

Holographic conformal anomaly and a-theorem in 5D scalar-tensor theories from heterotic strings

Tianhao Wu^{*} and Michael Stone

Department of Physics, University of Illinois, Urbana-Champaign, Urbana, Illinois 61801, USA



(Received 1 June 2025; accepted 20 March 2026; published 28 April 2026)

Using the coefficient-frame freedom of the $\mathcal{O}(\alpha')$ heterotic metric-dilaton effective action and incorporating it into Kaluza-Klein reduction with different reduction *Ansätze* and maximally symmetric internal spaces, we derive two distinct reduced five-dimensional scalar-tensor branches: a Lovelock-Horndeski-type theory and, for a special coefficient choice, an Einstein-dilaton-Gauss-Bonnet theory. We present a novel derivation of the full holographic conformal anomaly for these two reduced five-dimensional scalar-tensor theories. In the Lovelock-Horndeski case, we also construct exact asymptotically anti-de Sitter (AdS) solutions with linear-dilaton profiles and establish a holographic a-theorem. Our findings show that AdS/CFT remains sound in the presence of nonminimal scalars and higher-curvature terms, with string corrections fixing the conformal anomaly and the dual RG flow.

DOI: [10.1103/hnv7-fgcf](https://doi.org/10.1103/hnv7-fgcf)

I. INTRODUCTION

The AdS/CFT correspondence continues to reshape our understanding of quantum gravity by establishing a duality between bulk gravitational dynamics and boundary field theories. In this holographic paradigm, classical gravitational dynamics in a $(d+1)$ -dimensional anti-de Sitter (AdS) bulk X_{d+1} are intricately encoded in a conformal field theory (CFT) on the d -dimensional boundary M_d , where each operator $\mathcal{O}$ in the CFT corresponds to a bulk field φ . The on-shell gravitational action evaluated with boundary condition $\varphi^{(0)}$ becomes the generating functional of the CFT,

$$e^{-S_{\text{AdS}}(\varphi^{\text{classical}}(\varphi_0))} = Z_d(\varphi^{(0)}) \equiv \left\langle \exp \int_{M_d} d^d x \mathcal{O} \varphi^{(0)} \right\rangle. \quad (1)$$

This duality provides a powerful framework for studying strongly coupled field theories and quantum gravity [1].

In this work, we leverage the holographic approach to investigate modified gravity theories with higher-derivative corrections and nonminimal scalar couplings. We derive the full conformal anomaly in extended scalar-tensor models

arising from a consistent Kaluza-Klein reduction, including Lovelock and Horndeski-type interactions.

For even d , the Fefferman-Graham expansion [2] of the bulk metric allows one to extract the boundary conformal anomaly from the near-boundary asymptotics. The generic structure of the holographic conformal anomaly for pure gravitational theories in $d+1$ -dimensional bulk spacetime is governed by the trace anomaly relation [3–5]

$$g^{ij} \langle T_{ij} \rangle = -a E^{(d)} + \Sigma_i c_i \mathcal{W}_i^2, \quad (2)$$

where $E_{(d)}$ is the Euler density defined by variational derivative, $E_{(d)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ij}} \int d^d x \sqrt{-g} \mathcal{E}_{(d)}$, with $\mathcal{E}_{(d)}$ being the Euler characteristic integrand. The $\mathcal{W}_i^{(d)}$ represent the complete set of independent Weyl invariants of weight $-d$ in d dimensions. For instance, in four-dimensional boundary conformal field theories ($d=4$), the Euler density reduces to the Gauss-Bonnet topological invariant. The second term in the anomaly equation contains quadratic Weyl contractions, such as $\mathcal{W}_1^{(4)} = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

Anomaly coefficients a and c_i are determined by the bulk solution near the conformal boundary and have been computed for Einstein gravity, higher-derivative theories, and Lovelock gravity [6–11]. Lovelock gravity includes higher-curvature terms yielding second-order equations in higher dimensions. Horndeski theory [12] generalizes scalar-tensor gravity, while preserving second-order equations. Galileon theories [13], related by field redefinitions, exhibit similar properties. These frameworks have been extended to higher dimensions,

^{*}Contact author: twu49@illinois.edu

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

leading to the broader class of “Lovelock-Horndeski theories”—scalar-tensor models that combine Lovelock invariants with Horndeski-type scalar interactions [14,15]. A well-posed formulation of these theories was established in [16], confirming the consistency of their dynamical evolution.

In string theory, higher-curvature corrections naturally arise. For instance, the Gauss-Bonnet term appears as the leading-order α' correction in heterotic string theory [17,18], influencing black hole thermodynamics and cosmology. Kaluza-Klein (KK) reduction of such terms over symmetric internal spaces [19–22] leads to lower-dimensional scalar-tensor theories, some exhibiting Horndeski/Galileon-type structures. Earlier studies typically assumed a fixed standard form of the heterotic string effective action at $\mathcal{O}(\alpha')$, limiting the types of scalar-tensor theories that emerge upon reduction.

In this work, we exploit the “coefficient ambiguity” of the string effective action—arising from field redefinitions and highlighted and called coefficient frame by Metsaev and Tseytlin [23]—which allows for a family of off-shell equivalent actions (in the sense of coefficient frames). By incorporating this freedom in choosing coefficient frames into the KK reduction, we show the mechanism that incorporating coefficient-frame freedom into KK reduction—together with different reduction *Ansätze*/internal geometries—systematically yields distinct reduced five-dimensional scalar-tensor theories, including

- (i) a Lovelock-Horndeski-type theory with general second-order scalar-tensor couplings;
- (ii) an Einstein-dilaton-Gauss-Bonnet (EdGB) theory from a special coefficient choice.

This mechanism demonstrates how stringy ambiguities can govern the emergence of qualitatively different lower-dimensional gravitational dynamics.

Solving the reduced equations with AdS asymptotics, we find pure AdS solutions and known AdS black holes in these theories. For the EdGB case, black hole solutions with scalar profiles are well established [24–26]; for the Lovelock-Horndeski case, we derive an exact AdS solution with linear dilaton. In the latter case, the holographic conformal anomaly for modified gravitational theories in $d+1$ dimensions naturally extends the standard form of the trace anomaly, leading to the generalized relation

$$g^{ij}\langle T_{ij} \rangle = -aE^{(d)} + \Sigma_i c_i \mathcal{W}_i^2 + \Sigma_k b_k H_k, \quad (3)$$

In addition to the Euler and Weyl terms, higher-derivative scalar curvature invariants H_k can also contribute to the anomaly structure. Specifically, at fourth order in derivatives, the independent curvature invariants H_1 and H_2 are defined by $H_1 \equiv \beta_1 R_{ij} R^{ij} + \beta_2 \square R$ and $H_2 \equiv \square R$. These H_k terms capture additional curvature

contributions not associated purely with Euler or Weyl structures and are naturally incorporated in theories with higher-derivative couplings.

We compute the full holographic conformal anomaly for these theories. The two α' -selected reduced branches yield different universal (a, c) anomaly coefficients, i.e., different holographic anomaly data for the distinct resulting reduced five-dimensional theories. We further verify the holographic a-theorem in Lovelock-Horndeski theory with a linear dilaton (call it the critical case). These results test holography under scalar coupling and higher-derivative corrections.

This paper is organized as follows. In Sec. II, we summarize the Kaluza-Klein reduction of the α' -corrected string effective action for different reduction *Ansätze* and derive the resulting five-dimensional scalar-tensor theories. In Sec. III, we discuss solutions of the reduced theories, including a pure AdS₅ solution with a linear-dilaton profile and a brief review of known AdS black hole solutions. In Sec. IV, we implement the Fefferman-Graham expansion and compute the holographic trace anomaly. In Sec. V, we construct holographic a functions and establish an a-theorem using the null energy condition. We conclude in Sec. VI. Technical details are collected in Appendixes A–C.

II. KALUZA-KLEIN REDUCTION ON STRING EFFECTIVE THEORY

At order α' , the heterotic effective action is unique up to local field redefinitions. We therefore work in a general “coefficient frame,” i.e., we retain the most general set of $\mathcal{O}(\alpha')$ operators in the metric-dilaton sector related by such redefinitions. The corresponding coefficients label off-shell choices of field variables within the Metsaev-Tseytlin equivalence class and are not treated as independent physical couplings: on-shell observables (including the string S -matrix) are invariant under these redefinitions, and only the essential parameter combinations are fixed by amplitudes.

We work in the metric-dilaton sector, setting the remaining bosonic fields (e.g., the antisymmetric tensor and gauge fields) and their fluxes to zero and restricting to maximally symmetric internal spaces [27–29]. For this restricted class of backgrounds, the discarded fields are not sourced and can consistently be set to zero. This restricted setup suffices for the holographic conformal anomaly and a-theorem analyses below. Higher-derivative KK reductions and consistent truncations have been studied in closely related settings [19,20], as well as the heterotic four-derivative consistent truncation analyses [30,31].

With these conventions, the most general α' -corrected metric-dilaton action in D dimensions can be written as [23]

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} e^{-2\hat{\Phi}} \left(\hat{R} + 4(\nabla\hat{\Phi})^2 + \alpha' \left(\widehat{R^2}_{ABCD} + b_1 \hat{R}_{AB}^2 + b_2 \hat{R}^2 + b_3 \hat{R}^{AB} \nabla_A \hat{\Phi} \nabla_B \hat{\Phi} + b_4 \hat{R} (\nabla\hat{\Phi})^2 + b_5 \hat{R} \square \hat{\Phi} + b_6 (\square \hat{\Phi})^2 + b_7 (\nabla\hat{\Phi})^2 \square \hat{\Phi} + b_8 (\nabla\hat{\Phi})^4 \right) \right).$$

For the original theories with different coefficient combinations, we perform consistent KK reduction [19,22,29] with reduction *Ansatz* $d\hat{s}_D^2 = e^{2\alpha\phi} ds_d^2 + e^{2\beta\phi} d\Omega_n^2$, where α and β are arbitrary constants, the ds^2 denotes the metric of the d -dimensional spacetime, and $d\Omega_n^2$ is the metric of the internal n -dimensional space, which is maximally symmetric and characterized by the curvature parameter $\lambda = 0, \pm 1$ corresponding to toroidal, spherical, or hyperbolic geometries, respectively.

A. Einstein-dilaton-Gauss-Bonnet theory

Choosing the coefficient frame that gives the Gauss-Bonnet combination and vanishing nonlocal terms, we have the action in the form

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} e^{-2\hat{\Phi}} \left(\hat{R} + 4(\nabla\hat{\Phi})^2 - e^{\tilde{\lambda}\hat{\Phi}} \Lambda + \alpha' \left(\hat{\mathcal{L}}_{GB} + a_1 \hat{R}^{AB} \nabla_A \hat{\Phi} \nabla_B \hat{\Phi} + a_2 \hat{R} (\nabla\hat{\Phi})^2 + a_3 (\nabla\hat{\Phi})^2 \square \hat{\Phi} + a_4 (\nabla\hat{\Phi})^4 \right) \right) \quad (4)$$

with $\{a_i\}$ being undetermined coefficients and we include a dilaton potential term, consistent with effective string actions. Constant and dilaton-dressed vacuum energies at tree level [32,33] and at one loop in nonsupersymmetry heterotic models [34,35] establish that a Λ -term is admissible in the ten-dimensional heterotic action. One of the coefficients $\{a_i\}$ is prefixed due to the field redefinition relation and could be computed from string scattering amplitude. We need to cancel four coefficients after the KK reduction. However, the parameters α and β in the reduction *Ansatz* $d\hat{s}_D^2 = e^{2\alpha\phi} ds_d^2 + e^{2\beta\phi} d\Omega_n^2$ give us more freedom in the coefficients of the reduced theory. Upon the KK reduction, individual Horndeski terms arise. We put the details of emergence of the Horndeski terms in the Appendix C, where we used the identities $(\square\phi)^2 - \nabla_a \nabla_b \phi \nabla^a \nabla^b \phi = \mathcal{R}^{ab} \nabla_a \phi \nabla_b \phi + \nabla_a (\square\phi \nabla^a \phi - \nabla^a \nabla^b \phi \nabla_b \phi)$ and perform many integration by parts.

Once we fix the dimension d of the base/bulk manifold and the dimension n of the internal space, the parameters α and β are to be determined by matching the Einstein frame convention and cancellation of the coefficients fixed by the string S -matrix.

We found the corresponding five-dimensional theory given by

$$S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left(\mathcal{R} - \frac{1}{2} (\nabla\phi)^2 - e^{\lambda\phi} \Lambda + \tilde{\alpha} e^{-\gamma\phi} \mathcal{L}_{GB} \right). \quad (5)$$

B. Lovelock-Horndeski theory

We also consider another string effective theory giving rise to different lower-dimensional theories. Consider the following action:

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-\hat{g}} e^{-2\hat{\Phi}} \left(\hat{R} + 4(\nabla\hat{\Phi})^2 + \tilde{\alpha} \left(\hat{\mathcal{L}}_{GB} + a_1 \hat{R}^{\mu\nu} \nabla_\mu \hat{\Phi} \nabla_\nu \hat{\Phi} + a_2 \hat{R} (\nabla\hat{\Phi})^2 + a_3 (\nabla\hat{\Phi})^2 \square \hat{\Phi} + a_4 (\nabla\hat{\Phi})^4 \right) \right). \quad (6)$$

For the reduction *Ansatz* $d\hat{s}_D^2 = e^{2\alpha\phi} ds_d^2 + e^{2\beta\phi} d\Omega_n^2$ the variables $\alpha = 0$ and β are set to $n\beta = 2$ so that we can get rid of the exponential in front of the Lagrangian.

Imposing the flat internal space condition to eliminate the λ terms in last equation, we get the five-dimensional Lovelock-Horndeski action,

$$S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left(\mathcal{R} + 24(\nabla\phi)^2 + \alpha_1 \mathcal{L}_{GB} + \alpha_2 \mathcal{G}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \alpha_3 (\nabla\phi)^2 \square \phi + \alpha_4 (\nabla\phi)^4 \right), \quad (7)$$

where we have relabeled the coefficients by $\{\alpha_i\}_{i=1}^4$ and $\mathcal{G}^{\mu\nu}$ denotes the Einstein tensor. This is our desired form of lower-dimensional action, which leads to an exact AdS spacetime solution with linear dilaton. The metric equations of motion for theory (7) and its pure AdS solution are presented in the Appendix A.

III. SOLUTIONS TO LOWER-DIMENSIONAL THEORIES

A. Einstein-dilaton-Gauss-Bonnet theory

For the EdGB theory, there existed numerical black hole solutions representing stable asymptotically AdS black holes with scalar hair. Using standard boundary conditions

and suitable *Ansätze*, we solve numerically for black hole solutions in the effective theory (5). Similar setups have been explored in the literature [24,25], showing asymptotically AdS black holes with nontrivial scalar hair in dilatonic Einstein-Gauss-Bonnet gravity. The metric takes the form

$$ds_D^2 = -B(r)e^{-2\delta}dt^2 + \frac{dr^2}{B(r)} + r^2 h_{ij}dx^i dx^j \quad (8)$$

with asymptotic expansions

$$B(r) = \tilde{b}^2 r^2 - \frac{2\tilde{M}}{r^{\tilde{\mu}}}, \quad \phi(r) = \phi_0 + \frac{\phi_1}{r^{\tilde{\nu}}},$$

where $\tilde{b}^2$ is related to the AdS curvature scale and $\tilde{\mu}, \tilde{\nu}$ are scalar exponents with $\tilde{\mu} = \tilde{\nu} - 2$. In the effective potential picture, the dilaton evolves via the scalar equation of

motion. For $\lambda > 0$, the effective potential drives the dilaton to approach a finite constant ϕ_0 at infinity, preserving AdS asymptotics. Regularity at the horizon $r_H = 1$ is ensured by series expansion. Numerical integration via a shooting method determines the horizon and asymptotic data, including the dilaton values and gravitational mass, both in four and five dimensions [24,25]. The gravitational mass scales as $M_0 \propto r_H$ for fixed cosmological constant. Furthermore, the allowed parameter space in (γ, λ) is constrained by the AdS structure and a Breitenlohner-Freedman-type bound; in our case, it is governed by $\lambda \leq \gamma$.

B. Lovelock-Horndeski theory

Now we consider the Lovelock-Horndeski Theory. Since we have done KK reduction with $k = 0$ in the reduction *Ansatz*, the reduction takes on the form $X_{10} = AdS_5 \times T^5$.

The corresponding metric field equations of motion derived from theory (7) are

$$\begin{aligned} E_{\mu\nu} = & \mathcal{G}_{\mu\nu} + 24 \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right) \\ & - \alpha_1 \left(\frac{1}{2} (\mathcal{R}^2 - 4\mathcal{R}_{\gamma\delta} \mathcal{R}^{\gamma\delta} + \mathcal{R}_{\gamma\delta\lambda\sigma} \mathcal{R}^{\gamma\delta\lambda\sigma}) g_{\mu\nu} - 2\mathcal{R} \mathcal{R}_{\mu\nu} + 4\mathcal{R}_{\mu\gamma} \mathcal{R}_\nu^\gamma + 4\mathcal{R}_{\gamma\delta} \mathcal{R}_\mu^\gamma \delta_{\nu}^\delta - 2\mathcal{R}_{\mu\gamma\delta\lambda} \mathcal{R}_\nu^{\gamma\delta\lambda} \right) \\ & - \alpha_2 \left(\frac{1}{2} \partial_\mu \phi \partial_\nu \phi \mathcal{R} - 2\partial_\rho \phi \partial_{(\mu} \phi \mathcal{R}_{\nu)}^\rho - \partial_\rho \phi \partial_\sigma \phi \mathcal{R}_{\mu}^\rho \mathcal{R}_{\nu}^\sigma + (\nabla_\mu \nabla_\nu \phi) \square \phi + \frac{1}{2} \mathcal{G}_{\mu\nu} (\partial \phi)^2 \right. \\ & \left. - (\nabla_\mu \nabla^\rho \phi) (\nabla_\nu \nabla_\rho \phi) - g_{\mu\nu} \left(\frac{1}{2} (\square \phi)^2 - \frac{1}{2} (\nabla^\rho \nabla^\sigma \phi) (\nabla_\rho \nabla_\sigma \phi) - \partial_\rho \phi \partial_\sigma \phi \mathcal{R}^{\rho\sigma} \right) \right) \\ & + \alpha_3 \left(\nabla_\mu \phi \nabla_\nu \phi \square \phi + g_{\mu\nu} \nabla^\rho \phi \nabla^\sigma \phi \nabla_\rho \nabla_\sigma \phi - \frac{1}{2} \nabla_{(\mu} \phi \nabla_{\nu)} \nabla_\rho \phi \nabla^\rho \phi \right) \\ & + \alpha_4 \left(2\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right) (\nabla \phi)^2 = 0 \end{aligned} \quad (9)$$

and the scalar field equation is

$$E_\phi = \nabla^a \left(24 \nabla_a \phi + \alpha_2 G_{ab} \nabla^b \phi + 2\alpha_3 \nabla_a \phi \square \phi + 2\alpha_4 \nabla_a \phi (\nabla \phi)^2 \right) \quad (10)$$

$$\begin{aligned} & = 24 \square \phi^{(0)} + \alpha_2 \left(\mathcal{R}_{ab}^{(0)} \nabla^a \nabla^b \phi^{(0)} - \frac{1}{2} \mathcal{R}^{(0)} \square \phi^{(0)} \right) \\ & + \alpha_3 \left((\square \phi^{(0)})^2 - \mathcal{R}_{ab}^{(0)} \nabla^a \phi^{(0)} \nabla^b \phi^{(0)} - \nabla_a \nabla_b \phi^{(0)} \nabla^a \nabla^b \phi^{(0)} \right) \\ & + 2\alpha_4 \left((\nabla \phi^{(0)})^2 \square \phi^{(0)} + 2\nabla^a \nabla^b \phi^{(0)} \nabla_a \phi^{(0)} \nabla_b \phi^{(0)} \right) = 0. \end{aligned} \quad (11)$$

It is natural to consider the planar symmetric *Ansatz* for theory (7),

$$ds_{AdS_5}^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\vec{x}^2. \quad (12)$$

We take the following *Ansatz*:

$$F(r) = \frac{r^2}{\ell^2}, \quad \phi(r) = \chi \log r. \quad (13)$$

We found the following equations by substituting the functions back into the equations of motion, which are the pure AdS spacetime with linear-dilaton solution to the Lovelock-Horndeski theory,

$$\begin{aligned} 24\alpha_1 - 6\alpha_2\chi^2 + \alpha_4\chi^4 + 12\ell^2(2\chi^2 - 1) &= 0, \\ 12\ell^2 + 3\alpha_2 + 2\alpha_3\chi + \alpha_4\chi^2 &= 0. \end{aligned} \quad (14)$$

Previous works on Horndeski gravity with linear-dilaton profiles [36–39] have shown that both exact and numerical black hole solutions with scalar hair exist in such frameworks, with modified or enlarged $\{\alpha_i\}$ parameter space. Nonetheless, here we have shown that the Lovelock-Horndeski theory indeed contains a novel pure AdS solution with linear dilaton, thus validating the Fefferman-Graham expansion to be implemented.

IV. HOLOGRAPHIC CONFORMAL ANOMALY

The quantum effective action $W_{\text{CFT}}[g_{(0)}] \equiv -\log Z_{\text{CFT}}[g_{(0)}]$, defined as the generating functional of a CFT on a manifold M , depends critically on the boundary geometry. Although W_{CFT} is formally a functional of the boundary metric $g_{(0)}$, conformal invariance would classically constrain this dependence to the conformal class $[g_{(0)}]$. However, this symmetry is broken at the quantum level, as evidenced by the noninvariance of W_{CFT} under infinitesimal Weyl transformations $\delta g_{(0)ij} = 2\delta\sigma g_{(0)ij}$. The anomaly is captured by the transformation law [5]

$$\delta W_{\text{CFT}}[g_{(0)}] = \int_M d^d x \sqrt{\det g_{(0)}} \mathcal{A} \delta\sigma, \quad (15)$$

where $\mathcal{A}(x) \equiv \langle T^i_i \rangle$ is the trace anomaly density. For even-dimensional boundaries, this anomaly generically consists of A-type topological terms proportional to Euler densities and B-type Weyl invariants that are not topological [10]. To compute $\mathcal{A}$ holographically, we employ the Fefferman-Graham (FG) expansion [2,3] of the bulk metric in an asymptotically locally AdS $_{d+1}$ spacetime. In the FG gauge,

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{1}{\rho} g_{ij}(\rho, x) dx^i dx^j, \quad (16)$$

where ρ is the holographic radial coordinate ($\rho \rightarrow 0$ at the boundary), and ℓ is the AdS radius. The FG expansion for g_{ij} is given by [2]

$$g_{ij}(\rho, x) = g_{(0)ij}(x) + \rho g_{(1)ij}(x) + \rho^2 g_{(2)ij}(x) + \dots \quad (17)$$

with the inverse metric expanded as

$$\begin{aligned} g^{ij}(\rho, x) &= g^{ij}_{(0)}(x) - \rho g^{ij}_{(1)}(x) - \rho^2 \\ &\times \left(g^{ik}_{(1)}(x) g^j_{(1)k}(x) - g^{ij}_{(2)}(x) \right) \\ &+ \dots \end{aligned} \quad (18)$$

The bulk curvature tensors decompose accordingly. The Ricci tensor $\mathcal{R}_{\mu\nu}$ separates into boundary curvature components and ρ -dependent corrections. For even d , the expansion yields

$$\begin{aligned} \mathcal{R}_{ij} &= R_{ij} - \frac{2\rho}{\ell^2} g''_{ij} + \frac{1}{\ell^2} g^{lk} g'_{lk} g_{ij} + \frac{2\rho}{\ell^2} g^{kl} g'_{lj} g'_{ki} \\ &\quad - \frac{\rho}{\ell^2} g^{lk} g'_{lk} g'_{ij} + \frac{d-2}{\ell^2} g'_{ij} - \frac{d}{\ell^2} \frac{1}{\rho} g_{ij}, \\ \mathcal{R}_{\rho\rho} &= -\frac{d}{4\rho^2} - \frac{1}{2} g^{ij} g'_{ij} + \frac{1}{4} g^{ik} g^{jl} g'_{ij} g'_{kl}, \end{aligned} \quad (19)$$

and similar relations hold for the bulk Riemann tensor,

$$\begin{aligned} \mathcal{R}_{ijkl} &= \frac{1}{\rho} R_{ijkl} - \frac{1}{\rho^2 \ell^2} \left((g_{kl} - \rho g'_{jl})(g_{ij} - \rho g'_{ik}) \right. \\ &\quad \left. - (g_{jk} - \rho g'_{jk})(g_{il} - \rho g'_{il}) \right), \\ \mathcal{R}_{i\rho j\rho} &= -\frac{1}{4\rho^3} g_{ij} + \frac{1}{4\rho} g^{kl} g'_{ki} g'_{lj} - \frac{1}{2\rho} g''_{ij}, \end{aligned} \quad (20)$$

and scalar curvature

$$\begin{aligned} \mathcal{R} &= \frac{-d(d+1)}{\ell^2} + \rho R + \frac{2(d-1)\rho}{\ell^2} g^{ij} g'_{ij} + \frac{3\rho^2}{\ell^2} g^{ij} g^{kl} g'_{ik} g'_{jl} \\ &\quad - \frac{4\rho^2}{\ell^2} g^{ij} g'_{ij} - \frac{\rho^2}{\ell^2 g^{ij}} g^{kl} g'_{ij} g'_{kl}. \end{aligned} \quad (21)$$

The boundary Ricci tensor R_{ij} , derived from $g_{(0)ij}$, expands as

$$R_{ij} = R_{(0)ij} + \rho R_{(1)ij} + \mathcal{O}(\rho^2). \quad (22)$$

The first-order correction $R_{(1)}$ is expressed in terms of $g_{(1)kl}$ via

$$\begin{aligned} R_{(1)ij} &= -\frac{1}{2} \nabla_i \nabla_j \left(g^{kl}_{(0)} g_{(1)kl} \right) - \frac{1}{2} \nabla^k \nabla_k g_{(1)ij} \\ &\quad + g^{kl}_{(0)} \nabla_k \nabla_{(i} g_{j)l}, \end{aligned} \quad (23)$$

where ∇_i denotes the Levi-Civita connection associated with $g_{(0)ij}$. Taking the trace gives

$$R_{(1)} = g^{ij}_{(0)} R_{(1)ij} = -\square \text{Tr}\{g_{(1)}\} + \nabla^i \nabla^j g_{(1)ij} \quad (24)$$

with $\square = \nabla^i \nabla_i$. The leading term $\sqrt{-g_{(0)}} R_{(0)}$ contributes only a total derivative, reflecting its topological origin [11].

For theories that assume usual dilaton profile, the FG expansion of the bulk scalar field ϕ dual to a marginal boundary operator is given by

$$\phi(\rho, x) = \phi_{(0)}(x) + \rho\phi_{(1)}(x) + \rho^2\phi_{(2)}(x) + \mathcal{O}(\rho^3). \quad (25)$$

For Lovelock-Horndeski theory in the critical case, thus possessing linear dilaton [36,40,41], we need to consider the logarithmic term for ϕ , so it expands as [42]

$$\begin{aligned} \phi(\rho, x) = & \phi_s \log \rho + \phi_{(0)}(x) + \rho\phi_{(1)}(x) \\ & + \rho^2\phi_{(2)}(x) + \mathcal{O}(\rho^3). \end{aligned}$$

Substituting the FG expansions into the bulk action and integrating radially up to a cutoff $\rho = \epsilon$, one isolates divergences using heat kernel techniques [1,7]. The renormalized action acquires a logarithmic term,

$$\begin{aligned} S = & \frac{1}{16\pi G_5} \int d^4x \int d\rho \sqrt{-g} \mathcal{L}(g, \phi) \\ = & \frac{1}{16\pi G_5} \int d^4x \sqrt{-g_{(0)}} \int_\epsilon d\rho \left(\dots + \frac{\mathcal{A}}{\rho} + \dots \right) \\ = & \frac{1}{16\pi G_5} \int d^4x \sqrt{-g_{(0)}} \left(\dots + \mathcal{A} \log \epsilon + \dots \right), \quad (26) \end{aligned}$$

from which the conformal anomaly $\mathcal{A} = \langle T_i^i \rangle$ on the boundary M_d can be extracted.

A. Holographic conformal anomaly for Einstein-dilaton-Gauss-Bonnet theory

We consider that the EdGB theory with regular scalar field stabilizes asymptotically, and Lovelock-Horndeski theories that possess both linear-dilaton scalar field, which diverges asymptotically, and a usual scalar field.

For the EdGB theory with regular scalar field profile (5), we obtain the anomalous action $\mathcal{A}$ for theory (5) in four dimensions, and the anomalous action takes the form

$$\begin{aligned} \ell^{-1} \mathcal{A} = & \left(\alpha R_{(0)}^2 - 4\alpha R_{(0)ab} R_{(0)}^{ab} + \alpha R_{(0)abcd} R_{(0)}^{abcd} \right) e^{-\gamma\phi_{(0)}} - \left(\frac{108\alpha\gamma}{\ell^4} e^{-\gamma\phi_{(0)}} + \lambda\Lambda e^{\lambda\phi_{(0)}} \right) \phi_{(2)} \\ & + \frac{4\alpha}{\ell^2} \left(R_{(0)}^{ab} g_{(1)ab} - \frac{1}{2} R_{(0)} g_{(0)}^{ab} g_{(1)ab} \right) e^{-\gamma\phi_{(0)}} + \frac{12\alpha\gamma\phi_{(1)} R_{(0)}}{\ell^2} e^{-\gamma\phi_{(0)}} \\ & + \left(\frac{4}{\ell^2} + \frac{54\alpha\gamma^2}{\ell^4} e^{-\gamma\phi_{(0)}} - \frac{\lambda^2\Lambda}{2} e^{\lambda\phi_{(0)}} \right) \phi_{(1)}^2 + 2\nabla_a \phi_{(0)} \nabla^a \phi_{(1)} - g_{(1)ab} \nabla^a \phi_{(0)} \nabla^b \phi_{(0)} \\ & + \left(-\frac{6}{\ell^2} - \frac{18\alpha}{\ell^4} e^{-\gamma\phi_{(0)}} - \frac{\Lambda}{2} e^{\lambda\phi_{(0)}} \right) g_{(2)a}^a + \left(\frac{18\alpha\gamma}{\ell^4} e^{-\gamma\phi_{(0)}} - \frac{\lambda\Lambda}{2} e^{\lambda\phi_{(0)}} \right) \phi_{(1)} g_{(0)}^{ab} g_{(1)ab} \\ & + \left(\frac{2}{\ell^2} + \frac{7\alpha}{\ell^4} e^{-\gamma\phi_{(0)}} + \frac{\Lambda}{4} e^{\lambda\phi_{(0)}} \right) g_{(0)}^{ca} g_{(0)}^{db} g_{(1)cd} g_{(1)ab} + \left(\frac{-1}{2\ell^2} - \frac{5\alpha}{2\ell^4} e^{-\gamma\phi_{(0)}} - \frac{\Lambda}{8} e^{\lambda\phi_{(0)}} \right) \left(g_{(0)}^{ab} g_{(1)ab} \right)^2. \quad (27) \end{aligned}$$

Variation of the on-shell anomalous action $\mathcal{A}$ with respect to $\phi_{(2)}$ gives

$$\begin{aligned} 0 = & \frac{108\alpha\gamma}{\ell^4} e^{-\gamma\phi_{(0)}} + \lambda\Lambda e^{\lambda\phi_{(0)}} \\ \Rightarrow \Lambda = & -\frac{108\alpha\gamma}{\ell^4} \frac{1}{\lambda} e^{-(\gamma+\lambda)\phi_{(0)}}. \end{aligned}$$

Then the variation $\frac{\delta\mathcal{A}}{\delta g_{(2)a}^a}$ give us

$$\begin{aligned} 0 = & \frac{6}{\ell^2} + \frac{18\alpha}{\ell^4} e^{-\gamma\phi_{(0)}} + \frac{\Lambda}{2} e^{\lambda\phi_{(0)}} \\ \Rightarrow \frac{1}{\ell^2} = & \left(\frac{9\alpha\eta}{\ell^4} - \frac{3\alpha}{\ell^4} \right) e^{-\gamma\phi_{(0)}} = (3\eta - 1) \frac{3\alpha}{\ell^4} e^{-\gamma\phi_{(0)}}. \end{aligned}$$

Defining $\eta \equiv \gamma/\lambda$, we find $e^{-\gamma\phi_{(0)}} = \frac{\ell^2}{3\alpha(3\eta-1)}$. With $\eta = 1$, the effective AdS scale becomes $-3/\alpha$, yielding $\Lambda = -\frac{6}{\ell^2}$.

Further variations with respect to $g_{ab}^{(1)}$ and $\phi_{(1)}$ yield

$$\begin{aligned}\frac{\delta\mathcal{A}}{\delta\phi_{(1)}} &= \left(\frac{18\alpha\gamma}{\ell^4}e^{-\gamma\phi_{(0)}} - \frac{\lambda\Lambda}{2}e^{\lambda\phi_{(0)}}\right)g_{(1)a}^a + \left(\frac{4}{\ell^2} + \frac{54\alpha\gamma^2}{\ell^4}e^{-\gamma\phi_{(0)}} - \frac{\lambda^2\Lambda}{2}e^{\lambda\phi_{(0)}}\right)\phi_{(1)} + \frac{12\alpha\gamma}{\ell^2}R_{(0)}e^{-\gamma\phi_{(0)}} \\ \frac{\delta\mathcal{A}}{\delta g_{(1)ab}} &= 2\left(\frac{-1}{2\ell^2} - \frac{5\alpha}{2\ell^4}e^{-\gamma\phi_{(0)}} + \frac{\Lambda}{8}e^{\lambda\phi_{(0)}}\right)g_{(0)}^{ab}g_{(0)}^{cd}g_{(1)cd} + \left(\frac{2}{\ell^2} + \frac{7\alpha}{\ell^4}e^{-\gamma\phi_{(0)}} + \frac{\Lambda}{4}e^{\lambda\phi_{(0)}}\right)g_{(0)}^{ca}g_{(0)}^{db}g_{(1)cd} \\ &\quad + \frac{4\alpha}{\ell^2}\left(R_{(0)}^{ab} - \frac{1}{2}R_{(0)}g_{(0)}^{ab}\right)e^{-\gamma\phi_{(0)}} + \left(\frac{18\alpha\gamma}{\ell^4}e^{-\gamma\phi_{(0)}} - \frac{\lambda\Lambda}{2}e^{\lambda\phi_{(0)}}\right)\phi_{(1)}g_{(0)}^{ab}.\end{aligned}$$

Solving these yields $g^{(1)}$ and $g^{(2)}$ in terms of $g^{(0)}$ and $R^{(0)}$. Substituting these back into the anomaly expression leads to

$$\begin{aligned}\mathcal{A} &= C_1 R_{(0)ab}R^{(0)ab} + C_2 R_{(0)abcd}R^{abcd} + C_3 R_{(0)}^2 + C_4 \left(\nabla\phi_{(0)}\right)^2 + C_5 \square R_{(0)} + C_6 R_{(0)}\square\phi \\ &\quad + C_7 \nabla^a\phi\nabla_a R_{(0)} + C_8 R_{(0)}\left(\nabla\phi_{(0)}\right)^2 + C_{10}\left(\nabla\phi\right)^2\square\phi + C_9(\square\phi)^2 + C_{11}\nabla_a\phi\nabla^b\nabla_b\nabla^a\phi \\ &\quad + C_{12}\nabla^b\nabla_b\nabla^a\nabla_a\phi + C_{13}R_{(0)ab}\nabla^a\phi_{(0)}\nabla^b\phi_{(0)} + C_{14}\left(\left(\nabla\phi\right)^2\right)^2 + C_{15}\nabla^a\phi\nabla_b\nabla_a\phi\nabla^b\phi \\ &\quad + C_{16}R_{(0)}^{ab}\nabla_b\nabla_a\phi + C_{17}(\nabla_b\nabla_a\phi)(\nabla^b\nabla^a\phi),\end{aligned}\tag{28}$$

where the coefficients C_i are constants given in Appendix B. We deduced the central charges for this theory as

$$\begin{aligned}a &= \frac{\ell^3}{8\lambda(9\gamma - \lambda)}\left(27\gamma^2 - 66\gamma\lambda + 19\lambda^2\right), \\ c &= \frac{\ell^3}{24\lambda(9\gamma - \lambda)}\left(81\gamma^2 - 126\gamma\lambda + 49\lambda^2\right).\end{aligned}\tag{29}$$

The closed-form expressions for $a(\gamma, \lambda)$ and $c(\gamma, \lambda)$ supply, for the first time, to the best of our knowledge, the full Euler- and Weyl-anomaly coefficients of a five-dimensional EdGB theory with two independent couplings. They also allow one to delineate, analytically, the parameter subregion in which $a, c > 0$; hence, the dual four-dimensional theory satisfies the standard positivity bounds. These results open the way to systematic tests of causality, entanglement-entropy inequalities, and higher-spin constraints in a two-parameter family of strongly coupled CFT_4 duals.

B. Holographic conformal anomaly for Lovelock-Horndeski theory

The full anomalous action for Lovelock-Horndeski theory (7) contains the linear-dilaton solution where we consider the boundary theory to be a generalized conformal brane [42]. We note that the bulk gravity is still exactly AdS gravity, while the boundary field theory breaks some of the conformal symmetries [43].

We consider the Lovelock-Horndeski theory (7) of the critical case, that is, the case with linear-dilaton solution. The Fefferman-Graham expansion is now given by Eqs. (16), (17), (26),

$$\begin{aligned}ds^2 &= \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{1}{\rho}g_{ij}(\rho, x)dx^i dx^j, \\ g_{ij}(\rho, x) &= g_{(0)ij}(x) + \rho g_{(1)ij}(x) + \rho^2 g_{(2)ij}(x) + \dots, \\ \phi(\rho, x) &= \phi_s \log \rho + \rho\phi_{(1)}(x) + \rho^2\phi_{(2)}(x) + \dots\end{aligned}\tag{30}$$

We found the anomalous action,

$$\begin{aligned}
\ell^{-1}\mathcal{A} = & \alpha_1 \left(R_{(0)}^2 - 4R_{(0)ab}R_{(0)}^{ab} + R_{(0)abcd}R_{(0)}^{abcd} \right) + 4B_1\phi_s\phi_{(2)} + B_1(\phi_{(1)})^2 - \frac{4\beta\phi_s}{\ell^2}\phi_{(1)}R_{(0)} \\
& + \frac{96\phi_s}{\ell^2}\phi_{(1)}g_{(1)a}^a + B_2g_{(1)ab}\left(R_{(0)}^{ab} - \frac{1}{2}R_{(0)}g_{(0)}^{ab}\right) + B_3g_{(1)}^{ab}g_{(1)ab} + B_4\left(g_{(1)a}^a\right)^2 + B_5g_{(2)a}^a \\
& + \frac{12}{\ell^2}B_1\nabla_a\phi_{(0)}\nabla^a\phi_{(1)} + B_6g_{(1)ab}\nabla^a\phi_{(0)}\nabla^b\phi_{(0)} + \left(12 + \frac{\alpha_2}{\ell^2}\right)g_{(1)a}^a\left(\nabla\phi_{(0)}\right)^2 + \alpha_3\left(\nabla\phi_{(0)}\right)^2\Box\phi_{(0)} \\
& + \alpha_2\left(R_{(0)}^{ab} - \frac{1}{2}R_{(0)}g_{(0)}^{ab}\right)\nabla_a\phi_{(0)}\nabla_b\phi_{(0)} + \alpha_4\left(\nabla\phi_{(0)}\right)^4,
\end{aligned} \tag{31}$$

where the coefficients B_i and the variations are

$$\begin{aligned}
B_1 &= \frac{24}{\ell^4}\left(4\ell^2 + \beta\right), & B_2 &= \frac{2}{\ell^2}\left(2\alpha + \beta\phi_s^2\right) - 1, \\
B_3 &= \ell^2(2 - 24\phi_s^2) - 2\alpha + 4\beta\phi_s^2, \\
B_4 &= \frac{-\ell^2}{2}(1 - 24\phi_s^2) - \alpha - \beta\phi_s^2, \\
B_5 &= 6\left(\ell^2(8\phi_s^2 - 1) + 2(\alpha - \beta\phi_s^2)\right), \\
B_6 &= -4\left(6 + \frac{\beta}{\ell^2}\right).
\end{aligned} \tag{32}$$

Variation with respect to $g^{(2)}$ yields $\beta = -4\ell^2$, while variation with respect to $\phi^{(1)}$ gives

$$g^{(1)a}{}_a = \frac{-\ell^2}{6}R^{(0)}. \tag{33}$$

Then varying the action with respect to $g_{ab}^{(1)}$, we find the equation

$$\begin{aligned}
\frac{\delta\mathcal{A}}{\delta g_{ab}^{(1)}} = & \left(1 - 40\phi_s^2\right)R^{(0)ab} - \frac{1}{2}\left(1 - 40\phi_s^2\right)R^{(0)}g^{(0)ab} \\
& + \frac{2}{\ell^2}\left(1 - 24\phi_s^2\right)g^{(1)ab} - \frac{2}{\ell^2}\left(1 - 24\phi_s^2\right)g^{(1)a}{}_ag^{(0)ab} \\
& + 8\left(\nabla\phi^{(0)}\right)^2g^{(0)ab} - 8\nabla^a\phi^{(0)}\nabla^b\phi^{(0)} \\
& + \frac{96}{\ell^2}\phi_s\phi^{(1)}g^{(0)} = 0.
\end{aligned} \tag{34}$$

Contracting this equation with $g_{ab}^{(0)}$, we get

$$\phi^{(1)} = -\frac{\ell^2}{24}\phi_sR^{(0)} - \frac{\ell^2}{16\phi_s}\left(\nabla\phi^{(0)}\right)^2. \tag{35}$$

Substituting the expressions for $g_{ab}^{(1)}$ and $\phi^{(1)}$ into Eq. (15), we find

$$\begin{aligned}
g^{(1)ab} = & \ell^2\left(\left(\frac{1}{6} - 8\chi^2\right)R^{(0)}g^{(0)ab} - \left(1 - 40\chi^2\right)R^{(0)ab} + 8\nabla^a\phi^{(0)}\nabla^b\phi^{(0)}\right. \\
& \left. - 2\left(\nabla\phi^{(0)}\right)^2g^{(0)ab}\right)/2\left(1 - 24\chi\right).
\end{aligned} \tag{36}$$

After a lengthy algebra, we obtain the anomalous action,

$$\begin{aligned} \mathcal{A} = & \alpha_1 \left(R_{(0)}^2 - 4R_{(0)ab}R_{(0)}^{ab} + R_{(0)abcd}R_{(0)}^{abcd} \right) + 4B_1\phi_s\phi_{(2)} + B_1\left(\phi_{(1)}\right)^2 - \frac{4\beta\phi_s}{\ell^2}\phi_{(1)}R_{(0)} \\ & + \frac{96\phi_s}{\ell^2}\phi_{(1)}g_{(1)a}^a + B_2g_{(1)ab}\left(R_{(0)}^{ab} - \frac{1}{2}R_{(0)}g_{(0)}^{ab}\right) + B_3g_{(1)}^{ab}g_{(1)ab} + B_4\left(g_{(1)a}^a\right)^2 + B_5g_{(2)a}^a \\ & + \frac{12}{\ell^2}B_1\nabla_a\phi_{(0)}\nabla^a\phi_{(1)} + B_6g_{(1)ab}\nabla^a\phi_{(0)}\nabla^b\phi_{(0)} + \left(12 + \frac{\alpha_2}{\ell^2}\right)g_{(1)a}^a\left(\nabla\phi_{(0)}\right)^2 + \alpha_3\left(\nabla\phi_{(0)}\right)^2\Box\phi_{(0)} \\ & + \alpha_2\left(R_{(0)}^{ab} - \frac{1}{2}R_{(0)}g_{(0)}^{ab}\right)\nabla_a\phi_{(0)}\nabla_b\phi_{(0)} + \alpha_4\left(\nabla\phi_{(0)}\right)^4, \end{aligned} \quad (37)$$

where the coefficients B_i and the variations are listed above. Finally, we obtain the full holographic conformal anomaly in four dimensions,

$$\begin{aligned} \mathcal{A}_{\text{grav}} = & -\frac{\ell^2(7 - 320\chi^2 + 3840\chi^4)}{12(24\chi^2 - 1)}R_{(0)}^2 + \frac{\ell^2(9 - 400\chi^2 + 4672\chi^4)}{4(24\chi^2 - 1)}R_{(0)ab}R_{(0)}^{ab} \\ & + \frac{\ell^2}{2}\left(1 - 16\chi^2\right)R_{(0)abcd}R_{(0)}^{abcd}, \\ \mathcal{A}_m = & \frac{16\chi^2\ell^2}{24\chi^2 - 1}\left(4R_{(0)ab} - R_{(0)}g_{(0)ab}\right)\nabla^a\phi_{(0)}\nabla^b\phi_{(0)} + \alpha_3\left(\nabla\phi_{(0)}\right)^2\left(\Box\phi_{(0)}\right)^2 + \left(\alpha_4 + \frac{16\ell^2}{24\chi^2 - 1}\right)\left(\nabla\phi_{(0)}\right)^4, \end{aligned} \quad (38)$$

where the gravitational and matter sectors are separated as above. The central charges for this theory are

$$\begin{aligned} a = & \frac{\ell^3}{8\left(24\left(\phi_s^2 - 1\right)\right)}\left(5 - 240\phi_s^2 + 3136\phi_s^4\right), \\ c = & \frac{\ell^3}{8\left(24\left(\phi_s^2 - 1\right)\right)}\left(1 - 80\phi_s^2 + 1600\phi_s^4\right), \end{aligned} \quad (39)$$

and new b -type charges arise $b_1 = \frac{16\ell^3\phi_s^4}{3(24(\phi_s^2-1))}$ and $b_2 = \frac{8\ell^3\phi_s^3}{3(24(\phi_s^2-1))}$, where we have defined that $H_1 = \beta_1 R_{(0)}^2 + \beta_2 \Box R$, $H_2 = \Box R_{(0)}$ with coefficients being $\beta_1 = 1, \beta_2 = 0$. The linear-dilaton asymptotics break the full conformal invariance of the would-be boundary theory and lead instead to a generalized conformal structure [42]. In our critical Lovelock-Horndeski solution, the metric remains AdS_5 , while the nontrivial dilaton background breaks part of the AdS isometry; correspondingly, the boundary theory exhibits Poincaré invariance and scale invariance, but not the full conformal group [44,45]. This structure is precisely what allows additional (“ b -type”) invariants to appear in the trace anomaly beyond the standard Euler and Weyl terms.

V. HOLOGRAPHIC A-THEOREM

We construct a functions, which decrease monotonically from the UV to IR limit. Consider the *Ansatz* [46]

$$\begin{aligned} ds_5^2 = & dr^2 + e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) \\ \phi = & \phi(r). \end{aligned} \quad (40)$$

The AdS vacuum is given by $A(r) = r/\ell$, with ℓ being the AdS radius. The function $A(r)$ describes the flow to the AdS “fixed” point. For asymptotically AdS domain walls, we adopt the convention that the asymptotic region is located at $r \rightarrow \infty$. The relation between r and ρ (with flat boundary) is given by $r = -\frac{\ell}{2}\log \rho$.

Assuming the presence of a generic matter stress-energy tensor $T_{\mu\nu}^{\text{matter}}$, minimally coupled to Horndeski gravity, the Einstein equations become

$$E_{\mu\nu} = T_{\mu\nu}^{\text{matter}}.$$

Inserting the domain wall *Ansatz* and imposing the null energy condition (NEC) on the matter sector yields constraints on $A''(r)$, leading to conditions for monotonicity of $a(r)$.

For theory (7) with linear-dilaton solution, we insert the *Ansatz* (40) into the scalar equation and find that

$$e^{2A(r)}\phi'(r)\left(24 + 6\alpha_2 A'(r)^2 + \alpha_4 \phi'(r)^2 + \alpha_3 \phi''(r) + 4\alpha_3 A'(r)\phi'(r)\right) = C. \quad (41)$$

With the help of asymptotically $A(r) \rightarrow r/\tilde{L}$, focusing on the $\alpha_3 = 0$ slice, the scalar equation of motion gives

$$\phi'(r)\left(24 + 6\alpha_2 A'(r)^2 + \alpha_4 \phi'(r)^2\right) = 0. \quad (42)$$

Choosing suitable parameters, we found the constraint condition $A'(r)^2 + \phi'(r)^2 = \frac{1}{\ell^2}$. The null energy condition gives

$$\begin{aligned} (T^{\text{matter}})_t^t + (T^{\text{matter}})_r^r &= E_r - E_t^t \\ &= \left(32A'(r)^2 - \frac{24}{\ell^2} - 8\ell^2 A'(r)^4\right) \\ &\quad + \left(1 + (-6\ell^2 + 4\alpha_1)A'(r)^2\right) \\ &\quad \times A''(r) \geq 0. \end{aligned} \quad (43)$$

Imposing the asymptotic boundary condition that $A(r) \rightarrow r/\tilde{L}$, we notice that the a charge contains another constant χ . The scalar field ϕ can indeed also run along the RG flow. Thus, we propose a flow function

$$a(r) = \frac{\left(5 - 240\phi'(r)^2 + 3136\phi'(r)^4\right)}{8A'(r)^3\left(24\left(\phi'(r)^2 - 1\right)\right)}. \quad (44)$$

Then $a'(r) = (1915263 - 4379792\ell^2 A'(r)^2 + 3067136\ell^4 \times A'(r)^4 - 602112\ell^6 A'(r)^6)A''(r)/8A'(r)^4(49 - 48\ell^2 A'(r)^2)^2$ with the above constraint, we find that $a'(r) > 0$; thus, we established monotonicity of $a(r)$.

VI. CONCLUSION

We have presented a systematic framework for deriving scalar-tensor theories via dimensional reduction of string effective actions. The resulting lower-dimensional theories encompass a broad class of models with nonminimal scalar couplings, including Galileon theories and Einstein-dilaton-Gauss-Bonnet gravity as special cases. This approach provides a unified gravitational and holographic perspective on modified theories of gravity with higher-curvature corrections. We have shown that a single α' -corrected heterotic effective action, when combined with consistent Kaluza-Klein reductions and the standard field redefinition (coefficient-frame) freedom at order α' , gives rise to two distinct KK reduced five-dimensional branches of scalar-tensor theories: an EdGB branch and a Lovelock-Horndeski branch. It provides the first top-down arena in which distinct UV branches both exhibit the AdS_5 geometry. A further trivial S_1 compactification reduces the

EdGB theory to the familiar four-dimensional one that has been constrained by gravitational-wave observations, opening new tests for AdS/CFT and gravitational-wave phenomenology [47,48].

We computed the full holographic conformal anomalies for scalar-tensor theories with dilaton couplings, and for the first time to our knowledge, for a theory containing both Gauss-Bonnet and Horndeski-type interactions. Our closed-form anomaly $a(\lambda, \gamma)$ and $c(\lambda, \gamma)$ convert the universal collider bound $0 < c/a < 3/2$ into an explicit admissible region in the Gauss-Bonnet-Horndeski coupling plane. Thus, the anomaly not only anchors the dual CFT but also supplies model-independent priors for phenomenological effective field theory fits, excluding (λ, γ) values incompatible with fundamental causality and positivity.

Furthermore, we established holographic a theorems for Lovelock-Horndeski theory. Our results confirm the robustness of AdS/CFT in the presence of nonminimal scalar couplings and higher-derivative corrections. These results suggest a number of promising directions for future investigation. One natural extension involves including higher-order Lovelock terms in the dimensional reduction framework to capture broader classes of higher-curvature corrections. It would also be interesting to explore RG flows with non-AdS asymptotics and examine whether the structure of the conformal anomaly generalizes in such settings. For instance, the thermodynamic and entanglement properties of the dual CFTs, particularly those influenced by nontrivial dilaton profiles, deserve further attention. Finally, the presence of linear-dilaton configurations opens the door to potential applications in cosmology, where such scalars could play a role in early-Universe dynamics or dark energy models.

ACKNOWLEDGMENTS

The authors would like to thank Jorge Noronha, Helvi Witek, and Freddy Pardoe for many useful discussions.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: EQUATIONS OF MOTION

We now plug in the *Ansatz*

$$\begin{aligned} ds_{\text{AdS}_5}^2 &= -N(r)^2 F(r) dt^2 + \frac{dr^2}{F(r)} + r^2 d\vec{x}^2 \\ F(r) &= \frac{r^2}{\ell^2}, \quad \phi(r) = \chi \log r \end{aligned} \quad (\text{A1})$$

into the covariant equations of motion (9) and (10) in Sec. III. The full covariant equations of motion are given by

$$0 = \frac{F(r)N(r)}{4r^3} \left(F(r) \left(2r(24r^2\phi'(r)^2 - 6) \right) + F'(r) \left(24\alpha_1 - 9r^2\alpha_2\phi'(r)^2 - 2r^3\alpha_3\phi'(r)^3 \right) \right. \\ \left. + 2rF^2(r)\phi'(r)(r^2\alpha_4\phi'(r)^3 - 6r\alpha_2\phi''(r) - \phi'(r)(3\alpha_2 + 2r^2\alpha_3\phi''(r))) \right) - 6r^2F'(r) \quad (\text{A2})$$

$$0 = \left(6N(r) \left(2rF(r) + F'(r)(r^2 - 4\alpha_1F(r)) \right) + 12F(r)N'(r) \left(r^2 - 4\alpha_1F(r) \right) \right. \\ \left. + rF(r)\phi'(r)^2 \left(18r\alpha_2F(r)N'(r) + N(r)(18\alpha_2F(r) + r(48r + 9\alpha_2F'(r))) \right) \right. \\ \left. + 2r^2\alpha_3F(r)\phi'(r)^3 \left(rN(r)F'(r) + 2F(r)(3N(r) + rN'(r)) \right) + 6r^3\alpha_4F(r)^2N(r)\phi'(r)^4 \right) \frac{1}{4r^3F(r)N(r)} \quad (\text{A3})$$

$$0 = \left(6r^2F'(r)N'(r) + N(r)(8rF'(r) + F'(r)^2(r^2\alpha_2\phi'(r)^2 - 8\alpha_1) + 2r^2F''(r)) \right. \\ \left. + F(r)(N'(r)(8r + 5F'(r)(r^2\alpha_2\phi'(r)^2 - 8\alpha_1)) + N(r)(4 + 2r^2\alpha_3F'(r)\phi'(r)^3 - 8\alpha_1F''(r)) \right. \\ \left. + r\phi'(r)^2(8\alpha_2F'(r) + r\alpha_2F''(r) - 48r) + 2r^2\alpha_2F'(r)\phi'(r)\phi''(r)) \right. \\ \left. + 4r^2N''(r) + 2F(r)^2 \left(N''(r)(\alpha_2r^2\phi'(r)^2 - 8\alpha_1) + 2r\alpha_2N'(r)\phi'(r)(\phi'(r) + r\phi''(r)) \right. \right. \\ \left. \left. + N(r)\phi'(r)(4r\alpha_2\phi''(r) - r^2\alpha_4\phi'(r)^3 + \phi'(r)(\alpha_2 + 2r^2\alpha_3\phi''(r))) \right) \right) \frac{1}{4N(r)}, \quad (\text{A4})$$

together with the scalar equation of motion given as

$$0 = \frac{1}{2r^3N(r)^2} \left(\left(rF(r) \left(rN(r)N'(r)(48r + F'(r)(15\alpha_2 + 5r\alpha_3\phi'(r))) \right) \right. \right. \\ \left. \left. + r^2N(r)^2F'(r) \left(48r + F'(r)(3\alpha_2 + r\alpha_3\phi'(r)) \right) + \left(F'(r)(18\alpha_2 + r\phi'(r)(15\alpha_3 + 8r\alpha_4\phi'(r))) \right. \right. \right. \\ \left. \left. + rF''(r)(3\alpha_2 + r\alpha_3\phi'(r)) + 144r \right) N(r)^2 + 2 \left(6N(r)^2 \left(\frac{\alpha_2}{2} + r\phi'(r)(\alpha_3 + \alpha_4r\phi'(r)) \right) \right. \right. \\ \left. \left. + rN(r) \left(N'(r)(9\alpha_2 + 2r\phi'(r)(3\alpha_3 + \alpha_4r\phi'(r))) + r(3\alpha_2 + \alpha_3r\phi'(r)N''(r)) \right) F(r)^2 \right) \phi(r) \right. \\ \left. + rF(r)N(r)\phi''(r) \left(4rF(r)N'(r) \left(\frac{3\alpha_2}{2} + \alpha_3r\phi'(r) \right) + N(r) \left(4(48r + F'(r)(3\alpha_2 + 2\alpha_3r\phi'(r)) \right. \right. \right. \right. \\ \left. \left. \left. + 12F(r) \left(\frac{\alpha_2}{2} + r\phi'(r)(\alpha_3 + \alpha_4r\phi'(r)) \right) \right) \right) \right) \right) \quad (\text{A5})$$

with prime denoting derivative with respect to the radial coordinate. Then we consistently set the lapse function $N(r) = 1$ and present them in a clean way so that every term is grouped after $\{\alpha_i\}$'s. The equations of motion in terms of $F(r)$ and $\phi(r)$ and their derivatives are then found as

$$0 = \frac{6\alpha_1F(r)^2F'(r)}{r^3} + \frac{1}{2}\alpha_4F(r)^3\phi'(r)^4 + \frac{F(r)}{4r^3} \left(-6r^2F'(r) + 12rF(r)(-1 + 4r^2\phi'(r)^2) \right) \\ + \frac{\alpha_2F(r)}{4r^3} \left(-6rF(r)^2\phi'(r)^2 - 9r^2F(r)F'(r)\phi'(r)^2 - 12r^2F(r)^2\phi'(r)\phi''(r) \right) \\ - \frac{\alpha_3F(r)}{4r^3} \left(2r^3F(r)F'(r)\phi'(r)^3 + 4r^3F(r)^2\phi'(r)^2\phi''(r) \right) \quad (\text{A6})$$

$$\begin{aligned}
0 = & -\frac{1}{2}r^2\alpha_4F(r)^2\phi'(r)^4 + \frac{1}{4}\alpha_1\left(-8F'(r)^2 - 8F(r)F''(r)\right) \\
& + \frac{1}{4}\left(4F(r) + 8rF'(r) - 48r^2F(r)\phi'(r)^2 + 2r^2F''(r)\right) + \frac{1}{4}\alpha_2\left(2F(r)^2\phi'(r)^2 + 8rF(r)F'(r)\phi'(r)^2\right. \\
& + r^2F(r)\phi'(r)^2F''(r) + 8rF(r)^2\phi'(r)\phi''(r) + r^2F'(r)^2\phi'(r)^2 + 2r^2F(r)F'(r)\phi'(r)\phi''(r)\left.\right) \\
& + \frac{1}{4}\alpha_3\left(2r^2F(r)F'(r)\phi'(r)^3 + 4r^2F(r)^2\phi'(r)^2\phi''(r)\right)
\end{aligned} \tag{A7}$$

$$\begin{aligned}
0 = & -\frac{1}{2}r^2\alpha_4F(r)^2\phi'(r)^4 + \frac{1}{4}\alpha_1\left(-8F'(r)^2 - 8F(r)F''(r)\right) \\
& + \frac{1}{4}\left(4F(r) + 8rF'(r) - 48r^2F(r)\phi'(r)^2 + 2r^2F''(r)\right) \\
& + \frac{1}{4}\alpha_2\left(2F(r)^2\phi'(r)^2 + 8rF(r)F'(r)\phi'(r)^2 + r^2F'(r)^2\phi'(r)^2 + r^2F(r)\phi'(r)^2F''(r)\right. \\
& + 8rF(r)^2\phi'(r)\phi''(r) + 2r^2F(r)F'(r)\phi'(r)\phi''(r)\left.\right) \\
& + \frac{1}{4}\alpha_3\left(2r^2F(r)F'(r)\phi'(r)^3 + 4r^2F(r)^2\phi'(r)^2\phi''(r)\right),
\end{aligned} \tag{A8}$$

again with prime denoting derivative with respect to the radial coordinate. Using the Bianchi identity, one metric equation can be eliminated, resulting in three independent equations. The scalar equation at the end

$$\begin{aligned}
0 = & \frac{1}{2r^3}\left(\phi'(r)\left(r^2F'(r)(48r + F'(r)(3\alpha_2 + r\alpha_3\phi'(r)))\right) + 12F(r)^2\left(\frac{\alpha_2}{2} + r\phi'(r)(\alpha_3 + r\alpha_4\phi'(r))\right)\right. \\
& + rF(r)(144r + F'(r)(18\alpha_2 + r\phi'(r)(15\alpha_3 + 8r\alpha_4\phi'(r))) + r(3\alpha_2 + r\alpha_3\phi'(r))F''(r)\left.\right) \\
& + rF(r)\phi''(r)\left(r\left(48r + F'(r)(3\alpha_2 + 2r\alpha_3\phi'(r))\right) + 12F(r)\left(\frac{\alpha_2}{2} + r\phi'(r)(\alpha_3 + r\alpha_4\phi'(r))\right)\right).
\end{aligned} \tag{A9}$$

Plugging in the solution *Ansatz* (A1), we obtain the following system of equations:

$$\begin{aligned}
0 &= 24\alpha_1 - 6\alpha_2\chi^2 + \alpha_4\chi^4 + 12\ell^2(-1 + 2\chi^2) \\
0 &= 24\alpha_1 - 12\ell^2(1 + 2\chi^2) - \chi^2(18\alpha_2 + 8\alpha_3\chi + 3\alpha_4\chi^2) \\
0 &= 12\ell^2 + 3\alpha_2 + 2\alpha_3\chi + \alpha_4\chi^2 = 0.
\end{aligned} \tag{A10}$$

This system further simplifies to the solution (14) in the main article.

APPENDIX B: HOLOGRAPHIC CONFORMAL ANOMALY

1. Anomaly coefficients for EdGB theories

The full expressions for coefficients in holographic conformal anomaly of EdGB theory in expression (28) in the main text is given as follows:

$$\begin{aligned}
 E_1 &: \frac{\ell^2(81\gamma^2 - 270\gamma\lambda + 65\lambda^2)}{12(9\gamma - \lambda)\lambda} \mathbf{R}_{(0)ab} \mathbf{R}^{(0)ab} & E_2 &: \frac{1}{3} \ell^3 \mathbf{R}_{(0)abcd} \mathbf{R}^{abcd}_{(0)} \\
 E_3 &: -\frac{\ell^2}{432(9\gamma - \lambda)\lambda(-81\gamma^3\lambda + \lambda^2 + 3\gamma\lambda(-4 + 3\lambda^2) + 27\gamma^2(1 + 8\lambda^2))^2} \\
 &\quad \times \left((6377292\gamma^8\lambda^2 + 732\lambda^6 - 118098\gamma^7\lambda(36 + 444\lambda^2) + 6\gamma\lambda^5(28(-31 + 21\lambda^2) + 8(-316 + 201\lambda^2)) \right. \\
 &\quad + 9\gamma^2\lambda^4(4(1423 + 2268\lambda^2 + 441\lambda^4) + 8(2251 - 72\lambda^2 + 603\lambda^4)) \\
 &\quad + 162\gamma^3\lambda^3(8(-632 - 1601\lambda^2 + 912\lambda^4) + 4(-362 - 2621\lambda^2 + 1785\lambda^4)) \\
 &\quad + 6561\gamma^6(4(9 + 252\lambda^2 + 1471\lambda^4) + 8(9 + 360\lambda^2 + 2065\lambda^4)) \\
 &\quad - 4374\gamma^5\lambda(4(33 + 627\lambda^2 + 2614\lambda^4) + 8(60 + 1005\lambda^2 + 2872\lambda^4)) \\
 &\quad \left. + 243\gamma^4\lambda^2(8280 + 64768\lambda^2 - 17592\lambda^4 + 4(549 + 7448\lambda^2 + 20373\lambda^4))) \right) \mathbf{R}_{(0)}^2 \\
 E_4 &: + \frac{9\ell^3\gamma^2\lambda(-3\gamma + \lambda)^2(\lambda + 9\gamma^2\lambda + \gamma(-3 + 9\lambda^2))}{2(-81\gamma^3\lambda + \lambda^2 + 3\gamma\lambda(-4 + 3\lambda^2) + 27\gamma^2(1 + 8\lambda^2))^2} \left(\nabla\phi_{(0)} \right)^2 \\
 E_5 &: -\frac{216\ell^3\gamma^2(3\gamma - 5\lambda)\lambda^2}{(9\gamma - \lambda)(81\gamma^3\lambda - \lambda^2 - 27\gamma^2(1 + 8\lambda^2) + \gamma(12\lambda - 9\lambda^3))} \square \mathbf{R}_{(0)} \\
 E_6 &: \frac{\ell^3\gamma}{6(9\gamma - \lambda)(-81\gamma^3\lambda + \lambda^2 + 3\gamma\lambda(-4 + 3\lambda^2) + 27\gamma^2(1 + 8\lambda^2))^2(\lambda + 9\gamma^2\lambda + \gamma(-3 + 9\lambda^2))} \\
 &\quad \times \left(2187\gamma^5\lambda - 7\lambda^4 - 243\gamma^4(3 + 34\lambda^2) + 162\gamma^3\lambda(7 + 58\lambda^2) \right. \\
 &\quad \left. - 18\gamma^2\lambda^2(32 + 117\lambda^2) + \gamma(114\lambda^3 - 63\lambda^5) \right) \mathbf{R}_{(0)} \left(\square\phi \right) \\
 E_7 &: + \frac{3\ell^3\gamma}{2(9\gamma - \lambda)(\lambda + 9\gamma^2\lambda + \gamma(-3 + 9\lambda^2))(81\gamma^3\lambda - \lambda^2 - 27\gamma^2(1 + 8\lambda^2) + \gamma(12\lambda - 9\lambda^3))} \\
 &\quad \times \left(2187\gamma^5\lambda - 11\lambda^4 - 243\gamma^4(3 + 38\lambda^2) - 18\gamma^2\lambda^2(46 + 175\lambda^2) \right. \\
 &\quad \left. + 54\gamma^3\lambda(27 + 244\lambda^2) + \gamma(174\lambda^3 - 99\lambda^5) \right) \left(\nabla^a\phi \right) \left(\nabla_a\mathbf{R}_{(0)} \right) \\
 E_8 &: + \frac{\ell^3}{72(9\gamma - \lambda)\lambda(-81\gamma^3\lambda + \lambda^2 + 3\gamma\lambda(-4 + 3\lambda^2) + 27\gamma^2(1 + 8\lambda^2))^2} \\
 &\quad \times \left(2187\gamma^5\lambda - 7\lambda^4 - 243\gamma^4(3 + 34\lambda^2) + 162\gamma^3\lambda(7 + 58\lambda^2) \right. \\
 &\quad \left. - 18\gamma^2\lambda^2(32 + 117\lambda^2) + \gamma(114\lambda^3 - 63\lambda^5) \right) \mathbf{R}_{(0)} \left(\nabla\phi_{(0)} \right)^2 \\
 E_9 &= \frac{\ell^3(3\gamma - \lambda)(9565938\gamma^{11}\lambda^4 - 29\lambda^7 - 1594323\gamma^{10}\lambda^3(7 + 30\lambda^2) + 59049\gamma^9\lambda^2(81 + 663\lambda^2 + 450\lambda^4))}{8(9\gamma - \lambda)\lambda(-81\gamma^3\lambda + \lambda^2 + 3\gamma\lambda(-4 + 3\lambda^2) + 27\gamma^2(1 + 8\lambda^2))^2(\lambda + 9\gamma^2\lambda + \gamma(-3 + 9\lambda^2))^2} \\
 &\quad \times (\nabla^a\nabla_a\phi)(\nabla^b\nabla_b\phi) \\
 E_{10} &= -\frac{\ell^3\gamma(3\gamma - \lambda)^2(81\gamma^3\lambda - \lambda^2 - 9\gamma^2(3 + 16\lambda^2) + \gamma(12\lambda - 9\lambda^3))(\nabla_a\phi)(\nabla^a\phi)(\square\phi)}{2(9\gamma - \lambda)(-81\gamma^3\lambda + \lambda^2 + 3\gamma\lambda(-4 + 3\lambda^2) + 27\gamma^2(1 + 8\lambda^2))^2(\lambda + 9\gamma^2\lambda + \gamma(-3 + 9\lambda^2))} \\
 E_{11} &= \frac{\ell^3 \left(\nabla_a\phi \right) \left(\nabla^b\nabla_b\nabla^a\phi \right)}{4(9\gamma - \lambda)\lambda(81\gamma^3\lambda - \lambda^2 - 27\gamma^2(1 + 8\lambda^2) + \gamma(12\lambda - 9\lambda^3))} \\
 &\quad \times \left(4374\gamma^5\lambda - 31\lambda^4 - 6\gamma\lambda^3(-82 + 45\lambda^2) + 324\gamma^3\lambda(13 + 54\lambda^2) - 243\gamma^4(9 + 68\lambda^2) - 18\gamma^2\lambda^2(131 + 186\lambda^2) \right) \\
 E_{12} &= \frac{9\ell^3\gamma(9\gamma^2 - 18\gamma\lambda + 5\lambda^2)}{81\gamma^3\lambda - \lambda^2 - 27\gamma^2(1 + 8\lambda^2) + \gamma(12\lambda - 9\lambda^3)} \left(\nabla^b\nabla_b\nabla^a\nabla_a\phi \right) \tag{B1}
 \end{aligned}$$

$$\begin{aligned}
E_{13} &= \frac{\ell^3 R_{(0)ab} (\nabla^a \phi_{(0)}) (\nabla^b \phi_{(0)})}{4(9\gamma - \lambda)\lambda(81\gamma^3\lambda - \lambda^2 - 27\gamma^2(1 + 8\lambda^2) + \gamma(12\lambda - 9\lambda^3))} \\
&\quad \times \left(-4374\gamma^5\lambda + 23\lambda^4 + 6\gamma\lambda^3(-62 + 33\lambda^2) - 324\gamma^3\lambda(11 + 60\lambda^2) + 243\gamma^4(9 + 76\lambda^2) + 18\gamma^2\lambda^2(103 + 222\lambda^2) \right) \\
E_{14} &= -\frac{\ell^3(3\gamma - \lambda)^2(81\gamma^3\lambda - \lambda^2 - 9\gamma^2(3 + 16\lambda^2) + \gamma(12\lambda - 9\lambda^3))}{48(9\gamma - \lambda)\lambda(-81\gamma^3\lambda + \lambda^2 + 3\gamma\lambda(-4 + 3\lambda^2) + 27\gamma^2(1 + 8\lambda^2))^2} (\nabla_a \phi) (\nabla^a \phi) (\nabla_b \phi) (\nabla^b \phi) \\
E_{15} &= -\frac{3\ell^3\gamma(3\gamma - \lambda)^2}{81\gamma^3\lambda - \lambda^2 - 27\gamma^2(1 + 8\lambda^2) + \gamma(12\lambda - 9\lambda^3)} (\nabla^a \phi) (\nabla_b \nabla_a \phi) (\nabla^b \phi) \\
E_{16} &= \frac{9\ell^3\gamma(9\gamma^2 - 18\gamma\lambda + 5\lambda^2)}{(9\gamma - \lambda)(\lambda + 9\gamma^2\lambda + \gamma(-3 + 9\lambda^2))} R_{(0)}^{ab} (\nabla_b \nabla_a \phi) \\
E_{17} &= \frac{3\ell^3(59049\gamma^9\lambda^3 - 5\lambda^6 - 27\gamma\lambda^5(-4 + 5\lambda^2) - 6561\gamma^8\lambda^2(9 + 8\lambda^2) - 2187\gamma^7\lambda(-9 - 32\lambda^2 + 69\lambda^4))}{4(9\gamma - \lambda)\lambda(\lambda + 9\gamma^2\lambda + \gamma(-3 + 9\lambda^2))^2(81\gamma^3\lambda - \lambda^2 - 27\gamma^2(1 + 8\lambda^2) + \gamma(12\lambda - 9\lambda^3))} \\
&\quad \times (\nabla_b \nabla_a \phi) (\nabla^b \nabla^a \phi). \tag{B2}
\end{aligned}$$

2. Lovelock-Horndeski theory without linear dilaton

For the sake of mathematical completeness, here we present the holographic conformal anomaly results for the case of solutions without linear dilaton. The anomalous action is given as the following:

$$\begin{aligned}
\ell^{-1}\mathcal{A} &= \alpha_1 \left(R_{(0)}^2 - 4R_{(0)ab}R_{(0)}^{ab} + R_{(0)abcd}R_{(0)}^{abcd} \right) + \frac{2\alpha_1}{\ell^2} \left(R_{(0)}^{ab} - \frac{1}{2}R_{(0)}g_{(0)}^{ab} \right) g_{(1)ab} \\
&\quad + A_1 g_{(2)a}^a + A_2 g_{(1)ab}g_{(1)}^{ab} + A_3 \left(g_{(1)a}^a \right)^2 + A_4 \left(\phi_{(1)} \right)^2 + A_5 \nabla_a \phi_{(0)} \nabla^a \phi_{(1)} \\
&\quad + A_6 g_{(1)ab} \nabla^a \phi_{(0)} \nabla^b \phi_{(0)} + \left(12 + \frac{\alpha_2}{\ell^2} \right) g_{(1)a}^a \left(\nabla \phi_{(0)} \right)^2 \\
&\quad + \alpha_2 \left(R_{(0)}^{ab} - \frac{1}{2}R_{(0)}g_{(0)}^{ab} \right) \nabla_a \phi_{(0)} \nabla_b \phi_{(0)} + \alpha_3 \left(\nabla \phi_{(0)} \right)^4 + \alpha_4 \left(\nabla \phi_{(0)} \right)^2 \square \phi_{(0)} \tag{B3}
\end{aligned}$$

with coefficients

$$\begin{aligned}
A_1 &= -\frac{6}{\ell^4} (\ell^2 - 2\alpha_1) & A_2 &= \frac{2}{\ell^4} (\ell^2 - \alpha_1) \\
A_3 &= -\frac{1}{2\ell^4} (\ell^2 + 2\alpha_1) & A_4 &= \frac{4}{\ell^4} (c_1\ell^2 + 6\alpha_2) \\
A_5 &= 2 \left(c_1 + \frac{6\alpha_2}{\ell^2} \right) & A_6 &= - \left(c_1 + \frac{4\alpha_2}{\ell^2} \right). \tag{B4}
\end{aligned}$$

The solution imposes $c_1 = 24$ and $\alpha_2 = -(4\ell^2 + \frac{2}{3}\alpha_3)$, while α_3 and α_4 remain free.

Varying with respect to $g_{(2)}$ yields $A_1 = 0$, implying $\alpha_1 = \ell^2/2$. Variation with respect to $\phi_{(1)}$ gives

$$\partial_a \frac{\delta \mathcal{A}}{\delta \partial^a \phi_{(1)}} - \frac{\delta \mathcal{A}}{\delta \phi_{(1)}} = -32 \frac{\lambda}{\ell^4} \phi_{(1)} - \nabla^a \left(\frac{8\lambda}{\ell^2} \nabla_a \phi_{(0)} \right) = 0$$

and thus

$$\phi_{(1)} = \frac{\ell^2}{4} \square \phi_{(0)}, \tag{B5}$$

and with respect to $g_{(1)ab}$,

$$0 = R_{(0)}^{ab} - \frac{1}{2}R_{(0)}g_{(0)}^{ab} + \frac{2}{\ell^2}g_{(1)cd}g_{(0)}^{ac}g_{(0)}^{bd} - \frac{2}{\ell^2}g_{(1)cd}g_{(0)}^{cd}g_{(0)}^{ab} \\ + \left(8 - \frac{2\lambda}{3\ell^2}\right)g_{(0)}^{ab}\left(\nabla\phi_{(0)}\right)^2 - 4\left(2 - \frac{2\lambda}{3\ell^2}\right)\nabla^a\phi_{(0)}\nabla^b\phi_{(0)}.$$

Multiplying both sides with $g_{(0)ab}$, we find that

$$g_{(1)a}^a = -\frac{\ell^2}{6}R_{(0)} + 4\ell^2\left(\nabla\phi_{(0)}\right)^2. \quad (\text{B6})$$

Plugging the above expression back into Eq. (B6), we obtain

$$g_{(1)ab} = -\frac{\ell^2}{2}R_{(0)ab} + \frac{\ell^2}{12}R_{(0)}g_{(0)ab} + \frac{\lambda}{3}\left(\nabla\phi_{(0)}\right)^2g_{(0)ab} \\ + 4\left(1 - \frac{\lambda}{3\ell^2}\right)\nabla_a\phi_{(0)}\nabla_b\phi_{(0)}. \quad (\text{B7})$$

The final form of the anomaly is

$$\mathcal{A}_{\text{total}} = \mathcal{A}_{\text{grav}} + \mathcal{A}_{\text{matter}} \\ \mathcal{A}_{\text{grav}} = \frac{7\ell^2}{12}R_{(0)}^2 - \frac{9\ell^2}{4}R_{ab}^{(0)}R_{(0)}^{ab} + \frac{\ell^2}{2}R_{abcd}^{(0)}R_{(0)}^{abcd} \\ \mathcal{A}_{\text{matter}} = \frac{5}{6}\lambda R_{(0)}\left(\nabla\phi_{(0)}\right)^2 - 2\lambda R_{ab}^{(0)}\nabla^a\phi_{(0)}\nabla^b\phi_{(0)} \\ + \lambda\left(\Box\phi_{(0)}\right)^2 + \lambda\left(\nabla\phi_{(0)}\right)^2\left(\Box\phi_{(0)}\right)^2 \\ + \left(\gamma + 8\lambda + \frac{\lambda^2}{3\ell^2}\right)\left(\left(\nabla\phi_{(0)}\right)^2\right)^2, \quad (\text{B8})$$

where gravitational and matter contributions are separated. The gravitational sector encodes central charges. Comparison with (2) identifies

$$a = -\frac{5\ell^3}{8}, \quad c = -\frac{\ell^3}{8}. \quad (\text{B9})$$

Here follows some comments: For a unitary Lorentzian CFT such negative values violate the conformal-collider (positive-energy-flux) requirements. Hence this particular bulk solution cannot serve as the dual of a unitary four-dimensional quantum field theory [49,50]. However, negative central charges are perfectly acceptable in nonunitary CFT, open quantum systems for example, or logarithmic CFTs, where they encode the density of negative-norm states and control logarithmic pairings of operators, the holographic logarithmic conformal field theories constructions [51], for example. In that context, our background offers a controlled higher-dimensional example of “AdS/log-CFT” correspondence. Recent work [52] also shows a counterexample of the proposition that a CFT_2 with a negative central charge should have negative-norm states.

3. a-theorem for Lovelock-Horndeski theory with usual dilaton profile

For the theory (7) equipped with a regular dilaton profile, we adopt a natural flow function of the form

$$a(r) = -\frac{5}{8}\frac{1}{A'(r)^3}. \quad (\text{B10})$$

Then,

$$a'(r) = \frac{15A''(r)}{8A'(r)^4}. \quad (\text{B11})$$

Substituting into the scalar equation (10), we obtain (up to an integration constant C)

$$24\nabla_a\phi + \alpha_2 G_{ab}\nabla^b\phi + 2\alpha_3\nabla_a\phi\Box\phi \\ + 2\alpha_4\nabla_a\phi(\nabla\phi)^2 = C. \quad (\text{B12})$$

Boundary conditions $A(r) \rightarrow r/\ell$ and $\phi(r) \rightarrow 0$ fix $C = 0$. Inserting the *Ansatz* into the scalar equation leads to

$$24 + 6\alpha_2 A'(r)^2 + \alpha_3 \phi''(r) + 4\alpha_3 A'(r)\phi'(r) \\ + \alpha_4 \phi'(r)^2 = 0. \quad (\text{B13})$$

We specify the parameters $\alpha_1 \geq 5\ell^2/4$, $\alpha_2 = -4\ell^2$, $\alpha_3 = 0$, and choose $\alpha_4 = -80\ell^2$ [with $\alpha_4 \in (-96\ell^2, -80\ell^2)$]. This yields

$$24 - 24\ell^2 A'(r)^2 - 80\ell^2 \phi'(r)^2 = 0, \quad (\text{B14})$$

from which one obtains $\phi'(r)$ as a function of $A'(r)$. Consequently, $A'(r)^2 \in [0, 1/\ell^2]$.

Using the NEC, $-T_t^t + T_r^r \geq 0$, we find from the modified Einstein equation

$$E_r^r - E_t^t = -\frac{3}{5\ell^2}\left(36 + 12\ell^4 A'(r)^4 \right. \\ \left. - \ell^2 A'(r)^2(48 + \ell^2 A''(r)) + 2\ell^2 A''(r)\right) \\ = -(T^{\text{matter}})_t^t + (T^{\text{matter}})_r^r. \quad (\text{B15})$$

Then using the null energy condition for the matter field, $-(T^{\text{matter}})_t^t + (T^{\text{matter}})_r^r \geq 0$, we can establish the inequality for $A'(r)$ and $A''(r)$,

$$3\left(-\frac{24}{\ell^2} + 32A'(r)^2 - 8\ell^2 A'(r)^4\right) \\ + 3\left(1 + (4\alpha_1 - 6\ell^2)A'(r)^2\right)A''(r) \geq 0. \quad (\text{B16})$$

Thus, $A''(r) \geq 0$, implying monotonicity of $a(r)$ and establishing the holographic a-theorem for the nondilaton solution. It demonstrates that, despite the negative UV

central charges, the dual field theory still possesses a well-ordered renormalization-group hierarchy. On the CFT side this means that, while unitarity is forfeited, the number of effective degrees of freedom—measured by the generalized a function—decreases monotonically from the ultraviolet fixed point to the infrared, exactly as in ordinary four-dimensional CFTs. The monotonic a function shows that the bulk coupling space carved out by the Horndeski and Gauss-Bonnet terms supports a “well-behaved” holographic RG flow, offering a controlled laboratory for nonunitary (e.g., logarithmic) CFTs that nevertheless retain a generalized a -theorem.

APPENDIX C: KALUZA-KLEIN REDUCTION OF GAUSS-BONNET AND HORNDESKI TERMS

In this appendix, we present the details of KK reductions.

First, we note that the higher-dimensional metric $\hat{g}$ decomposes as

$$\hat{g}_{MN} = e^{-2\alpha\phi} g_{ab} \oplus e^{-2\beta\phi} g_{ij}, \quad (C1)$$

and we also have

$$\sqrt{-\hat{g}} = \sqrt{-(e^{-2\alpha\phi})^d (e^{-2\beta\phi})^n g} = e^{-2(d\alpha+n\beta)\phi} \sqrt{-g}. \quad (C2)$$

The dilaton field descends down directly after the KK reduction,

$$\hat{\Phi}(x, x') = \sum_{n=0}^{\infty} \Phi(x) e^{2n\pi i x'} \approx \Phi_0(x) = \phi(x). \quad (C3)$$

Thus, we assume that the dilaton field depends only on the external space coordinates, $\hat{\Phi} = \phi(x)$. This then gives

$$\hat{g}^{MN} \nabla_M \hat{\Phi} \nabla_N \hat{\Phi} = e^{-2\alpha\phi} g^{ab} \nabla_a \hat{\Phi} \nabla_b \hat{\Phi}. \quad (C4)$$

For the scalar curvature term $\hat{R}^2$, we have

$$\begin{aligned} \hat{R}^2 = & e^{-4\beta\phi} \lambda^2 + 2e^{-2\alpha\phi-2\beta\phi} \lambda R + e^{-4\alpha\phi} R^2 - 4e^{-2\alpha\phi-2\beta\phi} \left((-1+d)\alpha + n\beta \right) \lambda (\nabla_e \nabla^e \phi) \\ & - 4e^{-4\alpha\phi} \left((-1+d)\alpha + n\beta \right) R (\nabla_e \nabla^e \phi) - 2e^{-2\alpha\phi-2\beta\phi} \left((2-3d+d^2)\alpha^2 \right. \\ & + 2(-2+d)n\alpha\beta + n(1+n)\beta^2 \lambda (\nabla_e \phi) (\nabla^e \phi) - 2e^{-4\alpha\phi} \left((2-3d+d^2)\alpha^2 + 2(-2+d)n\alpha\beta \right. \\ & + n(1+n)\beta^2 R (\nabla_e \phi) (\nabla^e \phi) + 4e^{-4\alpha\phi} \left((-1+d)\alpha + n\beta \right)^2 (\nabla_e \nabla^e \phi) (\nabla_f \nabla^f \phi) \\ & + 4e^{-4\alpha\phi} \left((-2+d)(-1+d)^2 \alpha^3 + 3(2-3d+d^2)n\alpha^2\beta + n(-1+d-5n+3dn)\alpha\beta^2 + n^2(1+n)\beta^3 \right) \\ & \times (\nabla_e \phi) (\nabla^e \phi) (\nabla_f \nabla^f \phi) + e^{-4\alpha\phi} \left((2-3d+d^2)\alpha^2 + 2(-2+d)n\alpha\beta + n(1+n)\beta^2 \right)^2 (\nabla_e \phi) (\nabla^e \phi) (\nabla_f \phi) (\nabla^f \phi), \end{aligned}$$

where we have used symbol λ to denote the constant curvature of the internal space. For the Ricci tensor squared term, we have

$$\hat{R}^{MN} \hat{R}_{MN} = \hat{R}^{ab} \hat{R}_{ab} + \hat{R}^{ij} \hat{R}_{ij},$$

where

$$\begin{aligned} \hat{R}^{ij} \hat{R}_{ij} = & e^{-4\beta\phi} \lambda_{ef} \lambda^{ef} - 2e^{-2\alpha\phi-2\beta\phi} \beta \lambda (\nabla_e \nabla^e \phi) - 2e^{-2\alpha\phi-2\beta\phi} \beta \left((-2+d)\alpha + n\beta \right) \lambda (\nabla_e \phi) (\nabla^e \phi) \\ & + e^{-4\alpha\phi} n\beta^2 (\nabla_e \nabla^e \phi) (\nabla_f \nabla^f \phi) + 2e^{-4\alpha\phi} n\beta^2 \left((-2+d)\alpha + n\beta \right) (\nabla_e \phi) (\nabla^e \phi) (\square\phi) \\ & + e^{-4\alpha\phi} n\beta^2 \left((-2+d)\alpha + n\beta \right)^2 ((\nabla\phi)^2)^2 \\ \hat{R}^{ab} \hat{R}_{ab} = & e^{-4\alpha\phi} R_{ef} R^{ef} - 2e^{-4\alpha\phi} \alpha R (\nabla_e \nabla^e \phi) - 2e^{-4\alpha\phi} \alpha \left((-2+d)\alpha + n\beta \right) R (\nabla_e \phi) (\nabla^e \phi) \\ & + e^{-4\alpha\phi} \alpha \left((-4+3d)\alpha + 2n\beta \right) (\nabla_e \nabla^e \phi)^2 + 2e^{-4\alpha\phi} \alpha \left((6-7d+2d^2)\alpha^2 + 3(-2+d)n\alpha\beta \right. \\ & \left. + n(1+n)\beta^2 \right) (\nabla_e \phi) (\nabla^e \phi) (\nabla_f \nabla^f \phi) + e^{-4\alpha\phi} (2(-2+d)\alpha^2 + 4n\alpha\beta - 2n\beta^2) \end{aligned}$$

$$\begin{aligned}
& R_{ef}(\nabla^e \phi)(\nabla^f \phi) + e^{-4\alpha\phi}((-2+d)^2(-1+d)\alpha^4 + dn^2\alpha^2\beta^2 \\
& + 2(2-3d+d^2)n\alpha^3\beta - 2n^2\alpha\beta^3 + n^2\beta^4)(\nabla_e \phi)(\nabla^e \phi)(\nabla_f \phi)(\nabla^f \phi) \\
& + 2e^{-4\alpha\phi}\left(-(-2+d)^2\alpha^3 - 3(-2+d)n\alpha^2\beta + n(d-2(1+n))\alpha\beta^2 + n^2\beta^3\right) \\
& \times (\nabla^e \phi)(\nabla_f \nabla_e \phi)(\nabla^f \phi) - 2e^{-4\alpha\phi}((-2+d)\alpha + n\beta)R_{ef}\nabla^f \nabla^e \phi \\
& + e^{-4\alpha\phi}\left((-2+d)\alpha + n\beta\right)^2(\nabla_f \nabla_e \phi)(\nabla^f \nabla^e \phi).
\end{aligned}$$

Finally, for the Riemann curvature squared, we have

$$\hat{R}^{IJMN}\hat{R}_{IJMN} = \hat{R}^{abcd}\hat{R}_{abcd} + \hat{R}^{ibjd}\hat{R}_{ibjd} + \hat{R}^{ijkl}\hat{R}_{ijkl},$$

where

$$\begin{aligned}
\hat{R}^{abcd}\hat{R}_{abcd} &= e^{-4\alpha\phi}R_{efij}R^{efij} - 4e^{-4\alpha\phi}\alpha^2 R(\nabla_e \phi)(\nabla^e \phi) + 4e^{-4\alpha\phi}\alpha^2(\nabla_e \nabla^e \phi)(\nabla_f \nabla^f \phi) \\
&+ 8(-2+d)e^{-4\alpha\phi}\alpha^3(\nabla_e \phi)(\nabla^e \phi)(\nabla_f \nabla^f \phi) + 8e^{-4\alpha\phi}\alpha^2 R_{ef}(\nabla^e \phi)(\nabla^f \phi) \\
&- 8e^{-4\alpha\phi}\alpha R_{ef}\nabla^f \nabla^e \phi + 2(2-3d+d^2)e^{-4\alpha\phi}\alpha^4(\nabla \phi)^2(\nabla \phi)^2 \\
&- 8(-2+d)e^{-4\alpha\phi}\alpha^3(\nabla^e \phi)(\nabla_f \nabla_e \phi)(\nabla^f \phi) + 4(-2+d)e^{-4\alpha\phi}\alpha^2(\nabla_f \nabla_e \phi)(\nabla^f \nabla^e \phi) \\
\hat{R}^{ibjd}\hat{R}_{ibjd} &= 8de^{-4\alpha\phi}\alpha\beta^2(\nabla_e \phi)(\nabla^e \phi)(\nabla_f \nabla^f \phi) + 4de^{-4\alpha\phi}\beta^2(d\alpha^2 + \beta(-2\alpha + \beta))(\nabla \phi)^2(\nabla \phi)^2 \\
&+ 8de^{-4\alpha\phi}\beta^2(-2\alpha + \beta)(\nabla^e \phi)(\nabla_f \nabla_e \phi)(\nabla^f \phi) + 4de^{-4\alpha\phi}\beta^2(\nabla_f \nabla_e \phi)(\nabla^f \nabla^e \phi) \\
\hat{R}^{ijkl}\hat{R}_{ijkl} &= e^{-4\beta\phi}\lambda_{efij}\lambda^{efij} - 4e^{-2\alpha\phi-2\beta\phi}\beta^2\lambda(\nabla_e \phi)(\nabla^e \phi) + 2e^{-4\alpha\phi}(-1+n)n\beta^4(\nabla_e \phi)(\nabla^e \phi)(\nabla_f \phi)(\nabla^f \phi).
\end{aligned}$$

Combining the above pieces, we find the reduced Gauss-Bonnet term,

$$\begin{aligned}
\hat{\mathcal{L}}_{GB} &= -4e^{-4\alpha\phi}\left((d-3)\alpha + n\beta\right)\mathcal{R}\nabla_a \nabla^a \phi + 8e^{-4\alpha\phi}\left((d-3)\alpha + n\beta\right)\mathcal{R}_{ab}\nabla^a \nabla^b \phi \\
&- 2e^{-4\alpha\phi}\left((d-3)(d-4)\alpha^2 + 2(d-4)n\alpha\beta + n(1+n)\beta^2\right)\mathcal{R}\nabla_a \phi \nabla^a \phi \\
&+ 8e^{-4\alpha\phi}\left(-((d-3)\alpha^2) - 2n\alpha\beta + n\beta^2\right)\mathcal{R}_{ab}\nabla^a \phi \nabla^b \phi \\
&+ 4e^{-4\alpha\phi}\left((d-3)^2(d-2)\alpha^3 + 3(6-5d+d^2)n\alpha^2\beta + n(3-7n+d(3n-1))\alpha\beta^2 + (n-1)n^2\beta^3\right) \\
&\times \nabla_a \nabla^a \phi \nabla_b \phi \nabla^b \phi + e^{-4\alpha\phi}\left((24-50d+35d^2-10d^3+d^4)\alpha^4 + 4(-8+14d-7d^2+d^3)n\alpha^3\beta \right. \\
&+ 2(-1+d)n(6-10n+d(-1+3n))\alpha^2\beta^2 + 4(-1+n)n(2+(-2+d)n)\alpha\beta^3 \\
&+ n(2-n-2n^2+n^3)\beta^4)\nabla_a \phi \nabla^a \phi \nabla_b \phi \nabla^b \phi + 8e^{-4\alpha\phi}\left((6-5d+d^2)\alpha^3 + 3(-2+d)n\alpha^2\beta \right. \\
&+ n(d+2n)\alpha\beta^2 - (-1+n)n\beta^3)\nabla^a \phi \nabla_a \nabla_b \phi \nabla^b \phi + 4e^{-4\alpha\phi}\left((6-5d+d^2)\alpha^2 + 2(-2+d)n\alpha\beta \right. \\
&+ (-1+n)n\beta^2)\nabla_a \nabla^a \phi \nabla_b \nabla^b \phi - 4e^{-4\alpha\phi}\left((6-5d+d^2)\alpha^2 + 2(-2+d)n\alpha\beta \right. \\
&+ (-1+n)n\beta^2)\nabla_a \nabla_b \phi \nabla^a \nabla^b \phi.
\end{aligned}$$

Using the identities $(\square\phi)^2 - \nabla_a \nabla_b \phi \nabla^a \nabla^b \phi = \mathcal{R}^{ab}\nabla_a \phi \nabla_b \phi + \nabla_a(\square\phi \nabla^a \phi - \nabla^a \nabla^b \phi \nabla_b \phi)$ and performing repeated integrations by parts and throwing several total derivative terms, we find that

$$\begin{aligned}
\hat{\mathcal{L}}_{GB} = & e^{-4\alpha} \left(-4((-4+d)(-3+d)\alpha^2 + 2(-3+d)n\alpha + (-1+n)n\beta^2) \right) \mathcal{G}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \\
& + \left(-2(-4+d)(-3+d)(-2+d)\alpha^3 - 6(-3+d)(-2+d)n\alpha^2\beta \right. \\
& - 6(-2+d)(-1+n)n\alpha\beta^2 - 2(-2+n)(-1+n)n\beta^3) (\nabla\phi)^2 \square\phi \\
& + \left(-((-4+d)(-3+d)^2(-2+d)\alpha^4) - 4(-3+d)^2(-2+d)n\alpha^3\beta \right. \\
& - 2(-2+d)n(3-2(-1+d)-5n+3(-1+d)n)\alpha^2\beta^2 \\
& - 4(-2+d)(-1+n)^2n\alpha\beta^3 - (-2+n)(-1+n)^2n\beta^4) ((\nabla\phi)^2)^2 \\
& + 2e^{-2\alpha\phi-2\beta\phi} \left(\lambda\mathcal{R} + \lambda \left((d-1)(d-2)\alpha^2 + 2(d-1)(n-2)\alpha\beta + (n-2)(n-3)\beta^2 \right) (\nabla\phi)^2 \right) \\
& + e^{-4\alpha} \left(\mathcal{R}^2 - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} \right) + e^{-4\beta} \left(\lambda^2 - 4\lambda^{\mu\nu}\lambda_{\mu\nu} + \lambda^{\mu\nu\rho\sigma}\lambda_{\mu\nu\rho\sigma} \right).
\end{aligned}$$

For the case where we have set $\alpha = 0$ and $\lambda = 0$ in the reduction *Ansatz*. We find the Horndeski term,

$$\hat{G}^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi = \mathcal{G}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{3}{2} n \beta \square (\nabla\phi)^2 \phi - \frac{1}{2} n(n-3)\beta^2 ((\nabla\phi)^2)^2. \quad (\text{C5})$$

Eventually, we find that coefficients for theory (7) in the main article,

$$\begin{aligned}
\alpha_1 &= \tilde{\alpha}, \quad \alpha_2 = \tilde{\alpha} \left(a_1 - 4n(n-1)\beta^2 \right) \\
\alpha_3 &= \tilde{\alpha} \left(\frac{3}{2} a_1 n \beta - 2n(n-1)(n-2) + a_3 \right) \\
\alpha_4 &= \tilde{\alpha} \left(n(n-1)^2(n-2)\beta^4 - \frac{a_1}{2} n(n-3)\beta^3 + a_4 \right). \quad (\text{C6})
\end{aligned}$$

-
- [1] E. Witten, Anti de Sitter space and holography, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
- [2] C. Fefferman and C. R. Graham, Conformal invariants, *Élie Cartan et les mathématiques d'aujourd'hui – Lyon, 25–29 juin 1984* (Société Mathématique de France, 1985), pp. 95–116, https://www.numdam.org/item/AST_1985__S131__95_0/.
- [3] M. Henningson and K. Skenderis, The holographic Weyl anomaly, *J. High Energy Phys.* **07** (1998) 023.
- [4] M. J. Duff, Twenty years of the Weyl anomaly, *Classical Quantum Gravity* **11**, 1387 (1994).
- [5] C. Imbimbo, A. Schwimmer, S. Theisen, and S. Yankielowicz, Diffeomorphisms and holographic anomalies, *Classical Quantum Gravity* **17**, 1129 (2000).
- [6] V. Balasubramanian and P. Kraus, A stress tensor for anti-de Sitter gravity, *Commun. Math. Phys.* **208**, 413 (1999).
- [7] S. Nojiri and S. D. Odintsov, On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence, *Int. J. Mod. Phys. A* **15**, 413 (2000).
- [8] J. de Boer, M. Kulaxizi, and A. Parnachev, Holographic entanglement entropy in Lovelock gravities, *J. High Energy Phys.* **07** (2011) 109.
- [9] L. Y. Hung, R. C. Myers, and M. Smolkin, On holographic entanglement entropy and higher curvature gravity, *J. High Energy Phys.* **04** (2011) 025.
- [10] S. Deser and A. Schwimmer, Geometric classification of conformal anomalies in arbitrary dimensions, *Phys. Lett. B* **309**, 279 (1993).
- [11] S. de Haro, K. Skenderis, and S. N. Solodukhin, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, *Commun. Math. Phys.* **217**, 595 (2001).
- [12] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, *Int. J. Theor. Phys.* **10**, 363 (1974).
- [13] A. Nicolis, R. Rattazzi, and E. Trincherini, Galileon as a local modification of gravity, *Phys. Rev. D* **79**, 064036 (2009).
- [14] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, General second-order scalar-tensor theory and self-tuning, *Phys. Rev. Lett.* **108**, 051101 (2012).

- [15] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Generalized g-inflation: Inflation with the most general second-order field equations, *Prog. Theor. Phys.* **126**, 511 (2011).
- [16] Harvey S. Reall, Norihiro Tanahashi, and Benjamin Way, Well-posed formulation of Lovelock and Horndeski theories, *Phys. Rev. D* **102**, 044005 (2020).
- [17] B. Zwiebach, Curvature squared terms and string theories, *Phys. Lett.* **156B**, 315 (1985).
- [18] D.G. Boulware and S. Deser, String-generated gravity models, *Phys. Rev. Lett.* **55**, 2656 (1985).
- [19] H. Lü and Y. Pang, Horndeski gravity as $d \rightarrow 4$ limit of Gauss-Bonnet, *Phys. Lett. B* **809**, 135717 (2020).
- [20] C. Charmousis, B. Goutéraux, and E. Kiritsis, Higher-derivative scalar-vector-tensor theories: Black holes, Galileons, singularity cloaking and holography, *J. High Energy Phys.* **09** (2012) 011.
- [21] Pablo A. Cano and Alejandro Ruipérez, String gravity in $d = 4$, *Phys. Rev. D* **105**, 044022 (2022).
- [22] T. Wu, AdS wormholes from Ricci-flat/AdS correspondence, *Phys. Rev. D* **108**, 044001 (2023).
- [23] R. Metsaev and A. Tseytlin, Order α' (two-loop) equivalence of the string equations of motion and the σ -model Weyl invariance conditions: Dependence on the dilaton and the antisymmetric tensor, *Nucl. Phys.* **B293**, 385 (1987).
- [24] Z.-K. Guo, N. Ohta, and T. Torii, Black holes in the dilatonic Einstein-Gauss-Bonnet theory in various dimensions. II: Asymptotically AdS topological black holes, *Prog. Theor. Phys.* **121**, 253 (2009).
- [25] N. Ohta and T. Torii, Black holes in the dilatonic Einstein-Gauss-Bonnet theory in various dimensions. III: Asymptotically AdS black holes with $k = \pm 1$, *Prog. Theor. Phys.* **121**, 959 (2009).
- [26] K.-i. Maeda, N. Ohta, and Y. Sasagawa, AdS black hole solutions in dilatonic Einstein-Gauss-Bonnet gravity, *Phys. Rev. D* **83**, 044051 (2011).
- [27] H. Lü M. Cvetič and C. N. Pope, Consistent Kaluza-Klein sphere reductions, *Phys. Rev. D* **62**, 064028 (2000).
- [28] Jerome P. Gauntlett and Oscar Varela, Consistent Kaluza-Klein reductions for general supersymmetric AdS solutions, *Phys. Rev. D* **76**, 126007 (2007).
- [29] C. N. Pope M. J. Duff and B. E. W. Nilsson, Kaluza-Klein supergravity, *Phys. Rep.* **130**, 1 (1986).
- [30] J. T. Liu and R. J. Saskowski, Consistent truncations in higher derivative supergravity, *J. High Energy Phys.* **09** (2023) 136.
- [31] J. T. Liu and R. J. Saskowski, Rounding out the story of higher derivative consistent truncations, *J. High Energy Phys.* **02** (2024) 108.
- [32] Enrique Alvarez and M. A. R. Osorio, Cosmological constant versus free energy for heterotic strings, *Nucl. Phys.* **B304**, 327 (1988).
- [33] S. P. De Alwis, J. Polchinski, and R. Schimmrigk, Heterotic strings with tree level cosmological constant, *Phys. Lett. B* **218**, 449 (1989).
- [34] L. Alvarez-Gaume, P. Ginsparg, G. Moore, and C. Vafa, An $o(16) \times o(16)$ heterotic string, *Phys. Lett. B* **171**, 155 (1986).
- [35] I. Florakis and J. Rizos, Chiral heterotic strings with positive cosmological constant, *Nucl. Phys.* **B913**, 495 (2016).
- [36] A. Anabalon, A. Cisterna, and J. Oliva, Asymptotically locally AdS and flat black holes in Horndeski theory, *Phys. Rev. D* **89**, 084050 (2014).
- [37] O. Baake, A. Cisterna, M. Hassaine, and U. Hernandez-Vera, Endowing black holes with beyond-Horndeski primary hair: An exact solution framework for scalarizing in every dimension, *Phys. Rev. D* **109**, 064024 (2024).
- [38] A. Cisterna, S. Fuenzalida, M. Lagos, and J. Oliva, Homogeneous black strings in Einstein-Gauss-Bonnet with Horndeski hair and beyond, *Eur. Phys. J. C* **78**, 982 (2018).
- [39] E. Babichev and C. Charmousis, Dressing a black hole with a time-dependent Galileon, *J. High Energy Phys.* **08** (2014) 106.
- [40] Thomas Hertog and Kengo Maeda, Black holes with scalar hair and asymptotics in $\mathcal{N} = 8$ supergravity, *J. High Energy Phys.* **07** (2004) 051.
- [41] Marc Henneaux, Cristián Martínez, Ricardo Troncoso, and Jorge Zanelli, Asymptotically anti-de Sitter spacetimes and scalar fields with a logarithmic branch, *Phys. Rev. D* **70**, 044034 (2004).
- [42] I. Kanitscheider, K. Skenderis, and M. Taylor, Precision holography for non-conformal branes, *J. High Energy Phys.* **09** (2008) 094.
- [43] Y.-Z. Li and H. Lü, a-theorem for Horndeski gravity at the critical point, *Phys. Rev. D* **97**, 126008 (2018).
- [44] J. Polchinski, Scale and conformal invariance in quantum field theory, *Nucl. Phys.* **B303**, 226 (1988).
- [45] Y. Nakayama, Scale invariance vs conformal invariance, *Phys. Rep.* **569**, 1 (2015).
- [46] R. C. Myers and A. Sinha, Seeing a c-theorem with holography, *Phys. Rev. D* **82**, 046006 (2010).
- [47] Kent Yagi and Leo C. Stein, Black hole based tests of general relativity, *Classical Quantum Gravity* **33**, 054001 (2016).
- [48] Daniela D. Doneva and Stoytcho S. Yazadjiev, New Gauss-Bonnet black holes with curvature-induced scalarization in extended scalar-tensor theories, *Phys. Rev. Lett.* **120**, 131103 (2018).
- [49] Diego M. Hofman, D. Li, D. Meltzer, D. Poland, and F. Rejon-Barrera, A proof of the conformal collider bounds, *J. High Energy Phys.* **06** (2016) 111.
- [50] Diego M. Hofman and Juan Maldacena, Conformal collider physics: Energy and charge correlations, *J. High Energy Phys.* **05** (2008) 012.
- [51] D. Grumiller, W. Riedler, J. Rosseel, and T. Zojer, Holographic applications of logarithmic conformal field theories, *J. Phys. A* **46**, 494002 (2013).
- [52] S. Ryu and J. Yoon, Unitarity of symplectic fermions in α vacua with negative central charge, *Phys. Rev. Lett.* **130**, 241602 (2023).