

Extreme-mass-ratio inspirals in relativistic accretion discs

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We compute relativistic Lindblad torques for circular, equatorial extreme-mass-ratio inspirals (EMRIs) embedded in relativistic thin accretion discs, including spinning black hole configurations. We find that relativistic effects can amplify the magnitude of these torques by orders of magnitude in the strong-field regime, and that the torque can even reverse direction as the EMRI approaches the innermost stable circular orbit (ISCO). However, we show that the location of this reversal is highly spin-dependent, shifting progressively closer to the ISCO, where gravitational-wave emission completely dominates the inspiral, as the spin of the central black hole increases. Spin also modifies the radial dependence of the Lindblad torques. We investigate whether Lindblad torques can be approximated by parametrized power laws of the form $T_{\text{LR}} = A(r_s/10M)^{n_r}$ (or combinations thereof), and find significant spin- and disc-dependent variations in the slope parameter n_r . For instance, for spin $a/M = 0.9$, we find $n_r = 3.6$ in the strong-field regime, compared to the Newtonian value of $n_r = 4.5$. Given current forecasts of parameter recovery for “golden,” loud EMRIs in accretion discs ($\Delta n_r \sim 0.5$), we predict LISA could distinguish between different disc configurations through their relativistic Lindblad torque signatures, providing the first direct probe of the midplane structure of the inner region of accretion discs, which is inaccessible to electromagnetic observations.

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I. INTRODUCTION

Dense astrophysical environments at the centers of galaxies—such as nuclear clusters [1,2], the discs of active galactic nuclei (AGN) [3], and dark-matter overdensities [4–6]—have assumed growing interest in gravitational-wave (GW) astronomy. It is now well established that these structures can significantly influence the formation, evolution, and GW signature of compact binaries across the entire GW spectrum [7–21]. This increased activity is partially driven by the recognition that relativistic effects are crucial to accurately model the interaction between binaries and their astrophysical surroundings [22–34].

Extreme-mass-ratio inspirals (EMRIs) are particularly promising candidates to detect environmental effects in GWs. EMRIs consist of a *secondary* stellar-mass compact

object—typically a black hole (BH) with mass $\mu \sim 10\text{--}100M_\odot$ —slowly inspiralling into a *primary* massive black hole (MBH) with $M \gtrsim 10^5M_\odot$ at the center of a galaxy [35,36]. EMRIs are unique probes of strong-field gravity, as they execute tens of thousands of relativistic orbits while in the frequency band accessible to the Laser Interferometer Space Antenna (LISA), a space-based GW observatory planned for launch in 2035 [37], as well as to the proposed missions Taiji and TianQin [38]. The highly asymmetric mass ratio in EMRIs renders standard techniques developed for comparable-mass binaries inadequate. Instead, the relativistic two-body problem is addressed by the self-force program, which employs BH perturbation theory to expand the Einstein equations in the small mass ratio $q = \mu/M$ [39,40].

To extend the self-force framework to nonvacuum backgrounds, one can introduce an additional expansion parameter encoding the energy density of the surrounding medium [27–31]. The Einstein equations are then solved self-consistently alongside the equations governing the matter fields, enabling the systematic inclusion of processes such as accretion, dynamical friction, and back-reaction on the environment. Applications of this method to EMRIs in

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dark-matter overdensities have revealed that relativistic corrections substantially strengthen binary-environment interactions compared to Newtonian predictions [24,29,30].

The relativistic modeling of EMRIs in accretion discs remains far less developed. This is primarily due to the complexity of modeling strong gravity together with the challenging microphysics of discs, including radiation transport, viscosity, and magnetic fields [3]. Nonetheless, EMRIs in AGN discs represent a compelling science case for LISA, offering a potential window into accretion processes near MBHs [9,17–19,25,41,42]. Intriguingly, indirect evidence for such systems may already exist in the form of quasiperiodic eruptions (QPEs)—sharp, recurrent x-ray bursts observed in galactic centers on timescales of hours to days. QPEs have been interpreted as signatures of a compact object periodically crossing an accretion disc, exciting shocks or density perturbations that modulate the x-ray flux [43–45].

A key property of accretion discs is their differential rotation. At specific radii, particles within the disc can orbit with frequencies that are integer multiples of the time-varying gravitational potential induced by the EMRI. These resonant conditions enable an efficient exchange of angular momentum, giving rise to Lindblad resonances, whose associated torques play a central role in shaping the coupled evolution of the system. Lindblad torques have been extensively investigated in Newtonian gravity, particularly in the context of disc-planet interactions [46–51]. In such systems, the net torque generated by Lindblad resonances drives the evolution of the planet’s orbit, causing migration either inward or outward depending on the relative strengths of the inner and outer Lindblad torques. These Newtonian results have been directly applied to the study of EMRIs embedded in accretion discs. It was shown that they are the dominant environmental effect for these systems [7], and they could be measurable by LISA [9,25,52]. Neglecting these torques in waveform modeling can lead to substantial biases in parameter estimation, introducing systematic errors, causing signals to be missed or misinterpreted as deviations from general relativity (GR) [9,53].

Earlier studies already suggested that accurate EMRI waveform templates would require relativistic models. In 2011, Hirata [54,55] (henceforth referred to as Hirata I [54] and Hirata II [55]) extended the computation of Lindblad torques into the fully relativistic regime, calculating Lindblad resonances as a perturbative correction to the Kerr spacetime surrounded by an arbitrarily thin disc.¹ The disc is assumed to be a two-dimensional, *pressureless* collection of particles in circular geodesics around the primary MBH. Hirata’s analyses demonstrated that general

relativistic effects systematically enhance the magnitude of these torques compared to Newtonian predictions. However, this pioneering work focused primarily on the formalism and only calculated the torque mode by mode for small multipoles normalized by the local disc density. It did not compute the total disc torque exerted on the EMRI and its implications for GW observations. Another limitation of Hirata’s model is that it does not include disc pressure, which is responsible for maintaining the disc’s vertical structure and propagating density waves in the disc. As we will see, without pressure corrections, the total Lindblad torque exerted on the EMRI diverges.

In this work, we extend Hirata’s approach in three main ways: (i) we compute the Lindblad torques for higher-order multipoles (i.e., larger azimuthal mode numbers m), which are numerically challenging because the computational cost increases rapidly with m ; (ii) we explicitly include pressure effects, which regularize the total torque by introducing a minimum cutoff length scale below which disc material in the immediate vicinity of the secondary no longer contributes to resonant Lindblad torques; and (iii) we combine the relativistic computation of the Lindblad torques with a fully relativistic disc structure, adopting the Novikov-Thorne solution for the disc profile [56].

The paper is structured as follows. Sec. II summarizes the three main conclusions of our work. Section III reviews Hirata’s formalism and highlights the divergence of mode sums without pressure. Section IV introduces our pressure regularization and applies it to several disc models, yielding the results of Fig. 1. We conclude with implications for GW astronomy. Unless otherwise stated, we adopt $G = c = 1$, and use primes to denote radial derivatives.

II. EXECUTIVE SUMMARY

Our main results are summarized in Fig. 1, which compares the relativistic and Newtonian torque strengths across different disc models (the relativistic Novikov-Thorne profiles at various spins, and the Newtonian Shakura-Sunyaev profile, with the same accretion rate and viscosity).

We highlight three novel conclusions of our work:

- (1) Relativistic effects substantially amplify accretion torques. Both relativistic corrections to the binary-disc interaction and to the disc structure itself, which had not been considered before, can increase torque amplitudes by orders of magnitude;
- (2) The torque scaling with orbital radius is altered. Lindblad torques can still be described by power laws of the form $T_{\text{LR}} = A(r_s/r_0)^{n_r}$, but the slope n_r is spin-dependent and consistently smaller than the Newtonian prediction, particularly in the strong-field regime, implying Lindblad torques are more relevant for small orbital separations than previously expected (see Fig. 6);

¹The series’ first paper [54] does not specify that the primary BH is Kerr, or the disc being thin; it simply restricts to axisymmetric, time-stationary spacetime with a plane of symmetry.

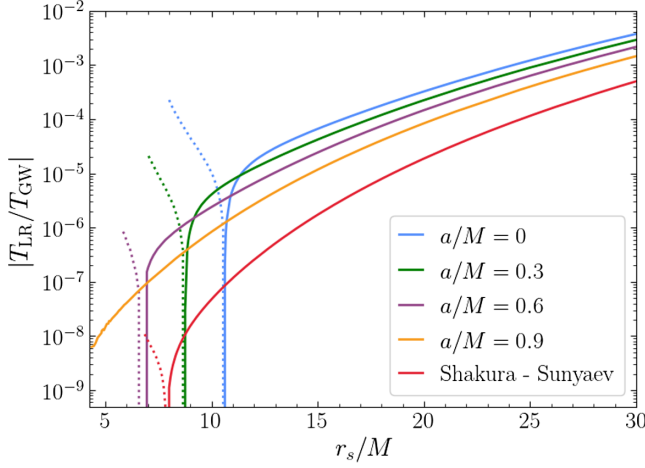


FIG. 1. Ratio of the relativistic Lindblad torques to the gravitational-wave torques for an equatorial, circular EMRI embedded in different disc configurations, shown as a function of orbital separation r_s and for representative disc models with the same viscosity and accretion rate (see Fig. 5). For spinning primaries, we use the relativistic Novikov-Thorne disc model. Dotted segments indicate regions where the Lindblad torque becomes negative.

- (3) Near the ISCO, torques can reverse sign (shown by the dotted lines in Fig. 1). This reversal is associated with the accumulation of inner Lindblad resonances closer to the secondary than the outer resonances as the orbit approaches the ISCO. The reversal radius depends on the MBH spin, occurring closer to the ISCO for higher spins.

Additional remarks on physical configurations. For clarity, we briefly comment on two extensions discussed in our response to the referee.

As we were finalizing this work, we became aware of independent efforts [57,58] pursuing the same problem through a (semi)analytical Hamiltonian approach, improving on Hirata’s original formulation. The methodologies differ: our work focuses on numerically evaluating small multipoles and extrapolating to large ones, whereas Refs. [57,58] derive analytical expressions at large multipoles and extrapolates downward. Our findings are consistent, reinforcing the robustness of these results.

III. RELATIVISTIC LINDBLAD TORQUES IN PRESSURELESS 2D DISCS

We start by summarizing Hirata’s seminal results on relativistic Lindblad resonances for pressureless 2D discs in axisymmetric and stationary spacetimes. Throughout, we assume that the disc is aligned with the spin of the central BH and neglect its backreaction on the Kerr geometry. Both assumptions are well justified in the inner disc region, where alignment is expected from the Bardeen-Petterson or magnetic spin-alignment mechanisms [59–61], and the

disc’s self-gravity is negligible [62]. Additionally, we focus on *prograde* EMRI circular orbits, since interactions with the disc are expected to align their angular momentum orientation and damp eccentricity before the inspiral reaches the inner regions relevant to the LISA band (although residual eccentricity could survive in-band [63,64]). We refer the reader to Refs. [54,55] for complete details.

A. Location of the resonances

Let us start by discussing the radial location of the resonance in the disc for a given radius of the circular orbit of the secondary. Assuming that disc particles follow approximately circular geodesics in the background space-time, a particle that is radially displaced will undergo oscillations about its original orbit at a frequency referred to as the (radial) epicyclic frequency. In the limit of small eccentricities, this epicyclic frequency coincides with the radial orbital frequency. Lindblad resonances arise at locations where the epicyclic frequency of the disc is commensurate with the perturbation frequency from the secondary object. More precisely, these resonances can be identified by the vanishing of the *resonant detuning function*

$$D[r] := m(\Omega[r] - \Omega_s) \mp \kappa[r], \quad m \in \mathbb{N} \quad (1)$$

where the subscript s refers to the secondary, Ω is the orbital frequency for circular equatorial, prograde geodesics in Kerr

$$\Omega[r] = \frac{M^{1/2}}{r^{3/2} + M^{1/2}a}, \quad (2)$$

with a the spin parameter, and κ is the (prograde) radial epicyclic frequency in Kerr,

$$\kappa[r] = \Omega[r] \sqrt{1 - \frac{6M}{r} + \frac{8aM^{1/2}}{r^{3/2}} - \frac{3a^2}{r^2}}. \quad (3)$$

The $-$ sign version of the detuning function identifies inner (with respect to the secondary) Lindblad resonances, whereas the $+$ version identifies outer Lindblad resonances. For any given Ω_s , there are infinitely many inner and outer Lindblad resonance positions (r_{ILR} and r_{OLR} respectively) that satisfy $D[r_{\text{LR}}] = 0$, one for each value of m . Outer Lindblad resonances are responsible for inward migration, while inner ones lead to outward migration.

In Fig. 2, we plot the locations of Lindblad resonances as a function of the secondary’s orbital radius r_s , for various values of m and primary spin parameter a . In the Newtonian regime (large r_s), the outer Lindblad resonances lie closer to the secondary than the inner ones. For a disc with a constant surface density profile, this naturally causes the

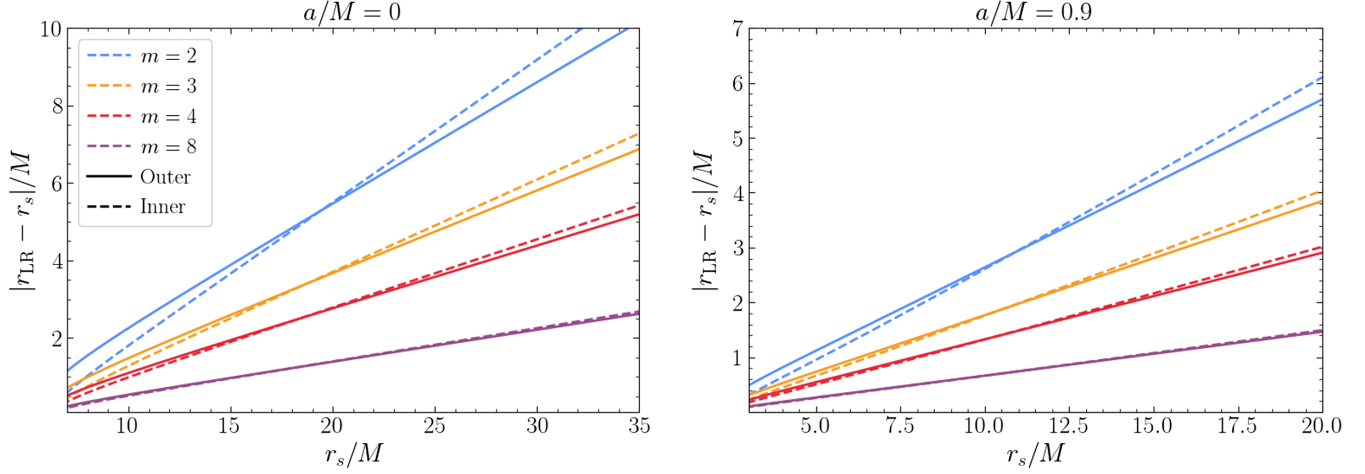


FIG. 2. Radial location of Lindblad resonances, r_{LR} , as a function of the secondary orbital radius, r_s , for different azimuthal mode numbers m and primary BH spin a . In the large- m limit, the inner and outer Lindblad resonances become equidistant and close to the secondary. For large orbital radii, outer Lindblad resonances are consistently located closer to the secondary than their inner counterparts (for the same m), but this hierarchy reverses at smaller r_s . The critical radius at which this switch occurs is spin-dependent, decreasing with increasing BH spin parameter a .

outer resonances to dominate at large radii, resulting in a net inward Lindblad torque on the EMRI—thus accelerating its inspiral compared to one driven by vacuum GW emission.

As the secondary moves inward and r_s decreases, this pattern reverses: the inner Lindblad resonances move closer to the secondary than the outer ones, and the net disc torque may become outward. If GW emission dominates the evolution, this slows down the inspiral with respect to vacuum. If migration torques compete with or dominate over GW emission, the inspiral can be stalled, forming a migration trap for the EMRI. The critical radius where the inner and outer resonance locations switch proximity is spin-dependent—it occurs at smaller r_s for higher a . Specifically, for a nonspinning primary, $a = 0$, this transition happens at $r_s \sim 18M$, while for a rapidly spinning one, $a = 0.9M$, it is shifted inward to approximately $r_s \sim 10M$. This feature is also present in the analysis of Refs. [57,58].

This switch radius also depends on the resonance index m : it shifts to smaller values for larger m , but saturates in the $m \rightarrow \infty$ limit, with the inner and outer Lindblad resonances becoming equidistant from the secondary. Note also that the inner resonances never fall below the ISCO. Additionally, higher- m resonances always lie closer to the secondary, which causes them to contribute more strongly to the total Lindblad torque (when summing over all m). Naively, this sum would diverge, but as we will discuss in later sections, it is regularized by including disc pressure effects, which were absent in Hirata’s original treatment.

B. Mode by mode torque amplitudes

The Lindblad torque corresponds to the cycle-averaged mechanical exchange of energy and angular momentum at

a resonance between the secondary and the disc, which is analogous to the Newtonian theory of disc-satellite coupling [46]. Hirata’s main result was to show this resonant exchange can be expressed in terms of solutions to the Teukolsky equation [65,66], which governs linear perturbations to the Weyl curvature tensor, i.e., provide a complete, gauge-invariant description of (linear) gravitational perturbations in Kerr. These solutions are typically associated with GW emission, as they describe the radiative degrees of freedom of gravity at both null infinity and the BH horizon. However, they also describe the components of the near-zone perturbed gravitational field, which is what is exploited in Hirata’s framework.

1. Definitions: The Kerr metric, circular geodesics, and Teukolsky formalism

To make this section self-contained, we introduce definitions of the Kerr metric [67] and the Teukolsky formalism [65,66]. For comprehensive reviews, see Refs. [40,68]. The explicit form of the line element of the Kerr metric in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ is

$$ds^2 = -\left(1 - \frac{2Mr}{|\rho|^2}\right)dt^2 - \frac{4Mar}{|\rho|^2}\sin^2\vartheta dtd\varphi + \frac{|\rho|^2}{\Delta}dr^2 + |\rho|^2 d\vartheta^2 + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\vartheta}{|\rho|^2} \sin^2\vartheta d\varphi^2, \quad (4)$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad |\rho|^2 = r^2 + a^2 \cos^2\vartheta. \quad (5)$$

From this, one can obtain the contravariant metric coefficients

$$g^{tt} = -\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta}{\Delta |\rho|^2}, \quad (6)$$

$$g^{t\varphi} = -\frac{2aMr}{\Delta |\rho|^2}, \quad g^{\varphi\varphi} = \frac{\Delta - a^2 \sin^2 \vartheta}{\Delta |\rho|^2 \sin^2 \vartheta}, \quad (7)$$

$$g^{rr} = \frac{\Delta}{|\rho|^2}, \quad g^{\vartheta\vartheta} = \frac{1}{|\rho|^2}. \quad (8)$$

Equatorial circular geodesics in Kerr spacetime are described by the four velocity w^α , with

$$w^t = \frac{dt}{d\tau_s} = -g^{tt}E + g^{t\varphi}L_z, \quad (9)$$

$$w^r = 0, \quad (10)$$

$$w^\vartheta = 0, \quad (11)$$

$$w^\phi = \frac{d\phi}{d\tau_s} = -g^{t\phi}E + g^{\phi\phi}L_z, \quad (12)$$

where τ_s is the secondary's proper time, and E and L_z are, respectively, its energy and angular momentum, given by

$$E = \frac{r^{3/2} - 2Mr^{1/2} + aM^{1/2}}{r^{3/4}\sqrt{r^{3/2} - 3Mr^{1/2} + 2aM^{1/2}}}, \quad (13)$$

$$L_z = \frac{M^{1/2}r^2 - aM^{1/2}(2r - M)}{r^{3/2} - 2Mr^{1/2} + aM^{1/2}}. \quad (14)$$

The emission of GWs by an EMRI in Kerr (to first-order in the mass ratio) is governed by the Teukolsky equation for the Newman-Penrose Weyl scalar ψ_4 [69]

$${}_{-2}\mathcal{O}[\psi_4] = T, \quad (15)$$

where ${}_{-2}\mathcal{O}$ is a second-order differential operator and T represents a point-particle source—the secondary or a particle in the disc—that scales linearly with the mass-ratio $q = \mu/M$. ψ_4 is directly related to the gravitational wave polarizations at infinity,

$$\psi_4 \rightarrow \frac{1}{2} \frac{d^2}{dt^2} (h_+ - ih_\times) \quad \text{as } r \rightarrow \infty. \quad (16)$$

The Teukolsky equation is separable in the frequency domain with the decomposition

$$\psi_4 = \frac{1}{\rho^4} \int_{-\infty}^{\infty} d\omega \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} R_{\ell m}(r; \omega) {}_{-2}S_{\ell m}(\vartheta; \chi) e^{i(m\varphi - \omega t)}, \quad (17)$$

where

$$\chi = a\omega, \quad \rho = r - ia \cos \vartheta. \quad (18)$$

Here, ${}_{-2}S_{\ell m}$ are the spin-weighted spheroidal harmonics with spin-weight $s = -2$, which satisfy the eigenvalue equation

$${}_{-2}\mathcal{O}_{\vartheta\varphi}[{}_{-2}S_{\ell m}] = {}_{-2}\mathcal{E}_{\ell m-2}S_{\ell m}, \quad (19)$$

where ${}_{-2}\mathcal{O}_{\vartheta\varphi}$ is a second-order differential operator containing only derivatives in ϑ and φ , while ${}_{-2}\mathcal{E}_{\ell m}$ are the corresponding eigenvalues, which in Kerr are frequency dependent.

The radial function $R_{\ell m}$ satisfies the inhomogeneous radial Teukolsky equation

$$\Delta^2 \frac{d}{dr} \left(\Delta^{-1} \frac{dR}{dr} \right) - VR = -\mathcal{T}, \quad (20)$$

where $\mathcal{T}$ represents the source term, $V = V(r; \omega, \lambda_{\ell m})$ is the effective potential, and

$$\lambda_{\ell m} = {}_{-2}\mathcal{E}_{\ell m} - 2m\varpi + \varpi^2 - 2, \quad (21)$$

with the frame-dragging shifted frequency

$$\varpi = \omega - m\Omega_H, \quad (22)$$

where Ω_H is the angular velocity of the horizon

$$\Omega_H = \frac{a}{2r_+}, \quad (23)$$

and r_+ is the event horizon radius

$$r_+ = M + \sqrt{M^2 - a^2}. \quad (24)$$

The physical solutions of Eq. (20) correspond to outgoing waves at infinity and ingoing ones at the horizon

$$R_{\ell m}^\infty(r) \rightarrow Z_{\ell m \omega}^\infty r^3 e^{i\omega r_*}, \quad r \rightarrow \infty, \quad (25)$$

$$R_{\ell m}^H(r) \rightarrow Z_{\ell m \omega}^H \Delta e^{-i\varpi r_*}, \quad r \rightarrow r_+, \quad (26)$$

where r_* is the tortoise radial coordinate

$$\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}. \quad (27)$$

For generic EMRIs (inclined and eccentric orbits), the amplitudes $Z_{\ell m \omega}^{\infty, H}$ admit a multimodal decomposition [70]

$$Z_{\ell m \omega}^{\infty, H} = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} Z_{\ell m n k}^{\infty, H} \delta(\omega - \omega_{mnk}), \quad (28)$$

where

$$\omega_{mnk} = m\Omega_\varphi + n\Omega_r + k\Omega_\theta, \quad (29)$$

and where $\Omega_\varphi, \Omega_r, \Omega_\theta$ are the fundamental frequencies for bound motion in the azimuthal, radial, and polar direction, respectively. For equatorial orbits, there is no motion in θ and thus we set $k = 0$. Similarly, for circular orbits $n = 0$. The power emitted in GWs to infinity relates to the Teukolsky amplitudes by [65,66]

$$P_\infty = \sum_{\ell, m, n} \frac{|Z_{\ell m n}^\infty|^2}{2\omega_{mn}^2}; \quad (30)$$

and the power absorbed at the horizon is

$$P_H = \sum_{\ell, m, n} \alpha \frac{|Z_{\ell m n}^H|^2}{2\omega_{mn}^2}, \quad (31)$$

where

$$\alpha = \frac{8192r_+ \varpi (\varpi^2 + \Gamma^2) (\varpi^2 + 4\Gamma^2) \omega^3}{|C|^2}, \quad (32)$$

$$\Gamma = \frac{2\sqrt{1-a^2}}{a^2 + r_+^2}, \quad (33)$$

$$|C|^2 = [(\lambda_{\ell m} + 2)^2 + 4m\varpi - 4\varpi^2][\lambda_{\ell m}^2 + 36m\varpi - 36\varpi^2] + 48(2\lambda_{\ell m} + 3)\varpi(2\varpi - m) + 144\omega^2(1 - a^2). \quad (34)$$

Hirata exploited a known relation [71] between ingoing and outgoing Teukolsky amplitudes, reducing the number of amplitudes that must be computed. The outgoing solution, Eq. (25), can be expressed in terms of its near-horizon amplitude via connection coefficients c_{13} and c_{14}

$$R_{\ell m}^\infty(r) \rightarrow Z_{\ell m \omega}^H \left(c_{13} r^3 e^{i\omega r_*} + c_{14} \frac{e^{-i\omega r_*}}{r} \right), \quad r \rightarrow \infty, \quad (35)$$

which are complex constants that can be determined numerically. The power of the outgoing wave, Eq. (30), is then

$$P_\infty = \sum_{\ell, m} \frac{|c_{13} Z_{\ell m \omega}^H|^2}{2\omega^2}. \quad (36)$$

2. Relativistic Lindblad torques formalism

The Hirata's Lindblad torque strength is given by [54]

$$T_{\text{LR}}^{(m)} = \mp \frac{2\pi^2 m}{|D'(r_{\text{LR}})|} r_{\text{LR}} w^t \mathcal{Z} \Sigma |S^{(m)}|^2, \quad (37)$$

where all quantities are evaluated at the resonance r_{LR} , and $\mathcal{Z}$ is the specific epicyclic impedance,

$$\mathcal{Z} = \sqrt{\frac{C_{020}}{C_{002}}} \Big|_{r=r_{\text{LR}}}, \quad (38)$$

with

$$C_{002} = \frac{g^{rr}}{w^t}, \quad (39)$$

$$C_{020} = g^{\varphi\varphi} \left(\frac{g^{\varphi\varphi} - g^{tt}}{(w^t)^3} \right) \left(\frac{dL_z}{dr} \right)^2 - \left(\frac{d\Omega}{dr} \right) \left(\frac{dL_z}{dr} \right). \quad (40)$$

Here, Σ is the surface density of the 2D disc (at the resonance), and $S^{(m)}$ are the resonant amplitudes.²

The resonant amplitudes can be related to a power amplitude [55], $\mathcal{P}_{\text{LR}}^{(m)}$,

$$S^{(m)} = \lim_{e \rightarrow 0} \frac{2i}{m\Omega_s \mu \mathcal{Z}_{\text{LR}} e r_{\text{LR}}} \mathcal{P}_{\text{LR}}^{(m)}(e), \quad (41)$$

where μ is the mass of the disc particle and e is the (small) eccentricity of the test particle in the disc. The power amplitude $\mathcal{P}_{\text{LR}}^{(m)}$ characterizes the cycle-averaged work that the metric perturbation from the secondary would do on a test particle on an orbit with a small eccentricity, e , when the epicyclic frequency is commensurate with the secondary's perturbation frequency. In the limit of small e , the power amplitude is linear in e , and the resonant amplitudes are independent of the actual value of e .

Hirata demonstrated, using a (relativistic) Hamiltonian framework, that this power amplitude $\mathcal{P}_{\text{LR}}^{(m)}$ can be expressed in terms of the same Teukolsky amplitudes mentioned above for computing GW fluxes. Importantly, this reformulation does not imply that Lindblad torques are a consequence of gravitational radiation. It should instead be interpreted as a linear gravitational coupling coefficient between the secondary's perturbation and the disc response, rather than as a radiative energy loss. The resonant interaction can therefore be computed through interference terms between Teukolsky modes sourced by the secondary at $r = r_s$ in circular motion (with $n = 0$) and a particle in the disc on a slightly eccentric orbit (with mode number

²Our definition of $T_{\text{LR}}^{(m)}$ is derived from Hirata I Eq. (92) integrated against r ; this should not be confused with T in Hirata I Eq. (92).

$n = \pm 1$) at $r = r_{\text{LR}}$, where the $+$ ($-$) corresponds to outer (inner) Lindblad resonances. Constructive interference occurs only at Lindblad resonance radii, where the perturbations induced by the EMRI are phase-coherent with the disc particle's orbital motion. In the Newtonian and near-zone limits, the EMRI's GW fluxes vanish, but the same interference structure continues to encode the local exchange of energy and angular momentum between the secondary and the disc.

Then, for inner Lindblad resonances, the power amplitude is the negative of the power due to constructive interference of Teukolsky modes at infinity and down the horizon; i.e.,

$$\sum_m \mathcal{P}_{\text{ILR}}^{(m)} = -\delta P_\infty - \delta P_{\text{H}}, \quad (42)$$

$$\delta P_\infty = \text{Re} \sum_{\ell m} c_{13} Z_{\ell m 0}^{\text{H}} (Z_{\ell m -1}^\infty)^*, \quad (43)$$

$$\delta P_{\text{H}} = \text{Re} \sum_{\ell m} \frac{\alpha Z_{\ell m 0}^{\text{H}} (Z_{\ell m -1}^{\text{H}})^*}{(m\Omega_s)^2}. \quad (44)$$

Here, the $n = 0$ solutions are evaluated with the secondary as source term, while the $n = -1$ solutions are evaluated with the test particle in a slightly eccentric orbit as source term. Using symmetry arguments between coefficients and amplitudes with $\pm m$, the power associated with inner Lindblad resonances can be written as

$$\mathcal{P}_{\text{ILR}}^{(m)} = -\frac{1}{(m\Omega_s)^2} \sum_{\ell \geq m} Z_{\ell m 0}^{\text{H}} (c_{13} Z_{\ell m -1}^{\infty*} + \alpha Z_{\ell m -1}^{\text{H}*}). \quad (45)$$

Following a similar procedure, the power associated with the outer Lindblad resonances is

$$\mathcal{P}_{\text{OLR}}^{(m)} = -\frac{1}{(m\Omega_s)^2} \sum_{\ell \geq m} Z_{\ell m 0}^\infty (Z_{\ell m 1}^{\infty*} - c_{13}^* Z_{\ell m 1}^{\text{H}*}). \quad (46)$$

One key aspect of Eq. (41) is that the energy (and angular momentum) in the disc is invariant under the Lindblad resonance at leading order. This follows from Hirata's assumptions that the disc particles follow circular geodesics of the Kerr spacetime, and the Lindblad resonances only perturb the orbit to linear order in eccentricity. Because, to linear order in e , circular and slightly eccentric orbits in Kerr have the same energy and angular momentum, Lindblad resonances do not change the energy and angular momentum of the disc under Hirata's assumptions.

Note that the power amplitude is not itself the power dissipated by a Lindblad resonance—that quantity arises only at second order, after the disc develops an eccentric response. This distinction explains why the torque strength, Eq. (37), scales as $|\mathcal{P}_{\text{LR}}^{(m)}|^2$: the first power amplitude factor

comes from the perturbing force, while the second comes from the disc particle's radial displacement, which is itself proportional to the forcing amplitude. As mentioned above, in this sense Hirata's "power amplitude" should be viewed as a linear coupling coefficient, whose square determines the true resonant torque.

We now have all the necessary ingredients to calculate the variable of interest, the torque strength, Eq. (37). It is often useful to calculate the normalized Lindblad resonance torque strength, $\mathcal{N}^{(m)}$, where the disc density is normalized out. This is given by [55]

$$\mathcal{N}^{(m)} = \frac{T_{\text{LR}}^{(m)}}{2\pi r_{\text{LR}} \Sigma(r_{\text{LR}})} = \pm \frac{\pi m \omega' Z_{\text{LR}}}{|D'(r_{\text{LR}})|} |\mathcal{S}^{(m)}|^2, \quad (47)$$

which is independent of r_s in the Newtonian limit, $r_s \gg r_+$.

3. Results

With the theoretical formalism established, the computation of Lindblad torques requires only a numerical Teukolsky solver. We employ the publicly available PYBHPT Python package [72], which provides all the necessary amplitudes and coefficients to evaluate Eq. (47). We verified that our results recover those presented in Table 2 of Hirata II [55]. In Eqs. (45) and (46), we typically sum up to $\ell_{\text{max}} = 10(m-1)$ to obtain converging results. In [73], we provide a simplified tutorial notebook with our numerical implementation, as described below.

In Fig. 3, we plot the normalized torque strengths as a function of the secondary orbital radius for different values of m and primary spin a . As noted in Hirata's original work, relativistic effects enhance the torque strength by a factor of a few with respect to the Newtonian limit (i.e., the constant value that $\mathcal{N}$ approaches at large r). Also, as anticipated in Sec. III A, the inner Lindblad torques become stronger than the outer ones at small r_s , as the location of the inner resonance moves closer to the secondary. However, note that this switch in torque strengths does not occur exactly at the switch in the resonant radius, which we attribute to relativistic effects in the torque amplitude. Also, from Eq. (47) we expect that the additional radial dependence in the actual torque T_{LR} (e.g., through the disc density) will also affect the switch radius. The difference between inner and outer torque strengths becomes smaller for larger m because, as mentioned, resonances tend to become equidistant from the secondary.

From the right panel of Fig. 3 we see that, on a given (prograde) orbit, the primary spin reduces the relativistic enhancement of the torque strengths. Note, however, that torque strengths will be further enhanced by the primary spin when the orbit is retrograde, see [55]. As also anticipated from the resonance locations in Fig. 2, the switch between outer and inner torques happens at smaller radii for rapidly spinning primaries.

Since the larger the m , the closer the resonant radius is to the secondary, the amplitude of the torques grows with m .

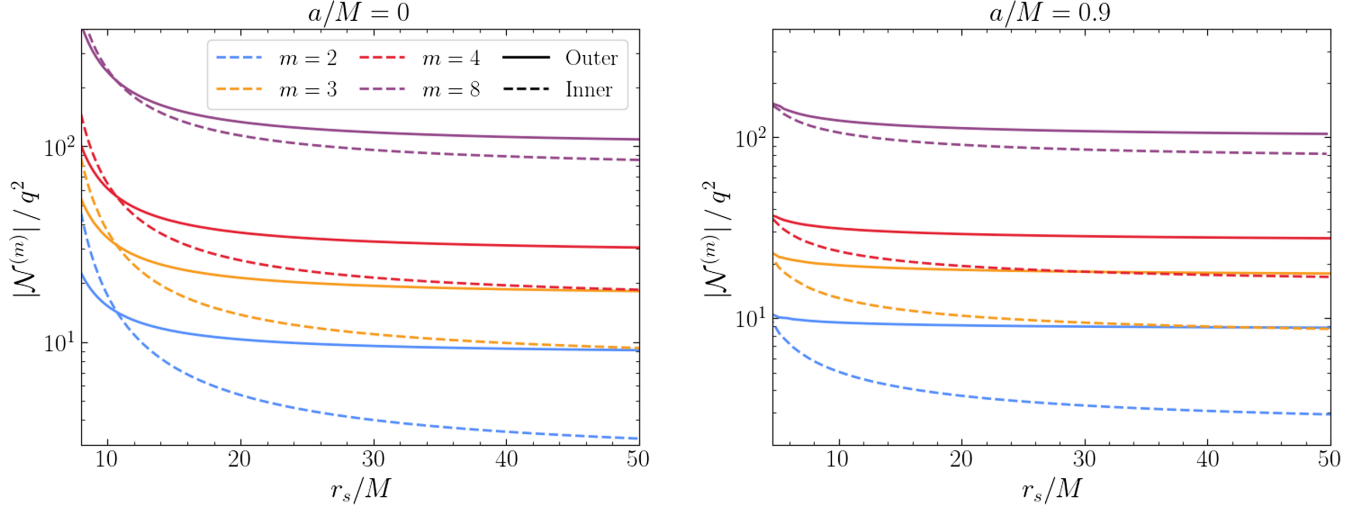


FIG. 3. Normalized Lindblad torques as a function of the EMRI radius for different modes m . For large $r_s \gg r_+$, $\mathcal{N}^{(m)}$ approaches its constant Newtonian limit. For smaller r_s , the inner resonances become stronger than the outer ones, as anticipated from their location in Fig. 2. For large m , the amplitudes of the inner and outer torques become comparable, as the resonances tend to become equidistant from the secondary. Increasing primary spin leads to smaller normalized Lindblad torques.

In Fig. 4, we present this growth for a representative orbital radius of the companion and primary BH spin. We find that the growth of the inner and outer Lindblad torques with m for fixed secondary radius is quadratic already at small m , and the total normalized Lindblad torque (i.e., their sum) is approximately linear (Refs. [57,58] presents an analytical derivation of this scaling in the large m limit). We verified this happens for any spin and even at larger orbital radii. This quadratic scaling provides a substantial computational advantage in evaluating the total Lindblad torque at a given radius (summed over m). Rather than computing high- m modes, which require prohibitively many ℓ -modes, it suffices to calculate the torques at small m , perform a quadratic fit, and extrapolate to larger m . Without this simplification, obtaining the total torques would be

numerically infeasible, as solving the Teukolsky equation at large ℓ is challenging.

It is evident that, even in the Newtonian limit, the sum over m for the total Lindblad torque diverges as $m \rightarrow \infty$. This divergence, well known in the migration literature, is nonphysical and is regularized when pressure effects are considered, as we discuss in the next section.

IV. RELATIVISTIC LINDBLAD TORQUES WITH PRESSURE CORRECTIONS

A. Pressure cutoff

Astrophysical thin accretion discs are not strictly two-dimensional but possess a vertical structure characterized by the scale height H (or, equivalently, the aspect ratio

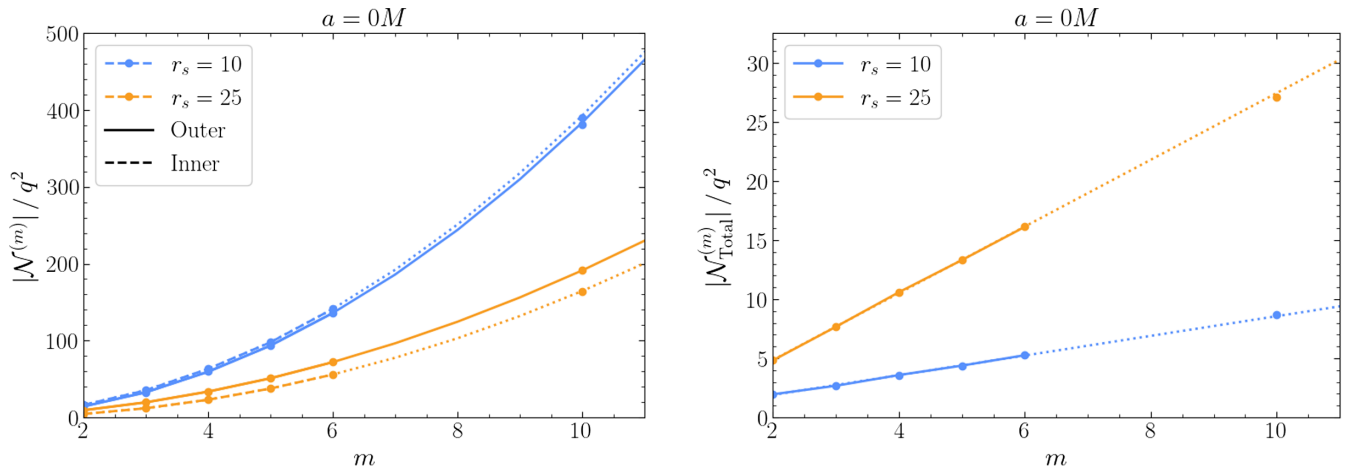


FIG. 4. Left panel: variation of the mode-by-mode inner and outer normalized Lindblad torques with m . The growth becomes quadratic for any secondary radius r_s and spin a at relatively small m . Right panel: same as in the left panel but for the total torque, i.e., the sum of the inner and outer contributions. In this case, the growth is approximately linear with m because the quadratic components of the outer and inner torques cancel each other.

$h = H/r$). This thickness is maintained by pressure support, which also governs the propagation of density waves throughout the disc. Lindblad resonances excite such density waves.

When summing over different Lindblad resonances at varying m , a key assumption is that their associated wavelengths and positions remain sufficiently distinct to avoid interference. Intuitively, this approximation fails once the wavelength of the perturbation becomes comparable to, or shorter than, the spacing between consecutive Lindblad resonances, leading to destructive interference and a weakened effective coupling between the perturber and the disc. In addition, pressure alters the angular velocity of disc particles, whose motion is no longer purely geodesic (or Keplerian in the Newtonian limit).

In the Newtonian regime, accounting for pressure modifies the resonance detuning function to [47,74]

$$D_{\text{Newt.}}^{\text{pressure}}[r] = m(\Omega[r] - \Omega_s) \mp \kappa_{\text{Newt.}}[r] \sqrt{1 + \xi[r]^2}, \quad (48)$$

where the dimensionless pressure parameter is

$$\xi = m \frac{c_s}{\Omega r} = m \frac{H}{r} = mh, \quad (49)$$

and c_s is the local sound speed. In the second equality, we used the fact that for a locally isothermal disc in vertical hydrostatic equilibrium, $c_s = H\Omega$.

As a result of this modification, Lindblad resonances no longer asymptotically converge to the secondary's orbital radius as $m \rightarrow \infty$. Instead, they saturate at a finite separation set by the disc scale height [47]

$$r_{\text{LR}} \rightarrow r_s \pm \frac{2H}{3}. \quad (50)$$

Relativistic effects are expected to slightly shift these saturation distances, but the correction should remain small since the effect is local to the secondary. For reference, thin accretion discs typically have aspect ratios $h \sim 0.01\text{--}0.1$ [75].

These pressure-induced modifications were first systematically studied in the seminal work of Goldreich and Tremaine on disc–planet interactions [76], and later refined by Artymowicz [47]. In practice, torque amplitudes for modes satisfying

$$\xi \gtrsim 1 \Leftrightarrow m \gtrsim 1/h \sim 10 - 100, \quad (51)$$

are strongly suppressed, due both to the displacement of the resonance locations and to pressure effects on the excitation amplitudes themselves.

This suppression produces a cutoff function for the torque that combines polynomial and exponential decay. In particular, we adopt the following mode-by-mode

prescription for the pressure cutoff³

$$T_{\text{LR,pressure}}^{(m)} = \left(\frac{2}{1 + e^\xi} \right) \left(\frac{1}{1 + 4\xi^2} \right) T_{\text{LR}}^{(m)}. \quad (52)$$

The numerical coefficients here may be tuned to match hydrodynamical simulations, though variations are expected to be only of order unity. With this decay, the total Lindblad torque summed over modes converges, enabling the use of Hirata's formalism to assess the implications of relativistic Lindblad torques for GW astronomy.

B. Disc models

To compute the total torque, we need to specify a model for the disc density and scale height profiles. In this work, we use the relativistic model for a thin disc by Novikov and Thorne [77]. For comparison, we also employ the non-relativistic Shakura–Sunyaev model [78].

1. Shakura–Sunyaev

The Shakura–Sunyaev α -disc describes a geometrically thin, optically thick, radiatively efficient disc with axisymmetric, stationary flow on Keplerian circular orbits. In this regime, the surface density Σ and the disc scale height (or aspect ratio) $h \equiv H/r$ can be represented by simple power laws

$$\Sigma(r) = \Sigma_0 \left(\frac{r}{10M} \right)^{-\Sigma_p}, \quad (53)$$

$$h(r) = h_0 \left(\frac{r}{10M} \right)^{(2\Sigma_p - 1)/4}. \quad (54)$$

The exponent Σ_p is set by the dominant pressure and opacity sources, while the normalizations Σ_0 and h_0 are fixed by global properties such as the accretion rate and viscosity. The aspect-ratio scaling is chosen so that the accretion rate is radially constant in steady state. In the inner disc ($r \lesssim 10^2 M$), where radiation pressure dominates and electron scattering sets the opacity, one has $\Sigma_p = -3/2$, implying $\Sigma \propto r^{3/2}$ and $h \propto r^{-1}$.

2. Novikov–Thorne

The Novikov–Thorne model is the general-relativistic extension of the Shakura–Sunyaev thin disc, incorporating orbital dynamics in the Kerr metric, including frame dragging, and adopting the ISCO as the inner boundary of the thin disc. In the standard prescription, the viscous torque is set to vanish at the ISCO, motivated by the rapid inward plunge; in practice, fluid elements can complete

³Alternative, one can introduce a cutoff at $\xi = 1$, as done in Refs. [57,58].

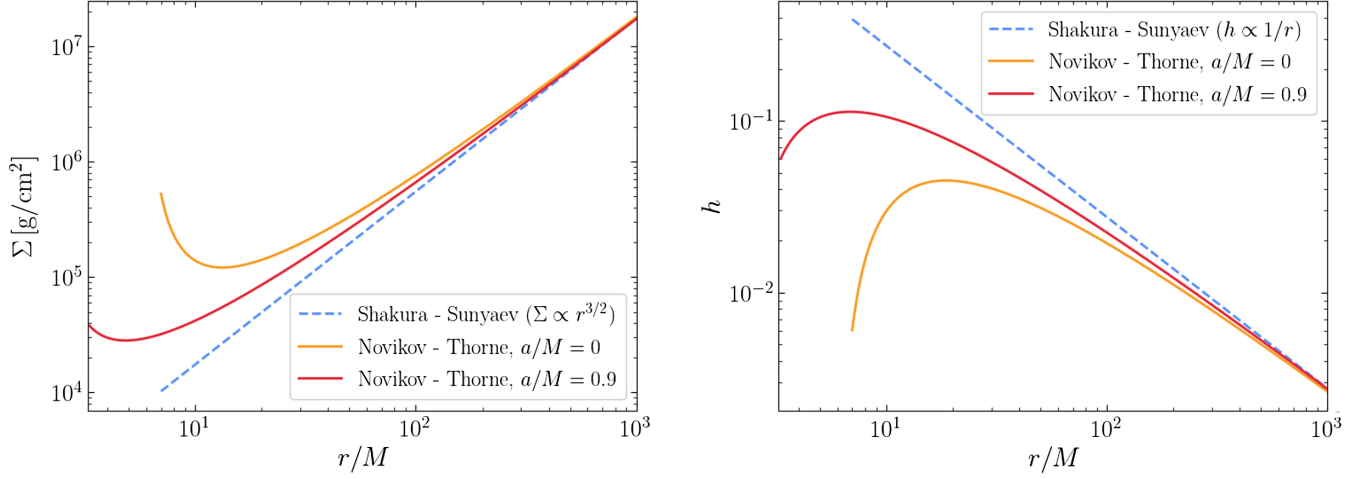


FIG. 5. Surface density (left) and aspect ratio (right) radial profiles for the Novikov–Thorne disc model at different BH spins. The physical parameters of the system are $M = 10^6 M_\odot$ for the primary mass, $\dot{M}/\dot{M}_{\text{Edd}} = 0.1$ for the accretion rate in terms of the Eddington limit, and $\alpha = 0.1$ for the phenomenological parameter controlling viscosity. Changing these values alters the overall normalization of the profiles but not their radial dependence. Dashed blue curves show the corresponding Shakura–Sunyaev power law profiles. Profiles are plotted to large radius, beyond the inner disc region—where radiation pressure dominates—illustrating that relativistic effects are noticeable already for $r \lesssim 250M$.

several orbits inside the ISCO, yielding a small but nonzero stress in the plunging region. For discs without a dynamically important net magnetic field, the Novikov–Thorne solution accurately describes the radial profiles outside the ISCO, in particular for $r \gtrsim r_{\text{ISCO}} + M$ [79].

The Novikov–Thorne inner-disc profiles are given explicitly in Sec. 5.3 of Ref. [3] (see Eq. (99) therein). We plot their radial behavior in Fig. 5 for two values of the central BH spin. We use as primary mass $M = 10^6 M_\odot$, accretion rate in terms of the Eddington ratio $\dot{M}/\dot{M}_{\text{Edd}} = 0.1$ and the viscosity parameter $\alpha = 0.1$. These choices are consistent with current observational constraints [80], which, however, only probe the surface/atmosphere density of AGN discs and not the midplane where the secondary is located. Changing these values alters the overall normalization of the profiles but not their radial dependence. We extend the plot to very large radii, to normalize all profiles to the same value in the Newtonian limit as $r \rightarrow \infty$. For comparison, we overlay the Shakura–Sunyaev power-law.

The figure shows that, across the radii relevant for EMRIs in the LISA band, relativistic corrections to the disc structure are already significant, and the Novikov–Thorne profiles should be used in place of purely Newtonian scalings when studying the effects of migration torques. In particular, the density profiles are not captured by simple power laws and thus neither should migration torques, as they occur in Newtonian theory. We also stress that these are steady-state profiles, but AGN discs are expected to exhibit periods of turbulent flows, which can severely alter their stationary matter distribution configuration and impact the torques derived within the linear theory [81].

C. Results on total torques

We compute the Lindblad torque acting on the EMRI using the density and aspect-ratio profiles shown in Fig. 5. The torque is obtained by evaluating Eq. (52) and summing over all azimuthal modes required for the pressure-corrected expression to converge (typically up to $m_{\text{max}} \sim 3/h$).

The results are displayed in Fig. 6. A key feature, also highlighted in the recent works [57,58], is that relativistic torques in relativistic discs can differ by orders of magnitude from their Newtonian counterparts. In our Fig. 6, part of this variation arises from the normalization of the disc profiles in Fig. 5: while these profiles coincide in the Newtonian limit ($r \rightarrow \infty$), they differ substantially in the strong-field regime. This discrepancy is not only due to differences in surface densities but also in scale heights, which determine the pressure cutoff and thus the number of contributing Lindblad resonances. Choosing an alternative normalization, e.g., matching both surface densities and aspect ratios at a smaller radius (such as the ISCO, $r = 10M$), still yields large differences in torque amplitudes at other radii.

The radial dependence of the torque within each configuration is also noteworthy. As anticipated in the previous section, the Lindblad torque can change sign when the inner Lindblad contribution dominates over the outer one. In this regime, the net migration torque drives outward motion (as also noted in Ref. [57]). For the disc models considered here, however, GW emission remains the dominant driver of evolution (see Fig. 1): although Lindblad torques act outward, the binary will still inspiral, albeit more slowly than in vacuum. Near the ISCO, the inner Lindblad torque diverges and could, in principle,

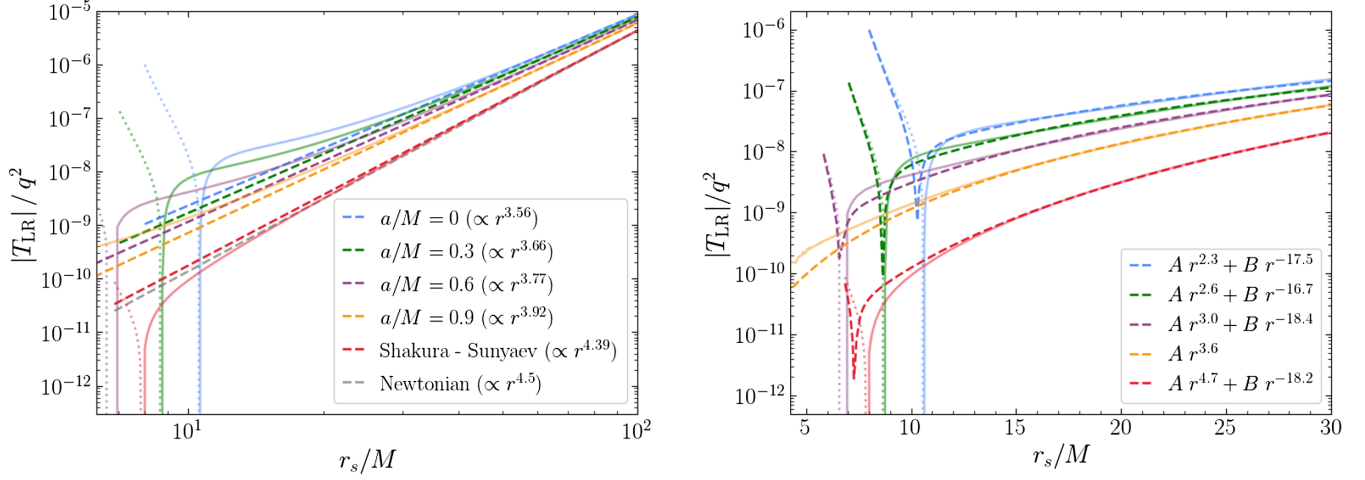


FIG. 6. Left panel: relativistic Lindblad torque exerted on an equatorial, circular EMRI with radius r_s embedded in different accretion disc profiles. Distinct spin values correspond to different Novikov-Thorne discs, normalized as in Fig. 5 (the same normalization is applied to the Shakura-Sunyaev profile). Dotted curves denote opposite sign values of T_{LR} , where inner resonances dominate over outer ones, driving outward migration (though the EMRI still inspirals, since GW emission dominates its evolution). Dashed lines show power-law fits of the form $T_{\text{LR}} = A(r/10M)^{n_A}$. Same as in the left panel, but restricted to numerical data in the strong-field regime ($r_s < 30M$). Here, the Lindblad torque is fitted with a superposition of two power laws, $T_{\text{LR}} = A(r/10M)^{n_A} + B(r/10M)^{n_B}$, to attempt capturing the transition between positive and negative T_{LR} .

exceed GW losses, but our disc and torque prescriptions are not valid at such radii. We therefore restrict our results to $r > r_{\text{ISCO}} + 2M$. The radius at which the torque changes sign depends on the disc profile, and in Novikov-Thorne discs, it also depends on the primary spin. Increasing spin shifts the inversion radius inward toward the ISCO; for $a/M = 0.9$ no inversion occurs within the range of radii shown (though one may appear at smaller separations).

We also explored whether Lindblad torques can be approximated by power laws of the form $T_{\text{LR}} = A(r_s/10M)^{n_r}$. Such parametrizations are valuable: given the wide range of possible physical models for environmental effects, it is unlikely that all physically motivated prescriptions can be implemented in LISA data analysis pipelines. Power-law fits provide a practical, model-agnostic description that can later be mapped onto more detailed models [9]. In Newtonian theory [48,50]

$$T_{\text{LR}}^{\text{Newt}} = c_1 M \Sigma \frac{r}{h^2}, \quad (55)$$

with c_1 a constant calibrated to simulations and dependent on density and height gradients. For the Shakura-Sunyaev profile (53)–(54), the inner-disc scaling ($\Sigma_p = -3/2$) yields $T_{\text{LR}}^{\text{Newt}} \propto r^{9/2}$.

To compare with this prediction, we fit our numerical torques to power laws in two regimes: (i) all radii with $r_s < 100M$ (left panel of Fig. 6), and (ii) the strong-field region $r_s < 30M$ (right panel), where we used a sum of two power-laws, $T_{\text{LR}} = A(r_s/10M)^{n_A} + B(r_s/10M)^{n_B}$, to capture the transition across the torque sign change. Fit parameters are listed in Table I.

For the Shakura-Sunyaev disc, neither fit reproduces the Newtonian index $9/2$ exactly, but the deviation is only at the $\sim 1\%$ level. In the strong-field case, the two-power-law model does not perfectly capture the transition radius, though less strongly than for Novikov-Thorne profiles. For the latter, fitting over $r_s < 100M$ can underestimate the strong-field torque by nearly an order of magnitude, especially at low spin, with power-law indices differing by $\sim 10\text{--}50\%$ depending on fit domain. The fitted indices also vary significantly with spin and are consistently smaller than the Newtonian value. Restricting to the strong-field region further reduces the index. For example,

TABLE I. Coefficients of power-law fits $T_{\text{LR}} = A(r/10M)^{n_A}$ (or $T_{\text{LR}} = A(r/10M)^{n_A} + B(r/10M)^{n_B}$) to the numerical data for Lindblad torques shown in Fig. 6. Shak-Suny. denotes the Shakura-Sunyaev model.

a/M	Fit domain	$A[10^{-9}]$	n_r^A	$-B[10^{-9}]$	$-n_r^B$
0	All r_s	2.33	3.57	-	-
	$r_s \leq 30M$	11.4	2.30	19.5	17.5
0.3	All r_s	1.69	3.66	-	-
	$r_s \leq 30M$	6.35	2.61	0.383	16.7
0.6	All r_s	1.16	3.77	-	-
	$r_s \leq 30M$	3.06	3.02	4.59E-4	18.4
0.9	All r_s	0.721	3.92	-	-
	$r_s \leq 30M$	1.13	3.56	-	-
Shak-Suny.	All r_s	0.176	4.39	-	-
	$r_s \leq 30M$	0.124	4.65	9.24E-5	18.17

in Novikov–Thorne discs we obtain $T_{\text{LR}} \propto r^{3.92}$ ($r^{3.56}$) for $a/M = 0.9$ ($a/M = 0$) when fitting all radii, and $T_{\text{LR}} \propto r^{3.96}$ ($r^{2.3}$) in the strong-field range.

Finally, we explored different values of viscosity and accretion rate, which set the overall normalization of the surface density and scale height. We found that modifying the relation between Σ and h does not affect the rising power-law dependence of the torques with r , but it can slightly shift the radius at which the torque changes sign. This shift, however, is only of order $\mathcal{O}(M)$.

D. Consequences for GW astronomy

The natural follow-up question to the previous section is how the relativistic enhancement of Lindblad torques in accretion discs impacts LISA observations of EMRIs. Building on earlier work, some qualitative conclusions can already be drawn.

Using state-of-the-art circular, equatorial EMRI waveforms in a fully Bayesian LISA data analysis, Ref. [25] showed that for a “golden” EMRI with $M = 10^6 M_\odot$, $\mu = 50 M_\odot$, $a/M = 0.9$, initial separation $r_0 \approx 15.5M$, and four years of observation before merger (at a sky location yielding $\text{SNR} = 50$), Newtonian Lindblad torques modeled as $T_{\text{LR}} = Ar^{n_r}$ could be constrained with relative uncertainties of $\sim 15\%$ on the amplitude A and $\sim 6\%$ on the slope n_r for typical astrophysical disc configurations. Since relativistic corrections systematically enhance Lindblad torques relative to GW emission, we expect these constraints to improve, allowing LISA to probe a broader range of disc models.

Importantly, the relative error on the slope n_r is comparable to the differences found in our strong-field power-law fits (right panel of Fig. 6). Moreover, because the slope depends on the spin of the central BH—measurable with exquisite precision in EMRI observations (errors $\sim 10^{-5}$)—LISA should be able to distinguish between relativistic Lindblad torques arising from different disc configurations. More quantitative conclusions require repeating the Bayesian inference studies of Refs. [9, 25] with relativistic torques injected into the waveforms and recovered with power-law models. We leave this investigation to future work.

A key caveat is the sign reversal of the Lindblad torque as the EMRI inspirals. Reference [9] (albeit with less accurate waveforms than Ref. [25]) found that for Newtonian torques, the parameters A and n_r are strongly correlated. This arises because the environmental effect is largest early in the inspiral, so GW observations are most sensitive to the combination $T_{\text{LR}} \approx Ar_0^{n_r}$. This assumption holds provided there is only one stage of the inspiral where the relative contribution of Lindblad torques peaks. In the relativistic case, however, the situation is more subtle: the torque can change sign and increase in magnitude near the ISCO, so a single power-law may not suffice. To assess this, Fig. 1 compares relativistic Lindblad torques with GW torques (computed using `few` [82]) as a function of orbital radius. For the golden EMRI described above, only in low-

spin Novikov–Thorne discs ($a/M = 0$ and $a/M = 0.3$) do the negative Lindblad torques near plunge become comparable (in relative magnitude) to those at the start of the inspiral ($r_s \gtrsim 15M$). For higher spins, we expect the conclusions of Ref. [9] to hold, while at low spin the torque evolution is more complex and may require inference with multiple power laws. Observations and theory both suggest that luminous AGNs are typically powered by rapidly spinning SMBHs ($a/M \gtrsim 0.7$), though low-spin configurations remain possible in scenarios dominated by chaotic accretion or recent SMBH mergers [83, 84].

V. DISCUSSION

We showed that relativistic corrections to Lindblad torques exerted on EMRIs embedded in accretion discs—both through modifications to the binary–disc interaction and to the disc structure—can amplify the torque amplitude by several orders of magnitude compared to Newtonian predictions. Moreover, relativistic torques display a different radial dependence, which itself varies with the primary BH spin. Together with previous studies on the detectability of accretion disc effects with LISA, our findings suggest that EMRI observations could, in principle, accurately measure disc properties and distinguish between different disc configurations.

As one of the first investigations of relativistic Lindblad torques on EMRIs—together with [57, 58], to which our model should be quantitatively compared in the future—our work has some limitations.

A. Disc modeling

The torque model we adopted, following Hirata’s work, assumes a pressureless disc. As we discussed, this assumption leads to a divergence in the total torque. To regularize it, we introduced a phenomenological cutoff function inspired by Newtonian studies. However, a more rigorous treatment of pressure effects in relativistic discs is needed for more accurate predictions of disc torques. One promising approach, explored in a simplified disc configuration in [85]—is to adapt the methodology developed in Ref. [30] for EMRIs in superradiant scalar clouds: first reconstruct the perturbed spacetime metric from solutions of the (sourced) Teukolsky equation, and then solve the relativistic hydrodynamics equations (e.g., the Euler equation) for the disc in this background. This yields fluid perturbations that encode Lindblad torques self-consistently, and, at the same time, capture additional effects absent from our treatment, such as accretion and coorbital resonant torques near the secondary, which can compete with Lindblad resonances.

We also assumed a stationary background disc model, whereas real AGN discs are expected to be turbulent and time-dependent. Strong turbulence could, in principle, invalidate linear perturbation theory estimates; but, for milder fluctuations, one can still capture the dominant physical

effects [81]. In such cases, the main consequence would be small biases in the inferred environmental parameters, rather than the vacuum EMRI ones [86]. Developing a better understanding of the regime of validity of linear estimates across the parameter space will be necessary for robust astrophysical inferences. Numerical simulation will play a key role in that.

Another interesting scenario is to consider retrograde discs, which although less favored astrophysically due to spin-alignment mechanisms [59], can still form in AGNs with episodes of chaotic accretion (i.e., with random matter inflow directions), or after recent merger episodes [87]. In these systems, EMRIs fall into two broad classes: (1) the EMRI orbits prograde with respect to the MBH but counterrotates relative to the disc; (2) the EMRI and disc are corotating, while both are retrograde with respect to the MBH spin. Case (1) is astrophysically disfavored for the same reasons given to consider only prograde systems in this work. The interaction with the disc as the EMRI migrates inward from the nuclear cluster tends to align its angular momentum with the disc. Moreover, Lindblad torques are strongly suppressed in counter-rotating systems, since the resonance condition cannot be satisfied; the interaction instead resembles gas dynamical friction, with a wake trailing the EMRI [88]. In case (2), the local EMRI-disc configuration is effectively prograde, and standard Lindblad resonances operate, which can be treated within Hirata's framework. Disc torques are therefore expected to be comparable in magnitude to those in prograde systems.

B. Orbital configurations

Our analysis was restricted to prograde, equatorial, circular orbits. Disc interactions are expected to damp inclination and drive circularization by the time the EMRI enters the LISA band. Nonetheless, some residual eccentricity may survive [63,64]. Newtonian studies have shown that eccentricity enhances the detectability of disc effects with LISA [41]. Extending the relativistic analysis to eccentric orbits would therefore be a natural next step. This is conceptually straightforward: the perturber's motion (and the associated source terms in the Teukolsky equation) can be decomposed into harmonics of the fundamental frequencies of the eccentric orbit, as done generically in Eqs. (28)–(29), and the disc response and torque can then be computed mode-by-mode and summed over harmonics. This has recently been explored in the context of three-body resonances [89].

C. Corotation torques

In this work we ignored the interaction of the EMRI with disc particles in its corotation region, which do not contribute to Lindblad torques due to the pressure cutoff effects discussed. However, they can still contribute to the so called corotation torques, which arise from gas executing horseshoe orbits in this region. They can be important when the disc has strong vortensity and/or entropy gradients

across the horseshoe region, provided that viscosity and/or thermal diffusion prevent *saturation* (i.e., phase mixing) [90]. In AGN discs, turbulence driven by the magneto-rotational instability is expected to provide efficient diffusion to maintain an unsaturated corotation torque. However, simulations also show that turbulence induces stochastic fluctuations on timescales comparable to the orbital period, which are much shorter than the libration time of disc particles across the horseshoe region (typically ~ 10 – 100 orbital periods). These fluctuations destroy phase coherence and effectively smooth vortensity/entropy gradients across the horseshoe region, weakening the net corotation torque even when it remains formally unsaturated [91,92]. This would make the magnitude of this effect for EMRIs in AGN discs strongly model-dependent, and thus we leave a detailed relativistic analysis of corotation torques for future work.

D. Implications for LISA data analysis

Finally, a crucial direction for future work is to determine how to incorporate environmental effects such as those studied here into LISA data analysis pipelines. A systematic survey across the relevant parameter space and for a range of disc models is needed to assess whether parametrized, agnostic waveform models can effectively capture the true physical signals. If the answer is positive, one could then establish a mapping between the agnostic waveforms and the physical models. Moreover, LISA is expected to detect $\sim 10^7$ overlapping signals, which must be extracted simultaneously through a global fit. An open question is whether small deviations from vacuum GR, including environmental effects, must be included directly in the global fit, or whether they can instead be studied in postprocessing using a catalog of sources obtained from vacuum-only templates. Addressing this will require accurate and physically well-motivated environmental models.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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