

Phase structure of $(3 + 1)$ -dimensional dense two-color QCD at $T = 0$ in the strong coupling limit with the tensor renormalization group

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We investigate the phase structure of the $(3 + 1)$ -dimensional strong coupling two-color QCD at zero temperature ($T = 0$) with finite chemical potential using the tensor renormalization group method. The chiral and diquark condensates and the quark number density are evaluated as a function of the chemical potential. We further determine the critical exponents associated with the diquark condensate, which suggest consistency with the predictions of mean-field theory.

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I. INTRODUCTION

The complex action problem with finite chemical potential μ does not allow the standard Monte Carlo simulation to investigate the phase structure of QCD in the finite baryon density at $T = 0$. On the other hand, the simulation of two-color QCD (QC_2D) with $\mu \neq 0$ is free from the complex action problem, so that it provides us an opportunity to understand the physics of matter under the finite density condition. In the strong coupling limit, the analytical methods have been used to investigate the phase structure. In the 1980s, there were several mean field (MF) studies on chiral and diquark condensates in the strong coupling limit of the finite density QC_2D with the Kogut-Susskind quark [1–3]. The results were later refined by Nishida, including the QCD case [4,5]. On the other hand, at finite coupling the standard Monte Carlo approaches have been employed since the mid-1980s [6]. Although large-scale simulations are attainable in recent years, systematic studies changing the lattice sizes reveal temperature dependence of the phase

structure [7–9]. It is manifest that larger lattice sizes are required toward zero temperature limit.

In this paper we investigate the phase structure in the strong coupling limit of the $(3 + 1)$ -dimensional $((3 + 1)d)$ finite density QC_2D at $T = 0$ using the tensor renormalization group (TRG) method [10]. With the use of the Kogut-Susskind quark, we measure the chiral and diquark condensates and the quark number density as a function of μ . There are two purposes in this work. First, we enlarge the lattice size up to 1024^4 with the TRG method to investigate the phase structure at $T = 0$, where the standard Monte Carlo approach is hardly accessible. It is useful to compare our results with the analytical ones obtained by the MF method and the $1/d_s$ expansion with d_s the spatial dimension [4]. Second, this is an appropriate preparatory study before exploring $(3 + 1)d$ finite density QCD, whose complex action problem prohibits the standard Monte Carlo approaches. The TRG method may be qualified to investigate the phase structure of finite density QCD: This method, which is in principle free from the sign problem or the complex action problem, has been successfully applied to various types of models with these problems [12,13,23–42]. So far there exist several TRG applications, particularly dealing with the $(1 + 1)d$ QCD-like theories [43,44], the $(1 + 1)d$ QCD in the strong coupling limit [36], and an effective field theory of the QCD at finite density [13]. In Ref. [44], the bond-weighted TRG (BTRG) algorithm [14,15] is employed to study the finite-density QC_2D . Although the tensor network formulation of the $(3 + 1)d$ QC_2D can be straightforwardly

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obtained by following Ref. [44], we need to resort to a different TRG algorithm rather than the BTRG, which assumes the $(1+1)d$ tensor network representation. A practically useful algorithm for the $(3+1)d$ QC₂D can be the anisotropic TRG algorithm [16,45], which has been already applied to the $(3+1)d$ finite density Nambu–Jona-Lasinio (NJL) model [13]. Although computational techniques treating four-fermi interactions are in common between the NJL model and the strong coupling QC₂D, the introduction of the color degree of freedom significantly increases the computational difficulty enlarging the size of the initial tensor by 2^{16} . Since this difficulty is further enhanced drastically from QC₂D to QCD we need to develop an efficient method to treat the large size of the initial tensor.

This paper is organized as follows. In Sec. II, we define the action of the QC₂D with finite chemical potential and give the tensor network representation. In Sec. III we measure the chiral and diquark condensates and the quark number density as a function of the chemical potential. The critical exponents are also determined. The results are compared with the analytical ones in Ref. [4]. Section IV is devoted to the summary and outlook.

II. FORMULATION AND NUMERICAL ALGORITHM

A. Strong coupling QC₂D

We consider the finite density QC₂D on a $(3+1)d$ lattice $\Lambda_{3+1} = \{(n_1, n_2, n_3, n_4) | n_{1,2,3} = 1, \dots, N_s, n_4 = 1, \dots, N_\tau\}$ whose volume is $V = N_s^3 \times N_\tau$ and temperature is defined as $T = 1/N_\tau$. The lattice spacing a is set to $a = 1$ unless necessary. The action consists of the gluonic part and the fermionic one. For the former, the simple plaquette action is given by

$$S_G = \frac{2N_c}{g^2} \sum_{n, \nu, \rho} \left\{ 1 - \frac{1}{N_c} \text{ReTr}(U_\nu(n) U_\rho(n + \hat{\nu})) \times U_\nu^\dagger(n + \hat{\rho}) U_\rho^\dagger(n) \right\} \quad (1)$$

with $N_c = 2$ and g the gauge coupling constant. The $SU(N_c)$ -valued link variable between the sites n and $n + \hat{\nu}$, with $\hat{\nu}$ the unit vector in the ν direction, is denoted by $U_\nu(n)$. For the latter we employ the Kogut-Susskind quark action with the finite chemical potential μ ,

$$S_F = \sum_n \left[\frac{1}{2} \sum_\nu \eta_\nu(n) \{ e^{\mu \delta_{\nu,4}} \bar{\chi}(n) U_\nu(n) \chi(n + \hat{\nu}) - e^{-\mu \delta_{\nu,4}} \bar{\chi}(n + \hat{\nu}) U_\nu^\dagger(n) \chi(n) \} + m \bar{\chi}(n) \chi(n) \right] \quad (2)$$

with m the quark mass. $\chi = (\chi^1, \chi^2, \dots, \chi^{N_c})$ is a N_c component Grassmann variable, and $\eta_\nu(n)$ is the staggered

sign function defined by $\eta_\nu(n) = (-1)^{n_1 + \dots + n_{\nu-1}}$ with $\eta_1(n) = 1$.

It is worthwhile to describe the symmetries in this theory. Let us start with the $m = \mu = 0$ case. Equation (2), which corresponds to four-flavor quarks in the continuum limit, is invariant under the following continuous vector and axial vector transformation,

$$\chi(n) \rightarrow e^{i\alpha} \chi(n), \quad \bar{\chi}(n) \rightarrow \bar{\chi}(n) e^{-i\alpha}, \quad (3)$$

$$\chi(n) \rightarrow e^{i\alpha e(n)} \chi(n), \quad \bar{\chi}(n) \rightarrow \bar{\chi}(n) e^{i\alpha e(n)} \quad (4)$$

with $\alpha \in \mathbb{R}$ and $e(n) = (-1)^{n_1 + n_2 + n_3 + n_4}$. The theory has the global $U(1)_V \times U(1)_A$ symmetry, which indicates the conservation of the baryon number and the axial charge. The two-color case is special in the sense that this symmetry is enlarged to the $U(2)$ symmetry [46] thanks to the pseudoreality of the $SU(2)$ group [47,48]. The introduction of the finite chemical potential $\mu \neq 0$ at $m = 0$ reduces the $U(2)$ symmetry to the $U(1)_V \times U(1)_A$ symmetry. In the case of $m \neq 0$ and $\mu \neq 0$ we are left with the $U(1)_V$ symmetry.

The partition function is defined as

$$Z = \int \left(\prod_{n \in \Lambda_{3+1}, \nu} d\chi(n) d\bar{\chi}(n) dU_\nu(n) \right) e^{-S_G - S_F - S_D}, \quad (5)$$

where we add an explicit $U(1)_V$ symmetry breaking term S_D to detect the diquark condensation [46,49],

$$S_D = -\frac{\lambda}{2} \{ \chi^\dagger \tau_2 \chi + \bar{\chi} \tau_2 \bar{\chi}^\dagger \} \quad (6)$$

with τ_2 the second Pauli matrix acting on the color space. In the strong coupling limit $g \rightarrow \infty$, the gluonic action vanishes because of the overall factor of $1/g^2$, and the link variables are left only in the fermionic action.

B. Grassman tensor network representation

In order to apply tensor network methods to this system, it is necessary to rewrite the partition function Z in Eq. (5) as a product of local tensors. Since the theory involves fermionic degrees of freedom, we must properly incorporate their anticommuting nature. To this end, we employ a Grassmann tensor network formulation [50], in which the local tensors are defined using auxiliary Grassmann variables.

For clarity, let us consider the case $\lambda = 0$, which simplifies the discussion. The extension to $\lambda \neq 0$ is straightforward. To obtain the Grassmann tensor network representation of the partition function in Eq. (5), we introduce two-component auxiliary Grassmann variables ζ_ν and ξ_ν ($\nu = 1, 2, 3, 4$) [50]. As a basic tool, for any

N -component Grassmann variable θ_i , one can use the identity

$$\int_{\bar{\theta}, \theta} := \int \prod_{i=1}^N d\bar{\theta}_i d\theta_i e^{\theta_i \bar{\theta}_i} = 1. \quad (7)$$

By inserting Eq. (7) into the hopping terms of S_F for each of the auxiliary variables ζ_ν and ξ_ν ,

$$\exp \left[-\frac{\eta_\nu(n)}{2} e^{\mu\delta_{\nu,4}} \bar{\chi}(n) U_\nu(n) \chi(n + \hat{\nu}) \right] = \int \int_{\bar{\zeta}_\nu(n), \zeta_\nu(n)} \exp \left[-\frac{\eta_\nu(n)}{2} e^{\mu\delta_{\nu,4}} \bar{\chi}(n) \zeta_\nu(n) + \bar{\zeta}_\nu(n) U_\nu(n) \chi(n + \hat{\nu}) \right] \quad (8)$$

$$\exp \left[\frac{\eta_\nu(n)}{2} e^{-\mu\delta_{\nu,4}} \bar{\chi}(n + \hat{\nu}) U_\nu^\dagger(n) \chi(n) \right] = \int \int_{\bar{\xi}_\nu(n), \xi_\nu(n)} \exp \left[\bar{\chi}(n + \hat{\nu}) U_\nu^\dagger(n) \bar{\xi}_\nu(n) - \frac{\eta_\nu(n)}{2} e^{-\mu\delta_{\nu,4}} \xi_\nu(n) \chi(n) \right], \quad (9)$$

we can define the local Grassmann tensor $\mathcal{T}$ by integrating out link variables $U_\nu(n)$ and the staggered fields $\chi(n), \bar{\chi}(n)$ at each lattice site n ,

$$\mathcal{T} = \int \left[\prod_\nu dU_\nu \right] d\chi d\bar{\chi} e^{-m\bar{\chi}\chi} \prod_\nu \left\{ \exp \left[-\frac{\eta_\nu(n)}{2} e^{\mu\delta_{\nu,4}} \bar{\chi} \zeta_\nu(n) + \bar{\zeta}_\nu(n - \hat{\nu}) U_\nu(n - \hat{\nu}) \chi \right] \right. \\ \left. \times \exp \left[\bar{\chi} U_\nu^\dagger(n - \hat{\nu}) \bar{\xi}_\nu(n - \hat{\nu}) - \frac{\eta_\nu(n)}{2} e^{-\mu\delta_{\nu,4}} \xi_\nu(n) \chi \right] \right\}. \quad (10)$$

$$= \sum_{\{i,j\}} \int \left[\prod_\nu dU_\nu \right] d\chi^1 d\bar{\chi}^1 d\chi^2 d\bar{\chi}^2 e^{-m\bar{\chi}\chi} \left[\prod_\nu \left(-\frac{\eta_\nu(n)}{2} e^{\mu\delta_{\nu,4}} \right)^{i_\nu^1 + i_\nu^2} \left(-\frac{\eta_\nu(n)}{2} e^{-\mu\delta_{\nu,4}} \right)^{j_\nu^1 + j_\nu^2} \right] \\ \times \left[\prod_\nu \left(\sum_{a_\nu} \bar{\zeta}_\nu^2 U_\nu^{2a_\nu} \chi^{a_\nu} \right)^{i_\nu^2} \right] \left[\prod_\nu \left(\sum_{b_\nu} \bar{\zeta}_\nu^1 U_\nu^{1b_\nu} \chi^{b_\nu} \right)^{i_\nu^1} \right] \left[\prod_\nu (\xi_\nu^1 \chi^1)^{j_\nu^1} \right] \left[\prod_\nu (\xi_\nu^2 \chi^2)^{j_\nu^2} \right] \\ \times \left[\prod_\nu \left(\sum_{c_\nu} \bar{\chi}^{c_\nu} U_\nu^{c_\nu 2^\dagger} \bar{\xi}_\nu^2 \right)^{j_\nu^2} \right] \left[\prod_\nu \left(\sum_{d_\nu} \bar{\chi}^{d_\nu} U_\nu^{d_\nu 1^\dagger} \bar{\xi}_\nu^1 \right)^{j_\nu^1} \right] \left[\prod_\nu (\bar{\chi}^1 \xi_\nu^1)^{i_\nu^1} \right] \left[\prod_\nu (\bar{\chi}^2 \xi_\nu^2)^{i_\nu^2} \right]. \quad (11)$$

In the above, we have introduced the fermion occupation numbers $i_\nu^1, i_\nu^2, j_\nu^1, j_\nu^2, i_\nu^1, i_\nu^2 \in \{0, 1\}$ ($\nu = 1, 2, 3, 4$), which arise from the Taylor expansion of Eq. (10), and the auxiliary integer variables $a_\nu, b_\nu, c_\nu, d_\nu \in \{1, 2\}$ to sum out the color indices of the staggered fields χ connected to the link variables U_ν . The summation $\sum_{\{i,j\}}$ denotes $(\prod_\nu \sum_{i_\nu^1, i_\nu^2, j_\nu^1, j_\nu^2, i_\nu^1, i_\nu^2})$. Here we omit the explicit space-time dependence except for $\eta_\nu(n)$. This implies that there are eight types of initial tensors, depending on the staggered sign [13].

To carry out the group integral over U_ν in Eq. (11), we introduce the indicator function $g(i) = \frac{1}{2} \delta_{i,0} + \delta_{i,1}$. Using

this function, the Grassmann variables and the link variables can be separated as follows:

$$\left(\sum_{a_\nu} \bar{\zeta}_\nu^2 U_\nu^{2a_\nu} \chi^{a_\nu} \right)^{i_\nu^2} = g(i_\nu^2) \sum_{a_\nu} \left[(\bar{\zeta}_\nu^2)^{i_\nu^2} (U_\nu^{2a_\nu})^{i_\nu^2} (\chi^{a_\nu})^{i_\nu^2} \right]. \quad (12)$$

Thus, within the sum of $\sum_{a_\nu, b_\nu, c_\nu, d_\nu}$, we can analytically integrate the link variables,

$$F_{i_\nu^2 i_\nu^1 j_\nu^2 j_\nu^1}^{a_\nu b_\nu c_\nu d_\nu} = g(i_\nu^2) g(i_\nu^1) g(j_\nu^2) g(j_\nu^1) \int dU_\nu (U_\nu^{2a_\nu})^{i_\nu^2} (U_\nu^{1b_\nu})^{i_\nu^1} (U_\nu^{c_\nu 2^\dagger})^{j_\nu^2} (U_\nu^{d_\nu 1^\dagger})^{j_\nu^1}, \quad (13)$$

where $F_{i_\nu^2 i_\nu^1 j_\nu^2 j_\nu^1}^{a_\nu b_\nu c_\nu d_\nu}$ indicates the results of the integration using Weingarten calculus [51]. After integrating out the gauge fields, all that remains is to integrate out χ and $\bar{\chi}$. A straightforward calculation yields the final form of the Grassmann tensor $\mathcal{T}$,

$$\begin{aligned}
& \mathcal{T}_{\Psi_1(n)\Psi_2(n)\Psi_3(n)\Psi_4(n)\bar{\Psi}_4(n-\hat{4})\bar{\Psi}_3(n-\hat{2})\bar{\Psi}_2(n-\hat{2})\bar{\Psi}_1(n-\hat{1})} \\
&= \sum_{i,j,i',j'} T_{(i_1^1,i_1^2,j_1^1,j_1^2)(i_2^1,i_2^2,j_2^1,j_2^2)(i_3^1,i_3^2,j_3^1,j_3^2)(i_4^1,i_4^2,j_4^1,j_4^2)(i_1^1,i_1^2,j_1^1,j_1^2)(i_2^1,i_2^2,j_2^1,j_2^2)(i_3^1,i_3^2,j_3^1,j_3^2)(i_4^1,i_4^2,j_4^1,j_4^2)} \\
&\quad \times \left[(\zeta_1^1)^{i_1^1} (\zeta_1^2)^{i_1^2} (\xi_1^1)^{j_1^1} (\xi_1^2)^{j_1^2} \right] \left[(\zeta_2^1)^{i_2^1} (\zeta_2^2)^{i_2^2} (\xi_2^1)^{j_2^1} (\xi_2^2)^{j_2^2} \right] \left[(\zeta_3^1)^{i_3^1} (\zeta_3^2)^{i_3^2} (\xi_3^1)^{j_3^1} (\xi_3^2)^{j_3^2} \right] \left[(\zeta_4^1)^{i_4^1} (\zeta_4^2)^{i_4^2} (\xi_4^1)^{j_4^1} (\xi_4^2)^{j_4^2} \right] \\
&\quad \times \left[(\bar{\zeta}_4^1)^{i_4^1} (\bar{\zeta}_4^2)^{i_4^2} (\bar{\xi}_4^1)^{j_4^1} (\bar{\xi}_4^2)^{j_4^2} \right] \left[(\bar{\zeta}_3^1)^{i_3^1} (\bar{\zeta}_3^2)^{i_3^2} (\bar{\xi}_3^1)^{j_3^1} (\bar{\xi}_3^2)^{j_3^2} \right] \left[(\bar{\zeta}_2^1)^{i_2^1} (\bar{\zeta}_2^2)^{i_2^2} (\bar{\xi}_2^1)^{j_2^1} (\bar{\xi}_2^2)^{j_2^2} \right] \left[(\bar{\zeta}_1^1)^{i_1^1} (\bar{\zeta}_1^2)^{i_1^2} (\bar{\xi}_1^1)^{j_1^1} (\bar{\xi}_1^2)^{j_1^2} \right], \quad (14)
\end{aligned}$$

where $\Psi_\nu = (\zeta_\nu^1, \zeta_\nu^2, \xi_\nu^1, \xi_\nu^2)$ and $\bar{\Psi}_\nu = (\bar{\zeta}_\nu^1, \bar{\zeta}_\nu^2, \bar{\xi}_\nu^1, \bar{\xi}_\nu^2)$ denote the composite Grassmann variables, and the coefficient tensor T is defined as

$$\begin{aligned}
& T_{(i_1^1,i_1^2,j_1^1,j_1^2)(i_2^1,i_2^2,j_2^1,j_2^2)(i_3^1,i_3^2,j_3^1,j_3^2)(i_4^1,i_4^2,j_4^1,j_4^2)(i_1^1,i_1^2,j_1^1,j_1^2)(i_2^1,i_2^2,j_2^1,j_2^2)(i_3^1,i_3^2,j_3^1,j_3^2)(i_4^1,i_4^2,j_4^1,j_4^2)} \\
&= \sum_{i,j} \left[\prod_{\nu} \left(-\frac{\eta_{\nu}(n)}{2} e^{\mu \delta_{\nu,4}} \right)^{i_{\nu}^1+i_{\nu}^2} \left(-\frac{\eta_{\nu}(n)}{2} e^{-\mu \delta_{\nu,4}} \right)^{j_{\nu}^1+j_{\nu}^2} \right] \left(\prod_{\nu} \sum_{a_{\nu}, b_{\nu}, c_{\nu}, d_{\nu}} \right) \left(\prod_{\nu} \tilde{F}_{i_{\nu}^2, i_{\nu}^1, j_{\nu}^2, j_{\nu}^1}^{a_{\nu}, b_{\nu}, c_{\nu}, d_{\nu}} \right) \\
&\quad \times \left(-\delta_{1, \sum_{\nu} (\delta^{1,a_{\nu}} i_{\nu}^2 + \delta^{1,b_{\nu}} i_{\nu}^1 + j_{\nu}^1)} \delta_{1, \sum_{\nu} (\delta^{2,a_{\nu}} i_{\nu}^2 + \delta^{2,b_{\nu}} i_{\nu}^1 + j_{\nu}^2)} \delta_{1, \sum_{\nu} (\delta^{1,c_{\nu}} j_{\nu}^2 + \delta^{1,d_{\nu}} j_{\nu}^1 + i_{\nu}^1)} \delta_{1, \sum_{\nu} (\delta^{2,c_{\nu}} j_{\nu}^2 + \delta^{2,d_{\nu}} j_{\nu}^1 + i_{\nu}^2)} \right. \\
&\quad + m \delta_{1, \sum_{\nu} (\delta^{1,a_{\nu}} i_{\nu}^2 + \delta^{1,b_{\nu}} i_{\nu}^1 + j_{\nu}^1)} \delta_{0, \sum_{\nu} (\delta^{2,a_{\nu}} i_{\nu}^2 + \delta^{2,b_{\nu}} i_{\nu}^1 + j_{\nu}^2)} \delta_{1, \sum_{\nu} (\delta^{1,c_{\nu}} j_{\nu}^2 + \delta^{1,d_{\nu}} j_{\nu}^1 + i_{\nu}^1)} \delta_{0, \sum_{\nu} (\delta^{2,c_{\nu}} j_{\nu}^2 + \delta^{2,d_{\nu}} j_{\nu}^1 + i_{\nu}^2)} \\
&\quad + m \delta_{0, \sum_{\nu} (\delta^{1,a_{\nu}} i_{\nu}^2 + \delta^{1,b_{\nu}} i_{\nu}^1 + j_{\nu}^1)} \delta_{1, \sum_{\nu} (\delta^{2,a_{\nu}} i_{\nu}^2 + \delta^{2,b_{\nu}} i_{\nu}^1 + j_{\nu}^2)} \delta_{0, \sum_{\nu} (\delta^{1,c_{\nu}} j_{\nu}^2 + \delta^{1,d_{\nu}} j_{\nu}^1 + i_{\nu}^1)} \delta_{1, \sum_{\nu} (\delta^{2,c_{\nu}} j_{\nu}^2 + \delta^{2,d_{\nu}} j_{\nu}^1 + i_{\nu}^2)} \\
&\quad \left. + m^2 \delta_{0, \sum_{\nu} (i_{\nu}^2 + i_{\nu}^1 + j_{\nu}^1 + j_{\nu}^2 + j_{\nu}^1 + i_{\nu}^1 + i_{\nu}^2)} \right) \\
&\quad \times (-1)^R \left(i_1^1, i_2^2, j_1^1, j_2^2 \right) \left(i_1^2, i_2^1, j_1^2, j_2^1 \right) \left(i_3^1, i_4^2, j_3^1, j_4^2 \right) \left(i_3^2, i_4^1, j_3^2, j_4^1 \right) \left(i_1^1, i_2^2, j_1^1, j_2^2 \right) \left(i_1^2, i_2^1, j_1^2, j_2^1 \right) \left(i_3^1, i_4^2, j_3^1, j_4^2 \right) \left(i_3^2, i_4^1, j_3^2, j_4^1 \right). \quad (15)
\end{aligned}$$

The tensor $\tilde{F}$ is a redefined version of F that incorporates a, b, c, d dependence of the sign factors arising from the reordering of $\chi^2, \chi^1, \bar{\chi}^2, \bar{\chi}^1$ in the integration procedure. The tensor R likewise includes sign factors originating from the rearrangement of the auxiliary Grassmann fields $\zeta, \xi, \bar{\zeta}, \bar{\xi}$ to the form of Eq. (14). The explicit definitions of $F, \tilde{F}$, and R are provided in the Appendix. Using these definitions, the partition function Z can be expressed as a Grassmann tensor network [29],

$$Z = \text{gTr} \left[\prod_n \mathcal{T}_{\Psi_1(n)\Psi_2(n)\Psi_3(n)\Psi_4(n)\bar{\Psi}_4(n-\hat{4})\bar{\Psi}_3(n-\hat{3})\bar{\Psi}_2(n-\hat{2})\bar{\Psi}_1(n-\hat{1})} \right]. \quad (16)$$

Here, gTr denotes the Grassmann trace, i.e., the integration over the auxiliary Grassmann variables Ψ and $\bar{\Psi}$ in the sense of Eq. (7).

To evaluate Eq. (16), we have employed the Grassmann-anisotropic TRG method [13,16] with the bond dimension D incorporating the multi-GPU parallelization technique proposed in Ref. [52].

III. NUMERICAL RESULTS

Before presenting the results for the physical quantities, we check the convergence behavior of the thermodynamic potential $f(m, \mu, \lambda, D) = \ln Z(m, \mu, \lambda, D)/V$ by defining the quantity

$$\delta f(m, \mu, \lambda, D) = \left| \frac{\ln Z(m, \mu, \lambda, D) - \ln Z(m, \mu, \lambda, D=60)}{\ln Z(m, \mu, \lambda, D=60)} \right| \quad (17)$$

on $V = 1024^4$. Figure 1 shows the D dependence of δf at $(m, \lambda) = (1.0, 0)$ with the choices of $\mu = 1.08$ and 1.12 , the former of which resides in the Silver Blaze region and the latter in the region of the diquark condensation, as shown below. We observe good convergence behaviors for both cases: δf reaches $O(10^{-4})$ at $D = 55$. Hereafter, we present the results at $D = 55$ and omit the argument D in the functions of f and Z .

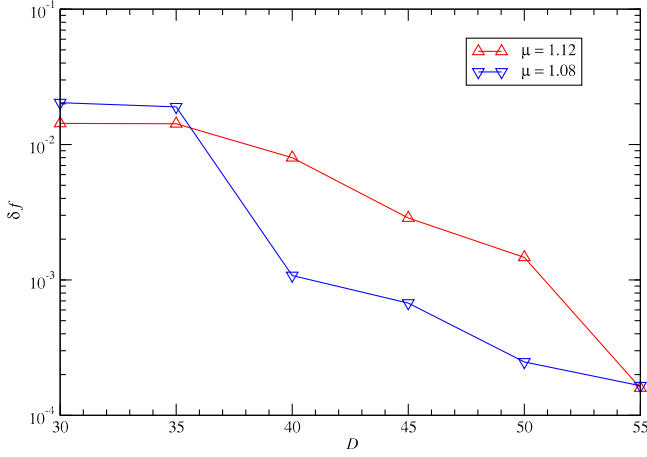


FIG. 1. Convergence behaviors of thermodynamic potential as a function of D at $(m, \lambda) = (1.0, 0)$ on a 1024^4 . $\mu = 1.08$ and 1.12 cases are plotted.

A. Chiral condensate and number density

We first show the results for the expectation values of the chiral condensate $\langle \bar{\chi}\chi \rangle$ and the quark number density $\langle n \rangle$ employing the numerical derivative of the thermodynamic potential,

$$\begin{aligned} \langle \bar{\chi}\chi \rangle &= \frac{1}{V} \frac{\partial \ln Z(m, \mu, \lambda = 0)}{\partial m} \\ &\simeq \frac{1}{V} \frac{\ln Z(m + \Delta m, \mu, \lambda = 0) - \ln Z(m - \Delta m, \mu, \lambda = 0)}{2\Delta m}, \end{aligned} \quad (18)$$

$$\begin{aligned} \langle n \rangle &= \frac{1}{V} \frac{\partial \ln Z(m, \mu, \lambda = 0)}{\partial \mu} \\ &\simeq \frac{1}{V} \frac{\ln Z(m, \mu + \Delta \mu, \lambda = 0) - \ln Z(m, \mu - \Delta \mu, \lambda = 0)}{2\Delta \mu} \end{aligned} \quad (19)$$

with $\Delta m = 0.02$ and $\Delta \mu = 0.02$.

Figure 2 plots $\langle \bar{\chi}\chi \rangle$ and $\langle n \rangle$, together with the diquark condensate $\langle \chi\chi \rangle$, as a function of μ at $m = 1.0$ on a 1024^4 lattice with the choice of $D = 55$. We find that the qualitative behaviors of $\langle \bar{\chi}\chi \rangle$ and $\langle n \rangle$ are similar to those for $m \neq 0$ given by the MF analysis in Ref. [4]: $\langle \bar{\chi}\chi \rangle$ stays constant up to $\mu = \mu_c^{\text{low}}$ and monotonically decreases up to the vanishing point of $\mu = \mu_c^{\text{up}}$, where $\mu_c^{\text{low}} \leq \mu \leq \mu_c^{\text{up}}$ is the region of $\langle \chi\chi \rangle \neq 0$; $\langle n \rangle/2$ starts to increase from zero at $\mu = \mu_c^{\text{low}}$ and saturates to $\langle n \rangle/2 = 1$ at $\mu \geq \mu_c^{\text{up}}$. Note that $\langle n \rangle > 2$ is not allowed due to the Pauli exclusion principle. In Fig. 3 we plot the results for $\langle \bar{\chi}\chi \rangle$ and $\langle n \rangle$ changing the volume. It is clear that the 1024^4 lattice size is large enough to obtain the results in the thermodynamic limit at the zero temperature.

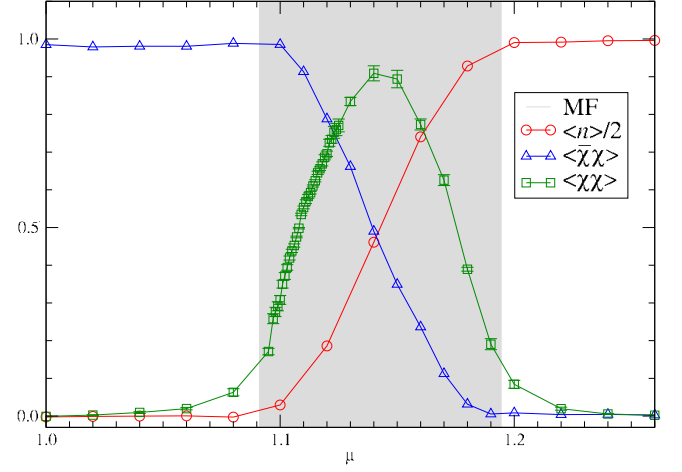


FIG. 2. μ dependence of $\langle \bar{\chi}\chi \rangle$, $\langle \chi\chi \rangle$ and $\langle n \rangle/2$ at $(m, \lambda) = (1.0, 0)$ on a 1024^4 lattice with $D = 55$.

B. Diquark condensate

The diquark condensate is defined by

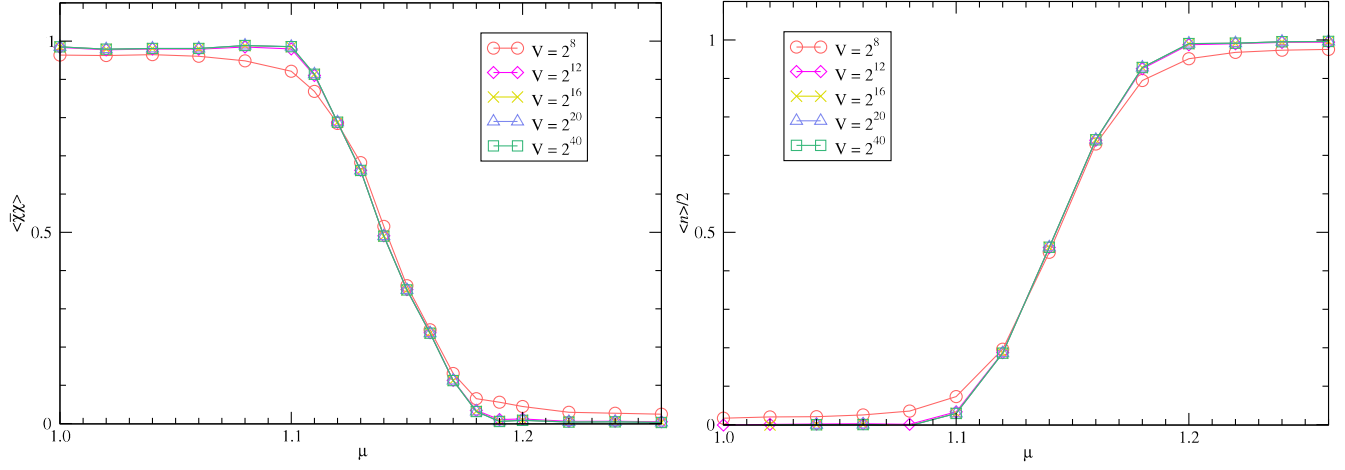
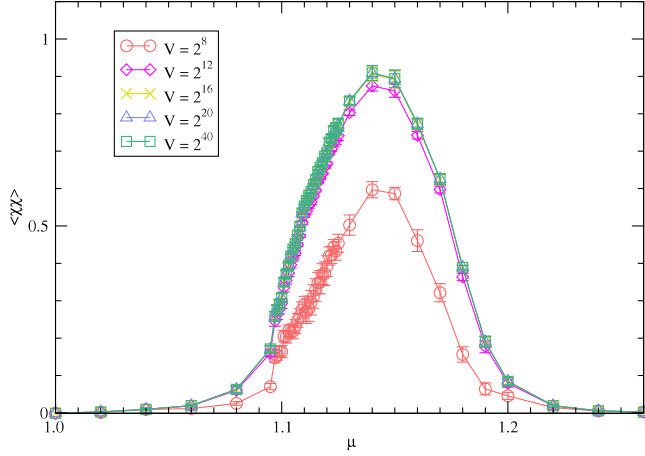
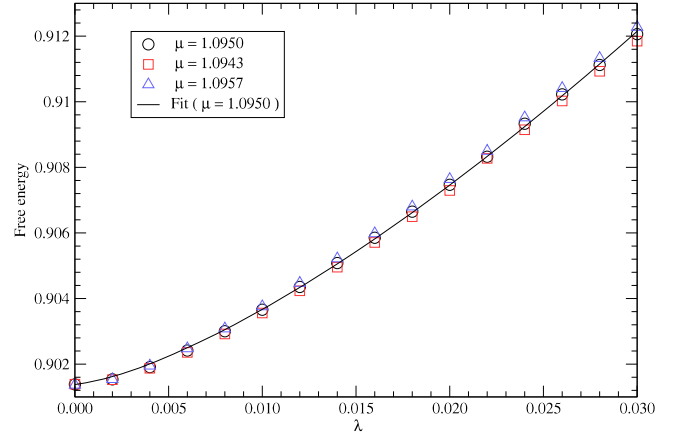
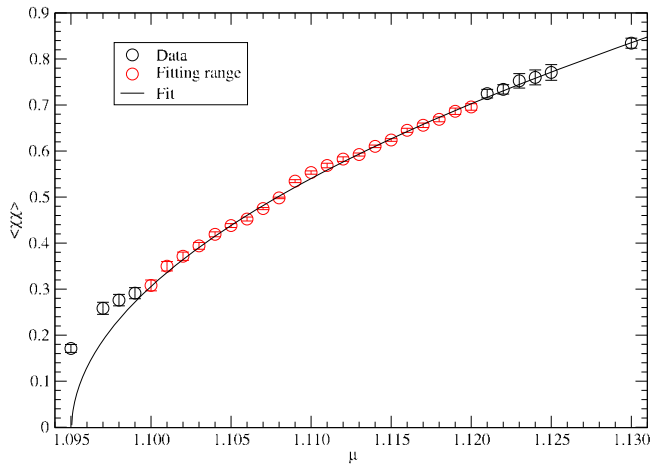
$$\langle \chi\chi \rangle = \frac{1}{V} \frac{\partial \ln Z(m, \mu, \lambda)}{\partial \lambda} \Big|_{\lambda \rightarrow 0}. \quad (20)$$

In order to avoid instabilities of the numerical derivative in terms of λ in the vicinity of $\lambda = 0$, we evaluate the diquark condensate by fitting $f(m, \mu, \lambda)$ with the following function:

$$f(m, \mu, \lambda) = b_1 \lambda^2 + b_2 |\lambda| + f(m, \mu, \lambda = 0), \quad (21)$$

where the coefficient b_2 gives the diquark condensate. The range of λ was chosen depending on μ : for $\mu = 1.095$, $\lambda \in [0, 0.030]$ with $\Delta \lambda = 0.002$, for $1.097 \leq \mu \leq 1.11$, $\lambda \in [0, 0.045]$ with $\Delta \lambda = 0.005$, and for other μ , $\lambda \in [0, 0.040]$ with $\Delta \lambda = 0.005$. We find that the μ dependence of the diquark condensate in Fig. 2 is qualitatively described by the MF analysis in Ref. [4]. The region of $\langle \chi\chi \rangle \neq 0$ shows rough consistency with the MF prediction, which is denoted by the gray band from $\mu_c^{\text{low}} = 1.0913$ to $\mu_c^{\text{up}} = 1.1944$. Figure 4 shows the volume dependence of the diquark condensate. We hardly observe the volume dependence for the results on $V \geq 2^{16}$. As in the case of $\langle \bar{\chi}\chi \rangle$ and $\langle n \rangle$, the diquark condensate in the thermodynamic limit at zero temperature is obtained on $V = 1024^4$.

Now let us determine the critical exponents β_m and δ . Figure 5 shows the μ dependence of $\langle \chi\chi \rangle$ with $D = 55$ near the critical point μ_c^{low} . A fit of $\langle \chi\chi \rangle$ in the region of $\mu \in [1.10, 1.12]$ assuming $\langle \chi\chi \rangle = A(\mu - \mu_c^{\text{low}})^{\beta_m}$ with A , μ_c^{low} and β_m the free parameters yields $A = 4.7(4)$, $\mu_c^{\text{low}} = 1.0950(7)$ and $\beta_m = 0.514(27)$. The value of the critical μ is close to the MF prediction of $\mu_c^{\text{low}} = 1.0913$ and β_m shows consistency with the MF prediction of $\beta_m = 0.5$. The lower limit of the fitting range is chosen to avoid the

FIG. 3. Volume dependence of $\langle \bar{\chi}\chi \rangle$ and $\langle n \rangle/2$ at $(m, \lambda) = (1.0, 0)$ with $D = 55$.FIG. 4. Volume dependence of $\langle \chi\chi \rangle$ at $(m, \lambda) = (1.0, 0)$ with $D = 55$.FIG. 6. λ dependence of $\langle \chi\chi \rangle$ at $m = 1.0$ on a 1024^4 lattice with $D = 55$. Circle, triangle and square symbols denote the results at $\mu = 1.0950, 1.0957$, and 1.0943 , respectively.FIG. 5. μ dependence of $\langle \chi\chi \rangle$ around μ_c^{low} at $(m, \lambda) = (1.0, 0)$ on a 1024^4 lattice with $D = 55$.

immediate vicinity of the critical point. Below $\mu = 1.10$ nonanalytic behaviors of $f(m, \mu, \lambda)$ in terms of λ becomes manifest. The robustness of the fit results against the change of the upper limit of the fitting range is indicated by the fact that the data in the region of $1.12 < \mu \leq 1.13$ is on the fit curve. We also estimate another critical exponent δ from the λ dependence of the free energy at $\mu_c^{\text{low}} = 1.0950(7)$ plotted in Fig. 6. The circle, triangle, and square symbols denote the results at $\mu = 1.0950, 1.0957$, and 1.0943 , respectively. The solid curve denotes the fit result at $\mu = 1.0950$ with the form $f(m, \mu, \lambda = 0) + b_0 \lambda^{1+1/\delta}$ choosing the fitting range of $\lambda \in [0, 0.03]$. The free parameters are determined to be $(b_0, \delta) = (1.51(6), 2.44(6))$ at $\mu = 1.0950$, $(b_0, \delta) = (1.42(5), 2.59(6))$ at $\mu = 1.0957$, and $(b_0, \delta) = (1.59(7), 2.32(6))$ at $\mu = 1.0943$. The scattering of the central value of δ due to the uncertainty in the determination of μ_c^{low} is considered to be the systematic

error. The value of δ is estimated as $\delta = 2.44(6)^{(+0.15)}_{(-0.12)}$, where the numbers in the second parentheses represent the systematic error. The discrepancy between our result of δ and the MF prediction of $\delta = 3$ may be attributed to the difficulty in the precise determination of μ_c^{low} . We find that the value of δ seems to become larger as μ increases in the vicinity of the critical point. Actually in case that δ is measured at $\mu = 1.0971$, which is 3σ away from $\mu_c = 1.0950$, we obtain $\delta = 2.91(6)$ resulting in almost consistent with the MF prediction.

IV. SUMMARY AND OUTLOOK

We have investigated the phase structure of the $(3 + 1)d$ finite density QC₂D at $m = 1.0$ on a 1024^4 lattice with $D = 55$. The volume is large enough to be regarded as the thermodynamic limit at zero temperature. The TRG results for the chiral and diquark condensates and the number density show similar μ dependences to the previous MF results. The value of the critical exponents β_m is comparable to the MF prediction, while δ shows a slightly smaller value.

As a next step, it may be worthwhile to incorporate gauge fields at finite coupling. Another possible direction is an extension to the Wilson quark.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: EXPLICIT FORM OF F AND R

We provide explicit form of the initial tensors. The Weingarten integration of link variables is expressed as follows:

$$\begin{aligned}
 F_{i_\nu^2 i_\nu^1 j_\nu^2 j_\nu^1}^{a_\nu b_\nu c_\nu d_\nu} &= g(i_\nu^2)g(i_\nu^1)g(j_\nu^2)g(j_\nu^1) \int dU_\nu (U_\nu^{2a_\nu})^{i_\nu^2} (U_\nu^{1b_\nu})^{i_\nu^1} (U_\nu^{c_\nu 2^\dagger})^{j_\nu^2} (U_\nu^{d_\nu 1^\dagger})^{j_\nu^1} \\
 &= \left[\delta_{0,i_\nu^2} \delta_{0,i_\nu^1} \delta_{0,j_\nu^2} \delta_{0,j_\nu^1} + \delta_{1,i_\nu^2} \delta_{1,i_\nu^1} \delta_{1,j_\nu^2} \delta_{1,j_\nu^1} \frac{1}{3} \left\{ \delta^{2,2} \delta^{a_\nu, c_\nu} \delta^{1,1} \delta^{b_\nu, d_\nu} - \frac{1}{2} (\delta^{2,2} \delta^{a_\nu, d_\nu} \delta^{1,1} \delta^{b_\nu, c_\nu}) \right\} \right. \\
 &\quad + \frac{1}{2} \delta_{1,i_\nu^2} \delta_{1,j_\nu^2} \delta_{0,i_\nu^1} \delta_{0,j_\nu^1} (\delta^{a_\nu, c_\nu}) + \frac{1}{2} \delta_{0,i_\nu^2} \delta_{0,j_\nu^2} \delta_{1,i_\nu^1} \delta_{1,j_\nu^1} (\delta^{b_\nu, d_\nu}) + \delta_{1,i_\nu^2} \delta_{1,i_\nu^1} \delta_{0,j_\nu^2} \delta_{0,j_\nu^1} \frac{1}{2!} (\epsilon^{21} \epsilon^{a_\nu b_\nu}) \\
 &\quad \left. + \delta_{0,i_\nu^2} \delta_{0,i_\nu^1} \delta_{1,j_\nu^2} \delta_{1,j_\nu^1} \frac{1}{2!} (\epsilon^{c_\nu d_\nu} \epsilon^{21}) \right] g(i_\nu^2)g(i_\nu^1)g(j_\nu^2)g(j_\nu^1). \tag{A1}
 \end{aligned}$$

The difficulty in the Grassmann integration in Eq. (11) is that we cannot simply align staggered fields $\chi^1, \chi^2, \bar{\chi}^1, \bar{\chi}^2$ to a set order. To see this, we rewrite Eq. (11) in the following:

$$\begin{aligned}
 \mathcal{T} &= \sum_{\{i,j\}} \int d\chi^1 d\bar{\chi}^1 d\chi^2 d\bar{\chi}^2 (1 - m\bar{\chi}^1 \chi^1 - m\bar{\chi}^2 \chi^2 + m^2 \bar{\chi}^1 \chi^1 \bar{\chi}^2 \chi^2) \left(\prod_\nu \sum_{a_\nu, b_\nu, c_\nu, d_\nu} \right) \left(\prod_\nu F_{i_\nu^2 i_\nu^1 j_\nu^2 j_\nu^1}^{a_\nu b_\nu c_\nu d_\nu} \right) \\
 &\quad \times \left[\prod_\nu (\bar{\xi}_\nu^2 \chi_\nu^{a_\nu})^{i_\nu^2} \right] \left[\prod_\nu (\bar{\xi}_\nu^1 \chi_\nu^{b_\nu})^{i_\nu^1} \right] \left[\prod_\nu (\xi_\nu^1 \chi_\nu^{j_\nu^1})^{j_\nu^1} \right] \left[\prod_\nu (\xi_\nu^2 \chi_\nu^{j_\nu^2})^{j_\nu^2} \right] \\
 &\quad \times \left[\prod_\nu (\bar{\chi}_\nu^{c_\nu} \bar{\xi}_\nu^2)^{j_\nu^2} \right] \left[\prod_\nu (\bar{\chi}_\nu^{d_\nu} \bar{\xi}_\nu^1)^{j_\nu^1} \right] \left[\prod_\nu (\bar{\chi}_\nu^1 \xi_\nu^1)^{i_\nu^1} \right] \left[\prod_\nu (\bar{\chi}_\nu^2 \xi_\nu^2)^{i_\nu^2} \right]. \tag{A2}
 \end{aligned}$$

Examining Eq. (A2), the ordering of the χ^1, χ^2 and $\bar{\chi}^1, \bar{\chi}^2$ depends on the choice of pairs of the primed variables i'_ν, j'_ν and the color indices $a_\nu, b_\nu, c_\nu, d_\nu$. Therefore, when performing the integration in the order $d\chi^1 d\bar{\chi}^1 d\chi^2 d\bar{\chi}^2$, a straightforward strategy is to absorb the corresponding sign factors into $F_{i'_\nu j'_\nu j'_\nu j'_\nu}^{a_\nu b_\nu c_\nu d_\nu}$. The final form is given by

$$\begin{aligned} \tilde{F}_{i'_\nu j'_\nu j'_\nu j'_\nu}^{a_\nu b_\nu c_\nu d_\nu} = & \left[\delta_{0,i'_\nu} \delta_{0,i'_\nu} \delta_{0,j'_\nu} \delta_{0,j'_\nu} + \delta_{1,i'_\nu} \delta_{1,i'_\nu} \delta_{1,j'_\nu} \delta_{1,j'_\nu} \frac{1}{3} \left\{ \delta^{2,2} \delta^{a_\nu, c_\nu} \delta^{1,1} \delta^{b_\nu, d_\nu} + \frac{1}{2} (\delta^{2,2} \delta^{a_\nu, d_\nu} \delta^{1,1} \delta^{b_\nu, c_\nu}) \right\} \right. \\ & + \frac{1}{2} \delta_{1,i'_\nu} \delta_{1,j'_\nu} \delta_{0,i'_\nu} \delta_{0,j'_\nu} (\delta^{a_\nu, c_\nu}) + \frac{1}{2} \delta_{0,i'_\nu} \delta_{0,j'_\nu} \delta_{1,i'_\nu} \delta_{1,j'_\nu} (\delta^{b_\nu, d_\nu}) + \delta_{1,i'_\nu} \delta_{1,i'_\nu} \delta_{0,j'_\nu} \delta_{0,j'_\nu} \frac{1}{2!} (\epsilon^{21} |\epsilon^{a_\nu b_\nu}|) \\ & \left. + \delta_{0,i'_\nu} \delta_{0,i'_\nu} \delta_{1,j'_\nu} \delta_{1,j'_\nu} \frac{1}{2!} (|\epsilon^{c_\nu d_\nu}| \epsilon^{21}) \right] g(i'_\nu) g(i'_\nu) g(j'_\nu) g(j'_\nu). \end{aligned}$$

By incorporating $\tilde{F}$ instead of F into $\mathcal{T}$, we can effectively account for the complicated sign factors that arise in the integration.

After performing the integration of link variables and staggered fields, the auxiliary Grassmann variables ζ, ξ appear in the order

$$\left[\prod_\nu (\zeta_\nu^2)^{i'_\nu} \right] \left[\prod_\nu (\bar{\zeta}_\nu^1)^{j'_\nu} \right] \left[\prod_\nu (\xi_\nu^1)^{j'_\nu} \right] \left[\prod_\nu (\xi_\nu^2)^{j'_\nu} \right] \left[\prod_\nu (\bar{\xi}_\nu^2)^{j'_\nu} \right] \left[\prod_\nu (\bar{\xi}_\nu^1)^{j'_\nu} \right] \left[\prod_\nu (\zeta_\nu^1)^{i'_\nu} \right] \left[\prod_\nu (\zeta_\nu^2)^{i'_\nu} \right]. \quad (\text{A3})$$

To reorder them into the form of Eq. (14), we must include sign factors $(-1)^R$, where

$$\begin{aligned} R_{(i'_1, j'_1, j'_1, j'_1)(i'_2, j'_2, j'_2, j'_2)(i'_3, j'_3, j'_3, j'_3)(i'_4, j'_4, j'_4, j'_4)(i'_1, j'_1, j'_1, j'_1)(i'_2, j'_2, j'_2, j'_2)(i'_3, j'_3, j'_3, j'_3)(i'_4, j'_4, j'_4, j'_4)} \\ = i'_1 \left(\sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \\ + i'_1 \left(\sum_{\nu=2}^4 i'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + j'_1 \left(\sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \\ + j'_1 \left(\sum_{\nu=2}^4 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + i'_2 \left(\sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=2}^4 j'_\nu + \sum_{\nu=2}^4 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \\ + i'_2 \left(\sum_{\nu=3}^4 i'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=2}^4 j'_\nu + \sum_{\nu=2}^4 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + j'_2 \left(\sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \\ + j'_2 \left(\sum_{\nu=3}^4 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + i'_3 \left(\sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=3}^4 j'_\nu + \sum_{\nu=3}^4 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \\ + i'_3 \left(i'_4 + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + \sum_{\nu=3}^4 j'_\nu + \sum_{\nu=3}^4 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + j'_3 \left(\sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \\ + j'_3 \left(j'_4 + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + i'_4 \left(\sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + j'_4 + j'_4 + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \\ + i'_4 \left(\sum_{\nu=1}^4 j'_\nu + \sum_{\nu=1}^4 j'_\nu + j'_4 + j'_4 + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + j'_4 \left(\sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + j'_4 \left(\sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \\ + j'_4 \left(\sum_{\nu=1}^3 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) + j'_4 \left(\sum_{\nu=1}^3 j'_\nu + \sum_{\nu=1}^3 j'_\nu + \sum_{\nu=1}^4 i'_\nu + \sum_{\nu=1}^4 i'_\nu \right) \end{aligned}$$

$$\begin{aligned}
& + i'^2_4 \left(\sum_{\nu=1}^3 i'^2_{\nu} \right) + i'^1_4 \left(\sum_{\nu=1}^3 i'^1_{\nu} + \sum_{\nu=1}^3 i'^2_{\nu} \right) + j'^2_3 \left(\sum_{\nu=1}^2 j'^2_{\nu} + \sum_{\nu=1}^3 i'^1_{\nu} + \sum_{\nu=1}^3 i'^2_{\nu} \right) \\
& + j'^1_3 \left(\sum_{\nu=1}^2 j'^1_{\nu} + \sum_{\nu=1}^2 j'^2_{\nu} + \sum_{\nu=1}^3 i'^1_{\nu} + \sum_{\nu=1}^3 i'^2_{\nu} \right) + i'^2_3 \left(\sum_{\nu=1}^2 i'^2_{\nu} \right) \\
& + i'^1_3 \left(\sum_{\nu=1}^2 i'^1_{\nu} + \sum_{\nu=1}^2 i'^2_{\nu} \right) + j'^2_2 \left(j'^2_1 + \sum_{\nu=1}^2 i'^1_{\nu} + \sum_{\nu=1}^2 i'^2_{\nu} \right) \\
& + j'^1_2 \left(j'^1_1 + j'^2_1 + \sum_{\nu=1}^2 i'^1_{\nu} + \sum_{\nu=1}^2 i'^2_{\nu} \right) + i'^2_2 i'^2_1 + i'^1_2 (i'^1_1 + i'^2_1) + j'^2_1 (i'^1_1 + i'^2_1) + j'^1_1 (i'^1_1 + i'^2_1).
\end{aligned}$$

Finally, we briefly comment on the numerical strategy for constructing the initial tensor in Eq. (15). Since the size of T is 16^8 , it is impractical to implement a naïve for-loop over all indices. However, careful inspection of the Kronecker delta constraints arising from the fermion and link variable integrations in Eq. (15) shows that they restrict nonzero contributions to only a small subset of entries, yielding a highly sparse and low-rank structure. Therefore, we performed for-loops only over the indices

allowed by the Kronecker delta constraints and constructed T in the form of a sparse matrix. To generate the fundamental tensors required in the ATRG method, we applied truncated SVD to T using Arpack [53]. Once the five-leg tensors required for the subsequent ATRG calculation are constructed, they are treated as dense matrices endowed with a graded $\mathbb{Z}_2$ symmetry originating from the Grassmann parity throughout the coarse-graining procedure.

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