

Nonsinglet vector current in lattice QCD: $O(a)$ -improvement from large volumes

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In previous work, we determined the improvement coefficients c_V and $c_{\bar{V}}$ required for the massless $O(a)$ -improvement of the local and point-split discretizations of the nonsinglet vector current for $N_f = 3$ nonperturbatively $O(a)$ -improved Wilson fermions and the Lüscher-Weisz gauge action, using ensembles of large-volume configurations generated by the coordinated lattice simulations (CLS) initiative. A new estimate for the mass-dependent improvement coefficient $\bar{b}_A^{\text{eff}}$ has recently become available, differing from the one used in our earlier study, and on which our implementation via a massive axial Ward identity relied. Here, we update our analysis of the mass-independent vector improvement coefficients based on the new axial current improvement coefficient, and analyze additional ensembles with a different chiral trajectory in order to validate our results at two values of the bare coupling. We find that using the new estimate of $\bar{b}_A^{\text{eff}}$ improves the consistency between the two chiral trajectories, as well as with a previous determination of the improvement coefficients directly in the massless limit on small volumes.

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I. INTRODUCTION

Numerical simulation of lattice field theory is one of the most successful and universal frameworks for studying strongly coupled quantum field theories from first principles. Lattice quantum chromodynamics (QCD) simulations have reached a maturity where they are essential for the precision study of the Standard Model of particle physics (SM) and in addition provide new insights into exotic states of matter. Such computations are often said to be systematically improvable, a feature which can be observed in the progress of precision lattice predictions over time, for example in those compiled by the Flavour Lattice Averaging Group [1].

One sense in which lattice simulations are systematically improvable is realized by Symanzik's improvement program [2,3]. According to effective field theory analysis, the approach to the continuum limit of renormalized on-shell matrix elements of local operators can be accelerated

by adding higher-dimension counterterms which cancel the discretization effects order by order in the lattice spacing a , so long as their coefficients are chosen appropriately. In this way, the uncertainty due to extrapolation to the continuum can be reduced in a fairly predictable way.

In Wilson's formulation of lattice QCD [4,5], due to the loss of chiral symmetry, the Symanzik analysis predicts leading linear effects for local operators, at least classically. In this case, the implementation of the $O(a)$ improvement programme is crucial to reach the required precision to test the SM with the currently achievable range of lattice spacings. The nonperturbative determination of improvement coefficients is mandatory in the range of couplings used for large-volume simulations, and chiral Ward identities provide natural and practical conditions for restoring chiral symmetry and removing the linear lattice effects [6]. In practice, the improvement of the Wilson fermion action is known for $N_f = 2, 3$ and 4 flavors and various gauge actions [7–11], but in addition to improving the action, each local operator must, in general, be improved by adding the necessary counterterms and coefficients, classified in Ref. [12] for fermionic bilinears. For $N_f = 3$ $O(a)$ -improved Wilson fermions, mass-independent improvement coefficients have been determined nonperturbatively for many such operators [13–17], including the nonsinglet

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axial and vector currents. At finite quark masses, the number of improvement coefficients proliferates, the so-called b -terms, although some combinations are known nonperturbatively in the $N_f = 3$ case [14,18–21]. In general, the determination of the improvement coefficients poses a challenge to obtaining physically interesting quantities in the continuum.

Correlation functions involving the quark electromagnetic current are of utmost importance to understand the interactions and structure of hadrons, fundamental quantities like the hadronic vacuum polarization and even isospin-breaking corrections. In particular, then, the $O(a)$ improvement of the vector current is extremely relevant input for current precision physics programmes. Following the proposal of Guagnelli and Sommer [22], there are two existing determinations of the mass-independent improvement coefficients c_V and $c_{\bar{V}}$ for the local and point-split discretizations for $N_f = 3$ $O(a)$ -improved Wilson fermions [14,15]. Although these determinations display a tension, they are compatible with higher-order ambiguities. In this work, we update the former analysis of the mass-independent improvement coefficients in light of new results for the mass-dependent improvement on which the axial Ward identity relies. This improves the consistency between the two determinations dramatically, and here we provide another determination at two values of the bare coupling using a different chiral trajectory, adding further confidence to the new determination.

II. BACKGROUND

For a theory with more than two quark flavors, a condition from which the improvement of the vector current can be obtained is the (continuum) chiral Ward identity [22]

$$\int d^3y \langle \delta S A_k^{23}(y) O^{31}(0) \rangle = \int d^3y \langle V_k^{13}(y) O^{31}(0) \rangle, \quad (1)$$

where the variation of the action under the flavored local axial transformations $\delta\psi^1(x) = -\alpha(x)\gamma_5\psi^2(x)$ and $\delta\bar{\psi}^2(x) = -\bar{\psi}^1(x)\alpha(x)\gamma_5$ is given by

$$\delta S = \int_{t_1}^{t_2} dx_0 \int d^3x \{ \partial_\mu A_\mu^{12}(x) - 2m^{12}P^{12}(x) \}, \quad (2)$$

and m^{rs} denotes the average bare mass of flavors r and s . We note that the product in the last term is renormalization-group invariant. In writing Eq. (2) we have chosen the transformation $\alpha(x)$ to be unity in the domain $t_1 < x_0 < t_2$ and null everywhere else. As the preceding Ward identity is violated in Wilson's lattice formulation by $O(a)$ cutoff effects, it can be used as a condition to fix one of the improvement coefficients which contribute, provided that the remaining coefficients are set appropriately to cancel the other effects at that order.

To this end, we must specify the discretization of the operators and integral in the chiral Ward identity. We write the massless $O(a)$ -improved axial and vector currents for $r \neq s$

$$A_{I,\mu}^{rs}(x) = A_\mu^{rs}(x) + ac_A \tilde{\partial}_\mu P^{rs}(x), \quad (3)$$

$$V_{I,\mu}^{rs}(x) = V_\mu^{rs}(x) + ac_V \tilde{\partial}_\nu T_{\mu\nu}^{rs}(x), \quad (4)$$

in terms of the bare currents

$$A_\mu^{rs} = \bar{\psi}^r \gamma_\mu \gamma_5 \psi^s, \quad V_\mu^{rs} = \bar{\psi}^r \gamma_\mu \psi^s, \quad (5)$$

and the pseudoscalar density and tensor operator

$$P^{rs} = \bar{\psi}^r \gamma_5 \psi^s, \quad T_{\mu\nu}^{rs} = \bar{\psi}^r \sigma_{\mu\nu} \psi^s \quad (6)$$

where the antisymmetric tensor matrix is defined as $\sigma_{\mu\nu} = -\frac{1}{2}[\gamma_\mu, \gamma_\nu]$. The finite difference operator $\tilde{\partial}_\mu$ denotes the symmetric $O(a)$ -improved definition.

In addition, we also compute the massless improvement for the point-split vector current

$$\hat{V}_\mu^{rs}(x) = \frac{1}{2} \{ \bar{\psi}^r(x + a\hat{\mu}) U_\mu^\dagger(x) (1 + \gamma_\mu) \psi^s(x) - \bar{\psi}^r(x) (1 - \gamma_\mu) U_\mu(x) \psi^s(x + a\hat{\mu}) \}, \quad (7)$$

or more precisely the site-centered version $\tilde{V}_\mu^{rs}(x) = \frac{1}{2}(\hat{V}_\mu^{rs}(x) + \hat{V}_\mu^{rs}(x - a\hat{\mu}))$, that has the same lattice symmetries as the local current, which allows us to use the same form for the improvement counterterm as for the local current in Eq. (4), but with coefficient $c_{\bar{V}}$ [23].

In the massive theory, there are additional mass-dependent counterterms which are proportional to the bare operator and therefore can be conveniently parametrized by the bare subtracted quark masses $m_q^r = m_0^r - m_c$, together with any multiplicative renormalization Z_J for massless-improved bilinears J_I^{rs} with $r \neq s$ as [12]

$$J_R^{rs} = \hat{Z}_J J_I^{rs} = Z_J(\tilde{g}_0^2)(1 + \bar{b}_J N_f a \bar{m}_q + b_J a m_q^{rs}) J_I^{rs}, \quad (8)$$

where $\bar{m}_q = \sum_{r=1}^{N_f} m_q^r$ is the average bare subtracted quark mass and $m_q^{rs} = (m_q^r + m_q^s)/2$. In the massive theory, there is one additional sticking point, which is that the bare coupling also needs to be improved

$$\tilde{g}_0^2 = g_0^2(1 + b_g a \bar{m}_q), \quad (9)$$

in order to fix the lattice spacing for different quark masses. Given that a shift in the bare coupling of the coefficients multiplying $O(a)$ -improvement terms will only contribute at higher orders in a , we only need to care about the dependence of the renormalization factor on this coupling. In fact, for the currents it is easy to see that defining

$\bar{b}_J^{\text{eff}} = \bar{b}_J + b_g/N_f \{d \ln Z_J / d \ln g_0^2\}$, we can absorb the dependence on b_g into $\bar{b}_J^{\text{eff}}$ [24] so that

$$\hat{Z}_J = Z_J(g_0^2)(1 + \bar{b}_J^{\text{eff}} N_f a \bar{m}_q + b_J a m_q^{rs}) \quad (10)$$

up to neglected $O(a^2)$ effects. Note that as the point-split current satisfies a vector Ward identity at finite lattice spacing, the entire normalization is constrained to be unity, i.e., $\hat{Z}_{\bar{V}} = 1$.

Inserting these definitions of the lattice operators in the chiral Ward identity, we arrive at a family of definitions for the improvement coefficients

$$c_J = \frac{a^3 \sum_{\vec{y}} \langle \delta S A_{R,k}^{23}(y) O^{31}(0) - \hat{Z}_J J_k^{13}(y) O^{31}(0) \rangle}{a^4 \sum_{\vec{y}} \langle \hat{Z}_J \tilde{\partial}_0 T_{k0}^{13}(y) O^{31}(0) \rangle} \quad (11)$$

for $J = V, \bar{V}$ and with

$$\begin{aligned} \delta S = & a^3 \sum_{\vec{x}} (A_{R,0}^{12}(t_2, \vec{x}) - A_{R,0}^{12}(t_1, \vec{x})) \\ & - a^4 \sum_{x_0=t_1}^{t_2} \sum_{\vec{x}} 2m^{12} P^{12}(x), \end{aligned} \quad (12)$$

which depend on t_1, t_2, y_0 , the choice of O^{31} and the masses m_q^1, m_q^2 , and m_q^3 . In the expression Eq. (12), m^{rs} is understood to be the bare current-quark mass defined via the partially conserved axial (PCAC) relation and determined in Ref. [14]. We note that, although the last term is renormalized and scale-independent, a contact term arises when it coincides with y which must be removed by sending m^{12} to zero. Therefore, with $N_f = 3$, different choices for m_q^3 in this limit, or correspondingly $\bar{m}_q$, give rise to different valid definitions of the improvement coefficients. A nonzero value of m_q^3 , however, requires that the axial current is improved in the massive theory and therefore the input of the correct mass-dependent improvement coefficients. While in earlier work we used an improvement condition with nonzero $\bar{m}_q$ in the limit of $m^{12} \rightarrow 0$, in this work we add a determination with $\bar{m}_q = 0$ for two of the values of the lattice spacing. This provides a strong cross-check on the validity of the implementation in the massive case.

III. NUMERICAL DETERMINATION

The definition of the improvement condition presented in the previous section is the same as in Ref. [14]. In addition, we choose the same values of the coordinates t_1, t_2 , and y_0 as used there. To ensure a smooth approach to the continuum limit, we fix the kinematic variables in the improvement condition in physical units to the same values used in the earlier work with $t_2 - t_1 \approx 0.7$ fm while the current was placed at distance $y_0 \approx 0.8$ fm from the

TABLE I. New ensembles analyzed in this work with $m_q^1 = m_q^2 = m_q^3$, in addition to those presented in Ref. [14] made available through the CLS initiative [25]. The ensembles labeled by “rqcd” were generated by the Regensburg group [26] with the BQCD code [27].

$6/g_0^2$	id	κ	$L_0/a \times (L/a)^3$	m_π (MeV)
3.46	rqcd29	0.136600	64×32^3	710
	rqcd30	0.1369587	64×32^3	320
	X450	0.136994	64×48^3	260
3.55	X250	0.137050	64×48^3	350
	X251	0.137100	64×48^3	270

external operator. The external operator is also unchanged, and in particular, we use the sum of the local vector current and tensor operator as in that work. First as in our previous work, we impose the improvement condition at vanishing m^{12} and with $\bar{m}_q = \bar{m}_q^{\text{phys}}$ and second we add a determination with $\bar{m}_q = 0$, by extrapolating from the ensembles with $m_q^1 = m_q^2 = m_q^3$ listed in Table I. The massive vector renormalization factor $\hat{Z}_V$ is the same as used in Ref. [14] determined by imposing a vector Ward identity.

A. Analysis using updated $\bar{b}_A^{\text{eff}}$

Compared with the previous analysis, we use the updated values for $\bar{b}_A^{\text{eff}}$ given in Table II from Refs. [26,28] based on the methodology of Ref. [18]. The values differ significantly from the set used in the previous work which ranged between $\bar{b}_A^{\text{eff}} = 0.4 - 2.0$ from the smallest to largest bare coupling. Even though a generous uncertainty was attributed to their value, it did not cover the range of the new determinations, which are now compatible with zero. Using these new values, we observe a significant decrease of the estimates for both improvement coefficients at finite mass compared with the previous determination. The $m^{12} \rightarrow 0$ limit is very flat, and we can obtain reasonable parametrizations either with a constant or linear behavior, see Fig. 1. The determination with $\bar{m}_q = 0$ (obtained from the red squares) is independent of the values of the mass-dependent improvement coefficients, and we checked that even if they are not set to their correct values, the limit is consistent, albeit reached with a larger slope. The resulting values from the trajectory with $\bar{m}_q = \text{const.}$ (circles) are in good agreement with the former. Both determinations with $m^{12} = 0$ are listed in Table III but in the following section we choose conservatively to interpolate the results from the linear extrapolation, which have a larger uncertainty.

IV. RESULTS AND DISCUSSION

The results obtained in the previous section are shown as a function of the bare coupling (blue and red points) in Fig. 2. They lie systematically between the ones obtained in

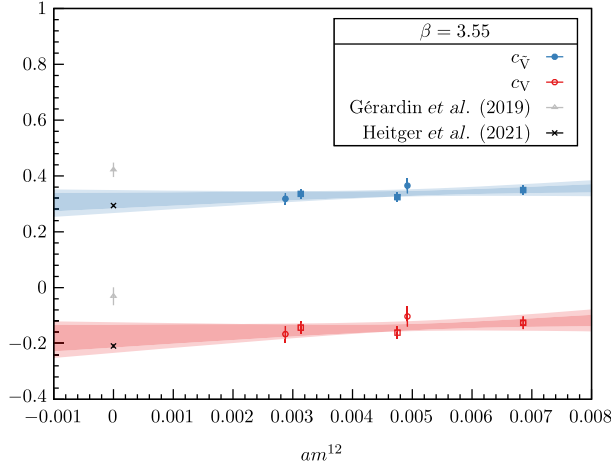


FIG. 1. Dependence on the estimator as a function of m^{12} at fixed bare coupling $\beta = 6/g_0^2 = 3.55$ for the local (red) and point-split (blue) discretizations of the vector current. The squares display the chiral trajectory with $m_q^1 = m_q^2 = m_q^3 = \bar{m}_q$ while the circles show the same with $\bar{m}_q = \text{const}$. The two trajectories are not constrained to coincide in the chiral limit. The earlier results from Ref. [14] are shown in the gray triangles while the results obtained from SF [15] are shown with crosses.

our earlier work (gray triangles) [14] and those obtained in small volumes with Schrödinger functional (SF) boundary conditions (black squares) [15]. In order to compare systematically with the latter, and because we observe a sharp decrease of the results over the range of bare couplings, we opt to use the same parametrization of the data as in that work, which we find to provide a very good description of the data displayed with the bands. Given the excellent consistency between the two independent chiral trajectories in both the $\beta = 3.46$ and $\beta = 3.55$ cases, we include the two estimations for each β in the fit to the final parametrization. In addition, the interpolation may be used within or close to the measured range of values and provides a smooth parametrization of the improvement coefficient toward the continuum limit.

The parametrization used for the local current is

$$c_V(g_0^2) = -0.0130 C_F g_0^2 \times [1 + (a + b g_0^2) \exp\{-(2b_0 g_0^2)^{-1}\}] \quad (13)$$

with $C_F = 4/3$ and $b_0 = 9/(4\pi)^2$ and for the point-split current

TABLE II. Values of the axial improvement coefficients used in this work [26,28].

$6/g_0^2$	3.4	3.46	3.55	3.7
b_A	1.244	1.239	1.232	1.221
$\bar{b}_A^{\text{eff}}$	-0.11(13)	0.08(11)	-0.03(13)	-0.047(75)

TABLE III. Values of the massless $O(a)$ -improvement coefficients c_V and $c_{\bar{V}}$ determined from linear and constant fits to the massive axial Ward identity. Both models provide good descriptions of the data. The values marked with the $\star$ indicate the chiral limit with $\bar{m}_q = 0$.

$6/g_0^2$	c_V		$c_{\bar{V}}$	
	Linear	Constant	Linear	Constant
3.4	-0.252(34)	-0.1888(97)	0.256(24)	0.3029(72)
3.46	-0.165(39)	-0.148(11)	0.322(27)	0.3334(78)
3.46 $\star$	-0.159(19)	-0.136(11)	0.324(14)	0.3412(82)
3.55	-0.183(51)	-0.134(16)	0.304(37)	0.342(12)
3.55 $\star$	-0.169(45)	-0.143(13)	0.317(33)	0.337(10)
3.7	-0.107(37)	-0.119(11)	0.367(26)	0.3569(87)

$$c_{\bar{V}}(g_0^2) = \frac{1}{2} - d g_0^2 [1 + c g_0^2 \exp\{-(2b_0 g_0^2)^{-1}\}] \quad (14)$$

and the resulting best fit parameters obtained are

$$a = -2513.80, \quad b = 2044.53, \quad (15)$$

for the local current; respectively, for the point-split one,

$$c = 316.96, \quad d = 0.024543. \quad (16)$$

The individual parameters are not significantly different from zero, so including the covariance matrix as given in Table IV is essential to providing a useful and accurate representation of the uncertainty. Typically, however, the improvement coefficient may be considered part of the definition of the lattice theory, and its statistical uncertainty

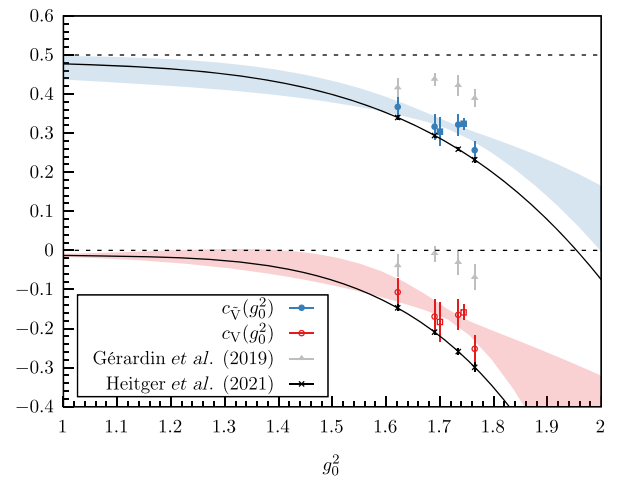


FIG. 2. Final results for c_V (red) and $c_{\bar{V}}$ (blue) as a function of the bare coupling, compared with the results of Heitger *et al.* [15] (black line). The old results of Ref. [14] are shown with the gray triangles.

TABLE IV. Covariance matrix of best fit parameters for the parametrizations for both improvement coefficients for the two discretizations of the vector current.

Cov	a	b	c	d
a	2.15×10^7	-1.24×10^7	-1.08×10^{10}	4.79×10^1
b	-1.24×10^7	7.18×10^6	6.23×10^9	-2.77×10^1
c	-1.08×10^{10}	6.23×10^9	4.81×10^{13}	-9.45×10^4
d	4.79×10^1	-2.77×10^1	-9.45×10^4	9.44×10^{-4}

may be considered part of its definition so long as they are determined correctly up to the expected $O(a)$ ambiguity. In our case, we checked that the difference between our determination and that of the SF is consistent with such an ambiguity, albeit with a large uncertainty.

We note that the difference between the improvement coefficients of the two vector-current discretizations is to a very good degree consistent with the leading order difference in g_0^2 , namely 0.5 over the whole range of couplings. The updated values of these improvement coefficients have already been put to use in Ref. [29], where good consistency was observed between the continuum limit of the integrated current correlator using either this determination of improvement coefficients or the ones of Heitger and Joswig [15].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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