [76–78]. (See also Ref. [79] for the old related work.)

The first aim of this work is therefore methodological: we provide a unified Hamiltonian construction of minimal-doubling fermions and systematically identify which discrete symmetries are preserved or broken in the resulting Hamiltonians. The second aim is to clarify the relation between the minimal-doubling Hamiltonian and the recently proposed BdG-based single-Weyl Hamiltonians, and to analyze the latter in terms of their chiral structure and possible symmetry-allowed deformations. From the lattice-gauge-theory perspective, the key question is whether the single-node phase can be maintained in the interacting theory. Throughout this paper, we use the term “node” to mean a gapless point (equivalently, a zero of the Bloch Hamiltonian / energy spectrum) in the Brillouin zone.

In this paper, we formulate minimal-doubling lattice Hamiltonians in $(3+1)$ dimensions and investigate their discrete symmetry patterns while classifying the Hamiltonians based on the location of the zeros (nodes) and the discrete symmetries. Then, we reexamine BdG-based single-Weyl Hamiltonians within the minimal-doubling framework and analyze their non-on-site chiral symmetry in terms of the Ginsparg-Wilson-like relation. Building on these results, we introduce a symmetry-preserving one-parameter deformation of the single-Weyl Hamiltonian. A key result is that, although the deformation preserves all the symmetries, it can drive the system out of the single-node regime by generating additional Weyl nodes once the parameter exceeds a critical threshold. Since this deformation is symmetry allowed, it can be generated in an interacting theory as a counterterm; consequently, maintaining the single-Weyl phase generically requires adjusting its coefficient.

This paper is organized as follows. In Sec. II, we provide a detailed derivation of the minimal-doubling Hamiltonians, derive their node structures, and classify their symmetry patterns. In Sec. III, we discuss the relation of the single-Weyl and the minimal-doubling Hamiltonians, introduce a one-parameter deformation, and show that the deformation can produce additional Weyl nodes. Section IV is devoted to the summary and discussion.

II. MINIMAL-DOUBLING LATTICE HAMILTONIANS

The minimal-doubling (minimally doubled) fermion is one of the ultralocal lattice fermion setups, which yields two species of fermions and preserves one exact chiral symmetry. As shown in Ref. [54], the minimal-doubling

Dirac operators in the Lagrangian formalism have four independent classes, including Karsten-Wilczek, Borici-Creutz, twisted-ordering and dropped-twisted-ordering fermions. In the Appendix, we review the Lagrangian formalism of minimal-doubling fermions, with emphasis on the Karsten-Wilczek fermion. Below, we will construct the $(3+1)$ -dimensional Hamiltonian formalism of minimal-doubling fermions and investigate their discrete symmetries.

A. Karsten-Wilczek type

The Karsten-Wilczek (KW) fermion [45] is the prototype of minimal-doubling fermions, achieved by selecting a preferred spatial direction to lift the degeneracy of species.

For exact chiral symmetry, the one-particle Hamiltonian h (Bloch Hamiltonian) must commute with γ_5 as $[h, \gamma_5] = 0$. Any term proportional to $\beta = \gamma_0$ anticommutes with γ_5 and is forbidden. In the restricted class considered here, namely, Hamiltonians continuously connected to the standard massless Dirac Hamiltonian, we restrict ourselves to the $\alpha_j = \gamma_0 \gamma_j$ ($j = 1, 2, 3$) structures.

1. Four-component fermion

We first consider a cubic spatial lattice with sites $\mathbf{x} = (x, y, z) \in \mathbb{Z}^3$ and a four-component spinor $\psi_{\mathbf{x}}$ at each site with the periodic boundary condition. Gamma matrices γ_{μ} or $\alpha_j = \gamma_0 \gamma_j$ act on the Dirac spinor. The position-space and momentum-space representations of the Dirac spinor have the relation,

$$\psi_{\mathbf{x}} = \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \psi_{\mathbf{p}}, \quad \mathbf{p} = (p_1, p_2, p_3), \quad (1)$$

where we implicitly consider an infinite lattice volume and continuous momenta.

The massless and free Karsten-Wilczek (KW) Hamiltonian for a 4-component Dirac spinor is

$$H_{\text{KW}}^{\text{D}} = \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} \psi_{\mathbf{p}}^{\dagger} h_{\text{KW}}^{\text{D}}(\mathbf{p}) \psi_{\mathbf{p}}, \quad (2)$$

with

$$h_{\text{KW}}^{\text{D}}(\mathbf{p}) = \alpha_1 \sin p_1 + \alpha_2 \sin p_2 + \alpha_3 \mathcal{F}_{\text{KW}}(\mathbf{p}), \quad (3)$$

$$\mathcal{F}_{\text{KW}}(\mathbf{p}) = \sin p_3 + r(2 - \cos p_1 - \cos p_2). \quad (4)$$

Here, r is a real parameter. The position-space representation of the Hamiltonian is

$$H_{\text{KW}}^{\text{D}} = \sum_{\mathbf{x}} \psi_{\mathbf{x}}^{\dagger} \left[\sum_{j=1}^3 \frac{\alpha_j}{2i} (\psi_{\mathbf{x}+\mathbf{e}_j} - \psi_{\mathbf{x}-\mathbf{e}_j}) + \alpha_3 \left(2\psi_{\mathbf{x}} - \sum_{k=1}^2 \frac{\psi_{\mathbf{x}+\mathbf{e}_k} + \psi_{\mathbf{x}-\mathbf{e}_k}}{2} \right) \right], \quad (5)$$

where $\mathbf{e}_j$ is the unit vector in the j direction. The energy dispersion of this fermion is $E^2(\mathbf{p}) = \sin^2 p_1 + \sin^2 p_2 + \mathcal{F}_{\text{KW}}^2(\mathbf{p})$. Vanishing energy requires $\sin p_1 = \sin p_2 = 0$, leading to four candidates in the momentum plane: $(p_1, p_2) \in \{(0, 0), (\pi, 0), (0, \pi), (\pi, \pi)\}$.

- (i) At $(0, 0)$: $\mathcal{F}_{\text{KW}} = \sin p_3$. This vanishes at $p_3 = 0$ and $p_3 = \pi$. $\Rightarrow$ two gapless modes.
- (ii) At $(\pi, 0)$ or $(0, \pi)$: $\mathcal{F}_{\text{KW}} = \sin p_3 + 2r$. For $|r| > 1/2$, these are never zeros, so gapped.
- (iii) At (π, π) : $\mathcal{F}_{\text{KW}} = \sin p_3 + 4r$. For $|r| > 1/4$, these are never zeros, so gapped.

Thus, provided $|r| > 1/2$, the KW Hamiltonian of the Dirac spinor in (2) has two Dirac nodes strictly at $\mathbf{p}_A = (0, 0, 0)$ and $\mathbf{p}_B = (0, 0, \pi)$, leading to two gapless Dirac modes. Near $\mathbf{p}_A = (0, 0, 0)$, it behaves as $h \approx \alpha \cdot \mathbf{p}$, while near $\mathbf{p}_B = (0, 0, \pi)$, expanding $\sin(\pi + k_3) \approx -k_3$, we find $h \approx \alpha_1 k_1 + \alpha_2 k_2 - \alpha_3 k_3$. The phase diagram on the number of species in the parameter space has been intensively studied in Ref. [59], where the number of species becomes six when a coefficient of the dimension-3 counterterm exceeds a certain threshold. Although the study was performed in the Euclidean Lagrangian formalism, the similar phase structure is also expected to exist in the present Hamiltonian formalism. This fact will be important in the discussion on the one-parameter deformation of single-Weyl Hamiltonian in Sec. III C.

Since the one-particle Hamiltonian depends only on α_j , it strictly commutes with the chiral operator γ_5 ,

$$[h_{\text{KW}}^{\text{D}}, \gamma_5] = 0 \Rightarrow e^{-i\gamma_5 \theta} h_{\text{KW}}^{\text{D}} e^{i\gamma_5 \theta} = h_{\text{KW}}^{\text{D}}, \quad (6)$$

which means

$$[H_{\text{KW}}^{\text{D}}, Q_5] = 0, \quad (7)$$

$$Q_5 = \sum_{\mathbf{x}} \psi_{\mathbf{x}}^\dagger \gamma_5 \psi_{\mathbf{x}} = \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} \psi_{\mathbf{p}}^\dagger \gamma_5 \psi_{\mathbf{p}}. \quad (8)$$

Thus, the many-body (field-theoretical) Hamiltonian possesses the $U(1)_A$ chiral symmetry, in addition to the $U(1)$ vectorlike symmetry given by $[H_{\text{KW}}^{\text{D}}, Q_0] = 0$ with $Q_0 = \sum_{\mathbf{x}} \psi_{\mathbf{x}}^\dagger \psi_{\mathbf{x}}$.

The term $\alpha_3(2 - \cos p_1 - \cos p_2)$ in (4) explicitly breaks the cubic rotational symmetry to its two-dimensional subgroup although the translational symmetry is automatically preserved. Furthermore, some of the discrete symmetries are broken in the Hamiltonian: here, we adopt the standard implementations of P, T, C symmetries (one may choose any representation; for Dirac basis they are $U_P = \gamma^0$, $U_T = i\gamma^1 \gamma^3$, $U_C = i\gamma^2 \gamma^0$) as

- (i) Parity symmetry:

$$\text{P: } U_P h(-\mathbf{p}) U_P^{-1} = h(\mathbf{p}), \quad U_P \alpha_i U_P^{-1} = -\alpha_i, \quad (9)$$

- (ii) Time reversal symmetry:

$$\text{T: } U_T h(-\mathbf{p})^* U_T^{-1} = h(\mathbf{p}), \quad U_T \alpha_i^* U_T^{-1} = -\alpha_i, \quad (10)$$

- (iii) Charge conjugation invariance:

$$\text{C: } -U_C h(\mathbf{p})^T U_C^{-1} = h(\mathbf{p}), \quad U_C \alpha_i^T U_C^{-1} = -\alpha_i. \quad (11)$$

It is worth noting that this charge conjugation invariance is the relativistic field-theoretical definition, not the particle-hole symmetry (accompanied by momentum flipping $\mathbf{p} \rightarrow -\mathbf{p}$) relevant in the condensed-matter physics. We here denote $s_i \equiv \sin p_i$, $c_i \equiv \cos p_i$ in the KW Hamiltonian (3). Then, for the Hamiltonian, one finds

- (i) P is broken: By applying (9) to (3), one finds

$$\begin{aligned} U_P h_{\text{KW}}^{\text{D}}(-\mathbf{p}) U_P^{-1} &= \alpha_1 s_1 + \alpha_2 s_2 \\ &\quad + \alpha_3 [s_3 - r(2 - c_1 - c_2)] \\ &\neq h_{\text{KW}}^{\text{D}}(\mathbf{p}), \end{aligned} \quad (12)$$

- (ii) T is broken: Similarly, from (10), one finds

$$\begin{aligned} U_T h_{\text{KW}}^{\text{D}}(-\mathbf{p})^* U_T^{-1} &= \alpha_1 s_1 + \alpha_2 s_2 \\ &\quad + \alpha_3 [s_3 - r(2 - c_1 - c_2)] \\ &\neq h_{\text{KW}}^{\text{D}}(\mathbf{p}), \end{aligned} \quad (13)$$

- (iii) C is preserved: Using (11), one finds

$$\begin{aligned} -U_C h_{\text{KW}}^{\text{D}}(\mathbf{p})^T U_C^{-1} &= \alpha_1 s_1 + \alpha_2 s_2 \\ &\quad + \alpha_3 [s_3 + r(2 - c_1 - c_2)] \\ &= h_{\text{KW}}^{\text{D}}(\mathbf{p}). \end{aligned} \quad (14)$$

Although P, T are each broken, the product PT is preserved: both P and T flip $\mathbf{p} \rightarrow -\mathbf{p}$, so the combined action leaves $\mathbf{p}$ unchanged. Since PT is antiunitary, we may write its operation as $U_{PT} K$ with K being the complex-conjugation operator. Hence, at the level of single-particle matrices, it acts as $h(\mathbf{p}) \mapsto U_{PT} h(\mathbf{p})^* U_{PT}^{-1}$. Using that $h_{\text{KW}}^{\text{D}}(\mathbf{p})$ has real coefficients and is a linear combination of α_i , one finds

$$\text{PT: } U_{PT} h_{\text{KW}}^{\text{D}}(\mathbf{p})^* U_{PT}^{-1} = h_{\text{KW}}^{\text{D}}(\mathbf{p}). \quad (15)$$

Therefore, from the condensed-matter viewpoint, this model describes a phase analogous to a *PT-symmetric Dirac semimetal*. Since h_{KW}^{D} is a 4×4 Hamiltonian built from three mutually anticommuting α matrices, its spectrum is $\pm E(\mathbf{p})$ with each energy level being twofold degenerate. The PT symmetry is consistent with this structure, while the chiral symmetry gives the $E \leftrightarrow -E$ spectral symmetry. In addition to the standard discrete

symmetries, one can define the special paritylike symmetry, which can be regarded as a Hamiltonian version of mirror fermion symmetry discussed for the KW Dirac operator [65],

$$\tilde{P}: \mathbf{p} \mapsto (-p_1, -p_2, -p_3 + \pi), \quad \psi \mapsto \alpha_3 \psi. \quad (16)$$

Using $\alpha_3 \alpha_{1,2} \alpha_3 = -\alpha_{1,2}$, one obtains the exact identity,

$$\alpha_3 h_{\text{KW}}^D(-p_1, -p_2, -p_3 + \pi) \alpha_3 = h_{\text{KW}}^D(p_1, p_2, p_3). \quad (17)$$

Thus $\tilde{P}$ is an exact symmetry of the single-particle Hamiltonian $h_{\text{KW}}^D(\mathbf{p})$.

In lattice gauge theory with Lagrangian formalism of minimally doubled fermions, it has been argued that the breaking of discrete symmetries allows radiative corrections, leading to the necessity of tuning the parameters in the dimension-3 and -4 counterterms [49–51, 56, 61, 64, 65]. In our Hamiltonian formalism, however, we have the combined PT symmetry, and it is an interesting question whether or not it prohibits the generation of dimension-3 counterterms.

2. Two-component fermion

We now consider a two-component spinor χ_x at each site. Pauli matrices $\sigma_{1,2,3}$ act on the Weyl spinor. The position-space and momentum-space representations are related as $\chi_x = \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} e^{ip \cdot x} \chi_p$. The massless and free KW Hamiltonian of the 2-component Weyl spinor is

$$H_{\text{KW}}^W = \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} \chi_p^\dagger h_{\text{KW}}^W(\mathbf{p}) \chi_p, \quad (18)$$

with

$$h_{\text{KW}}^W(\mathbf{p}) = \sigma_1 \sin p_1 + \sigma_2 \sin p_2 + \sigma_3 \mathcal{F}_{\text{KW}}(\mathbf{p}), \quad (19)$$

with σ_j being the Pauli matrices and $\mathcal{F}_{\text{KW}}(\mathbf{p})$ being defined (4). The position-space representation of the Hamiltonian is

$$H_{\text{KW}}^W = \sum_x \chi_x^\dagger \left[\sum_{j=1}^3 \frac{\sigma_j}{2i} (\chi_{x+e_j} - \chi_{x-e_j}) + \sigma_3 \left(2\chi_x - \sum_{k=1}^2 \frac{\chi_{x+e_k} + \chi_{x-e_k}}{2} \right) \right]. \quad (20)$$

Again, provided $|r| > 1/2$, the KW Hamiltonian for Weyl lattice fermions in (18) contains two Weyl nodes strictly at $\mathbf{p}_A = (0, 0, 0)$ and $\mathbf{p}_B = (0, 0, \pi)$, leading to the two gapless Weyl modes. Near $\mathbf{p}_A = \mathbf{0}$, it behaves as $h \approx \boldsymbol{\sigma} \cdot \mathbf{p}$, while near $\mathbf{p}_B = (0, 0, \pi)$ it behaves as $h \approx \sigma_1 p_1 + \sigma_2 p_2 - \sigma_3 p_3$.

The exact flavor-chiral symmetry possessed by this Weyl setup is the $U(1)$ on site symmetry,

$$[h_{\text{KW}}^W, \mathbf{1}_\sigma] = 0 \Rightarrow e^{-i\theta} h_{\text{KW}}^W e^{i\theta} = h_{\text{KW}}^W, \quad (21)$$

which means

$$[H_{\text{KW}}^W, Q_0] = 0, \quad (22)$$

$$Q_0 = \sum_x \chi_x^\dagger \chi_x = \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} \chi_p^\dagger \chi_p. \quad (23)$$

Here, $\mathbf{1}_\sigma$ denotes the 2×2 identity matrix acting on the Weyl-spinor space. This Hamiltonian breaks the cubic rotational symmetry to the two-dimensional subgroup and also breaks both Parity and Time-reversal invariances. In this sense, this Hamiltonian corresponds to the *P-broken magnetic Weyl semimetal* in the condensed-matter context. In comparison to the case of the Dirac formalism, the Weyl KW formalism does not possess the combined PT invariance, while it still has the special paritylike symmetry $\tilde{P}: \mathbf{p} \mapsto (-p_1, -p_2, -p_3 + \pi), \chi \mapsto \sigma_3 \chi$.

Let us analyze the low-energy effective theory in this setup. We have the two Weyl nodes in momentum space at $\mathbf{p}_A$ and $\mathbf{p}_B$. As with the case for the Lagrangian formalism introduced in the Appendix, by using the approximate projection operator $P_p^A = (1 + \cos p_3)/2$, $P_p^B = (1 - \cos p_3)/2$ one can define point-split fields [22, 55] as

- (i) Sector A ($\mathbf{p} \approx \mathbf{p}_A$): $\chi_R(\mathbf{k}) \equiv P_{\mathbf{k}+\mathbf{p}_A}^A \chi_{\mathbf{k}+\mathbf{p}_A}$ obeys $i\partial_\mu \chi_R(\mathbf{k}) = \boldsymbol{\sigma} \cdot \mathbf{k} \chi_R(\mathbf{k})$,
- (ii) Sector B ($\mathbf{p} \approx \mathbf{p}_B$): $\chi_L(\mathbf{k}) = \sigma_3 P_{\mathbf{k}+\mathbf{p}_B}^B \chi_{\mathbf{k}+\mathbf{p}_B}$ obeys $i\partial_\mu \chi_L(\mathbf{k}) = -\boldsymbol{\sigma} \cdot \mathbf{k} \chi_L(\mathbf{k})$.

These are regarded as Weyl equations with opposite chiralities (helicities). Thus, we can define a four-component Dirac spinor,

$$\Psi_x \equiv \begin{pmatrix} \chi_R(\mathbf{k}) \\ \chi_L(\mathbf{k}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \chi_x + \frac{\chi_{x+e_3} + \chi_{x-e_3}}{2} \\ (-1)^{x_3} \sigma_3 \left(\chi_x - \frac{\chi_{x+e_3} + \chi_{x-e_3}}{2} \right) \end{pmatrix}, \quad (24)$$

which means that the low-energy theory obtained by combining the two patches can be organized into a single massless Dirac fermion (a pair of Weyl fermions of opposite chirality). It is, however, notable that the on site $U(1)$ symmetry possessed by the Hamiltonian (21) corresponds to the $U(1)$ vector symmetry of the single Dirac fermion composed of the two Weyl species (24), while $U(1)_A$ chiral symmetry should be expressed as a non-on-site transformation, thus be broken even for the free theory in this setup. In the interacting theory with gauge field, this breaking of $U(1)_A$ is expected to yield the chiral anomaly. This point will later be important in the construction of the single-Weyl formulation in Sec. III.

As shown in the Appendix, one can introduce the species-splitting mass term to the minimal-doubling fermion based on the flavor representation of point-split fields [22]. Such a term gaps one of the two Weyl nodes while preserving the ordinary on site fermion-number $U(1)$ symmetry. For instance, by introducing the species-splitting mass term into the KW two-component

Hamiltonian (19), one reaches a single-Weyl Hamiltonian,
$$h_{\text{KW}}^{\text{split}}(\mathbf{p}) = \sigma_1 \sin p_1 + \sigma_2 \sin p_2 + \sigma_3 \mathcal{F}_{\text{KW}}(\mathbf{p}) + 1 - \cos p_3, \quad (25)$$

which keeps the Weyl node at $\mathbf{p}_A = (0, 0, 0)$ massless while gaps the other at $\mathbf{p}_B = (0, 0, \pi)$. In this setup, however, the on site fermion-number $U(1)$ symmetry does not work to protect the remaining gapless node. Thus, although Eq. (25) yields a single gapless Weyl node in the free theory, it can be gapped due to the radiative correction in the interacting theory; thus, it does not provide a symmetry-protected chiral single-Weyl construction. In the chiral-symmetric Weyl setup discussed in Sec. III, the analogous species-splitting term is introduced not in flavor space but in the Bogoliubov-de Gennes (BdG) representation on the Nambu (particle/hole) space, where the on site $U(1)$ is explicitly broken.

B. Twisted-ordering type

In the following two subsections, we construct other classes of minimal-doubling Hamiltonians. The first one is the twisted-ordering (TO) fermion [54], which is defined on an orthogonal lattice but utilizes a cyclic twist to maintain minimal doubling. The twisted-ordering Hamiltonian for the Dirac spinor is

$$\begin{aligned} h_{\text{TO}}^{\text{D}}(\mathbf{p}) &= \sum_{j=1}^3 \alpha_j [\sin p_j + \cos p_{j+1} - 1] \\ &= \alpha_1 (\sin p_1 + \cos p_2 - 1) \\ &\quad + \alpha_2 (\sin p_2 + \cos p_3 - 1) \\ &\quad + \alpha_3 (\sin p_3 + \cos p_1 - 1), \end{aligned} \quad (26)$$

where we use the index $j+1$ as a mod-3 number, for instance, $3+1 \rightarrow 1$. This Hamiltonian has two nodes (zeros) at $\mathbf{p} = (0, 0, 0)$ and $\mathbf{p} = (\pi/2, \pi/2, \pi/2)$, yielding two gapless Dirac modes. It possesses the $U(1)_A$ chiral symmetry in addition to the $U(1)$ vector symmetry as in the KW Dirac Hamiltonian in Sec. II A. Regarding the discrete rotational symmetry, this model breaks cubic symmetry but, in this case, preserves the Z_3 subgroup. Regarding P, T, C symmetries, the Hamiltonian breaks each of P and T, but preserves the combined PT symmetry while it does not have the special paritylike symmetry $\tilde{\text{P}}$ possessed by the KW Dirac Hamiltonian.

The twisted-ordering Hamiltonian for Weyl spinors is

$$\begin{aligned} h_{\text{TO}}^{\text{W}}(\mathbf{p}) &= \sum_{j=1}^3 \sigma_j [\sin p_j + \cos p_{j+1} - 1] \\ &= \sigma_1 (\sin p_1 + \cos p_2 - 1) \\ &\quad + \sigma_2 (\sin p_2 + \cos p_3 - 1) \\ &\quad + \sigma_3 (\sin p_3 + \cos p_1 - 1). \end{aligned} \quad (27)$$

It has two Weyl nodes at $\mathbf{p}_A = (0, 0, 0)$ and $\mathbf{p}_B = (\pi/2, \pi/2, \pi/2)$, yielding two gapless (massless) Weyl modes. They carry opposite physical chiralities (helicities) and therefore can be combined into a single massless Dirac fermion in the low-energy theory. While it breaks P and T symmetries, it possesses the $U(1)$ on site symmetry, which corresponds to $U(1)$ vector symmetry of the single Dirac fermion composed of the two Weyl modes at $\mathbf{p}_A, \mathbf{p}_B$ as in the KW Weyl Hamiltonian in Sec. II A 2.

C. Borici-Creutz type

The Borici-Creutz fermion [47,48] places the two zeros along the main diagonal of the Brillouin zone, preserving an S_3 permutation subgroup of the cubic symmetry. The Borici-Creutz Hamiltonian for Dirac spinors is

$$h_{\text{BC}}^{\text{D}}(\mathbf{p}) = \sum_{j=1}^3 [\alpha_j \sin(p_j + \pi/4) - \alpha'_j \sin(p_j - \pi/4) - \alpha_j], \quad (28)$$

with

$$\alpha'_j = A_{ji} \alpha_i, \quad A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}. \quad (29)$$

The two nodes are located at $\mathbf{p} = (\pi/4, \pi/4, \pi/4)$ and $\mathbf{p} = -(\pi/4, \pi/4, \pi/4)$, giving two massless Dirac modes. It preserves the $U(1)_A$ chiral symmetry and the $U(1)$ vector-like symmetry. Regarding the discrete rotational symmetry, the model breaks the cubic symmetry but preserves the S_3 subgroup, which is the largest rotational symmetry in the known minimal-doubling Hamiltonian. It breaks each of P and T, but preserves the combined PT symmetry as with the other minimal-doubling Hamiltonians, while it does not have the special paritylike symmetry.

The Borici-Creutz Hamiltonian for Weyl spinors is

$$h_{\text{BC}}^{\text{W}}(\mathbf{p}) = \sum_j [\sigma_j \sin(p_j + \pi/4) - \sigma'_j \sin(p_j - \pi/4) - \sigma_j], \quad (30)$$

with $\sigma'_j = A_{ji} \sigma_i$. It has two Weyl nodes at $\mathbf{p} = (\pi/4, \pi/4, \pi/4)$ and $\mathbf{p} = -(\pi/4, \pi/4, \pi/4)$, giving two massless Weyl modes with the $U(1)$ on site symmetry preserved and P, T symmetries broken. Again, the two Weyl modes carry opposite physical chiralities and therefore, can be combined into a single massless Dirac fermion in the low-energy theory.

D. Classification and generalization

In this subsection, we pursue classification and generalization of the minimal-doubling Hamiltonians. As in the Lagrangian formalism of minimal-doubling fermions in

Ref. [54], one can classify the expressions of minimal-doubling Hamiltonians into two types:

The first type is expressed as

$$h_1 = \sum_{j=1}^3 \alpha_j \sin p_j + r \sum_{i=2,3} \sum_{k=1,2} \alpha_i R_{ik} (\cos p_k - 1), \quad (31)$$

where we take $i = 2, 3$ and $k = 1, 2$ while $j = 1, 2, 3$. This type breaks the cubic rotational symmetry severely. For example, let us consider the following R :

$$R = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad (32)$$

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (33)$$

With (32), the Hamiltonian (31) reduces to Karsten-Wilczek one in (3). With (33) ($r = 1$), it reduces to the type, called the dropped twisted-ordering fermion [54],

$$h_1 = \alpha_1 \sin p_1 + \alpha_2 (\sin p_2 + \cos p_1 - 1) + \alpha_3 (\sin p_3 + \cos p_2 - 1), \quad (34)$$

which has two gapless nodes at $\mathbf{p} = (0, 0, 0)(0, 0, \pi)$. Of course, we need to set r so as to keep minimal doubling.

The second type is expressed as

$$h_2 = \sum_{j=1}^3 [\alpha_j \sin(p_j + \beta_j) - \alpha'_j \sin(p_j - \beta_j)] - \alpha, \quad (35)$$

where $\alpha'_j = A_{ji} \alpha_i$. Here, β_j and α are related with A as $\alpha = \sum_j \alpha_j \sin 2\beta_j = \sum_j \alpha'_j \sin 2\beta_j$, which means that $\sin 2\beta_j$ is an eigenvector of A . Thus, once A is fixed, β_j and α are determined up to an overall factor. By imposing these conditions on A , β_j , and α , the Hamiltonian can be minimal doubling, where $\pm\beta_j$ correspond to locations of two nodes. The Hamiltonian (35) reduces to Borici-Creutz one (28) by choosing (29) for A , which fixes $\beta_j = \pi/4$ and $\alpha = \sum_{j=1}^3 \alpha_j$. The Hamiltonian (35) also yields the twisted-ordering one by choosing A as

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (36)$$

which fixes $\beta_j = \pi/4$ and $\alpha = \sum_{j=1}^3 \alpha_j$. This agrees with the expression (26) by shifting $p_j \rightarrow p_j - \pi/4$. This type can maintain a relatively large subgroup of the cubic rotational symmetry such as S_3 or Z_3 .

III. CHIRAL-SYMMETRIC SINGLE-WEYL HAMILTONIANS

We now turn to a single Weyl fermion formulation, which uses the BdG formalism to attempt to remove the doubler, leaving a single Weyl node, in the KW Hamiltonian (3). In this section, we first show the relation of the Karsten-Wilczek minimal-doubling Hamiltonian and the single-Weyl model proposed by Gioia and Thorngren [68], and study its properties in terms of chiral symmetry. Finally, we propose one-parameter deformation of the single-Weyl Hamiltonian, who keeps all the symmetries of the original Hamiltonian, and show that it can destabilize the single-Weyl situation, leading to potential necessity of tuning parameters in an interacting theory.

A. Review of Gioia-Thorngren setup

In this subsection we recall in some detail the construction of the single-Weyl model [68], with focus on its relation to Karsten-Wilczek minimally doubled Hamiltonian in the previous section.

The starting point is a time-reversal-broken two-band tight-binding model on a cubic lattice, a magnetic Weyl semimetal, described by the second-quantized Hamiltonian. It is completely equivalent to the Karsten-Wilczek minimal-doubling Weyl Hamiltonian in (19),

$$H_{\text{KW}}^{\text{W}} = \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} \chi_p^\dagger h_{\text{KW}}^{\text{W}}(\mathbf{p}) \chi_p, \quad (37)$$

$$h_{\text{KW}}^{\text{W}}(\mathbf{p}) = \sigma_1 \sin p_1 + \sigma_3 \sin p_2 + \sigma_2 \mathcal{F}_{\text{KW}}(\mathbf{p}), \quad (38)$$

with $\mathcal{F}_{\text{KW}}(\mathbf{p}) = \sin p_3 + r(2 - \cos p_1 - \cos p_2)$. Here, we have performed the appropriate unitary transformation on the original KW Hamiltonian in (19) to match the notation to that in [68]. As we have discussed in Sec. II, for $|r| > 1/2$ the spectrum contains two Weyl nodes at

$$\mathbf{p}_A = (0, 0, 0), \quad \mathbf{p}_B = (0, 0, \pi), \quad (39)$$

and the model has an on site $U(1)$ symmetry generated by the charge $Q_0 = \sum_x \chi_x^\dagger \chi_x$, which acts as $\chi_x \mapsto e^{i\theta} \chi_x$.

To obtain a single Weyl fermion in the IR, the above $U(1)$ symmetry must be broken in order to gap one of the nodes. To implement the symmetry breaking in a convenient way, Ref. [68] passes to a Bogoliubov-de Gennes (BdG) representation, where one introduces the Nambu spinor,

$$\psi_p^{\text{BdG}} = \begin{pmatrix} \chi_p \\ \chi_{-p}^* \end{pmatrix}, \quad h_{\text{KW}}^{\text{BdG}}(\mathbf{p}) = \frac{1}{2} \begin{pmatrix} h_{\text{KW}}^{\text{W}}(\mathbf{p}) & \mathbf{0} \\ \mathbf{0} & -h_{\text{KW}}^{\text{W}}(-\mathbf{p})^* \end{pmatrix}. \quad (40)$$

Note that this slightly differs from the conventional definition of the Nambu spinor such as $\psi_p^{\text{BdG}} = (\chi_p, -i\sigma_2 \chi_{-p}^*)^T$, but they are unitary equivalent. Then, one writes the

single-particle BdG Hamiltonian (40) as

$$h_{\text{KW}}^{\text{BdG}}(\mathbf{p}) = \frac{1}{2}[\sigma_1 \sin p_1 + \sigma_3 \sin p_2 + \tau_3 \sigma_2 \sin p_3 + \sigma_2 r(2 - \cos p_1 - \cos p_2)], \quad (41)$$

with the total Hamiltonian $H_{\text{KW}}^{\text{BdG}} = \int \frac{d^3 p}{(2\pi)^3} (\psi_{\mathbf{p}}^{\text{BdG}})^\dagger \times h_{\text{KW}}^{\text{BdG}}(\mathbf{p}) \psi_{\mathbf{p}}^{\text{BdG}}$. Here, $\tau_{1,2,3}$ are Pauli matrices acting on the fictitious Nambu-doubling index (particle at $\mathbf{p}$ vs. hole at $-\mathbf{p}$). Consistency with the BdG redundancy or the particle-hole symmetry requires $\tau_1 h_{\text{KW}}^{\text{BdG}}(\mathbf{p})^* \tau_1 = -h_{\text{KW}}^{\text{BdG}}(-\mathbf{p})$, which is indeed satisfied by (41).

What we have done so far is just rewriting the KW Hamiltonian from the standard representation to the BdG representation, thus it keeps all the properties including the symmetries. For example, the $U(1)$ on site symmetry is expressed in this representation as

$$[h_{\text{KW}}^{\text{w}}, \tau_3] = 0, \quad (42)$$

which means

$$[H_{\text{KW}}^{\text{BdG}}, Q_0] = 0, \quad (43)$$

$$Q_0 = \sum_{\mathbf{x}} (\psi_{\mathbf{x}}^{\text{BdG}})^\dagger \tau_3 \psi_{\mathbf{x}}^{\text{BdG}} = \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} (\psi_{\mathbf{p}}^{\text{BdG}})^\dagger \tau_3 \psi_{\mathbf{p}}^{\text{BdG}}. \quad (44)$$

To gap the Weyl node at $\mathbf{p}_B = (0, 0, \pi)$ while keeping the one at $\mathbf{p}_A$ gapless, one adds a term,

$$\delta h(\mathbf{p}) = \frac{1}{2} \tau_1 \sigma_2 (1 - \cos p_3), \quad (45)$$

which breaks the on site $U(1)$ generated by τ_3 in the Nambu space. It is a sort of species-splitting mass term, but not in the flavor space but in the Nambu (particle/hole) space. The resulting single-Weyl BdG Hamiltonian is

$$h_{\text{single}}^{\text{BdG}}(\mathbf{p}) = \frac{1}{2}[\sigma_1 \sin p_1 + \sigma_3 \sin p_2 + \tau_3 \sigma_2 \sin p_3 + \sigma_2 r(2 - \cos p_1 - \cos p_2) + \tau_1 \sigma_2 (1 - \cos p_3)]. \quad (46)$$

For $|r| > 1$, this Hamiltonian has a single Weyl node at $\mathbf{p}_A = \mathbf{0}$ and is fully gapped elsewhere in the Brillouin zone. It has relatively small discrete symmetries; translational symmetry, 2d subgroup of cubic rotational symmetry, and BdG particle-hole redundancy. The present single-Weyl Hamiltonian, however, has the exact non-on-site $U(1)$ symmetry as we will discuss later, which may have potential to prevent the generation of counterterms originating in the discrete symmetry breaking. This will be the main topic of Sec. III C. For later convenience, we name the part of the Hamiltonian acting on the Nambu space nontrivially as

$$N(\mathbf{p}) \equiv \frac{\sigma_2}{2} [\tau_3 \sin p_3 + \tau_1 (1 - \cos p_3)] = \sigma_2 \sin \frac{p_3}{2} \left[\tau_3 \cos \frac{p_3}{2} + \tau_1 \sin \frac{p_3}{2} \right], \quad (47)$$

which is interpreted as the effective one-dimensional Hamiltonian in the p_3 direction defined as $N(\mathbf{p}) \equiv h_{\text{single}}^{\text{BdG}}(p_1 = 0, p_2 = 0, p_3)$. We note that it satisfies $N(\mathbf{p})^2 = (\sin \frac{p_3}{2})^2 \mathbf{1}$.

To clarify how the Hamiltonian acts on the Nambu spinor $\psi_{\mathbf{p}}^{\text{BdG}}$, one can rewrite it as a matrix acting on the Nambu space,

$$h_{\text{single}}^{\text{BdG}}(\mathbf{p}) = \frac{1}{2} \begin{pmatrix} h_{\text{KW}}^{\text{w}}(\mathbf{p}) & \Delta(\mathbf{p}) \\ \Delta^\dagger(\mathbf{p}) & -h_{\text{KW}}^{\text{w}}(-\mathbf{p})^* \end{pmatrix}, \quad (48)$$

with the species-splitting mass $\Delta(\mathbf{p})$,

$$\Delta(\mathbf{p}) = \sigma_2 (1 - \cos p_3). \quad (49)$$

This species-splitting mass term is a sort of Majorana mass term, giving the $\chi^T \chi$ term, who obviously breaks the $U(1)$ on site symmetry.

B. Non-on-site chiral symmetry and GW relation

The key observation of Ref. [68] is that the single-Weyl Hamiltonian (46) commutes with a momentum-dependent generator,

$$S_\chi(\mathbf{p}) = \frac{1}{2} [\tau_3 (1 + \cos p_3) + \tau_1 \sin p_3] = \cos \frac{p_3}{2} \left[\tau_3 \cos \frac{p_3}{2} + \tau_1 \sin \frac{p_3}{2} \right]. \quad (50)$$

This generator is proportional to $N(\mathbf{p})$, which is the only part acting on the Nambu space nontrivially in the Hamiltonian (47); thus, it is obvious that it commutes with the Hamiltonian (46). One can rephrase the situation as follows: the species-splitting mass term (45) rotates the $U(1)$ symmetric direction from τ_3 to $\tau_3 \cos \frac{p_3}{2} + \tau_1 \sin \frac{p_3}{2}$ in the Nambu $SU(2)$ space, with gapping the one of the nodes. Such a way of realization of chiral symmetry and doubler control on the lattice is found in the twisted-Wilson fermion [80,81] and the higher-Clifford-algebra fermion [63].

The “chiral” symmetry is explicitly expressed as

$$[S_\chi(\mathbf{p}), h_{\text{single}}^{\text{BdG}}(\mathbf{p})] = 0 \quad \text{for all } \mathbf{p}, \quad (51)$$

and S_χ obeys

$$S_\chi(\mathbf{p})^2 = \left(\cos \frac{p_3}{2} \right)^2 \mathbf{1}, \quad (52)$$

which means that eigenvalues of this chiral generator is not quantized but has continuum spectrum. Indeed, at the surviving Weyl node $\mathbf{p}_A = (0, 0, 0)$ one has $S_\chi(\mathbf{p}_A) = \tau_3$, which coincides with the original charge operator in the BdG basis, while at the gapped node $\mathbf{p}_B = (0, 0, \pi)$, one finds $S_\chi(\mathbf{p}_B) = 0$.

The associated many-body charge is

$$\begin{aligned} Q_\chi &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} (\psi_{\mathbf{p}}^{\text{BdG}})^\dagger S_\chi(\mathbf{p}) \psi_{\mathbf{p}}^{\text{BdG}} \\ &= \frac{1}{2} \sum_{\mathbf{x}} (\chi_{\mathbf{x}}^\dagger \chi_{\mathbf{x}} + \chi_{\mathbf{x}+\mathbf{e}_3}^\dagger \chi_{\mathbf{x}} - i \chi_{\mathbf{x}+\mathbf{e}_3}^\dagger \chi_{\mathbf{x}}) + \text{H.c.}, \end{aligned} \quad (53)$$

which satisfies $[H_{\text{single}}^{\text{BdG}}, Q_\chi] = 0$. This position-space expression involves on site and nearest-neighbor hopping terms along the third axis. In this sense the symmetry generated by Q_χ is manifestly non-on-site, yet it reduces to the usual axial $U(1)_A$ in the low-energy theory around $\mathbf{p}_A$.

The construction defines a consistent free field theory, but gauging it fails. The Majorana-like mass term $\chi^\dagger \chi$ carries charge $2e$ under $U(1)$ gauge transformations, and this is the source where the gauge anomaly emerges in this formulation. As discussed in [68], one needs to introduce flavors to cancel the gauge anomaly.

The chiral generator (50) is interpreted as the Hamiltonian analogue of a Ginsparg-Wilson chiral operator: it is momentum dependent, commutes with the Hamiltonian, and has a nonquantized spectrum constrained by (52). Indeed, for the effective one-dimensional Hamiltonian (47) and the chiral generator (50), there is the nontrivial relation,

$$N(\mathbf{p})^2 + S_\chi(\mathbf{p})^2 = \mathbf{1}. \quad (54)$$

It corresponds to the Hamiltonian version of Ginsparg-Wilson relation proposed in Ref. [69], where it was already pointed

out that the chiral generator for the Hamiltonian formalism needs to have the continuum spectrum. There have been the proposal of the improved Ginsparg-Wilson relation to give the quantized spectrum [70,74]. It is an interesting avenue to consider the Hamiltonians satisfying such an improved relation.

C. One-parameter deformation, extra nodes, and need for tuning

In this subsection, we discuss a natural one-parameter deformation of the single-Weyl BdG Hamiltonian and show that it can destabilize the “single-Weyl” situation by generating additional gapless nodes. We then interpret this deformation as a radiatively generated counterterm in an interacting theory and explain how a tuning of the parameter could be generically required.

Let us remind ourselves that the characteristic feature of the single-Weyl Hamiltonian (46) is that all τ dependence sits in the single spin channel σ_2 . Therefore, a natural one-parameter deformation to the Hamiltonian (46) is to add another σ_2 -channel term with the same τ structure as $S_\chi(\mathbf{p})$ in (50),

$$\delta h_\mu(\mathbf{p}) = \frac{\mu}{2} \sigma_2 [\tau_3 (1 + \cos p_3) + \tau_1 \sin p_3] = \mu \sigma_2 S_\chi(\mathbf{p}). \quad (55)$$

This deformation preserves the chiral symmetry as

$$[h_{\text{single}}^{\text{BdG}}(\mathbf{p}) + \delta h_\mu(\mathbf{p}), S_\chi(\mathbf{p})] = 0 \quad \text{for all } \mathbf{p}, \quad (56)$$

as well as all the other symmetries. Physically, (55) shifts the Weyl node position $\mathbf{p}_A$ along the p_3 direction without introducing a conventional mass term. We now write down the one-parameter-deformed Hamiltonian with collecting the τ -dependent part in the σ_2 channel as

$$\begin{aligned} h(\mathbf{p}) &\equiv h_{\text{single}}^{\text{BdG}}(\mathbf{p}) + \delta h_\mu(\mathbf{p}) = \sigma_1 \sin p_1 + \sigma_3 \sin p_2 \\ &\quad + \sigma_2 [rM(p_1, p_2) + a(p_3)\tau_3 + b(p_3)\tau_1], \end{aligned} \quad (57)$$

with

$$\begin{aligned} M(p_1, p_2) &= 2 - \cos p_1 - \cos p_2, \\ a(p_3) &= \sin p_3 + \mu(1 + \cos p_3), \\ b(p_3) &= (1 - \cos p_3) + \mu \sin p_3. \end{aligned} \quad (58)$$

Because $\sigma_1, \sigma_2, \sigma_3$ anticommute and the τ matrices commute with the σ matrices, one finds

$$4h(\mathbf{p})^2 = \sin^2 p_1 + \sin^2 p_2 + [rM\mathbf{1}_\tau + a\tau_3 + b\tau_1]^2. \quad (59)$$

Thus, the eigenvalues of $4h(\mathbf{p})^2$ are

$$\sin^2 p_1 + \sin^2 p_2 + (rM \pm \sqrt{a^2 + b^2})^2. \quad (60)$$

A necessary and sufficient condition for a gapless point is

$$\begin{aligned} \sin p_1 &= 0, & \sin p_2 &= 0, \\ rM(p_1, p_2) &= \pm \sqrt{a(p_3)^2 + b(p_3)^2}. \end{aligned} \quad (61)$$

Thus, p_1, p_2 are restricted to the four values $(p_1, p_2) \in \{(0, 0), (\pi, 0), (0, \pi), (\pi, \pi)\}$. At $(p_1, p_2) = (0, 0)$, we have $M(p_1, p_2) = 0$; hence, the gapless condition reduces to $a(p_3) = 0, b(p_3) = 0$. Using the parametrization $t = \tan(p_3/2)$ so that $\sin p_3 = \frac{2t}{1+t^2}$ and $\cos p_3 = \frac{1-t^2}{1+t^2}$, one finds

$$a(p_3) = 0 \iff t = -\mu, \quad \text{and then } b(p_3) = 0. \quad (62)$$

Therefore, the Weyl node is shifted from $p_3 = 0$ to

$$\tan \frac{p_3^*}{2} = -\mu \iff p_3^* = -2 \arctan \mu \pmod{2\pi}. \quad (63)$$

Importantly, the would-be node at $p_3 = \pi$ remains gapped because $b(p_3 = \pi) = 2$ for any μ .

The central issue is whether additional solutions of (61) appear at the other points in (p_1, p_2) . To analyze this, we first simplify $\sqrt{a^2 + b^2}$. With $t = \tan(p_3/2)$, (58) becomes $a(p_3) = \frac{2(t+\mu)}{1+t^2}, b(p_3) = \frac{2t(t+\mu)}{1+t^2}$; hence,

$$\begin{aligned} \sqrt{a(p_3)^2 + b(p_3)^2} &= 2\sqrt{1+\mu^2} \left| \sin \left(\frac{p_3}{2} + \phi \right) \right|, \\ \phi &\equiv \arctan \mu. \end{aligned} \quad (64)$$

In particular,

$$0 \leq \sqrt{a^2 + b^2} \leq 2\sqrt{1+\mu^2}. \quad (65)$$

The first possible extra nodes are $(p_1, p_2) = (\pi, 0)$ and $(0, \pi)$. For these points, we have $M(p_1, p_2) = 2$, so the gapless condition (61) reads

$$2r = \sqrt{a(p_3)^2 + b(p_3)^2}. \quad (66)$$

By (65), (66) has a solution in p_3 if and only if

$$\begin{aligned} 2r &\leq 2\sqrt{1+\mu^2} \iff r \leq \sqrt{1+\mu^2} \iff |\mu| \\ &\geq \sqrt{r^2 - 1} (|r| > 1). \end{aligned} \quad (67)$$

For generic parameters with strict inequality, $|\mu| > \sqrt{r^2 - 1}$; there are in fact two solutions for p_3 at each of $(\pi, 0)$ and $(0, \pi)$, because (64) gives

$$\begin{aligned} \left| \sin \left(\frac{\hat{p}_3}{2} + \phi \right) \right| &= \frac{r}{\sqrt{1+\mu^2}} \Rightarrow \hat{p}_3 \\ &= -2\phi \pm 2 \arcsin \left(\frac{r}{\sqrt{1+\mu^2}} \right) \pmod{2\pi}. \end{aligned} \quad (68)$$

Thus, once the bound $|\mu| < \sqrt{r^2 - 1}$ is violated, the system is no longer in the phase with a single Weyl node: additional gapless nodes are created at $(\pi, 0, \hat{p}_3)$ and $(0, \pi, \hat{p}_3)$.

Further possible nodes can be at $(p_1, p_2) = (\pi, \pi)$. At (π, π) , we have $M(p_1, p_2) = 4$, and the condition becomes $4r = \sqrt{a^2 + b^2}$, which can be satisfied only for even larger $|\mu|$,

$$4r \leq 2\sqrt{1+\mu^2} \iff |\mu| \geq \sqrt{4r^2 - 1}. \quad (69)$$

Hence, the first instability of the single-Weyl regime is governed by (67).

In summary, the μ deformation (55) shifts the physical Weyl node as in (63), but it also opens a channel for new gapless nodes to appear once

$$|\mu| \geq \sqrt{r^2 - 1}. \quad (70)$$

It is notable that the number of species becomes five for $\sqrt{r^2 - 1} < |\mu| < \sqrt{4r^2 - 1}$ since $\hat{p}_3$ are twofolded. It is notable that the Karsten-Wilczek Dirac operator has six species for a certain parameter domain [59], and one of the species is now gapped due to the species-splitting mass.

The deformation (55) is not an arbitrary modification: it is a local, symmetry-allowed operator in the same σ_2 channel as the original τ structure, and it commutes with the chiral generator $S_\chi(\mathbf{p})$. In an interacting gauge theory, any local operator not forbidden by symmetries is expected to be generated by radiative corrections. From this viewpoint, μ should be treated as a counterterm coefficient,

$$\mu_R = \mu_0 + \delta\mu(g^2, r, a), \quad (71)$$

where μ_0 is the bare parameter in the lattice action and $\delta\mu$ is the radiative correction. By the same logic as the additive mass renormalization in Wilson fermions, $\delta\mu$ is not parametrically protected and may be $O(1)$ in lattice units unless an additional symmetry suppresses it.

It is notable that the counterterm (55) explicitly breaks the cubic rotational symmetry, P and T symmetries, which are already broken in the undeformed Hamiltonian. Thus, we can interpret that the radiative correction (55) in the interacting theory originates in the breaking of those symmetries. It is a similar situation to the necessity of tuning the dimension-3 and -4 counterterms in the lattice

gauge simulation with the Lagrangian minimal-doubling formalism [49–51,56,61,62,64,65].

The analysis above shows that the “single-Weyl” regime is not guaranteed merely by enforcing the chiral symmetry $S_\chi(\mathbf{p})$: it requires the renormalized parameter to remain within the single-Weyl domain,

$$|\mu_R| < \sqrt{r^2 - 1}. \quad (72)$$

Therefore, to maintain a single Weyl node in the interacting theory, one must in general

- (i) include the μ term (55) from the outset as an allowed counterterm, and
 - (ii) adjust μ_0 (as a function of the gauge coupling and lattice spacing) so that (72) holds,
- unless an extra symmetry is imposed that forbids (55) altogether.

As we have discussed, the single-Weyl Hamiltonian itself cannot directly be gauged because of the gauge anomaly and the gauging requires introduction of flavor degrees of freedom [68]. We emphasize, however, that the emergence of extra nodes in interacting theories can occur broadly in multiflavor anomaly free setups and other exactly chiral-symmetric Weyl formulations.

In the end of the section, we make a comment on the recent work [75]. In the work, the authors in detail argued the classification of the single-Weyl Hamiltonians including the proposal of Gioia-Thorngren [68]. They studied the change of the node configuration by varying parameters corresponding to a magnetic field (r in our notation) and mass or chemical potential, but did not consider the symmetry-preserving “chiral-generator” deformation we propose here. It is an interesting question how the node configuration is affected if these independent deformations are combined in the variety of single-Weyl Hamiltonians.

IV. CONCLUSION AND DISCUSSION

In this work, we developed a systematic Hamiltonian framework for minimally doubled lattice fermions in three spatial dimensions. We treated both four-component Dirac and two-component Weyl constructions, and classified the resulting Hamiltonians by their nodal structure and discrete symmetry patterns. For representative examples, we derived explicit Bloch Hamiltonians, determined the node locations, and identified the discrete symmetries that remain intact and those that are broken.

We then reconsidered BdG-based “single-Weyl” Hamiltonians within our minimally doubled framework. By introducing a Nambu particle/hole doubling and an appropriate species-splitting “Majorana” mass term into the Karsten-Wilczek minimal-doubling Hamiltonian, we showed that one Weyl node can be gapped out while the other remains gapless in the free theory. The protection mechanism is encoded in a momentum-dependent conserved

generator acting in Nambu space: it commutes with the Hamiltonian, yet its spectrum is not quantized, which naturally suggests the Hamiltonian Ginsparg-Wilson-type structure. From this viewpoint, we studied a symmetry-preserving one-parameter deformation that keeps the full symmetry intact, yet can destabilize the single-node regime by generating additional Weyl nodes once the deformation exceeds a critical threshold. Concretely, the deformation opens a channel for extra gapless solutions that were gapped in the undeformed model, and the single-Weyl regime survives only inside a finite parameter window.

An important implication for the interacting theories is that the destabilizing deformation is not an artificial modification: it is not forbidden by the exact non-on-site conservation law, and accordingly, once gauge interactions are introduced, radiative corrections are expected to generate this term as an allowed counterterm with a renormalized coefficient that is not parametrically suppressed in lattice units, in close analogy with additive mass renormalization in Wilson fermion. The stability of the single-Weyl regime therefore could require a moderate tuning of the corresponding bare parameter so that the renormalized coefficient remains inside the single-node domain. In this sense, the exact non-on-site conservation alone does not guarantee radiative stability of the single-Weyl phase.

Finally, let us comment on the relation of the present constructions to the Nielsen-Ninomiya no-go theorem. Under its standard assumptions, including locality, Hermiticity, translation invariance, and conserved on site charges with discrete spectra, an isolated Weyl mode is forbidden. In the minimal-doubling constructions, the four-component Hamiltonian, which possesses two conserved on site charges, yields two massless Dirac fermions, while the two-component Hamiltonian, with one conserved on site charge, yields one massless Dirac fermion composed of two Weyl fermions. These constructions therefore remain within the standard scope of the theorem. By contrast, the BdG-based single-Weyl Hamiltonian possesses only a non-on-site momentum-dependent conserved generator and hence, lies outside that scope.

Several directions are suggested by our analysis. First, it is natural to ask whether one can engineer single-node BdG constructions starting from minimal-doubling Hamiltonians beyond the Karsten-Wilczek prototype, in such a way that the protecting symmetry is strengthened or that the discrete rotational structure is improved. From the minimally doubled viewpoint, this becomes a concrete design problem: identify which minimal-doubling classes admit a Nambu-space “species-splitting” mechanism while preserving the largest possible remnant lattice symmetry. Second, our results call for explicit numerical studies in the interacting theory. Given a candidate single-Weyl Hamiltonian (with or without deformations), one can directly verify the single-node condition by scanning the Brillouin zone for gapless points and by monitoring how the node count changes as the parameter vary.

Such computations would provide a sharp, nonperturbative test of the analytic stability bounds and would clarify how robust the single-Weyl regime remains once interactions are included. Third, it is important to map out the phase structure in the enlarged parameter space, in the spirit of earlier analyses of minimally doubled fermions in Lagrangian formulations [59]. A systematic “phase diagram” of node configurations, including the critical surfaces where extra nodes are created or annihilated, would both quantify the required tuning and help identify parameter regions where radiative stability might be enhanced by additional accidental symmetries or by improved lattice point-group structure.

From the condensed-matter viewpoint, single-Weyl Hamiltonians may serve as useful lattice models of Weyl-semimetal- or gapless-superconductor-type systems for studying how an isolated Weyl node can appear, how it can be destabilized by symmetry-allowed perturbations, and how additional nodes can emerge as parameters are varied.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: MINIMAL-DOUBLING DIRAC OPERATORS

In this appendix, based on Refs. [22,55], we review a four-dimensional Euclidean lattice formulation of minimally doubled fermions, with focus on the point-split flavor fields and the flavored-mass terms. For recent numerical and analytical developments with the point-split fields of minimal-doubling fermions, see Refs. [82–88]. We here set the lattice spacing to unity and consider a four-component Dirac spinor $\psi(x)$. A Karsten-Wilczek minimally doubled Dirac operator in momentum space is

$$D_{\text{KW}}(p) = i \sum_{j=1}^4 \gamma_j \sin p_j + i\gamma_4(3 - \cos p_1 - \cos p_2 - \cos p_3). \quad (\text{A1})$$

One easily checks that

$$D_{\text{KW}}(p) = 0 \iff p_A = (0, 0, 0, 0) \quad \text{or} \quad p_B = (0, 0, 0, \pi), \quad (\text{A2})$$

while all other would-be doublers are lifted by the Wilson-like term.

At the two zeros the Dirac structure is inequivalent. The gamma matrices at p_A are related to those at p_B by a similarity transformation,

$$\gamma'_\mu = \Gamma^{-1} \gamma_\mu \Gamma, \quad \Gamma = i\gamma_4 \gamma_5. \quad (\text{A3})$$

This difference will be taken into account through a point-splitting procedure that identifies the two poles as independent flavors.

To make the two minimally doubled species manifest, we introduce momentum-dependent factors that project onto the neighborhoods of p_A and p_B . Since in our convention the poles lie at these two values, it is natural to use

$$P_A = \frac{1 + \cos p_4}{2}, \quad P_B = \frac{1 - \cos p_4}{2}. \quad (\text{A4})$$

Near p_A , one has $\cos p_4 \simeq 1$ and hence, $P_A \simeq 1$, $P_B \simeq 0$ ($p_4 \simeq 0$), whereas near p_B , one has $\cos p_4 \simeq -1$, so that $P_A \simeq 0$, $P_B \simeq 1$ ($p_4 \simeq \pi$). We then define two point-split Dirac fields in momentum space by

$$u(p - p_A) = P_A \psi(p), \quad d(p - p_B) = \Gamma P_B \psi(p), \quad (\text{A5})$$

where Γ is the matrix in (A3). By construction, $u(p - p_A)$ has support near the pole at $p_4 = 0$, whereas $d(p - p_B)$ has support near $p_4 = \pi$. In position space, these fields correspond to linear combinations of $\psi(x)$ and its nearest neighbors in the four-direction.

We now assemble the two point-split fields into a flavor doublet,

$$\Psi_f(p) = \begin{pmatrix} u(p - p_A) \\ d(p - p_B) \end{pmatrix}, \quad (\text{A6})$$

on which Pauli matrices τ_i act in flavor (species) space. Because of the factor Γ in (A5), the Dirac structure at the two poles is brought to a common representation, and the action of the usual axial generator γ_5 becomes

$$\gamma_5 \psi(p) \longrightarrow \begin{pmatrix} +\gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix} \Psi_f(p) = (\gamma_5 \otimes \tau_3) \Psi_f(p). \quad (\text{A7})$$

That is, in the two-flavor description the exact lattice chiral symmetry of the KW action is realized as a flavored $U(1)$ generated by $\gamma_5 \otimes \tau_3$, rather than by the naive singlet $\gamma_5 \otimes \mathbf{1}$.

The point-split fields u and d allow us to define flavored mass terms that assign different masses to the two species. In the flavor basis, the simplest such term is

$$m \int \frac{d^4 p}{(2\pi)^4} \bar{\Psi}_f(p) (\mathbf{1} \otimes \tau_3) \Psi_f(p) = m \int \frac{d^4 p}{(2\pi)^4} [\bar{u}(p - p_A) u(p - p_A) - \bar{d}(p - p_B) d(p - p_B)], \quad (\text{A8})$$

with a real parameter m . In terms of the original field $\psi(p)$, this becomes a momentum-dependent “flavored” mass. Using the definitions in (A5) and the fact that $\Gamma^\dagger \Gamma = \mathbf{1}$, one finds $\bar{u}(p - p_A) u(p - p_A) = \bar{\psi}(p) P_A^2 \psi(p)$, $\bar{d}(p - p_B) d(p - p_B) = \bar{\psi}(p) P_B^2 \psi(p)$, so that

$$\bar{u}(p - p_A) u(p - p_A) - \bar{d}(p - p_B) d(p - p_B) = [P_A^2 - P_B^2] \bar{\psi}(p) \psi(p) = \cos p_4 \bar{\psi}(p) \psi(p), \quad (\text{A9})$$

and hence, the flavored mass term in the original Dirac field representation, which keeps the mode at p_A massless and makes the other mode at p_B massive, reads

$$S_m^{\text{flavor}} = m \int \frac{d^4 p}{(2\pi)^4} (1 - \cos p_4) \bar{\psi}(p) \psi(p). \quad (\text{A10})$$

Near the two poles this behaves as $\cos p_4 \sim +1$ ($p_4 \simeq 0$) and $\cos p_4 \sim -1$ ($p_4 \simeq \pi$), so that one of the species acquires bare mass $2m$ while the other remains massless. In position space, the factor $\cos p_4$ corresponds to a symmetric nearest-neighbor hopping in the four-direction. Introducing the translation operators $(T_{\pm 4}\psi)(x) = U_{x,\pm 4}\psi(x \pm e_4)$, one obtains

$$S_m^{\text{flavor}} = m \sum_x \bar{\psi}(x) [1 - (T_{+4} + T_{-4})/2] \psi(x). \quad (\text{A11})$$

This is a local, gauge-covariant operator involving only nearest neighbors along the temporal direction.

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