

Charged pion decay in a laser field

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We study the decays $\pi^+ \rightarrow l^+ \nu_l$ in the presence of a background electromagnetic plane wave with circular polarization. We find that, in the presence of this background, the number of hadronic form factors required to describe the pion-to-vacuum amplitude increases from one to four. Three of these form factors are associated with the axial vector current and one with the vector piece, which is in general different from zero. We obtain analytical expressions for the corresponding contributions to the decay width, considering in particular some range of values of interest for the frequency and amplitude of the external wave. For laser frequencies ω of the order of the eV, it is seen that the main correction to the result obtained in absence of the external field is proportional to the vector form factor, carrying a suppression factor of order $\sim (\omega/m_\pi)^{4/3}$. In addition, we show that our expressions can be connected with the Kroll-Watson formula, which provides the differential cross section for the scattering of a charged particle by some potential in the presence of a strong external electromagnetic wave. Our analytical results are also compatible with previous numerical calculations obtained in the limit of low amplitude laser fields, offering new insight on some characteristics of these results.

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I. INTRODUCTION

The recent progress in laser technology, in terms of both its intensity and its sources, is paving the way for the experimental study of quantum effects in the presence of a very large electromagnetic wave field [1–3]. From the theoretical point of view this subject has a rather long history. For example, already in the mid 1960s, the effect of some electromagnetic field configurations on leptonic decays of charged pions was considered [4–6]. Over the years, several other decay processes of interest in nuclear and particle physics, e.g., muon weak decays [7], β decays of nuclei [8,9], transmutation of protons [10], α or proton emissions from nuclei [11–15], etc., have been also analyzed in this context. More recent applications, as well as related references, can be found in Refs. [1,16]. Whether noticeable

effects on charged particle weak decays may be observed in present or forthcoming experimental facilities is an issue that shows some controversy. For example, it has been claimed [17] that the muon lifetime can change dramatically in the presence of laser fields having an electric field amplitude of 10^6 V/cm. This result has been questioned in Ref. [18] stating that such a field is in fact too weak to produce an observable effect, and this argument has been confirmed in a more quantitative way in Refs. [19,20].

Due in part to these developments, in Ref. [21] the weak decay of charged pions has been revisited. In that article, as in Ref. [5], the authors write the pion-to-vacuum hadronic current in terms of a single form factor (the usual pion decay constant in vacuum), considering laser fields with photon energies $E_\gamma = 1\text{--}2$ eV and electric field amplitudes $\bar{E} = 10^7\text{--}10^8$ V/cm. They conclude that within this range of laser parameters the calculated decay width basically coincides with the one obtained in vacuum. One important observation is that to get this result one has to sum over the contributions of a rather large number of exchanged photons. Since in Ref. [21] such a sum is performed numerically, the analysis of the process for higher electric

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field amplitudes lies beyond their computing capacity. Forthcoming experimental facilities, e.g., LUXE [22,23] or NSF-OPAL [24] are, however, expected to reach electric field amplitudes about five orders of magnitude larger ($\bar{E} \sim 10^{13}$ V/cm) with about the same laser frequency. Thus, one might wonder whether the conclusions of Ref. [21] would remain valid for these parameters. In addition, as shown in Ref. [25], the presence of a background electromagnetic field opens the possibility of the existence of other nonzero pion-to-vacuum transition form factors in addition to the pion decay constant considered in Refs. [5,21]. These form factors have been shown to play an important role in the analysis of the weak decay width of charged pions in the presence of a constant magnetic field [26]. Given this context, the scope of our work is twofold. First, considering the presence of an external plane wave (PW) electromagnetic field, we determine the form of the most general model-independent hadronic matrix elements, written in terms of gauge covariant Lorentz structures. In particular, we show that in the presence of a PW one can in principle define four independent form factors. Second, given the general form of the hadronic matrix elements, we obtain a general expression for the charged pion decay width and show that this expression can be conveniently approximated so as to obtain reliable numerical results for relevant modern laser parameters.

This article is organized as follows. In Sec. II we review $\pi^+ \rightarrow l^+ \nu_l$ decays in vacuum and discuss the main features of the external field. Then, in Secs. III and IV we obtain general expressions for the relevant matrix elements and for the corresponding decay width, respectively. In Sec. V we derive approximate analytical expressions for the contributions to the decay width, taking into account relevant parameter ranges for the laser field. The order of magnitude of these contributions is discussed, and it is shown that in the limit of low amplitude laser fields the results are compatible with previous numerical calculations. It is also seen that our analytical results can be related to the Kroll-Watson formula. Finally, in Sec. VI we present a summary of our work, together with our main conclusions. Specifications about the characteristics of pion and lepton fields in the presence of the external background are included in Appendix A, while in Appendix B we detail some mathematical steps followed in our calculations.

II. PRELIMINARIES

We are interested in the two-body decays of a charged pion into a charged lepton (electron or muon) and the corresponding neutrino, in the presence of an electromagnetic plane wave background field that can be understood as an idealization of a laser field. In particular, for simplicity, we consider the case in which the electromagnetic wave is circularly polarized and the pion moves in the same direction as the laser beam. We use natural

Heaviside-Lorentz units with $\hbar = c = 1$ and fine structure constant $\alpha = e^2/4\pi \simeq 1/137$.

A. $\pi^\pm \rightarrow l^\pm \nu_l$ decays in vacuum

Let us start by analyzing leptonic pion decays in absence of the external field, i.e., in vacuum. For concreteness we consider the process $\pi^+(q) \rightarrow l^+(p) \nu_l(t)$, where $l = e, \mu$, denoting by q , p , and t the four-momenta of the pion, charged lepton, and neutrino, respectively.

The decay is driven by the Fermi Lagrangian density

$$\mathcal{L}_F(x) = -\frac{G_F}{\sqrt{2}} V_{ud} \sum_l [J_h^{\dagger\mu}(x) J_{l\mu}(x) + J_h^\mu(x) J_{l\mu}^\dagger(x)], \quad (2.1)$$

where G_F is the Fermi decay constant, V_{ud} is the corresponding element of the Cabibbo-Kobayashi-Maskawa matrix, and $J_l^{\dagger\mu}(x)$, $J_h^\mu(x)$ are, respectively, the leptonic and hadronic currents involved in the $\pi^+(q) \rightarrow l^+(p) \nu_l(t)$ process. These currents are given by

$$\begin{aligned} J_l^{\dagger\mu}(x) &= \bar{\psi}_{\nu_l}(x) \gamma^\mu (1 - \gamma^5) \psi_l(x), \\ J_h^\mu(x) &= \bar{\psi}_d(x) \gamma^\mu (1 - \gamma^5) \psi_u(x). \end{aligned} \quad (2.2)$$

In vacuum, the decay amplitude can be written in terms of the matrix elements

$$\begin{aligned} &\langle \bar{l}(p, s) \nu_l(t, r) | \bar{\psi}_{\nu_l}(x) \gamma^\mu (1 - \gamma^5) \psi_l(x) | 0 \rangle \\ &= e^{it \cdot x} \bar{u}_{\nu_l}(t, r) \gamma^\mu (1 - \gamma^5) e^{ip \cdot x} v_l(p, s) \end{aligned} \quad (2.3)$$

for the leptonic current (here r and s label polarization states), and

$$\begin{aligned} \langle 0 | \bar{\psi}_d(x) \gamma^\mu \gamma^5 \psi_u(x) | \pi^+(q) \rangle &= -\sqrt{2} f(q^2) \partial^\mu e^{-iq \cdot x} \\ &= i\sqrt{2} f(q^2) q^\mu e^{-iq \cdot x} \end{aligned} \quad (2.4)$$

for the hadronic one. Here, the charged lepton and the neutrino are described as elementary pointlike particles, whereas the pion is treated as a composite object. Since the pion is a pseudoscalar particle, only the axial vector piece of the hadronic current contributes to the amplitude. The internal structure of the pion is taken into account through the form factor $f(q^2)$, whose on-shell value is the pion decay constant,

$$f(m_\pi^2) = f_\pi. \quad (2.5)$$

From the above amplitudes, the decay rate for the process $\pi^+ \rightarrow l^+ \nu$ is found to be given by the well-known expression

$$\Gamma_{\pi^+ \rightarrow l^+ \nu_l}^{(\text{vac})} = \frac{G_F^2 |V_{ud}|^2}{4\pi} f_\pi^2 m_\pi m_l^2 \left(1 - \frac{m_l^2}{m_\pi^2} \right)^2. \quad (2.6)$$

B. Background laser field

We proceed now to describe the features of the external background field. We consider the case in which it is given by an electromagnetic plane wave of wave vector k^μ , i.e., the electromagnetic field A^μ has the form

$$A^\mu(x) = e_1^\mu a_1(\zeta) + e_2^\mu a_2(\zeta), \quad (2.7)$$

where $\zeta = k_\mu x^\mu$, with $k^2 = 0$. Here, $a_i(\zeta)$ are arbitrary functions of ζ , while $e_i^\mu = (0, \vec{e}_i)$ are orthogonal polarization vectors (i.e., $\vec{e}_1 \cdot \vec{e}_2 = 0$) that satisfy the transversality condition $k \cdot e_i = 0$. The electromagnetic strength tensor is given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = k^\mu A'^\nu - k^\nu A'^\mu, \quad (2.8)$$

where

$$A'^\mu = e_1^\mu \frac{da_1}{d\zeta} + e_2^\mu \frac{da_2}{d\zeta}. \quad (2.9)$$

As stated above, we focus on the description of the pion decay in the presence of a background laser field. For a laser pulse of finite duration, the coefficients $a_i(\zeta)$ in Eq. (2.7) would be zero for values of ζ beyond the interaction region. Nevertheless, here we will assume that the actual length of the laser pulse is relatively large, in such a way that it can be approximately described by an infinitely extended PW. Moreover, for the sake of simplicity we will choose a monochromatic plane wave. Two possible photon polarizations are usually considered in the literature: a circularly polarized plane wave (CPPW) and a linearly polarized plane wave (LPPW). For a CPPW one has

$$a_1(\zeta) = a \cos \zeta, \quad a_2(\zeta) = -a \sin \zeta, \quad (2.10)$$

while for a LPPW one has

$$a_1(\zeta) = a_{\text{LP}} \cos \zeta, \quad a_2(\zeta) = 0. \quad (2.11)$$

As it will be discussed in the next section, the calculation of the decay width we are interested in becomes greatly simplified for a CPPW. Thus, in what follows we will concentrate in this particular case.

We can write now $k^\mu = \omega(1, \vec{n})$, with $\vec{n}^2 = 1$, $\vec{n} \cdot \vec{e}_i = 0$. Thus, a (or a_{LP}) stands for the wave amplitude, while $\lambda_\gamma = 2\pi/\omega$ is the corresponding wave length. We use the vectors $n^\mu = (1, \vec{n})/\sqrt{2}$, $\tilde{n}^\mu = (1, -\vec{n})/\sqrt{2}$, e_1^μ and e_2^μ as an orthonormal basis for spacetime. In addition, we choose the spatial axes in such a way that $\vec{n}$, $\vec{e}_1$, and $\vec{e}_2$ point to the $\hat{z}$, $\hat{x}$ and $\hat{y}$ directions, respectively. Then, for a CPPW the electromagnetic field reads

$$A^\mu = a(0, \cos \zeta, -\sin \zeta, 0). \quad (2.12)$$

Alternatively, one can introduce the useful notation [27]

$$A^\mu(x) = a \operatorname{Re}(\mathbf{\Omega}^\mu e^{-i\zeta}), \quad (2.13)$$

with

$$\mathbf{\Omega}^\mu = (0, 1, -i, 0). \quad (2.14)$$

It is easily seen that the external electromagnetic field is a function of $x^- = (x^0 - x^3)/\sqrt{2}$. As a consequence, the system under study breaks translational invariance only in the x^- direction, and it is advantageous to describe it using light-front coordinates. The coordinate x^- can be interpreted as the light front time, while $(x^+, \vec{x}_\perp)$, where $x^+ = (x^0 + x^3)/\sqrt{2}$ and $\vec{x}_\perp$ lies in the spacial plane perpendicular to $\vec{n}$ (defined by the vectors $\vec{e}_i$), are intended to be the “light-front space coordinates.” Concerning the momentum components, as x^- is the “light-front time coordinate,” its canonical conjugate $p^+ = (p^0 + p^3)/\sqrt{2}$ can be referred to as the “light-front Hamiltonian,” and the canonical conjugates of the “light-front space coordinates” $(x^+, \vec{x}_\perp)$ will be the “light-front momentum coordinates” $(p^-, \vec{p}_\perp)$, where $p^- = (p^0 - p^3)/\sqrt{2}$. For a particle of mass m on its mass shell we have $p^+ = (m^2 + \vec{p}_\perp^2)/(2p^-)$.

To fully characterize the charged pion decay in the presence of this background field we also need to specify the pion four-momentum. Several dimensionless, scalar and gauge invariant parameters have been defined in the literature for the description of this system. A discussion about the parameters of interest and the corresponding orders of magnitude can be found in Refs. [1,3,28]. We follow Ref. [1] for a manifestly gauge-invariant definition of these parameters, changing the electron mass used in Ref. [1] for the pion mass. Thus, we define the classical nonlinearity parameter ξ_π as

$$\xi_\pi = \frac{Q_\pi}{m_\pi k \cdot q} \langle -q_\mu F^{\mu\nu} q^\alpha F_{\alpha\nu} \rangle^{1/2}, \quad (2.15)$$

where q^μ is the momentum of the decaying pion, Q_π is the pion electric charge and the angle brackets indicate a phase cycle average. On the other hand, the relation between the energy of the probe particle and the plane wave background is parametrized by the linear quantum parameter

$$\eta_\pi = \frac{k \cdot q}{m_\pi^2}. \quad (2.16)$$

In our case, the background field can be characterized by two parameters: its amplitude, a , and the frequency of the laser field in the lab frame, ω (or, equivalently, the $+$ -component of the wave vector, k^+). These parameters can be written in terms of the dimensionless quantities

given in Eqs. (2.15) and (2.16) as

$$Q_\pi a = \xi_\pi m_\pi, \quad \sqrt{2} \omega q^- = \eta_\pi m_\pi^2. \quad (2.17)$$

In terms of experimental accessible quantities, ξ_π can be written as

$$\begin{aligned} \xi_\pi &= 1.41 \times 10^{-13} \frac{\bar{E}[\text{V/cm}]}{E_\gamma[\text{eV}]} \\ &= 2.21 \times 10^{-12} \sqrt{I[\text{W/cm}^2] \lambda_\gamma[\mu\text{m}]}, \end{aligned} \quad (2.18)$$

where $E_\gamma = \hbar\omega$ is the photon energy, $\bar{E}$ is the electric field amplitude and I is the associated laser field intensity. In the same way, η_π can be written as

$$\eta_\pi = \frac{E_\gamma}{E_\pi^0} \left(\frac{E_\pi}{E_\pi^0} + \sqrt{\left(\frac{E_\pi}{E_\pi^0} \right)^2 - 1} \right), \quad (2.19)$$

where $E_\pi^0 = m_\pi c^2$. Notice that for a pion at rest one has $\eta_\pi = E_\gamma/E_\pi^0$, while for a pion energy $E_\pi = 1 \text{ GeV}$ one has $\eta_\pi = 14.3 E_\gamma/E_\pi^0$.

In addition, as stated above, we will consider the particular case in which the pion momentum points in the direction of the wave vector, i.e., we will take $\vec{q}_\perp = 0$.

III. LEPTONIC AND HADRONIC MATRIX ELEMENTS IN THE PRESENCE OF THE EXTERNAL FIELD

A. Pion-to-vacuum hadronic amplitude

In the presence of the external electromagnetic field the decay process $\pi^+ \rightarrow l^+ \nu_l$ can still be written in terms of a Fermi Lagrangian density as the one in Eq. (2.1). However, charged particle fields have to be modified, and a parametrization of the pion-to-vacuum hadronic amplitude in terms of several form factors is expected.

The hadronic current $J_h^\mu(x)$ in Eq. (2.2) includes vector and axial vector pieces. In vacuum, the vector piece does not contribute to the pion-to-vacuum amplitude, while the contribution of the axial vector piece is given by Eq. (2.4). Now, in the presence of an electromagnetic background field A^μ , new tensors are available and can be used to obtain the desired Lorentz structure. In the most general situation, due to gauge symmetry, the derivative ∂^μ has to be replaced by the covariant derivative,

$$D^\mu = \partial^\mu + i Q_\pi A^\mu. \quad (3.1)$$

In addition, one has other two gauge invariant tensors, namely, the electromagnetic strength tensor and its dual,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \quad (3.2)$$

The vector and axial vector contributions to the pion-to-vacuum amplitude have to be built from contractions between these objects. From the strength tensor and its dual one can obtain the relativistic invariant quantities

$$\begin{aligned} \mathcal{S} &\equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{1}{2} (E^2 - B^2), \\ \mathcal{P} &\equiv \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = \vec{E} \cdot \vec{B}, \end{aligned} \quad (3.3)$$

where $\vec{E}$ and $\vec{B}$ are the external electric and magnetic fields. Other useful relations are the following: taking two tensors and contracting one Lorentz index one gets

$$\begin{aligned} F^{\mu\alpha} F_{\alpha\nu} &= \tilde{F}^{\mu\alpha} \tilde{F}_{\alpha\nu} - 2 \mathcal{S} \delta_\nu^\mu, \\ \tilde{F}^{\mu\alpha} F_{\alpha\nu} &= \mathcal{P} \delta_\nu^\mu, \end{aligned} \quad (3.4)$$

while covariant derivatives satisfy the commutation relations

$$\begin{aligned} [D_\mu, D_\nu] &= i Q_\pi F^{\mu\nu}, \\ [D_\alpha, F^{\mu\nu}] &= \partial_\alpha F^{\mu\nu}. \end{aligned} \quad (3.5)$$

Let us consider, in particular, an external electromagnetic PW. It is easy to see that in this case one has $\mathcal{S} = \mathcal{P} = 0$. Taking into account the previous relations, from the available tensors one can set up just three independent vector operators,

$$D^\mu, \quad Q_\pi F^{\mu\nu} D_\nu, \quad Q_\pi^2 F^{\mu\alpha} F_{\alpha\nu} D^\nu, \quad (3.6)$$

and one axial vector operator,

$$Q_\pi \tilde{F}^{\mu\nu} D_\nu. \quad (3.7)$$

In addition, in the case of an external electromagnetic PW, another available vectorlike physical quantity is the wave vector k^μ .

On the other hand, particle fields get modified in the presence of the external field. The form of the wave functions $\phi_q^\pm(x)$ for charged pions is discussed in Appendix A. In the case of an electromagnetic PW, they can be written as

$$\phi_q^\pm(x) = e^{\mp i q \cdot x} \varphi_q^\pm(\zeta), \quad (3.8)$$

where $q^\mu = (E_q, \vec{q})$, with $E_q = \sqrt{\vec{q}^2 + m_\pi^2}$, and the functions $\varphi_q^\pm(\zeta)$ are given by Eqs. (A5) and (A7).

Thus, the vector and axial vector pieces of the pion-to-vacuum amplitude can be written as

$$\langle 0 | \bar{\psi}_d(x) \gamma^\mu \psi_u(x) | \pi^+(q) \rangle = \sqrt{2} f_V Q_\pi \tilde{F}^{\mu\nu} D_\nu \phi_q^+(x), \quad (3.9)$$

$$\begin{aligned} & \langle 0 | \bar{\psi}_d(x) \gamma^\mu \gamma^5 \psi_u(x) | \pi^+(q) \rangle \\ &= -\sqrt{2} [f_{A1} D^\mu + i f_{A2} Q_\pi F^{\mu\nu} D_\nu + f_{A3} Q_\pi^2 F^{\mu\alpha} F_{\alpha\nu} D^\nu] \phi_q^+(x), \end{aligned} \quad (3.10)$$

where Q_π is equal to the positron charge e . In principle, in Eq. (3.10) one could include a further term proportional to k^μ ; however, as shown below, the corresponding form factor can be absorbed through a redefinition of f_{A3} .

The form factors f_V and f_{Ai} , $i = 1, 2, 3$, are scalar functions of the Lorentz invariant differential operators that can be built using the gauge invariant tensors in Eqs. (3.1) and (3.2) together with the wave vector k^μ . In fact, it can be seen that only three independent nontrivial scalar operators can be found. They can be chosen to be, e.g.,

$$D^\mu D_\mu, \quad Q_\pi^2 F^{\mu\nu} F_{\alpha\nu} D^\alpha D_\mu, \quad k^\mu D_\mu. \quad (3.11)$$

Taking into account the relation

$$\begin{aligned} D_\mu \phi_q^+(x) &= \left[-i q_\mu - \frac{i}{k \cdot q} \left(Q_\pi A(\zeta) \cdot q - \frac{1}{2} Q_\pi^2 A(\zeta)^2 \right) k_\mu \right. \\ &\quad \left. + i Q_\pi A_\mu(\zeta) \right] \phi_q^+(x) \end{aligned} \quad (3.12)$$

(see Appendix A), it is seen that the action of these scalar operators on the pion wave function gives

$$\begin{aligned} D^\mu D_\mu \phi_q^+(x) &= -q^2 \phi_q^+(x), \\ Q_\pi^2 F^{\mu\nu} F_{\alpha\nu} D^\alpha D_\mu \phi_q^+(x) &= -Q_\pi^2 (k \cdot q)^2 A'^2 \phi_q^+(x), \\ k^\mu D_\mu \phi_q^+(x) &= -i k \cdot q \phi_q^+(x). \end{aligned} \quad (3.13)$$

Hence, the form factors will be in general functions of the Lorentz invariant quantities q^2 , $k \cdot q$ and $Q_\pi^2 (k \cdot q)^2 A'^2$.

From Eqs. (3.9), (3.10) and (3.12) the pion-to-vacuum matrix element of the hadronic $V - A$ current is given by

$$\begin{aligned} \langle 0 | J_h^\mu(x) | \pi^+(q) \rangle &= \langle 0 | \bar{\psi}_d(x) \gamma^\mu (1 - \gamma^5) \psi_u(x) | \pi^+(q) \rangle \\ &= \sqrt{2} e^{-iq \cdot x} \varphi_q^+(\zeta) \mathbb{F}^\mu, \end{aligned} \quad (3.14)$$

where $\varphi_q^+(\zeta)$ is given by Eq. (A5), and the vector $\mathbb{F}^\mu$ is defined as

$$\mathbb{F}^\mu = -i(f_V g_V^\mu + f_{A1} g_{A1}^\mu + i f_{A2} g_{A2}^\mu + f_{A3} g_{A3}^\mu), \quad (3.15)$$

with

$$\begin{aligned} g_V^\mu &= Q_\pi \epsilon^{\mu\nu\rho\sigma} k_\rho A'_\sigma (q_\nu - Q_\pi A_\nu), \\ g_{A1}^\mu &= q^\mu + \frac{Q_\pi}{k \cdot q} A \cdot \left(q - Q_\pi \frac{A}{2} \right) k^\mu - Q_\pi A^\mu, \\ g_{A2}^\mu &= Q_\pi A' \cdot (q - Q_\pi A) k^\mu - Q_\pi k \cdot q A'^\mu, \\ g_{A3}^\mu &= -Q_\pi^2 A' \cdot A' k \cdot q k^\mu. \end{aligned} \quad (3.16)$$

In this way, as pointed out in Ref. [25], it is seen that the presence of an electromagnetic background field enables the existence of new form factors for pion decays. In particular, in our case we have a vector form factor f_V and three axial form factors f_{Ai} . In the $A^\mu \rightarrow 0$ limit the coefficients accompanying the new form factors f_V , f_{A2} and f_{A3} vanish, whereas $g_{A1}^\mu \rightarrow q^\mu$ and $f_{A1} \rightarrow f_\pi$.

In the particular case of a CPPW, it is seen that Eqs. (3.16) get further simplified, since one has $A \cdot A' = 0$, $A^2 = A'^2 = -a^2$. Moreover, if we consider the case in which $\vec{q}_\perp = 0$, we also have $q \cdot A = q \cdot A' = 0$. Then Eqs. (3.16) reduce to

$$\begin{aligned} g_V^\mu &= Q_\pi \epsilon^{\mu\nu\rho\sigma} k_\rho A'_\sigma (q_\nu - Q_\pi A_\nu), \\ g_{A1}^\mu &= q^\mu + \frac{Q_\pi^2 a^2}{2k \cdot q} k^\mu - Q_\pi A^\mu, \\ g_{A2}^\mu &= -Q_\pi k \cdot q A'^\mu, \\ g_{A3}^\mu &= Q_\pi^2 a^2 k \cdot q k^\mu. \end{aligned} \quad (3.17)$$

Assuming that the pion is on shell, the invariants in Eqs. (3.13) are given by

$$q^2 = m_\pi^2, \quad k \cdot q = m_\pi^2 \eta_\pi, \quad -Q_\pi^2 (k \cdot q)^2 A'^2 = (\xi_\pi \eta_\pi m_\pi^3)^2, \quad (3.18)$$

from which one gets the relations in Eq. (2.17).

If one considers, instead, the case of a LPPW, Eq. (3.16) read

$$\begin{aligned} g_V^\mu|_{\text{LP}} &= Q_\pi \epsilon^{\mu\nu\rho\sigma} k_\rho A'_{\text{LP}\sigma} (q_\nu - Q_\pi A_{\text{LP}\nu}), \\ g_{A1}^\mu|_{\text{LP}} &= q^\mu + \frac{Q_\pi^2 a_{\text{LP}}^2 \cos^2 \zeta}{2k \cdot q} k^\mu - Q_\pi A_{\text{LP}}^\mu, \\ g_{A2}^\mu|_{\text{LP}} &= -Q_\pi k \cdot q A'_{\text{LP}}^\mu - Q_\pi^2 a_{\text{LP}}^2 \sin \zeta \cos \zeta k^\mu, \\ g_{A3}^\mu|_{\text{LP}} &= Q_\pi^2 a_{\text{LP}}^2 \sin^2 \zeta k \cdot q k^\mu. \end{aligned} \quad (3.19)$$

Comparing Eqs. (3.19) and (3.17) it is seen that in the case of a LPPW the equations include a new term of order Q_π^2 , and some different dependences on ζ . This is due to the fact that now A_{LP}^2 and $A_{\text{LP}}'^2$ are not constant but depend on ζ , and the product $A'_{\text{LP}} \cdot A_{\text{LP}}$ is different from zero. As a consequence, one gets new contributions to the π^+ decay amplitude that correspond to a two-photon absorption or emission vertex. Since in principle these contributions do not act in a coherent way, their consideration implies a significant increase in the complexity of the calculation. On this basis we will restrict our analysis to the more simple case of a CPPW background.

B. Leptonic amplitude

In the case of the charged leptons, the form of the corresponding wave functions in the presence of an electromagnetic PW background is given in Appendix A 2,

where some definitions are introduced. Taking this into account, we get

$$\begin{aligned} & \langle \bar{l}(p, s) \nu_l(t, r) | \bar{\Psi}_{\nu_l}(x) \gamma^\mu (1 - \gamma^5) \Psi_l(x) | 0 \rangle \\ &= e^{it \cdot x} \bar{u}_\nu(t, r) \gamma^\mu (1 - \gamma^5) E_p^c(x) v_l(p, s), \end{aligned} \quad (3.20)$$

where

$$E_p^c(x) = e^{ip \cdot x} \varphi_p^-(\zeta) \left(1 - \frac{Q_l}{2k \cdot p} \not{k} A(\zeta) \right), \quad (3.21)$$

with $\varphi_p^-(\zeta)$ given by Eq. (A7). Here, $E_p^c(x)$ is the Ritus matrix [3,29] associated with the creation of the l^+ particle in the final state, and (with our convention for particles and antiparticles) $Q_l = -e$. In fact, for convenience, we have removed in Eq. (3.20) a prefactor $(2\pi)^{-3} (2E_p 2E_l)^{-1/2}$, which will be taken into account later through the phase space factors in the decay width.

IV. $\pi^+ \rightarrow l^+ \nu_l$ DECAY AMPLITUDE AND WIDTH

A. $\pi^+ \rightarrow l^+ \nu_l$ decay amplitude

The transition amplitude for the process $\pi^+(q) \rightarrow l^+(p) \nu_l(t)$ is given by

$$\begin{aligned} S_{\pi^+ \rightarrow l^+ \nu_l} &= -i \frac{G_F V_{ud}}{\sqrt{2}} \int d^4 x \langle 0 | J_h^\mu(x) | \pi^+(q) \rangle \\ &\times \langle \bar{l}(p, s) \nu_l(t, r) | J_{l\mu}^\dagger(x) | 0 \rangle. \end{aligned} \quad (4.1)$$

As discussed in the previous subsections, in the case of an electromagnetic CPPW background the pion-to-vacuum matrix element of the hadronic current can be written as in Eq. (3.14), while the leptonic matrix element is given by Eq. (3.20). One has in this way

$$S_{\pi^+ \rightarrow l^+ \nu_l} = -i \int d^4 x e^{i(p+t-q) \cdot x} I_{pq}(\zeta) \mathbb{M}_{(p,s)(t,r)}(\zeta), \quad (4.2)$$

where we have defined

$$I_{pq}(\zeta) = \varphi_p^{(-)}(\zeta) \varphi_q^{(+)}(\zeta) \quad (4.3)$$

and

$$\begin{aligned} \mathbb{M}_{(p,s)(t,r)}(\zeta) &= G_F V_{ud} \bar{u}_\nu(t, r) \mathbb{F}(1 - \gamma^5) \\ &\times \left(1 - \frac{Q_l}{2k \cdot p} \not{k} A(\zeta) \right) v_l(p, s), \end{aligned} \quad (4.4)$$

with $\varphi_p^-(\zeta)$, $\varphi_q^+(\zeta)$ given by Eqs. (A5) and (A7), and $\mathbb{F}^\mu$ given by Eqs. (3.15) and (3.17).

We follow now the procedure of evaluation carried out in Ref. [27]. In the case of a CPPW, taking $\vec{q}_\perp = 0$, we have

$$\begin{aligned} \varphi_q^{(+)}(\zeta) &= \exp \left[-\frac{i}{k \cdot q} \int_{-\infty}^{\zeta} d\zeta' \frac{1}{2} e^2 a^2 \right], \\ \varphi_p^{(-)}(\zeta) &= \exp \left[\frac{i}{k \cdot p} \int_{-\infty}^{\zeta} d\zeta' \left(e A(\zeta') \cdot p + \frac{1}{2} e^2 a^2 \right) \right]. \end{aligned} \quad (4.5)$$

Notice that in the case of a LPPW the squared electromagnetic field A_{LP}^2 is not constant. Therefore, the A_{LP}^2 terms in $\varphi_p^-(\zeta)$ and $\varphi_q^+(\zeta)$ [see Eqs. (A5) and (A7)] would lead to contributions to the amplitude that correspond to uncorrelated two-photon vertices.

Using the expression in Eq. (2.13) for the electromagnetic field, and assuming that the latter is switched on/off adiabatically, we can perform the above integrals. We obtain

$$I_{pq}(\zeta) = \exp \left[-\frac{e a}{2} \left(\frac{\mathbf{Q} \cdot p}{k \cdot p} e^{-i\zeta} - \frac{\mathbf{Q}^* \cdot p}{k \cdot p} e^{i\zeta} \right) + i \frac{e^2 a^2}{2} \left(\frac{1}{k \cdot p} - \frac{1}{k \cdot q} \right) \zeta \right], \quad (4.6)$$

which can also be written as

$$I_{pq}(\zeta) = \exp [-i d_1 \sin(\zeta - \alpha) + i d_2 \zeta], \quad (4.7)$$

with the definitions

$$d_1 = e a \frac{|\mathbf{Q} \cdot p|}{k \cdot p}, \quad \alpha = \arg(-\mathbf{Q} \cdot p), \quad d_2 = \frac{e^2 a^2}{2} \left(\frac{1}{k \cdot p} - \frac{1}{k \cdot q} \right). \quad (4.8)$$

Using polar coordinates in the perpendicular plane, i.e.,

$$p^\perp = p_\perp \cos \theta, \quad p^2 = p_\perp \sin \theta, \quad (4.9)$$

and taking into account the parameters introduced in Eqs. (2.15) and (2.16), we have

$$d_1 = \frac{\xi_\pi}{\eta_\pi} \frac{k \cdot q}{k \cdot p} \frac{p_\perp}{m_\pi}, \quad \alpha = -\theta, \quad d_2 = \frac{\xi_\pi^2}{2\eta_\pi} \left(\frac{k \cdot q}{k \cdot p} - 1 \right). \quad (4.10)$$

Now, we can use the generating function of the Bessel functions

$$e^{iz \sin \beta} = \sum_{\ell=-\infty}^{+\infty} e^{i\ell\beta} J_\ell(z) \quad (4.11)$$

to express I_{pq} as

$$I_{pq}(\zeta) = e^{id_2\zeta} \sum_{\ell=-\infty}^{+\infty} e^{-i\ell(\zeta-\alpha)} J_\ell(d_1). \quad (4.12)$$

Replacing in Eq. (4.2), we obtain

$$S_{\pi^+ \rightarrow l^+ \nu_l} = -i \int d^4x e^{i(p+t-q) \cdot x} e^{id_2\zeta} \times \sum_{\ell=-\infty}^{+\infty} e^{-i\ell(\zeta+\theta)} J_\ell(d_1) \mathbb{M}_{(p,s)(t,r)}(\zeta). \quad (4.13)$$

Next, we evaluate the fermionic contribution $\mathbb{M}_{(p,s)(t,r)}(\zeta)$, given in Eq. (4.4), where $Q_l = -e$ and $\mathbb{F}^\mu$ is given by Eqs. (3.15) and (3.17). After some calculation we arrive at the expression

$$\mathbb{M}_{(p,s)(t,r)}(\zeta) = G_F V_{ud} (\mathbb{M}^{(0)} + \mathbb{M}^{(+1)} e^{-i\zeta} + \mathbb{M}^{(-1)} e^{i\zeta}), \quad (4.14)$$

where $\mathbb{M}^{(i)}$ are ζ -independent spinor matrix elements given by

$$\begin{aligned} \mathbb{M}^{(0)} &= i \bar{u}_\nu(t, r) \left\{ -f_{A1} \not{q} + \frac{e^2 a^2}{2} \left[\left(\frac{1}{k \cdot p} - \frac{1}{k \cdot q} \right) f_{A1} + \frac{k \cdot q}{k \cdot p} f_{VA-} - 2f_{VA3} \right] \not{k} \right\} (1 - \gamma^5) v_l(p, s), \\ \mathbb{M}^{(+1)} &= i \frac{e a}{2} \left[f_{A1} \frac{k \cdot (p - q)}{k \cdot p} + k \cdot q f_{VA+} \right] \bar{u}_\nu(t, r) \not{\epsilon} (1 - \gamma^5) v_l(p, s), \\ \mathbb{M}^{(-1)} &= i \frac{e a}{2} [f_{A1} + k \cdot q f_{VA-}] \bar{u}_\nu(t, r) \not{\epsilon}^* (1 - \gamma^5) v_l(p, s), \end{aligned} \quad (4.15)$$

with

$$\begin{aligned} f_{VA+} &= f_V + f_{A2}, \\ f_{VA-} &= f_V - f_{A2}, \\ f_{VA3} &= f_V + k \cdot q f_{A3}. \end{aligned} \quad (4.16)$$

Finally, changing $\ell \rightarrow \ell \pm 1$ conveniently in the sums, we see that the $\pi^+ \rightarrow l^+ \nu_l$ transition amplitude can be written as

$$\begin{aligned} S_{\pi^+ \rightarrow l^+ \nu_l} &= \sum_{\ell=-\infty}^{+\infty} \int d^4x e^{i(p+t-q) \cdot x} e^{id_2\zeta} e^{-i\ell\zeta} \mathbb{M}_\ell \\ &= \sum_{\ell=-\infty}^{+\infty} (2\pi)^4 \delta^{(4)}(p + t - q - (\ell - d_2)k) \mathbb{M}_\ell, \end{aligned} \quad (4.17)$$

where

$$\begin{aligned} \mathbb{M}_\ell &= -i G_F V_{ud} e^{-i\ell\theta} [J_\ell(d_1) \mathbb{M}^{(0)} + e^{i\theta} J_{\ell-1}(d_1) \mathbb{M}^{(+1)} \\ &\quad + e^{-i\theta} J_{\ell+1}(d_1) \mathbb{M}^{(-1)}]. \end{aligned} \quad (4.18)$$

In this way, it is seen that the total transition amplitude can be understood as the sum of amplitudes that describe the

disintegration of the pion into the charged lepton and its neutrino together with a number ℓ of background photons. The energy-momentum conservation gets expressed by the condition

$$p + t - q - (\ell - d_2)k = 0. \quad (4.19)$$

B. $\pi^+ \rightarrow l^+ \nu_l$ decay width

The decay width $\Gamma(\pi^+ \rightarrow l^+ \nu_l)$ can be calculated from the transition amplitude $S_{\pi^+ \rightarrow l^+ \nu_l}$ obtained in the previous subsection. Taking into account the ℓ dependence of the momentum conservation delta in Eq. (4.17), it is seen that only terms of the form $|\mathbb{M}_\ell|^2$ can appear in the differential decay width. Thus, we have

$$\begin{aligned} d\Gamma(\pi^+ \rightarrow l^+ \nu_l) &= \sum_{\ell=-\infty}^{\infty} (2\pi)^4 \delta^{(4)}(p + t - q - (\ell - d_2)k) \\ &\quad \times \frac{1}{2E_q (2\pi)^3 2E_t (2\pi)^3 2E_p} \overline{|\mathbb{M}_\ell|^2}, \end{aligned} \quad (4.20)$$

where $\overline{|\mathbb{M}_\ell|^2} = \sum_{r,s} |\mathbb{M}_\ell|^2$. It is convenient to express the phase space factor in terms of light front coordinates,

defining

$$d\Phi_\ell = (2\pi)^4 \delta^{(4)}(p + t - q - (\ell - d_2)k) \frac{dt^- d^2 t_\perp dp^- d^2 p_\perp}{(2\pi)^3 2t^- (2\pi)^3 2p^-}, \quad (4.21)$$

where $d^2 p_\perp = dp^1 dp^2$, $d^2 t_\perp = dt^1 dt^2$. One has then

$$d\Gamma(\pi^+ \rightarrow l^+ \nu_l) = \frac{1}{2E_q} \sum_{\ell=-\infty}^{\infty} d\Phi_\ell |\overline{\mathbb{M}_\ell}|^2. \quad (4.22)$$

It is also useful to introduce a variable x that denotes the fraction of the light-front momentum of the pion (q^-) that is carried away by the charged lepton, i.e.,

$$x = \frac{p^-}{q^-}, \quad (4.23)$$

with $0 \leq x \leq 1$. After some algebra we obtain

$$\begin{aligned} d\Phi_\ell &= \frac{1}{4(2\pi)^2} dx d\theta dp_\perp^2 \delta(p_\perp^2 - (1-x)(xm_\pi^2(1+2\ell\eta_\pi) \\ &\quad - (1-x)m_\pi^2 \xi_\pi^2 - m_l^2)) \\ &\quad \times dt^- \delta(t^- + p^- - q^-) d^2 t_\perp \delta^{(2)}(\vec{t}_\perp + \vec{p}_\perp). \end{aligned} \quad (4.24)$$

In terms of $\vec{p}_\perp$, x and q^- , particle momenta can be written as

$$\begin{aligned} p^+ &= \frac{m_l^2 + p_\perp^2}{2xq^-}, \quad t^- = (1-x)q^-, \\ \vec{t}_\perp &= -\vec{p}_\perp, \quad t^+ = \frac{p_\perp^2}{2(1-x)q^-}, \quad q^+ = \frac{m_\pi^2}{2q^-}. \end{aligned} \quad (4.25)$$

The conditions $p_\perp^2 \geq 0$ and $0 \leq x \leq 1$ imply a lower limit for the sum over the integer index ℓ , as will be seen below.

Using the momentum conservation, Eq. (4.19), together with the relations $(\not{p} + m_l)v_l(p, s) = 0$ and $\bar{u}_{\nu_l}(t, r)\not{r} = 0$, we obtain

$$\begin{aligned} \mathbb{M}_\ell &= G_F V_{ud} e^{-i\ell\theta} \bar{u}_\nu(t, r)(1 + \gamma^5) \\ &\quad \times (m_l J_\ell(d_1) f_A^{(1)} + \not{Q}) v_l(p, s), \end{aligned} \quad (4.26)$$

where we have introduced the vector

$$\begin{aligned} Q^\mu &= J_\ell(d_1) \left(\ell f_{A1} + \frac{\xi_\pi^2 m_\pi^2}{2x} f_{VA-} - \xi_\pi^2 m_\pi^2 f_{VA3} \right) k^\mu \\ &\quad + e^{i\theta} J_{\ell-1}(d_1) \frac{\xi_\pi m_\pi}{2} \left[\left(1 - \frac{1}{x} \right) f_{A1} + \eta_\pi m_\pi^2 f_{VA+} \right] \mathbf{C}^\mu \\ &\quad + e^{-i\theta} J_{\ell+1}(d_1) \frac{\xi_\pi m_\pi}{2} (f_{A1} + \eta_\pi m_\pi^2 f_{VA-}) \mathbf{C}^{*\mu}. \end{aligned} \quad (4.27)$$

This leads to

$$\begin{aligned} |\overline{\mathbb{M}_\ell}|^2 &= G_F^2 |V_{ud}|^2 \text{tr}[\not{r}(1 + \gamma^5)(m_l J_\ell(d_1) f_{A1} + \not{Q})(\not{p} - m_l) \\ &\quad \times (m_l J_\ell(d_1) f_{A1}^* + \not{Q}^*)(1 - \gamma^5)]. \end{aligned} \quad (4.28)$$

After a long and straightforward calculation we arrive at the expression

$$|\overline{\mathbb{M}_\ell}|^2 = 8 G_F^2 |V_{ud}|^2 \sum_{\Sigma, \Sigma'} \epsilon_{\Sigma, \Sigma'} \mathbb{N}_\ell^{(\Sigma, \Sigma')} \text{Re}[f_\Sigma f_{\Sigma'}^*], \quad (4.29)$$

where the indices Σ and Σ' run over $A1$, $VA+$, $VA-$, and $VA3$. A factor $\epsilon_{\Sigma, \Sigma'}$, equal to 1 for $\Sigma = \Sigma'$ and to 1/2 for $\Sigma \neq \Sigma'$, has been included so as to avoid double counting of cross terms. The functions $\mathbb{N}_\ell^{(\Sigma, \Sigma')}(x, p_\perp)$, which satisfy $\mathbb{N}_\ell^{(\Sigma, \Sigma')}(x, p_\perp) = \mathbb{N}_\ell^{(\Sigma', \Sigma)}(x, p_\perp)$, are given by

$$\mathbb{N}_\ell^{(A1, A1)} = \frac{m_l^2}{2x} \left\{ m_l^2 (1-x) J_\ell(d_1)^2 + \frac{1}{1-x} [p_\perp J_\ell(d_1) - \xi_\pi m_\pi (1-x) J_{\ell+1}(d_1)]^2 \right\}, \quad (4.30)$$

$$\mathbb{N}_\ell^{(A1, VA+)} = -m_l^2 \eta_\pi \xi_\pi m_\pi^3 p_\perp J_\ell(d_1) J_{\ell-1}(d_1), \quad (4.31)$$

$$\mathbb{N}_\ell^{(A1, VA-)} = -m_l^2 \eta_\pi \xi_\pi m_\pi^3 \left[p_\perp J_{\ell+1}(d_1) J_\ell(d_1) - \xi_\pi m_\pi \frac{(1-x)}{x} (J_{\ell+1}(d_1)^2 - J_\ell(d_1)^2) \right], \quad (4.32)$$

$$\mathbb{N}_\ell^{(A1, VA3)} = 2 m_l^2 \eta_\pi \xi_\pi^2 m_\pi^4 (1-x) J_\ell(d_1)^2, \quad (4.33)$$

$$\mathbb{N}_\ell^{(VA+, VA+)} = \frac{1}{2} \eta_\pi^2 \xi_\pi^2 m_\pi^6 \frac{x}{1-x} p_\perp^2 J_{\ell-1}(d_1)^2, \quad (4.34)$$

$$\mathbb{N}_\ell^{(VA-, VA-)} = \frac{1}{2} \eta_\pi^2 \xi_\pi^2 m_\pi^6 \frac{(1-x)}{x} \left[m_l^2 J_{\ell+1}(d_1)^2 + (p_\perp J_{\ell+1}(d_1) - \xi_\pi m_\pi J_\ell(d_1))^2 \right], \quad (4.35)$$

$$\mathbb{N}_\ell^{(VA3,VA3)} = 2\eta_\pi^2 \xi_\pi^4 m_\pi^8 x(1-x)J_\ell(d_1)^2, \quad (4.36)$$

$$\mathbb{N}_\ell^{(VA+,VA-)} = -\eta_\pi^2 \xi_\pi^2 m_\pi^6 p_\perp (p_\perp J_{\ell-1}(d_1)J_{\ell+1}(d_1) - \xi_\pi m_\pi J_\ell(d_1)J_{\ell-1}(d_1)), \quad (4.37)$$

$$\mathbb{N}_\ell^{(VA+,VA3)} = -2\eta_\pi^2 \xi_\pi^3 m_\pi^7 x p_\perp J_\ell(d_1)J_{\ell-1}(d_1), \quad (4.38)$$

$$\mathbb{N}_\ell^{(VA-,VA3)} = 2\eta_\pi^2 \xi_\pi^3 m_\pi^7 (1-x)(p_\perp J_{\ell+1}(d_1)J_\ell(d_1) - \xi_\pi m_\pi J_\ell(d_1)^2), \quad (4.39)$$

with $d_1 = \xi_\pi p_\perp / (\eta_\pi m_\pi x)$. Throughout the calculation we have used the Bessel function recurrence relation

$$J_{\ell-1}(d_1) + J_{\ell+1}(d_1) = \frac{2\ell}{d_1} J_\ell(d_1). \quad (4.40)$$

Thus, from the above expressions the $\pi^+ \rightarrow l^+ \nu_l$ decay width is found to be given by

$$\Gamma(\pi^+ \rightarrow l^+ \nu_l) = \frac{1}{2E_q} 8 G_F^2 |V_{ud}|^2 \sum_{\Sigma, \Sigma'} \epsilon_{\Sigma, \Sigma'} \text{Re}(f_\Sigma f_{\Sigma'}^*) \mathcal{G}^{(\Sigma, \Sigma')}, \quad (4.41)$$

where

$$\begin{aligned} \mathcal{G}^{(\Sigma, \Sigma')} &= \frac{1}{4} \sum_\ell \int \frac{dx d\theta dp_\perp^2}{(2\pi)^2} \delta(p_\perp^2 - \bar{p}_{\perp\ell}^2) \mathbb{N}_\ell^{(\Sigma, \Sigma')}(x, p_\perp) \\ &= \frac{1}{8\pi} \sum_\ell \int dx \mathbb{N}_\ell^{(\Sigma, \Sigma')}(x, \bar{p}_{\perp\ell}), \end{aligned} \quad (4.42)$$

with

$$\bar{p}_{\perp\ell}^2 = m_\pi^2(1-x)[x(1+2\ell\eta_\pi) - (1-x)\xi_\pi^2 - m_l^2/m_\pi^2]. \quad (4.43)$$

In Eq. (4.42) we have used the fact that $\mathbb{N}_\ell^{(\Sigma, \Sigma')}$ does not depend on θ . The integration domain for x is restricted by the conditions

$$0 \leq x \leq 1, \quad 0 \leq \bar{p}_{\perp\ell}^2, \quad (4.44)$$

from which one gets

$$\frac{\xi_\pi^2 + m_l^2/m_\pi^2}{1 + \xi_\pi^2 + 2\ell\eta_\pi} \leq x \leq 1 \quad (4.45)$$

and

$$-\frac{m_\pi^2 - m_l^2}{2\eta_\pi m_\pi^2} \leq \ell < \infty. \quad (4.46)$$

Notice that, according to the discussion in Sec. III A, the form factors f_Σ turn out to be in general functions of m_π^2 , ξ_π , and η_π .

V. APPROXIMATED ANALYTICAL EXPRESSIONS FOR RELEVANT LASER PARAMETERS

As stated in the previous sections, for the physical situations of interest the photon energies are expected to be of the eV order in the lab frame, which implies that the dimensionless Lorentz-invariant parameter η_π should be of the order of 10^{-8} to 10^{-7} for pion energies $E_q \lesssim 1$ GeV. Concerning the parameter ξ_π , as discussed in the Introduction we consider a range of electric field amplitudes from about 10^8 to 10^{13} eV/cm, which for $E_\gamma \sim \mathcal{O}(1)$ eV correspond to a range of values of ξ_π from $\sim 10^{-5}$ up to $\sim \mathcal{O}(1)$ [see Eq. (2.18)]. Hence, for the cases of interest, the argument $d_1 = \xi_\pi \bar{p}_{\perp\ell} / (\eta_\pi m_\pi x)$ of the Bessel functions in the above defined quantities $\mathbb{N}_\ell^{(\Sigma, \Sigma')}$ will be in general very large, allowing us to approximate Bessel functions through oscillatory asymptotic expressions. In what follows we work out in this limit the results for the contributions to the $\pi^+ \rightarrow l^+ \nu_l$ decay width obtained in the previous section, obtaining approximate analytic expressions and discussing the corresponding orders of magnitude.

A. Leading order contributions to the decay width

Let us analyze the leading order contributions to the $\pi^+ \rightarrow l^+ \nu_l$ decay width for the considered ranges of the parameters ξ_π and η_π . As it will be discussed in Sec. V C, the sums over the number of exchanged photons ℓ in Eq. (4.41) turn out to be strongly dominated by the contributions of terms for which $\ell \sim \mathcal{O}(\xi_\pi \eta_\pi^{-1})$ or even larger. This means that in Eqs. (4.30) to (4.39) we have to deal with Bessel functions of both large argument and large index. It is convenient to introduce a new variable

$$z = \frac{d_1}{\ell} = \frac{\xi_\pi}{\ell \eta_\pi x m_\pi} \bar{p}_{\perp\ell}, \quad (5.1)$$

so as to write $J_\ell(d_1) = J_\ell(\ell z)$, with $z \sim \mathcal{O}(\xi_\pi / (\ell \eta_\pi))$. Then, the function $J_\nu(\nu z)$, for large positive (real) values of ν and positive z , can be approximated using the Debye asymptotic expansions (see Eqs. 8.452.1 and 8.453.1 of Ref. [30])

$$J_\nu(\nu z) \simeq \sqrt{\frac{1}{2\nu\pi \tanh \alpha}} \exp(\nu \tanh \alpha - \nu \alpha) [1 + \mathcal{O}(\nu^{-1})] \quad \text{if } z < 1, \quad \text{with } \cosh \alpha = 1/z, \quad (5.2)$$

$$J_\nu(\nu z) \simeq \sqrt{\frac{2}{\nu\pi \tan \beta}} \cos\left(\nu \tan \beta - \nu \beta - \frac{\pi}{4}\right) [1 + \mathcal{O}(\nu^{-1})] \quad \text{if } z > 1, \quad \text{with } \cos \beta = 1/z. \quad (5.3)$$

These relations show that $J_\ell(\ell z)$ exhibits a sharp exponential decrease if $z < 1$, whereas it is highly oscillating if $z > 1$. Now, notice that the functions $\mathcal{G}^{(\Sigma, \Sigma')}$ include an integral over the variable x [see Eq. (4.42)], while the integrands $\mathbb{N}_\ell^{(\Sigma, \Sigma')}$ involve products of two Bessel functions. Given the values of ℓ we are interested in, we can safely neglect the contribution to the integrals that comes from the $z < 1$ region and focus on the average value of the product of oscillatory Bessel functions, approximated by Eq. (5.3), within the $z > 1$ region.

If relevant contributions to the sum over ℓ are those corresponding to large values of ℓ ($\ell \sim \xi_\pi \eta_\pi^{-1}$ becomes larger than 10^2 even for $\xi_\pi \sim 10^{-5}$), the sum can be approximated, at the lowest order in powers of η_π , by an integral over $\mathfrak{z} = \ell \eta_\pi$, which we can take as a continuous variable. Thus, we have

$$\mathcal{G}^{(\Sigma, \Sigma')} \simeq \frac{1}{8\pi\eta_\pi} \int d\mathfrak{z} \int dx \mathbb{N}_\ell^{(\Sigma, \Sigma')}, \quad (5.4)$$

where the functions $\mathbb{N}_\ell^{(\Sigma, \Sigma')}$ can be approximated using the asymptotic expression given in Eq. (5.3). For now, let us consider the case of positive ℓ ; the case of negative ℓ can then be addressed using the relation $J_{-\ell}(d_1) = (-1)^\ell J_\ell(d_1)$. In the region where $z > 1$ we have, in terms of the variables $\mathfrak{z}$ and z (notice that z depends on both $\mathfrak{z}$ and x),

$$J_\ell(\ell z) \simeq \sqrt{\frac{2\eta_\pi}{\mathfrak{z}\pi\sqrt{z^2-1}}} \left\{ \cos\left[\frac{\mathfrak{z}}{\eta_\pi}(\sqrt{z^2-1} - \arccos z^{-1}) - \frac{\pi}{4}\right] \right\}, \quad (5.5)$$

$$\begin{aligned} & J_{\ell\pm 1}(\ell z) \\ &= J_{\ell\pm 1}\left((\ell \pm 1)\frac{\ell z}{(\ell \pm 1)}\right) \\ &\simeq \sqrt{\frac{2\eta_\pi}{\mathfrak{z}\pi\sqrt{z^2-1}}} \left\{ \frac{1}{z} \cos\left[\frac{\mathfrak{z}}{\eta_\pi}(\sqrt{z^2-1} - \arccos z^{-1}) - \frac{\pi}{4}\right] \right. \\ &\quad \left. \pm \frac{\sqrt{z^2-1}}{z} \sin\left[\frac{\mathfrak{z}}{\eta_\pi}(\sqrt{z^2-1} - \arccos z^{-1}) - \frac{\pi}{4}\right] \right\}. \end{aligned} \quad (5.6)$$

The integrals over $\mathfrak{z}$ and x in Eq. (5.4) can now be approximated by averaging the products of the oscillating Bessel functions, i.e., by replacing

$$\begin{aligned} J_\ell(d_1)^2 &\rightarrow \frac{\eta_\pi}{\pi \mathfrak{z} \sqrt{z^2-1}}, \\ J_\ell(d_1) J_{\ell\pm 1}(d_1) &\rightarrow \frac{\eta_\pi}{\pi \mathfrak{z} z \sqrt{z^2-1}}, \\ J_{\ell\pm 1}(d_1)^2 &\rightarrow \frac{\eta_\pi}{\pi \mathfrak{z} \sqrt{z^2-1}}, \\ J_{\ell-1}(d_1) J_{\ell+1}(d_1) &\rightarrow \frac{\eta_\pi(2-z^2)}{\pi \mathfrak{z} z^2 \sqrt{z^2-1}}. \end{aligned} \quad (5.7)$$

At this point it is advantageous to first perform the integral over $\mathfrak{z}$. Taking into account the relations (5.1) and (4.43), and performing the replacements given in Eqs. (5.7), we obtain

$$\mathcal{G}^{(A1, A1)} = \frac{1}{8\pi} \int_{\bar{x}}^1 dx \frac{1}{2} m_l^2 (m_\pi^2 - m_l^2) I_0, \quad (5.8)$$

$$\mathcal{G}^{(A1, V A \pm)} = -\frac{1}{8\pi} \int_{\bar{x}}^1 dx \eta_\pi m_\pi^4 m_l^2 x I_1, \quad (5.9)$$

$$\mathcal{G}^{(A1, V A3)} = \frac{1}{8\pi} \int_{\bar{x}}^1 dx 2\eta_\pi \xi_\pi^2 m_l^2 m_\pi^4 (1-x) I_0, \quad (5.10)$$

$$\mathcal{G}^{(VA+, VA+)} = \frac{1}{8\pi} \int_{\bar{x}}^1 dx \frac{\eta_\pi^2}{2} \xi_\pi^2 m_\pi^8 x [(x - \xi_\pi^2(1-x) - m_l^2/m_\pi^2) I_0 + 2x I_1], \quad (5.11)$$

$$\mathcal{G}^{(VA-, VA-)} = \frac{1}{8\pi} \int_{\bar{x}}^1 dx \frac{\eta_\pi^2}{2} \xi_\pi^2 m_\pi^8 (1-x) [(1-x + \xi_\pi^2(2-x) + m_l^2/m_\pi^2) I_0 - 2x I_1], \quad (5.12)$$

$$\mathcal{G}^{(VA3,V A3)} = \frac{1}{8\pi} \int_{\bar{x}}^1 dx 2\eta_\pi^2 \xi_\pi^4 m_\pi^8 x(1-x) I_0, \quad (5.13)$$

$$\mathcal{G}^{(VA+,VA-)} = -\frac{1}{8\pi} \int_{\bar{x}}^1 dx \eta_\pi^2 m_\pi^8 [-\xi_\pi^2(1-x)(x - \xi_\pi^2(1-x) - m_l^2/m_\pi^2) I_0 - \xi_\pi^2(3-2x)x I_1 + 2x^2 I_2], \quad (5.14)$$

$$\mathcal{G}^{(VA+,V A3)} = -\frac{1}{8\pi} \int_{\bar{x}}^1 dx 2\eta_\pi^2 \xi_\pi^2 m_\pi^8 x^2 I_1, \quad (5.15)$$

$$\mathcal{G}^{(VA-,V A3)} = \frac{1}{8\pi} \int_{\bar{x}}^1 dx 2\eta_\pi^2 \xi_\pi^2 m_\pi^8 (1-x)(x I_1 - \xi_\pi^2 I_0), \quad (5.16)$$

where we have defined

$$I_n = \frac{1}{\eta_\pi} \int_{\bar{z}_-}^{\bar{z}_+} d\bar{z} \frac{\eta_\pi \bar{z}^n}{\pi \sqrt{(\bar{z} - \bar{z}_-)(\bar{z}_+ - \bar{z})}}. \quad (5.17)$$

The integration limits $\bar{x}$ and $\bar{z}_\pm$ are determined by the condition $z^2 \geq 1$. One has

$$\bar{x} = \frac{m_l^2}{m_\pi^2} \quad (5.18)$$

and

$$\bar{z}_\pm = \frac{\xi_\pi^2(1-x) \pm \xi_\pi \sqrt{(1-x)(x - m_l^2/m_\pi^2)}}{x}. \quad (5.19)$$

The integrals I_n can be easily performed, obtaining

$$I_0 = 1, \quad I_1 = \frac{\xi_\pi^2(1-x)}{x}, \quad I_2 = \frac{\xi_\pi^2(1-x)}{2x^2} [x + 2\xi_\pi^2(1-x) - m_l^2/m_\pi^2]. \quad (5.20)$$

After integration over x we obtain

$$\mathcal{G}^{(A1,A1)} = \frac{1}{16\pi} m_l^2 m_\pi^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2, \quad (5.21)$$

$$\begin{aligned} \mathcal{G}^{(A1,VA+)} &= \mathcal{G}^{(A1,VA-)} = -\frac{1}{2} \mathcal{G}^{(A1,V A3)} \\ &= -\frac{1}{16\pi} \eta_\pi^2 \xi_\pi^2 m_l^2 m_\pi^4 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2, \end{aligned} \quad (5.22)$$

$$\begin{aligned} \mathcal{G}^{(VA+,VA+)} &= \mathcal{G}^{(VA-,VA-)} \\ &= \frac{1}{96\pi} \eta_\pi^2 \xi_\pi^2 m_\pi^8 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2 \\ &\quad \times \left[2 + \xi_\pi^2 + (1 + 2\xi_\pi^2) \frac{m_l^2}{m_\pi^2}\right], \end{aligned} \quad (5.23)$$

$$\begin{aligned} \mathcal{G}^{(VA+,V A3)} &= \mathcal{G}^{(VA-,V A3)} = -\mathcal{G}^{(V A3,V A3)} = -2\mathcal{G}^{(VA+,VA-)} \\ &= -\frac{1}{24\pi} \eta_\pi^2 \xi_\pi^4 m_\pi^8 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2 \left(1 + \frac{2m_l^2}{m_\pi^2}\right). \end{aligned} \quad (5.24)$$

Finally, replacing in Eq. (4.41), and turning back to the form factors f_V, f_{A_i} according to the relations in Eq. (4.16), we arrive at the result

$$\begin{aligned} \Gamma(\pi^+ \rightarrow l^+ \nu_l) &= \frac{G_F^2 |V_{ud}|^2}{4\pi E_q} m_\pi^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2 \\ &\quad \times \sum_{\sigma, \sigma'} \epsilon_{\sigma, \sigma'} C_{\sigma, \sigma'} \text{Re}(f_{\sigma, \sigma'}^* f_{\sigma'}), \end{aligned} \quad (5.25)$$

where σ, σ' run over the indices V and A_i , $i = 1, 2, 3$ and $\epsilon_{\sigma, \sigma'}$ is defined as in Eq. (4.29). At leading orders in powers of η_π , the coefficients $C_{\sigma, \sigma'}$ are found to be given by

$$C_{A1,A1} = m_l^2, \quad (5.26)$$

$$C_{A2,A2} = C_{V,V} = \frac{2}{3} \eta_\pi^2 \xi_\pi^2 m_\pi^6 \left(1 + \frac{m_l^2}{2m_\pi^2}\right), \quad (5.27)$$

$$C_{A3,A3} = \frac{2}{3} \eta_\pi^4 \xi_\pi^4 m_\pi^{10} \left(1 + \frac{2m_l^2}{m_\pi^2}\right), \quad (5.28)$$

$$C_{A1,\sigma} \sim m_l^2 \times \mathcal{O}(\eta_\pi^{1+\gamma_{A1,\sigma}}) \quad \text{for } \sigma = A2, V, \quad (5.29)$$

$$C_{A1,A3} = 2\eta_\pi^2 \xi_\pi^2 m_\pi^4 m_l^2, \quad (5.30)$$

$$C_{A2,V} \sim \mathcal{O}(\eta_\pi^{2+\gamma_{A2,V}}), \quad (5.31)$$

$$C_{\sigma,A3} \sim \mathcal{O}(\eta_\pi^{3+\gamma_{\sigma,A3}}) \quad \text{for } \sigma = A2, V, \quad (5.32)$$

where, in general, $0 < \gamma_{\sigma, \sigma'} \leq 1$.

Some comments regarding these results are in order. First, we point out that the expressions in Eqs. (5.26) to (5.32), corresponding to the terms of the sum in Eq. (5.25), are of different orders in powers of η_π . At the leading order, the width is given just by the contribution of the $|f_{A1}|^2$ term, which is essentially the result for the width in vacuum, see Eq. (2.6). Thus, it is essential to study the next-to-leading order contribution. Notice that from Eq. (5.22) one would expect the coefficients of the cross terms $f_{A1}^* f_V$ and $f_{A1}^* f_{A2}$ to be of order η_π . Nevertheless, it is seen that these $\mathcal{O}(\eta_\pi)$

contributions get canceled: one has

$$C_{A1,V} \propto \mathcal{G}^{(A1,V A+)} + \mathcal{G}^{(A1,V A-)} + \mathcal{G}^{(A1,V A3)}, \quad (5.33)$$

$$C_{A1,A2} \propto \mathcal{G}^{(A1,V A+)} - \mathcal{G}^{(A1,V A-)}, \quad (5.34)$$

which are zero according to the relations in Eq. (5.22) (in fact, it can be seen that these cancelations occur even before performing the integral over x). The natural question at this point is whether we are able to determine the dominant contributions to these terms. Although at first sight one could guess that the lowest nonzero contributions should be $\mathcal{O}(\eta_\pi^2)$, it can be seen that contributions of $\mathcal{O}(\eta_\pi^{4/3})$ show up. A detailed analysis of this issue is given in the next subsection. Notice that cancelations also occur for other coefficients of higher order in η_π , as stated in Eqs. (5.31) and (5.32).

It is also worth mentioning that the contributions involving f_{A1} vanish in the limit $m_l = 0$. Consequently, in the case of decays to electrons, the coefficients $C_{A2,A2}$ and $C_{V,V}$ would be competitive for external fields of very large frequencies or amplitudes. From Eqs. (5.26) and (5.27), the ratio between the contributions of $|f_{A2}|^2$ and $|f_{A1}|^2$ terms to the $\pi^+ \rightarrow e^+ \nu_e$ decay width is given by

$$\frac{C_{A2,A2}|f_{A2}|^2}{C_{A1,A1}|f_{A1}|^2} \simeq \frac{2\eta_\pi^2 \xi_\pi^2 m_\pi^6}{3m_e^2} \frac{|f_{A2}|^2}{|f_{A1}|^2} \sim 5 \times 10^4 \eta_\pi^2 \xi_\pi^2 \frac{m_\pi^4 |f_{A2}|^2}{|f_{A1}|^2}. \quad (5.35)$$

On the other hand, from Refs. [31,32] one can expect the form factors to be of order $|f_{A1}| \sim f_\pi$, $|f_{A2}| \sim 1/f_\pi$. Hence,

one would get sizeable corrections for values of $\eta_\pi \xi_\pi \sim \mathcal{O}(10^{-2})$. A similar situation occurs for the $|f_V|^2$ contribution.

B. Dominant subleading contribution in powers of η_π

As stated, the dominant subleading contribution to the total $\pi^+ \rightarrow l^+ \nu_l$ width is expected to be given by the cross terms $f_{A1}^* f_V$ and $f_{A1}^* f_{A2}$. The estimation of the corresponding coefficients requires some detailed analysis of the behavior of Bessel functions $J_\nu(\nu z)$ for large ν in the region $z \simeq 1$, which represents a complex but well studied problem [33].

The Debye asymptotic expansions, contrary to its appearance in Eqs. (5.2) and (5.3), are not expansions in inverse powers of the large index ν , but turn out to be uniform expansions in inverse powers of the large parameter $\nu^{2/3}(z-1)/z$ for $z > 1$ and the large parameter $\nu^{2/3}(1-z)$ for $z < 1$ [34]. Fractional powers of ν arise, in particular, when one approaches the value $z = 1$. In fact, for values of z around 1, both Debye asymptotic expansions fail. In the transition region one can instead use the expansion

$$J_\nu(\nu(1 + a\nu^{-2/3})) = \frac{2^{1/3}}{\nu^{1/3}} \text{Ai}(-2^{1/3}a)[1 + \mathcal{O}(\nu^{-2/3})], \quad (5.36)$$

where $\text{Ai}(x)$ is the Airy function. In the left panel of Fig. 1 we show, for $\nu = 100$, the behaviors of the exact $J_\nu(\nu z)$ function (solid black line), the Debye approximation given by Eq. (5.5) (red dashed line), and the approximated result for

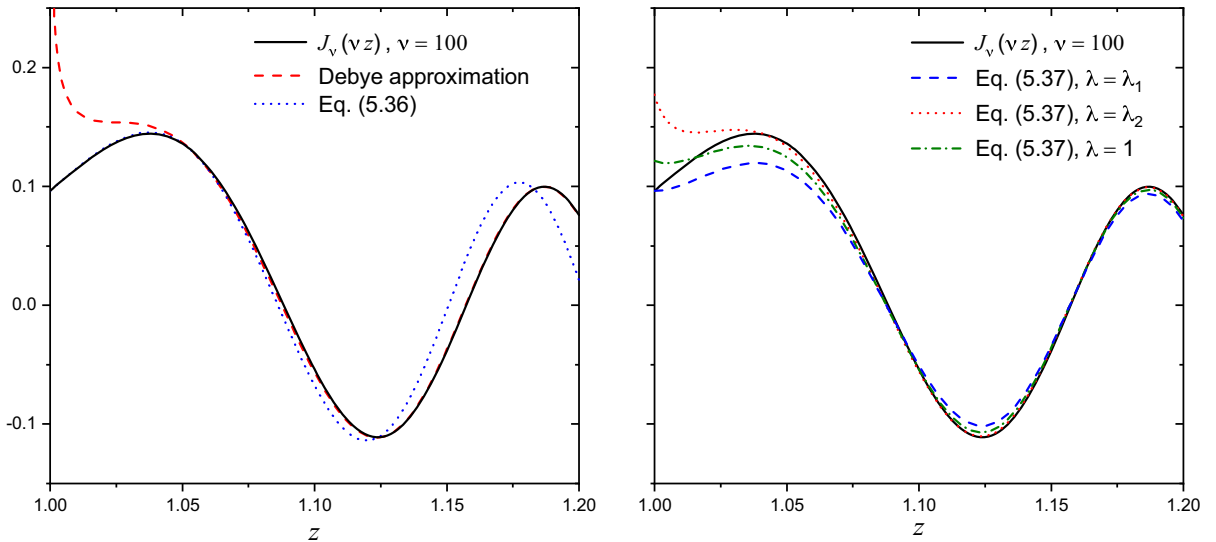


FIG. 1. Bessel function $J_\nu(\nu z)$ and approximate expressions for $\nu = 100$. The black curves in both panels correspond to the function $J_\nu(\nu z)$. In the left panel, the red dashed line corresponds to the Debye approximation given by Eq. (5.5), whereas the blue dotted line corresponds to the expansion in the $z \sim 1$ region, Eq. (5.36). In the right panel, the dashed, dotted, and dashed dotted lines correspond to the regularized approximation in Eq. (5.37) for $\lambda = \lambda_1$, $\lambda = \lambda_2$, and $\lambda = 1$, respectively.

the $z \simeq 1$ in Eq. (5.36) (blue dotted line). It is seen that Debye expansion fails in the region of z close to 1, where it diverges. Though this divergence is smooth enough to ensure that the integral over z is finite, its effects are worth to be taken into account.

The unwanted divergence at $z = 1$ can be regularized by introducing a cutoff λ in the Debye approximation. Here we propose to implement such a regularization by considering the approximate expression

$$J_\nu(\nu z) \simeq \sqrt{\frac{2}{\nu \pi \sqrt{z^2 - 1} + \lambda \nu^{-2/3}}} \times \cos \left[\nu(\sqrt{z^2 - 1} - \arccos z^{-1}) - \frac{\pi}{4} \right]. \quad (5.37)$$

One should now choose some criterion to determine an appropriate value for λ . A simple option is to impose coincidence between the regularized function and the exact Bessel function at $z = 1$. This implies to take

$$\lambda = \lambda_1 = \frac{1}{[2^{1/3} \sqrt{\pi} \text{Ai}(0)]^4} = 2.531. \quad (5.38)$$

A second possibility is to impose coincidence between the regularized function and the exact Bessel function just at the first maximum. The position of this maximum is defined by the condition $\sqrt{z^2 - 1} - \arccos(1/z) \simeq \pi/(4\nu)$, which implies $z \simeq 1 + [3\pi/(8\sqrt{2})]^{2/3} \nu^{-2/3}$. Thus, this criterion leads to

$$\lambda = \lambda_2 = \frac{2^{2/3}}{[\sqrt{\pi} \text{Ai}(-(3\pi/8)^{2/3})]^4} - \left(\frac{3\pi}{4}\right)^{2/3} = 0.222. \quad (5.39)$$

The behavior of the approximate expression in Eq. (5.37) for different values of λ is illustrated in the right panel of Fig. 1. It can be seen that the choice $\lambda = \lambda_1$ (blue dashed line) underestimates the Bessel function, whereas the choice $\lambda = \lambda_2$ (red dotted line) overestimates it. Since our results for the functions $\mathcal{G}^{(A1,VA\pm)}$ and $\mathcal{G}^{(A1,VA3)}$ ultimately depend on $\sqrt{\lambda}$ (see below), an intermediate value can be obtained from $\sqrt{\lambda} = (\sqrt{\lambda_1} + \sqrt{\lambda_2})/2$. The result can be approximated by $\lambda = 1$, which corresponds to the green dotted-dashed line in the right panel of Fig. 1. Taking a given value of λ , we have checked that the regularization method is stable regarding the effect of next-to-leading order corrections to the approximated expressions for Bessel functions.

The introduction of the regularized approximate function in Eq. (5.37) leads to corrections to our results that can be readily evaluated. Essentially, the integrals I_n , $n = 0, 1, 2$, defined in Eq. (5.17) have to be replaced by regularized integrals given by

$$I_n^r = \frac{1}{\eta_\pi} \int_{\mathfrak{z}_-}^{\mathfrak{z}_+} d\mathfrak{z} \frac{\eta_\pi \mathfrak{z}^n}{\pi \sqrt{(\mathfrak{z} - \mathfrak{z}_-)(\mathfrak{z}_+ - \mathfrak{z}) + \lambda \eta_\pi^{2/3} \mathfrak{z}^{4/3}}}. \quad (5.40)$$

As shown in Appendix B, at the leading order in η_π one obtains

$$I_n^r - I_n = -\frac{2}{\pi} \eta_\pi^{1/3} \sqrt{\lambda} \frac{1}{(\mathfrak{z}_+ - \mathfrak{z}_-)} (\mathfrak{z}_-^{n+2/3} + \mathfrak{z}_+^{n+2/3}), \quad (5.41)$$

where I_n , $n = 0, 1, 2$, stand for our previous results, given in Eq. (5.20).

Now we can go back to Eqs. (5.9) and (5.10). After integration over x , the above expressions lead to

$$\mathcal{G}^{(A1,VA\pm)} = -\frac{1}{8\pi} \eta_\pi m_\pi^4 m_l^2 \left[\frac{1}{2} \xi_\pi^2 (1 - \bar{x})^2 - (1 - \bar{x})^2 (1 + 2\bar{x}) \eta_\pi^{1/3} \frac{1}{\pi} \sqrt{\lambda} \xi_\pi^{5/3} f(\xi_\pi, \bar{x}) \right], \quad (5.42)$$

$$\mathcal{G}^{(A1,VA3)} = \frac{1}{8\pi} 2 \eta_\pi m_l^2 m_\pi^4 \left[\frac{1}{2} \xi_\pi^2 (1 - \bar{x})^2 - (1 - \bar{x})^2 (1 + 2\bar{x}) \eta_\pi^{1/3} \frac{1}{\pi} \sqrt{\lambda} \xi_\pi^{5/3} g(\xi_\pi, \bar{x}) \right], \quad (5.43)$$

where

$$f(\xi_\pi, \bar{x}) = \frac{1}{\xi_\pi^{8/3} (1 - \bar{x})^2 (1 + 2\bar{x})} \int_{\bar{x}}^1 dx \frac{x^2}{\sqrt{(1-x)(x-\bar{x})}} (\mathfrak{z}_+^{5/3} + \mathfrak{z}_-^{5/3}),$$

$$g(\xi_\pi, \bar{x}) = \frac{1}{\xi_\pi^{2/3} (1 - \bar{x})^2 (1 + 2\bar{x})} \int_{\bar{x}}^1 dx \frac{x \sqrt{1-x}}{\sqrt{x-\bar{x}}} (\mathfrak{z}_+^{2/3} + \mathfrak{z}_-^{2/3}). \quad (5.44)$$

It can be seen that the combination $\mathcal{G}^{(A1,VA+)} + \mathcal{G}^{(A1,VA-)} + \mathcal{G}^{(A1,VA3)}$ in Eq. (5.33) is now different from zero. Hence, we get a subleading contribution to the coefficient $C_{A1,V}$ given by

TABLE I. Values of the functions $f(\xi_\pi, \bar{x})$, $g(\xi_\pi, \bar{x})$ and $f(\xi_\pi, \bar{x}) - g(\xi_\pi, \bar{x})$ for $\pi^+ \rightarrow e^+ \nu_e$ and $\pi^+ \rightarrow \mu^+ \nu_\mu$ decays, considering different values of ξ_π .

		$f(\xi_\pi, \bar{x})$	$g(\xi_\pi, \bar{x})$	$f(\xi_\pi, \bar{x}) - g(\xi_\pi, \bar{x})$
$\pi^+ \rightarrow e^+ \bar{\nu}_e$	$\xi_\pi = 10^{-5}$	1.454	0.873	0.582
	$\xi_\pi = 1$	1.630	1.227	0.404
$\pi^+ \rightarrow \mu^+ \bar{\nu}_\mu$	$\xi_\pi = 10^{-5}$	1.294	0.776	0.517
	$\xi_\pi = 1$	1.669	1.362	0.307

$$C_{A1,V} = \frac{4}{\pi} \eta_\pi^{4/3} m_\pi^2 m_l^2 \left(1 + 2 \frac{m_l^2}{m_\pi^2} \right) \xi_\pi^{5/3} \sqrt{\lambda} [f(\xi_\pi, \bar{x}) - g(\xi_\pi, \bar{x})], \quad (5.45)$$

i.e., the power of η_π for the cross term $f_{A1}^* f_V$ in Eq. (5.29) is given by $1 + \gamma_{A1,V}$ with $\gamma_{A1,V} = 1/3$. On the other hand, the cancellation between $\mathcal{G}^{(A1,VA+)} - \mathcal{G}^{(A1,VA-)}$ still holds valid, leading to $\gamma_{A1,A2} \geq 1$ for the coefficient of the $f_{A1}^* f_{A2}$ term. We conclude from this analysis that in the limit $\eta_\pi \ll 1$ the π^+ decay width can be written as

$$\Gamma(\pi^+ \rightarrow l^+ \nu_l) = \Gamma_{\pi^+ \rightarrow l^+ \nu_l}^{(\text{vac})} (1 + \delta_{A1,V} + \mathcal{O}(\eta_\pi^2)), \quad (5.46)$$

where $\delta_{A1,V}$ is of order $\eta_\pi^{4/3}$, and we have assumed $f_{A1} \simeq f_\pi$.

The above conclusions do not change qualitatively with the chosen value of λ , which, as stated, should be taken to be of order unity in order to get a good approximation to the exact Bessel functions. Moreover, the functions $f(\xi_\pi, \bar{x})$ and $g(\xi_\pi, \bar{x})$ have been defined in such a way that they vary just smoothly for values of ξ_π within a range from 10^{-5} to 1. This is illustrated in Table I, where numerical values of both functions, as well as the difference $f(\xi_\pi, \bar{x}) - g(\xi_\pi, \bar{x})$, are quoted for $\xi_\pi = 10^{-5}$ and $\xi_\pi = 1$. In particular, in the limit $\xi_\pi \ll 1$, using the change of variables $x = y(1 - \bar{x}) + \bar{x}$ it is seen that Eqs. (5.44) can be written as

$$\begin{aligned} f(\xi_\pi, \bar{x}) &\simeq \frac{10}{3(1+2\bar{x})} \int_0^1 dy \left(y + \frac{\bar{x}}{(1-\bar{x})} \right)^{1/3} \frac{(1-y)^{5/6}}{y^{1/6}}, \\ g(\xi_\pi, \bar{x}) &\simeq \frac{3}{5} f(\xi_\pi, \bar{x}). \end{aligned} \quad (5.47)$$

According to the above estimations for the functions $f(\xi_\pi, \bar{x})$ and $g(\xi_\pi, \bar{x})$, the corrections to the leading contributions are found to be of the same order of magnitude for the decays into positrons and muons. The dominant correction in Eq. (5.46) is estimated to be

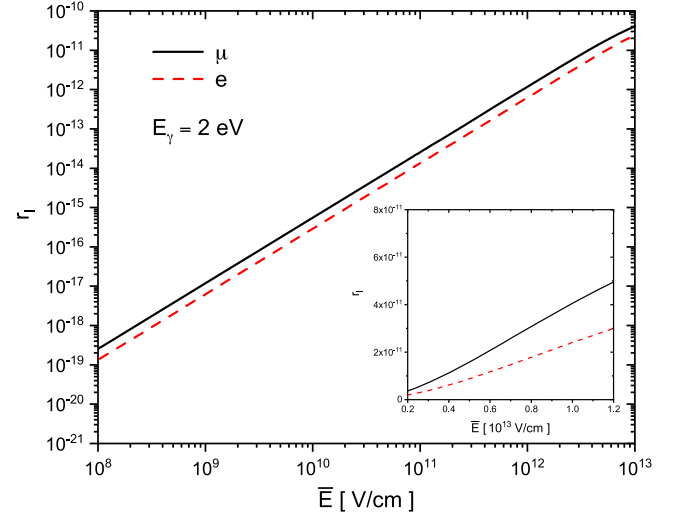


FIG. 2. Values of the factor r_l [see Eqs. (5.48) and (5.49)] as a function of the electric field amplitude $\bar{E}$, for a photon energy $E_\gamma = 2$ eV.

$$\delta_{A1,V} = \frac{2 C_{A1,V} \text{Re}(f_{A1}^* f_V)}{C_{A1,A1} |f_{A1}|^2} = m_\pi^2 \text{Re} \left(\frac{f_V}{f_{A1}} \right) \sqrt{\lambda} r_l, \quad (5.48)$$

where

$$r_l = \frac{8}{\pi} \eta_\pi^{4/3} \xi_\pi^{5/3} \left(1 + 2 \frac{m_l^2}{m_\pi^2} \right) [f(\xi_\pi, \bar{x}) - g(\xi_\pi, \bar{x})], \quad (5.49)$$

with $l = e, \mu$. As mentioned in Sec. VA, according to the discussion in Refs. [31,32] one expects the ratio $|f_V/f_{A1}|$ to be of order $\sim 1/f_\pi^2$. Thus, since $\sqrt{\lambda}$ is expected to lie within a range from ~ 0.5 to 1.6, it is seen that the order of magnitude of the correction $\delta_{A1,V}$ is essentially given by the factor r_l . In Fig. 2 we show our numerical results for the values of r_l ($l = \mu, e$) as a function of the electric field amplitude $\bar{E}$, assuming that the pion is at rest and taking a photon energy $E_\gamma = 2$ eV (this corresponds to $\eta_\pi = 1.43 \times 10^{-8}$). It is seen that the correction to the width can vary within a range of eight orders of magnitude for $\bar{E}$ ranging from 10^8 to 10^{13} V/cm, which correspond to ξ_π varying from $\mathcal{O}(10^{-5})$ to $\mathcal{O}(1)$. Notice that in the main plot we use logarithmic scales, while in the detailed graph for $\bar{E} \sim 10^{13}$ eV we move to linear scales. In fact, in the region where $\xi_\pi \ll 1$ the approximated expressions in Eqs. (5.47) lead to the power relation

$$r_l \simeq \begin{cases} 2.83 \eta_\pi^{4/3} \xi_\pi^{5/3} = 1.50 \times 10^{-32} \bar{E} [\text{V/cm}]^{5/3} / E_\gamma [\text{eV}]^{1/3} & \text{for } l = \mu, \\ 1.48 \eta_\pi^{4/3} \xi_\pi^{5/3} = 0.79 \times 10^{-32} \bar{E} [\text{V/cm}]^{5/3} / E_\gamma [\text{eV}]^{1/3} & \text{for } l = e. \end{cases} \quad (5.50)$$

C. Contributions from relative low values of ℓ

In our calculation we have assumed that the main contributions to the decay width correspond to values of $|\ell|$ of $\mathcal{O}(\xi_\pi \eta_\pi^{-1})$ or even larger. Let us now consider the contributions corresponding to a relatively low number of exchanged photons, i.e., to $|\ell| \leq \ell_0$, with $\ell_0 \eta_\pi \ll \xi_\pi$. We recall that we are interested in physical situations for which η_π is of order 10^{-7} to 10^{-8} , while the considered values of ξ_π range from 10^{-5} to 1. Since, as discussed in Sec. VA, the argument d_1 of the Bessel functions in Eqs. (4.30) to (4.39) will be in general a large number, the arguments of the integrals over x in Eq. (4.42) will still be highly oscillatory. Therefore, we can approximate the Bessel functions taking into account the asymptotic behavior [35]

$$J_\ell(d_1) \simeq \sqrt{\frac{2}{\pi d_1}} \cos\left(d_1 - \ell \frac{\pi}{2} - \frac{\pi}{4}\right), \quad \ell \leq \ell_0. \quad (5.51)$$

In fact, for these small ℓ contributions we can still consider the Debye approximation used in our calculation, Eq. (5.5), in the limit in which z^2 is much larger than one. Hence, the integrals in Eq. (4.42) can be estimated by introducing in Eqs. (4.30) to (4.39) similar replacements as in Eqs. (5.7), taking $z^2 = (d_1/\ell)^2 \gg 1$,

$$\begin{aligned} J_\ell(d_1)^2 &\rightarrow \frac{1}{\pi d_1}, \\ J_\ell(d_1) J_{\ell \pm 1}(d_1) &\rightarrow \frac{\ell}{\pi d_1^2} \sim 0, \\ J_{\ell \pm 1}(d_1)^2 &\rightarrow \frac{1}{\pi d_1}, \\ J_{\ell-1}(d_1) J_{\ell+1}(d_1) &\rightarrow -\frac{1}{\pi d_1}. \end{aligned} \quad (5.52)$$

For definiteness, let us consider the case of $\mathcal{G}^{(A1,A1)}$, denoting by $\mathcal{G}_{\ell_0}^{(A1,A1)}$ the total contribution to $\mathcal{G}^{(A1,A1)}$ arising from terms with $|\ell| \leq \ell_0$. After the above replacements, at the leading order in powers of η_π we obtain

$$\mathcal{G}_{\ell_0}^{(A1,A1)} = \sum_{\ell=-\ell_0}^{\ell_0} \frac{1}{8\pi} \int_{\bar{x}_0}^1 dx \frac{\eta_\pi}{2\pi \xi_\pi} m_\pi m_l^2 (m_\pi^2 - m_l^2) \frac{x}{\bar{p}_{\perp 0}}, \quad (5.53)$$

where

$$\bar{x}_0 = \frac{\xi_\pi^2 + m_l^2/m_\pi^2}{1 + \xi_\pi^2}, \quad \bar{p}_{\perp 0}^2 = m_\pi^2 (1 + \xi_\pi^2) (1 - x)(x - \bar{x}_0). \quad (5.54)$$

This integral can be analytically performed, leading to

$$\begin{aligned} \mathcal{G}_{\ell_0}^{(A1,A1)} &= \sum_{\ell=-\ell_0}^{\ell_0} \frac{1}{32\pi} \frac{\eta_\pi}{\xi_\pi (1 + \xi_\pi^2)^{3/2}} m_l^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right) \\ &\times [(1 + 2\xi_\pi^2)m_\pi^2 + m_l^2]. \end{aligned} \quad (5.55)$$

Notice that the expression in Eq. (5.55) is $\mathcal{O}(\eta_\pi \xi_\pi^{-1})$, and the terms in the sum do not depend on ℓ . Therefore, the full contribution of the sum turns out to be negligible when compared with the total value of $\mathcal{G}^{(A1,A1)}$, which, as shown in the previous subsections, is expected to be $\mathcal{O}(\eta_\pi^0)$. In other words, the sum of contributions of the form given in Eq. (5.55) will be negligible as long as ℓ does not reach values of order $\xi_\pi \eta_\pi^{-1}$. A similar reasoning is shown to be valid for all contributions to $\mathcal{G}^{(\Sigma,\Sigma')}$ satisfying $\ell \eta_\pi \ll \xi_\pi$.

D. Bounds for the number of exchanged photons

As stated in Sec. VA, for the computation of the total width we have found it convenient to perform first the sum (approximated by an integral) over the number of exchanged photons and then the integral over the phase space variable x . However, it is also interesting to consider the alternative procedure, i.e., to integrate over x for a given value of ℓ , as expressed in Eq. (4.42).

Since the usage of the Debye expansion of Bessel functions discussed in Sec. VA is always justified, the substitutions in Eqs. (5.7) can still be introduced, and once again the phase space turns out to be constrained by the condition $z^2 > 1$. For a given ℓ , the integral over $x = p^-/q^-$ is shown to be limited to $x_- < x < x_+$, where

$$x_{\pm}(\ell) = \frac{1 + 2\ell\eta_\pi + \xi_\pi^2(1 + \Delta^2) \pm \sqrt{(1 + 2\ell\eta_\pi - m_l^2/m_\pi^2)^2 - 4\ell^2\eta_\pi^2\Delta^2}}{2[1 + (\ell\eta_\pi + \xi_\pi^2)/\xi_\pi^2]}, \quad (5.56)$$

with

$$\Delta = \sqrt{1 + \frac{m_l^2}{\xi_\pi^2 m_\pi^2}}. \quad (5.57)$$

In turn, the sum over ℓ is found to be restricted to $\ell_{\min} < \ell < \ell_{\max}$, where

$$\ell_{\min} = -\frac{1 - m_l^2/m_\pi^2}{2\eta_\pi(\Delta + 1)}, \quad \ell_{\max} = \frac{1 - m_l^2/m_\pi^2}{2\eta_\pi(\Delta - 1)}. \quad (5.58)$$

That is to say, one finds upper and lower cutoffs for the number of photons to be exchanged with the electromagnetic field. If one approximates the sum over ℓ by an integral, the above defined coefficients $\mathcal{G}^{(\Sigma, \Sigma')}$ can be written in general as

$$\mathcal{G}^{(\Sigma, \Sigma')} = \int_{\ell_{\min}}^{\ell_{\max}} d\ell \int_{x_-(\ell)}^{x_+(\ell)} dx g^{(\Sigma, \Sigma')}(\ell, x). \quad (5.59)$$

The existence of these bounds has been observed from numerical calculations in Ref. [21]. In that reference the authors consider charged pion decays to muons, taking values of η_π similar to those considered in our work and values of ξ_π of order 10^{-6} to 10^{-5} , as well as some kinematical cuts in the muon momentum. It can be seen, indeed, that the obtained numerical results are consistent with the approximate limits $\ell \simeq \pm \xi_\pi(1 - m_l^2/m_\pi^2)m_\pi/(2\eta_\pi m_l)$ that follow from Eqs. (5.58) for such low values of ξ_π . This situation is illustrated in Fig. 3. In the left panel we show the integration region corresponding to Eq. (5.59) for pion decays to muons. We have taken a photon energy $E_\gamma = 2$ eV, which corresponds to $\eta_\pi \simeq 1.43 \times 10^{-8}$ for a pion at rest, and $\xi_\pi = 10^{-5}$, corresponding to an electric field amplitude $\bar{E} = 1.42 \times 10^8$ [see Eq. (2.18)]. It is seen that in this case the number of exchanged photons is not larger than ~ 200 . In addition, in the right panel of Fig. 3 we

show the behavior of the function $g^{(A1, A1)}(\ell, x)$, corresponding to the dominant contribution to the width. From Eq. (5.8) one has

$$g^{(A1, A1)}(\ell, x) = \frac{1}{16\pi^2} m_l^2 m_\pi^2 (1 - m_l^2/m_\pi^2) \times \frac{\eta_\pi}{\sqrt{(\ell\eta_\pi - \mathfrak{z}_-)(\mathfrak{z}_+ - \ell\eta_\pi)}}, \quad (5.60)$$

which after integration over the phase space gives the result in Eq. (5.21), namely,

$$\mathcal{G}^{(A1, A1)} = \frac{1}{16\pi} m_l^2 m_\pi^2 (1 - m_l^2/m_\pi^2)^2. \quad (5.61)$$

The curve in the right panel of Fig. 3 shows the behavior of the function $g^{(A1, A1)}(\ell, x)$ normalized by $\mathcal{G}^{(A1, A1)}$, for a chosen value $x = 0.8$. The form of the function, which grows asymptotically at the extreme values of ℓ , is qualitatively similar for other values of x within the allowed range.

Notice that the feasibility of the numerical analysis carried out in Ref. [21] is favored by the low value of ξ_π , which restricts the number of exchanged photons to values that does not exceed a few hundreds. In our work we reproduce those results on the basis of a more general ground, given by the analytical expressions in Eqs. (5.25) to (5.32). This allows us to extend the validity of our calculations up to values of ξ_π of order unity, which could be reached by present experimental facilities. In the left panel of Fig. 4 we show the integration region corresponding to Eq. (5.59), taking now $\bar{E}_\gamma = 2$ eV and $\xi_\pi = 1$ (i.e., $\bar{E} = 1.42 \times 10^{13}$). It is seen that in this case the maximum number of exchanged photons becomes enhanced up to $\simeq 6 \times 10^7$, challenging computational capabilities. In addition, it is worth noticing that upper and lower limits of ℓ are no longer symmetric. On the other hand, as shown in the

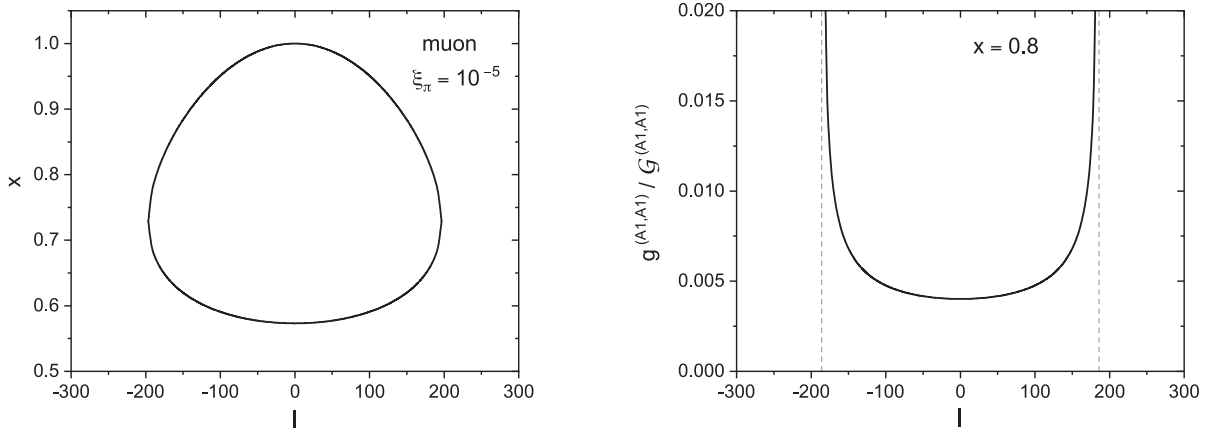


FIG. 3. Left panel: phase space for the decay $\pi^+ \rightarrow \mu^+ \nu_\mu$, for $\xi_\pi = 10^{-5}$. Right panel: behavior of the function $g^{(A1, A1)}(\ell, x)$ (normalized by $\mathcal{G}^{(A1, A1)}$) as a function of the number of exchanged photons ℓ , for $x = 0.8$.

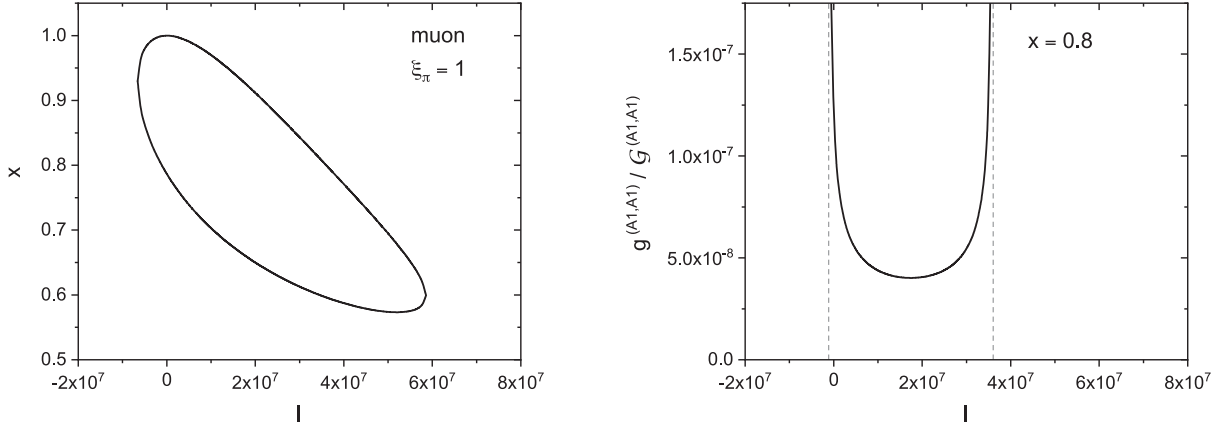


FIG. 4. Left panel: phase space for the decay $\pi^+ \rightarrow \mu^+ \nu_\mu$, for $\xi_\pi = 1$. Right panel: behavior of the function $g^{(A1,A1)}(\ell, x)$ (normalized by $\mathcal{G}^{(A1,A1)}$) as a function of the number of exchanged photons ℓ , for $x = 0.8$.

right panel of Fig. 4, the normalized function $g^{(A1,A1)}(\ell, x)$ is still heavily peaked at the extreme values of ℓ .

Finally, we remark that our analytical calculations can also be applied to the decays to electrons. The integration regions obtained taking same values of $\bar{E}_\gamma$ as in the case of muons are shown in Figs. 5 and 6, which correspond to $\xi_\pi = 10^{-5}$ and $\xi_\pi = 1$, respectively. In both figures the regions have been split into two graphs, so as to show the results for the region of low x using logarithmic scales. As can be seen from the figure, even for low values of ξ_π the number of exchanged photons can reach values as large as $\mathcal{O}(10^5)$. Moreover, for ξ_π of order unity the difference $\Delta - 1$ gets suppressed, and the upper limit for ℓ becomes enhanced up to values of $\mathcal{O}(10^{12})$. The normalized function $g^{(A1,A1)}(\ell, x)$ is found to show a similar behavior as in the case of the decays to muons.

E. Connection with the Kroll-Watson formula

At this point it is interesting to observe that there is a connection between our expression for the $\pi^+ \rightarrow l^+ \nu_l$

decay width and the Kroll-Watson formula [36]. The latter corresponds to the differential cross section, in the non-relativistic limit, for the scattering of a charged particle by some scattering potential in the presence of a strong external electromagnetic wave, considering the exchange of a given number of photons.

Let $\vec{p}_0$ and $\vec{p}(\ell)$ be the initial and final momenta of the charged particle, respectively, ℓ being the number of emitted ($\ell < 0$) or absorbed ($\ell > 0$) photons. The Kroll-Watson formula states that the differential cross section $d\sigma_\ell(p(\ell), p_0)/d\Omega$ can be written as

$$\frac{d\sigma_\ell(p(\ell), p_0)}{d\Omega} = \frac{|\vec{p}(\ell)|}{|\vec{p}_0|} J_\ell(x_{\text{KW}})^2 \frac{d\sigma_{\text{el}}}{d\Omega}, \quad (5.62)$$

where $d\sigma_{\text{el}}/d\Omega$ stands for the differential elastic scattering cross section in the absence of the electromagnetic field, and x_{KW} is defined as

$$x_{\text{KW}} = -Q \frac{\vec{A} \cdot (\vec{p}(\ell) - \vec{p}_0)}{m\omega}, \quad (5.63)$$

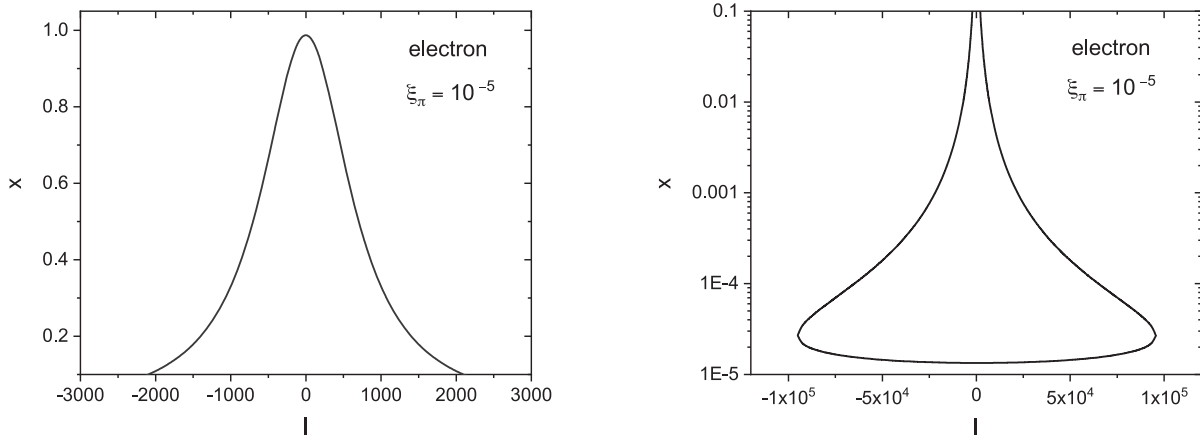


FIG. 5. Phase space for the decay $\pi^+ \rightarrow e^+ \nu_e$, for $\xi_\pi = 10^{-5}$. In the right panel we use a logarithmic scale in order to show the region corresponding to low values of x .

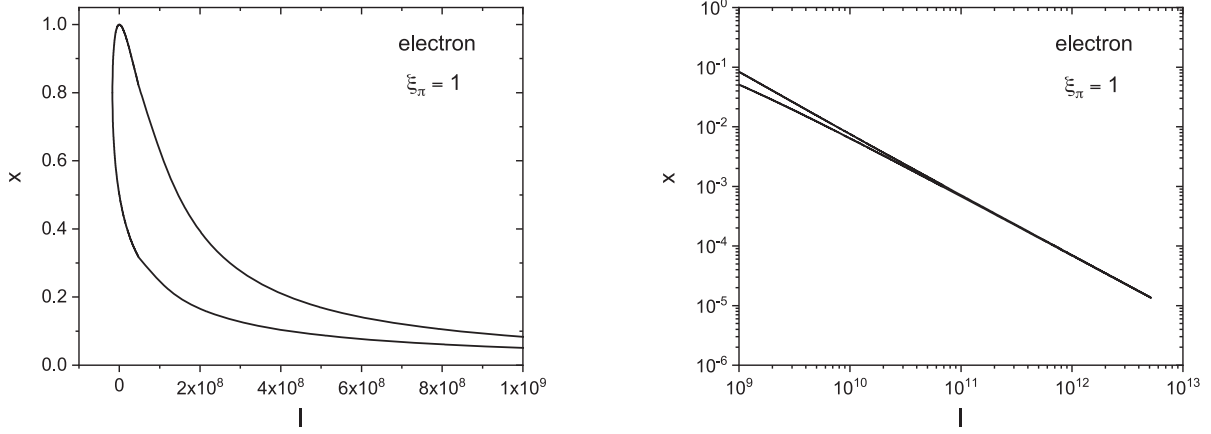


FIG. 6. Phase space for the decay $\pi^+ \rightarrow e^+ \nu_e$, for $\xi_\pi = 1$. In the right panel we use logarithmic scales for both x and ℓ in order to show the region corresponding to low values of x .

where Q is the particle charge and $\vec{A}$ and ω are the parameters that characterize the electromagnetic wave. According to Ref. [36], Eq. (5.62) is valid when the scattering potential is weak or when the wave frequency is small, which is given by the condition

$$\left| \frac{\ell \omega m}{Q \vec{A} \cdot (\vec{p}(\ell) - \vec{p}_0)} \right| = \left| \frac{\ell}{x_{\text{KW}}} \right| < 1. \quad (5.64)$$

Although Ref. [36] deals with a scattering problem, the result in Eq. (5.62) can be connected with our result for the charged pion decay. First of all, it can be seen that with our notation and conventions the quantity x_{KW} corresponds to our d_1 , defined as $d_1 = e a |\mathbf{A} \cdot \mathbf{p}| / (k \cdot p)$ in Eq. (4.8). We can take Eq. (4.30) and perform the replacements [see Eqs. (5.7)]

$$\begin{aligned} J_\ell(d_1) J_{\ell \pm 1}(d_1) &\rightarrow \frac{\ell}{d_1} J_\ell(d_1)^2, \\ J_{\ell+1}(d_1)^2 &\rightarrow J_\ell(d_1)^2, \end{aligned} \quad (5.65)$$

to obtain

$$\mathbb{N}_\ell^{(A1,A1)} = \frac{m_l^2}{2} (m_\pi^2 - m_l^2) J_\ell(d_1)^2. \quad (5.66)$$

Thus, identifying the form factor f_{A1} with the pion decay constant f_π , at the leading order $\mathcal{O}(\eta_\pi^0)$ the differential width can be written as $d\Gamma/dx = \sum_\ell d\Gamma_\ell/dx$, with

$$\begin{aligned} \frac{d\Gamma_\ell}{dx} &= \frac{1}{4\pi E_q} G_F^2 V_{ud}^2 m_l^2 (m_\pi^2 - m_l^2) f_\pi^2 J_\ell(d_1)^2 \\ &= \frac{d\Gamma^{(0)}}{dx} J_\ell(d_1)^2, \end{aligned} \quad (5.67)$$

where $d\Gamma^{(0)}/dx = G_F^2 V_{ud}^2 m_l^2 (m_\pi^2 - m_l^2) f_\pi^2 / (4\pi E_q)$ is the differential width in absence of the external field. In this

way, our result turns out to be analogous to the Kroll-Watson formula. The factor $|\vec{p}(\ell)|/|\vec{p}_0|$ in Eq. (5.62) is related to the fact that for the scattering process the differential cross section is defined with respect to the solid angle, while in our case the differential decay width is defined with respect to the variable $x = p^-/q^-$.

Finally, notice that the result given in Eq. (5.67) is valid for “small ℓ ” (i.e., $|\ell| < |x_{\text{KW}}|$), but also for large values of ℓ , due to the fact that one is dealing with a weak potential. In this way, at order η_π^0 we can evaluate the differential width $d\Gamma/dx$ by performing the sum

$$\frac{d\Gamma}{dx} = \sum_{\ell=-\infty}^{\infty} \frac{d\Gamma_\ell}{dx} = \frac{d\Gamma^{(0)}}{dx} \sum_\ell J_\ell(d_1)^2 = \frac{d\Gamma^{(0)}}{dx}. \quad (5.68)$$

Here we have used the relation $\sum_\ell J_\ell(d_1)^2 = I_0 = 1$, obtained in Sec. VA using the asymptotic expressions for Bessel functions. After integration over x , one recovers the result for the total width in absence of the external field, Eq. (2.6).

VI. SUMMARY AND CONCLUSIONS

We have obtained analytical expressions for the $\pi^+ \rightarrow l^+ \nu_l$ decay width in the presence of a background electromagnetic plane wave with circular polarization. Our first step has been to determine the number of form factors involved in the decay process. It is shown that for an electromagnetic plane wave background the hadronic $V-A$ pion-to-vacuum amplitude can be written in terms of four form factors, three of them corresponding to the axial-vector current and one to the vector current. This result coincides with the one obtained in Ref. [25] for the case of a background given by a static uniform magnetic field. We have also shown that, in general, the form factors are functions of the pion mass and the Lorentz-invariant quantities ξ_π and η_π , which are related to the

amplitude and the frequency of the external wave, respectively.

We have seen that the most adequate kinematic variables for the description of the decay process are the fraction of the light front momentum of the pion ported by the charged lepton, $x = p^-/q^-$, and the number of involved photons, ℓ . In terms of these quantities we have obtained the full analytic expression for the decay width, given by Eq. (4.41). As discussed in Sec. V, this expression can be approximated taking into account the values of the parameters related to the external field for the physical situations of interest, namely, $\eta_\pi \sim 10^{-8}$ to 10^{-7} and $\xi_\pi \sim 10^{-5}$ to 1, which correspond to photon energies of the order of a few eV and electric field amplitudes within the range from about 10^8 to 10^{13} V/cm. Transforming the sum over ℓ into an integral and expanding in powers of η_π , we have been able to perform the phase space integrals, arriving at the analytical expression for the decay width given by Eq. (5.25).

It is seen that the terms of the sum in Eq. (5.25), which correspond to different combinations of hadronic form factors, are of different orders in powers of η_π [see Eqs. (5.26) to (5.30)]. The dominant contribution is found to be given by the form factor f_{A1} , which can be identified with the pion decay constant f_π . With this identification, the leading order expression for the decay width reduces to the result obtained in absence of the external field.

In fact, the expected order of the corrections arising from each combination of form factors can be inferred from the general form of the pion-to-vacuum amplitude [see Eqs. (3.9) and (3.10)], since each term carrying an electromagnetic strength tensor will introduce a factor $QF^{\mu\nu} \propto \xi_\pi \eta_\pi m_\pi^2$. Nevertheless, it is found that several combinations get exactly canceled at the leading order. In particular, this is what happens with the contributions of order η_π , which arise from the crossed terms $f_{A1}^* f_V$ and $f_{A1}^* f_{A2}$. Due to the peculiar behavior of Bessel functions that appear in the decay amplitude, it is found that the largest subleading contribution to the width is the one provided by the $f_{A1}^* f_V$ term, which is shown to be of order $\eta_\pi^{4/3}$.

Our analytical results confirm previous numerical calculations obtained for low values of the parameter ξ_π (i.e., low values of the amplitude of the external electromagnetic wave) [21]. In particular, at leading order in powers of η_π , we have found analytical expressions for the upper and lower bounds on the number of exchanged photons; these kinematical cuts are shown to arise essentially from the properties of the Bessel functions involved in the decay amplitude. We remark that our expressions are valid for the full studied range of values of ξ_π . In particular, it is seen that the upper bound for the

number of exchanged photons gets highly increased for values of ξ_π of order unity. In addition, it is shown that, at the leading order in η_π , our analytical results for the contribution of a given number of exchanged photons are analogous to those obtained from the Kroll-Watson formula [36], which describes the cross section for the scattering of a charged particle by some weak potential in the presence of a strong external electromagnetic wave.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: PARTICLE FIELDS IN THE PRESENCE OF AN ELECTROMAGNETIC PW BACKGROUND

We work here under the assumption that the laser field can be described as a PW background field. Hence, it has to be treated at all orders in perturbation theory. This can be achieved taking into account the Furry interaction picture, in which the background field is considered as part of the unperturbed system [37]. To obtain the particle asymptotic states in this picture it is necessary to find the solutions of the Klein-Gordon and Dirac equations in the presence of the background, namely

$$[(\partial^\mu + i Q_\pi A^\mu)(\partial_\mu - i Q_\pi A_\mu) + m^2]\Phi^+(x) = 0, \quad (\text{A1})$$

$$[i\partial\!\!\!/ - Q_l \not{A}(x) - m]\Psi(x) = 0, \quad (\text{A2})$$

where A^μ is given by Eq. (2.7). The solutions of these equations can be found in Ref. [3,27,28]. To fix our notation, in what follows we extract the main results.

1. Spin 0 field

In the presence of the electromagnetic PW background given by Eq. (2.7), a charged spin 0 field can be expanded as

$$\begin{aligned}\Phi^+(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} [\phi_p^+(x) a_{\vec{p}} + \phi_p^-(x) b_{\vec{p}}^\dagger], \\ \Phi^-(x) &= \Phi^+(x)^\dagger = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} [\phi_p^+(x)^* a_{\vec{p}}^\dagger + \phi_p^-(x)^* b_{\vec{p}}],\end{aligned}\quad (\text{A3})$$

where $E_p = \sqrt{m^2 + \vec{p}^2}$ and $p^\mu = (E_p, \vec{p})$. The wave function $\phi_p^+(x)$ is given by

$$\phi_p^+(x) = e^{-ip \cdot x} \varphi_p^+(\zeta), \quad (\text{A4})$$

where

$$\varphi_p^+(\zeta) = \exp \left\{ -\frac{i}{k \cdot p} \int_{\zeta_0}^{\zeta} d\zeta' \left[QA(\zeta') \cdot p - \frac{1}{2} Q^2 A(\zeta')^2 \right] \right\}, \quad (\text{A5})$$

with $\zeta = k \cdot x$. By definition, the negative energy solution can be obtained from the positive energy as

$$\phi_p^-(x) = \phi_p^+(x)^*, \quad \text{with the change } Q \rightarrow -Q. \quad (\text{A6})$$

In our case, this is equivalent to the transformation $p^\mu \rightarrow -p^\mu$, for which one has

$$\begin{aligned}\phi_p^{(-)}(x) &= \phi_{-p}^{(+)}(x) = e^{ip \cdot x/\hbar} \varphi_p^{(-)}(\zeta), \\ \varphi_p^{(-)}(\zeta) &= \varphi_{-p}^{(+)}(\zeta) \\ &= \exp \left\{ -\frac{i}{k \cdot p} \int_{\zeta_0}^{\zeta} d\zeta' \left[QA(\zeta') \cdot p + \frac{1}{2} Q^2 A(\zeta')^2 \right] \right\}.\end{aligned}\quad (\text{A7})$$

The wave functions satisfy the orthogonality and completeness properties

$$\begin{aligned}\int d^4x \phi_{p'}^{(+)}(x)^* \phi_p^{(+)}(x) &= (2\pi)^4 \delta^4(p - p'), \\ \int \frac{d^4p}{(2\pi)^4} \phi_p^{(+)}(x')^* \phi_p^{(+)}(x) &= \delta^4(x - x'),\end{aligned}\quad (\text{A8})$$

where the momentum can be extended off the mass shell.

The commutation relations between creation and annihilation operators read

$$\begin{aligned}[a_{\vec{p}}, a_{\vec{p}'}^\dagger] &= [b_{\vec{p}}, b_{\vec{p}'}^\dagger] = \delta^3(\vec{p} - \vec{p}'), \\ [a_{\vec{p}}, b_{\vec{p}'}] &= [b_{\vec{p}}, b_{\vec{p}'}] = [a_{\vec{p}} b_{\vec{p}'}] = [a_{\vec{p}}, b_{\vec{p}'}^\dagger] = 0.\end{aligned}\quad (\text{A9})$$

A π^+ particle state of given momentum $\vec{p}$ is defined by

$$|\pi^+(\vec{p})\rangle = a_{\vec{p}}^\dagger |0\rangle. \quad (\text{A10})$$

2. Spin 1/2 field

A spin 1/2 field can be written as

$$\begin{aligned}\Psi(x) &= \sum_{s=1,2} \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} [E_p(x) u(\vec{p}, s) c_{\vec{p},s} + E_p^c(x) v(\vec{p}, s) d_{\vec{p},s}^\dagger], \\ \bar{\Psi}(x) &= \sum_{s=1,2} \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} [\bar{u}(\vec{p}, s) \bar{E}_p(x) c_{\vec{p},s}^\dagger + \bar{v}(\vec{p}, s) \bar{E}_p^c(x) d_{\vec{p},s}],\end{aligned}\quad (\text{A11})$$

where $c_{\vec{p},s}$ ($d_{\vec{p},s}^\dagger$) annihilates (creates) a fermion (antifermion) of quantum numbers $\vec{p}$, s . The spinors $u(\vec{p}, s)$ and $v(\vec{p}, s)$ are the usual free bispinors [3]. We use the bispinor normalization

$$\bar{u}(\vec{p}, s) \gamma_0 u(\vec{p}, r) = \bar{v}(\vec{p}, s) \gamma_0 v(\vec{p}, r) = 2E_0 \delta_{sr}, \quad (\text{A12})$$

which corresponds to the standard normalization

$$\bar{u}(\vec{p}, s) u(\vec{p}, r) = -\bar{v}(\vec{p}, s) v(\vec{p}, r) = 2m \delta_{sr}. \quad (\text{A13})$$

With this normalization condition one has the relations

$$\begin{aligned}\sum_{s=1,2} u(\vec{p}, s)_\sigma \bar{u}(\vec{p}, s)_\tau &= (\not{p} + m)_{\sigma\tau}, \\ \sum_{s=1,2} v(\vec{p}, s)_\sigma \bar{v}(\vec{p}, s)_\tau &= (\not{p} - m)_{\sigma\tau}.\end{aligned}\quad (\text{A14})$$

The Ritus matrices in Eqs. (A11) are

$$\begin{aligned} \mathbf{E}_p(x) &= \phi_p^{(+)}(x) \left(1 + \frac{Q}{2k \cdot p} \not{k} A(\zeta) \right), & \mathbf{E}_p^c(x) &= \phi_p^{(-)}(x) \left(1 - \frac{Q}{2k \cdot p} \not{k} A(\zeta) \right), \\ \bar{\mathbf{E}}_p(x) &= \left(1 + \frac{Q}{2k \cdot p} A(\zeta) \not{k} \right) \phi_p^{(+)}(x)^*, & \bar{\mathbf{E}}_p^c(x) &= \left(1 - \frac{Q}{2k \cdot p} A(\zeta) \not{k} \right) \phi_p^{(-)}(x)^*, \end{aligned} \quad (\text{A15})$$

where we have introduced the definition $\bar{\mathbf{E}} = \gamma^0 \mathbf{E}^\dagger \gamma^0$. In Eqs. (A11), the negative energy solution can be obtained from the positive energy one performing the transformations

$$\begin{aligned} \mathbf{E}_p^c(x) &= C[\bar{\mathbf{E}}_p(x)]^T C^{-1}, & \text{with } Q \rightarrow -Q, \\ \mathbf{E}_p(x) &= C[\bar{\mathbf{E}}_p^c(x)]^T C^{-1}, & \text{with } Q \rightarrow -Q, \end{aligned} \quad (\text{A16})$$

where $C = i\gamma^2 \gamma^0 = -C^\dagger = -C^{-1}$. The transformation $p^\mu \rightarrow -p^\mu$ leads to the relations

$$\mathbf{E}_p^c(x) = \mathbf{E}_{-p}(x), \quad \bar{\mathbf{E}}_p^c(x) = \bar{\mathbf{E}}_{-p}(x). \quad (\text{A17})$$

In addition, the Ritus matrices satisfy [28]

$$\begin{aligned} \int d^4x \bar{\mathbf{E}}_p(x) \mathbf{E}_q(x) &= (2\pi)^4 \delta^4(p - q), \\ \int \frac{d^4p}{(2\pi)^4} \bar{\mathbf{E}}_p(x) \mathbf{E}_p(x') &= \delta^4(x - x'), \end{aligned} \quad (\text{A18})$$

where the momentum is extended off the mass shell.

The anticommutation relations for creation and annihilation operators read

$$\begin{aligned} \{c_{\vec{p},s}, c_{\vec{p}',r}^\dagger\} &= \{d_{\vec{p},s}, d_{\vec{p}',s}^\dagger\} = \delta^3(\vec{p} - \vec{p}') \delta_{s,r}, \\ \{c_{\vec{p},s}, c_{\vec{p}',s}\} &= \{d_{\vec{p},s}, d_{\vec{p}',s}\} = \{c_{\vec{p},s}, d_{\vec{p}',s}\} = \{c_{\vec{p},s}, d_{\vec{p}',s}^\dagger\} = 0. \end{aligned} \quad (\text{A19})$$

A lepton state of momentum $\vec{p}$ and polarization state s is given by

$$|l(\vec{p}, s)\rangle = c_{\vec{p},s}^\dagger |0\rangle. \quad (\text{A20})$$

APPENDIX B: SUBLEADING CONTRIBUTION TO THE INTEGRALS I_n^r

In this appendix we outline the calculation of the lowest order correction to the regularized integrals I_n^r in powers of η_π , Eq. (5.41). Taking the definition of I_n^r in Eq. (5.40), we start by splitting the integral into three regions,

$$\begin{aligned} I_{n-}^r &= \frac{1}{\eta_\pi} \int_{\mathfrak{z}_-}^{\mathfrak{z}_- + \epsilon_-} d\mathfrak{z} \frac{\eta_\pi \mathfrak{z}^n}{\pi \sqrt{(\mathfrak{z} - \mathfrak{z}_-)(\mathfrak{z}_+ - \mathfrak{z}) + \lambda \eta_\pi^{2/3} \mathfrak{z}^{4/3}}}, \\ I_{nc}^r &= \frac{1}{\eta_\pi} \int_{\mathfrak{z}_- + \epsilon_-}^{\mathfrak{z}_+ - \epsilon_+} d\mathfrak{z} \frac{\eta_\pi \mathfrak{z}^n}{\pi \sqrt{(\mathfrak{z} - \mathfrak{z}_-)(\mathfrak{z}_+ - \mathfrak{z}) + \lambda \eta_\pi^{2/3} \mathfrak{z}^{4/3}}}, \\ I_{n+}^r &= \frac{1}{\eta_\pi} \int_{\mathfrak{z}_+ - \epsilon_+}^{\mathfrak{z}_+} d\mathfrak{z} \frac{\eta_\pi \mathfrak{z}^n}{\pi \sqrt{(\mathfrak{z} - \mathfrak{z}_-)(\mathfrak{z}_+ - \mathfrak{z}) + \lambda \eta_\pi^{2/3} \mathfrak{z}^{4/3}}}, \end{aligned} \quad (\text{B1})$$

with the assumptions

$$\epsilon_\pm \ll (\mathfrak{z}_+ - \mathfrak{z}_-) \quad (\text{B2})$$

and

$$\epsilon_\pm (\mathfrak{z}_+ - \mathfrak{z}_-) \gg \lambda \eta_\pi^{2/3} \mathfrak{z}_\pm^{4/3}. \quad (\text{B3})$$

Taking into account the relation in Eq. (B3), it is seen that the integral I_{nc}^r can be expanded in powers of $\eta_\pi^{2/3}$. One has

$$I_{nc}^r = \int_{\mathfrak{z}_- + \epsilon_-}^{\mathfrak{z}_+ - \epsilon_+} d\mathfrak{z} \frac{\mathfrak{z}^n}{\pi \sqrt{(\mathfrak{z} - \mathfrak{z}_-)(\mathfrak{z}_+ - \mathfrak{z})}} + \mathcal{O}(\eta_\pi^{2/3}), \quad (\text{B4})$$

where the integral can be performed analytically. At leading order in $\epsilon_\pm$ we have

$$I_{nc}^r = I_n - \frac{2}{\pi \sqrt{\mathfrak{z}_+ - \mathfrak{z}_-}} (\mathfrak{z}_-^n \sqrt{\epsilon_-} + \mathfrak{z}_+^n \sqrt{\epsilon_+}) + \mathcal{O}(\eta_\pi^{2/3}), \quad (\text{B5})$$

where I_n , $n = 0, 1, 2$, are given in Eq. (5.20).

Now, to evaluate the integral I_{n-}^r let us perform the change

$$\mathfrak{z} = \mathfrak{z}_- + \frac{1}{(\mathfrak{z}_+ - \mathfrak{z}_-)} y. \quad (\text{B6})$$

At leading order in ϵ_- , one has

$$\begin{aligned}
I_{n-}^r &\simeq \frac{1}{\pi} \frac{\mathfrak{z}_-^n}{(\mathfrak{z}_+ - \mathfrak{z}_-)} \int_0^{\epsilon_-(\mathfrak{z}_+ - \mathfrak{z}_-)} dy \frac{1}{\sqrt{y + \lambda \eta_\pi^{2/3} \mathfrak{z}_-^{4/3}}} \\
&\simeq \frac{2}{\pi} \mathfrak{z}_-^n \left[\frac{\sqrt{\epsilon_-}}{\sqrt{\mathfrak{z}_+ - \mathfrak{z}_-}} - \sqrt{\lambda} \frac{\mathfrak{z}_-^{2/3}}{(\mathfrak{z}_+ - \mathfrak{z}_-)} \eta_\pi^{1/3} \right] + \mathcal{O}(\eta_\pi^{2/3}).
\end{aligned}
\tag{B7}$$

In the same way, for the integral I_{n+}^r we can perform the change $\mathfrak{z} = \mathfrak{z}_+ - y/(\mathfrak{z}_+ - \mathfrak{z}_-)$. This leads to

$$I_{n+}^r \simeq \frac{2}{\pi} \mathfrak{z}_+^n \left[\frac{\sqrt{\epsilon_+}}{\sqrt{\mathfrak{z}_+ - \mathfrak{z}_-}} - \sqrt{\lambda} \frac{\mathfrak{z}_+^{2/3}}{(\mathfrak{z}_+ - \mathfrak{z}_-)} \eta_\pi^{1/3} \right] + \mathcal{O}(\eta_\pi^{2/3}).
\tag{B8}$$

Summing the expressions in Eqs. (B5), (B7), and (B8) the dependences on $\epsilon_\pm$ cancel (as they should do), leading to the result in Eq. (5.41).

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