

Supersymmetric quantum mechanics from wrapped D4-branes

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We find a large class of holographic solutions describing D4-branes wrapped on four-manifolds $\mathcal{M}_4$ with constant curvature leading to gravity duals of supersymmetric quantum mechanics in the IR via twisted compactifications. The manifolds $\mathcal{M}_4$ considered here are four-dimensional spheres and hyperbolic spaces, products of two Riemann surfaces, and Kahler four-cycles. The solutions are obtained from the maximal gauged supergravity in six dimensions with $CSO(p, q, 5 - p - q)$ and $CSO(p, q, 4 - p - q) \ltimes \mathbb{R}^4$ gauge groups. These gauged supergravities can be embedded in type IIA theory via consistent truncations on $H^{p,q} \times \mathbb{R}^{5-p-q}$ and $H^{p,q} \times \mathbb{R}^{4-p-q} \times S^1$, respectively. The solutions take the form of $t \times \mathcal{M}_4$ -sliced domain walls interpolating between locally flat domain walls and singular geometries in the IR. Upon uplifted to type IIA theory, many solutions admit physical IR singularities and could holographically describe supersymmetric quantum mechanics arising from twisted compactifications of D4-branes on $\mathcal{M}_4$.

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I. INTRODUCTION

For almost three decades, the AdS/CFT correspondence [1–3] and the generalization to nonconformal field theories called DW/QFT correspondence [4–6] have attracted much attention in the holographic study of strongly coupled field theories using gravity duals. Both $\text{AdS}_{d+1}/\text{CFT}_d$ and $\text{DW}_{d+1}/\text{QFT}_d$ dualities have been used to study various aspects of field theories in different dimensions with $d \geq 2$. Unlike higher dimensional counterparts, $\text{AdS}_2/\text{CFT}_1$ and DW_2/QFT_1 correspondences have been far less explored to date. The dual $(0+1)$ -dimensional field theories in this case are superconformal quantum mechanics and supersymmetric quantum mechanics. Some examples of the latter arise naturally in string/M theory in the form of matrix models [7,8]. These models are expected to play an important role in string theory, D-brane physics, and noncommutative geometries, see [9,10] for recent reviews.

Most of the important results in $\text{AdS}_{d+1}/\text{CFT}_d$ or $\text{DW}_{d+1}/\text{QFT}_d$ dualities are generally obtained by studying various types of solutions to gauged supergravity in $d+1$ dimensions. Since the construction of two-dimensional gauged supergravity has appeared only recently in [11], see also [12] for earlier results, the study of DW_2/QFT_1 correspondence within the framework of gauged supergravity has only been carried out in a handful of previous works [13–15], see also [16] for another approach using spherical branes. These provide gravity duals of matrix quantum mechanics on the world-volume of D0-branes. Alternatively, QFT_1 or supersymmetric quantum mechanics can also arise from Dp-branes wrapped on a p -dimensional compact manifold. These configurations can preserve some amount of the original supersymmetry by a topological twist [17,18].

In the present paper, we are interested in the case of D4-branes wrapped on a four-manifold $\mathcal{M}_4$ with constant curvature, see [19–26] for examples of other D-branes and M-branes wrapped on a four-manifold and, in particular, [27] for a solution describing supersymmetric quantum mechanics from D6-branes wrapped on six-cycles. The $\mathcal{M}_4$ manifolds under consideration here are given by a Riemannian four-manifold of constant curvature, a product of two Riemann surfaces, and a Kahler four-cycle. In the DW/QFT correspondence, the dynamics of the maximal five-dimensional gauge theory on the world-volume of

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D4-branes in type IIA theory can be explored by studying the effective maximal gauged supergravity in six dimensions [28]. Accordingly, we study solutions of D4-branes wrapped on $\mathcal{M}_4$ by performing topological twists on the (flat) domain wall solutions dual to five-dimensional super Yang-Mills (SYM) theory along $\mathcal{M}_4$. A large number of flat domain walls from maximal gauged supergravity in six dimensions has been found in [29–31], and solutions describing D4-branes wrapped on two- and three-manifolds have also studied recently in [32].

We consider six-dimensional gauged supergravity with $CSO(p, q, 5 - p - q)$ and $CSO(p, q, 4 - p - q) \ltimes \mathbb{R}^4$ gauged groups. Both of these gauged supergravities can be obtained from an S^1 reduction of the maximal gauged supergravity in seven dimensions with $CSO(p, q, 5 - p - q)$ and $CSO(p, q, 4 - p - q)$ gauge groups [33]. It has been shown in [34] that seven-dimensional $CSO(p, q, 4 - p - q)$ gauged supergravity can be obtained from type IIA theory via a consistent truncation on $H^{p,q} \times \mathbb{R}^{4-p-q}$. Accordingly, further truncation on S^1 leads to six-dimensional gauged supergravity with $CSO(p, q, 4 - p - q) \ltimes \mathbb{R}^4$, or equivalently, a truncation of type IIA theory on $H^{p,q} \times \mathbb{R}^{4-p-q} \times S^1$ leads to $CSO(p, q, 4 - p - q) \ltimes \mathbb{R}^4$ gauged supergravity in six dimensions. On the other hand, the maximal $CSO(p, q, 5 - p - q)$ gauged supergravity in seven dimensions can also be obtained as a truncation of 11-dimensional supergravity on $H^{p,q} \times \mathbb{R}^{5-p-q}$ [35]. Another dimensional reduction on S^1 gives $CSO(p, q, 5 - p - q)$ gauged supergravity in six dimensions as a consistent truncation of 11-dimensional supergravity on $H^{p,q} \times \mathbb{R}^{5-p-q} \times S^1$. Together with the fact that type IIA theory can be obtained from an S^1 reduction of M theory, the six-dimensional $CSO(p, q, 5 - p - q)$ gauged supergravity can be considered as a truncation of type IIA theory on $H^{p,q} \times \mathbb{R}^{5-p-q}$ after a truncation of M theory on S^1 . The full truncation *Ansatz* for the case of $S^4 \sim H^{5,0}$ has been constructed in [36]. However, the complete truncation *Ansätze* for both type IIA theory on $H^{p,q} \times \mathbb{R}^{4-p-q}$ or 11-dimensional supergravity on $H^{p,q} \times \mathbb{R}^{5-p-q}$ have not been worked to date although the relevant twist matrices and the truncation *Ansätze* for the metric and internal components of the form fields have already been given in [34,35]. Therefore, the complete truncations to six dimensions are currently not available either. In any case, all solutions of these six-dimensional gauged supergravities can, in principle, be embedded in type IIA theory as described above.

The solutions of interest in this paper take the form of $\mathbb{R} \times \mathcal{M}_4$ -sliced domain walls with the $\mathbb{R}$ factor being the time coordinate. We look for solution interpolating between asymptotically locally flat domain walls dual to maximal SYM in five dimensions and a curved domain wall with $t \times \mathcal{M}_4$ -sliced world-volume. Holographically, these solutions would describe renormalization group (RG) flows across dimensions from maximal five-dimensional SYM to

one-dimensional SYM or supersymmetric quantum mechanics in the IR. In general, due to the presence of topological twists, the latter is obtained from twisted compactifications of the former on $\mathcal{M}_4$. We also point out that, in holographic studies involving nonconformal field theories, supergravity solutions generically contain singular geometries in the IR.

To determine whether the resulting solutions are physically acceptable, we employ the condition introduced in [18]. According to this condition, the (00)-component of the type IIA metric $\hat{g}_{00}$ should not diverge near the IR singularities since fixed proper energy excitations in supergravity should correspond to lower and lower energy excitations in the dual field theory. Upon uplifted to type IIA theory, the solutions with physical IR singularities would describe different configurations involving D4-branes wrapped on $\mathcal{M}_4$. It should also be noted that although the complete *Ansätze* for the aforementioned truncations are not available, we can determine the ten-dimensional metric component $\hat{g}_{00}$ using partial results given in [34,35] and check whether a given singularity is physical or not. For convenience, we also give the relevant expressions for $\hat{g}_{00}$ for all of the solutions in Appendix B. In addition, it should be pointed out that there is another criterion to determine whether a given singularity is physically acceptable or not directly within the framework of lower-dimensional gauged supergravity theories. This is the Gubser's criterion given in [37]. According to this criterion, a singularity in gauged supergravity is physical if the scalar potential is bounded above when the singularity is approached. In order to apply this criterion, we need to construct the corresponding two-dimensional Lagrangian by a compactification of the Lagrangian of the six-dimensional gauged supergravity on $\mathcal{M}_4$ and identify the effective scalar potential in two dimensions, see, for example, the Appendix of [18]. However, due to some complications in transforming the effective two-dimensional Lagrangian to the Einstein frame, it is not clear how to implement this criterion in the present case. Accordingly, we do not consider Gubser's criterion in this paper.

The paper is organized as follows. Section II briefly reviews maximal $N = (2, 2)$ gauged supergravity in six dimensions. Then, the details of gaugings under $GL(5) \subset SO(5, 5)$ are given together with the embedding of $CSO(p, q, 5 - p - q)$ and $CSO(p, q, 4 - p - q) \ltimes \mathbb{R}^4$ gauge groups in the global symmetry group $SO(5, 5)$. The solutions describing D4-branes wrapped on $\mathcal{M}_4$ from the resulting gauged supergravities are, respectively, given in Secs. III and IV. Conclusions and discussions are given in Sec. V. For convenience, we collect some useful formulas and relations in Appendixes A and B together with a number of numerical solutions in Appendixes C and D.

II. SIX-DIMENSIONAL $N=(2,2)$ GAUGED SUPERGRAVITY WITH $CSO(p, q, 5-p-q)$ AND $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ GAUGE GROUPS

We first review the general structure of six-dimensional $N=(2,2)$ gauged supergravity in the embedding tensor formalism constructed in [28]. We mainly setup the notations and collect relevant formulas for finding supersymmetric solutions. More detailed construction can be found in [28].

A. Maximal gauged supergravity in six dimensions

Maximal supersymmetry only allows for the supergravity multiplet. In six dimensions, the bosonic field content consists of the vielbein $e_\mu^\hat{\mu}$, two-form potentials $B_{\mu\nu,m}$ and $B_{\mu\nu}^m$ ($m=1, \dots, 5$), three-forms $C_{\mu\nu\rho,A}$ ($A=1, \dots, 16$), vector fields A_μ^A , and scalars $V_A^{\alpha\dot{\alpha}}$ ($\alpha=1, \dots, 4$ and $\dot{\alpha}=\dot{1}, \dots, \dot{4}$) that parametrize the coset space $SO(5,5)/SO(5) \times SO(5)$. Curved and flat spacetime indices are denoted by μ, ν and $\hat{\mu}, \hat{\nu}$, respectively. Lower and upper m indices, respectively, label fundamental and antifundamental representations of $GL(5) \subset SO(5,5)$. The index A describes Majorana–Weyl spinors of the $SO(5,5)$ duality symmetry, while the indices α and $\dot{\alpha}$ are, respectively, two sets of $SO(5)$ spinor indices in the $SO(5) \times SO(5)$ local symmetry.

Fermionic field content contains the gravitini $\psi_{+\mu\alpha}$ and $\psi_{-\mu\dot{\alpha}}$, and the spin- $\frac{1}{2}$ fields $\chi_{+a\dot{\alpha}}$ and $\chi_{-\dot{a}\alpha}$ where $a=1, \dots, 5$ and $\dot{a}=\dot{1}, \dots, \dot{5}$, respectively, denote two sets of $SO(5)$ vector indices in $SO(5) \times SO(5)$. We use $\pm$ to indicate spacetime chiralities of the spinors. In addition, all spinors are symplectic-Majorana–Weyl. With all these, all component fields can be written as

$$\left(e_\mu^\hat{\mu}, B_{\mu\nu,M}, C_{\mu\nu\rho,A}, A_\mu^A, V_A^{\alpha\dot{\alpha}}, \psi_{+\mu\alpha}, \psi_{-\mu\dot{\alpha}}, \chi_{+a\dot{\alpha}}, \chi_{-\dot{a}\alpha} \right). \quad (1)$$

We also note that due to the self-duality of the two-form fields, the Lagrangian of the ungauged supergravity is only invariant under the $GL(5)$ subgroup of the full duality symmetry $SO(5,5)$. In this case, only the electric two-forms $B_{\mu\nu,m}$ appear in the Lagrangian, while the magnetic duals $B_{\mu\nu}^m$ can be introduced on shell.

In the embedding tensor formalism, gaugings can be described by introducing the minimal coupling of various fields via the covariant derivative,

$$D_\mu = \partial_\mu - g A_\mu^A \Theta_A^{MN} t_{MN}, \quad (2)$$

where g is a gauge coupling constant, and Θ_A^{MN} is the embedding tensor. The corresponding gauge generators are given by $X_A = \Theta_A^{MN} t_{MN}$. In general, these are particular linear combinations of the $SO(5,5)$ generators t_{MN} and form a closed Lie algebra of the gauge group

$G_0 \subset SO(5,5)$. The latter implies that the embedding tensor needs to satisfy the quadratic constraint of the form

$$[X_A, X_B] = -(X_A)_B^C X_C, \quad (3)$$

with $(X_A)_B^C$ denoting gauge generators in chiral spinor representation of $SO(5,5)$ given below.

In addition, consistency with supersymmetry requires the embedding tensor to transform as $\mathbf{144}_c$ representation of $SO(5,5)$. Accordingly, Θ_A^{MN} can be parametrized in terms of a vector-spinor θ^{AM} of $SO(5,5)$ as

$$\Theta_A^{MN} = -\theta^{B[M} (\Gamma^{N]})_{BA} \equiv (\Gamma^{[M} \theta^{N]})_A, \quad (4)$$

with θ^{AM} subject to the constraint

$$(\Gamma_M)_{AB} \theta^{BM} = 0, \quad (5)$$

where $(\Gamma_M)_{AB}$ is an $SO(5,5)$ gamma matrix. In terms of θ^{AM} , the quadratic constraint reduces to the following two conditions:

$$\theta^{AM} \theta^{BN} \eta_{MN} = 0 \quad \text{and} \quad \theta^{AM} \theta^{B[N} (\Gamma^{P]})_{AB} = 0. \quad (6)$$

η_{MN} is the off-diagonal $SO(5,5)$ invariant tensor given by

$$\eta_{MN} = \eta^{MN} = \begin{pmatrix} 0 & \mathbb{1}_5 \\ \mathbb{1}_5 & 0 \end{pmatrix}. \quad (7)$$

With $SO(5,5)$ generators in vector and spinor representations given by

$$(t_{MN})_P^Q = 4\eta_{P[M} \delta_{N]}^Q \quad \text{and} \quad (t_{MN})_A^B = (\Gamma_{MN})_A^B, \quad (8)$$

the corresponding gauge generators take the forms

$$(X_A)_M^N = 2(\Gamma_M \theta^N)_A + 2(\Gamma^N \theta_M)_A \\ \text{and} \quad (X_A)_B^C = (\Gamma^M \theta^N)_A (\Gamma_{MN})_B^C. \quad (9)$$

Unlike the ungauged supergravity, the most general gauged supergravity can involve a symmetry that lies outside the off-shell $GL(5)$ symmetry. Under $GL(5)$, θ^{AM} can be decomposed as $\theta^{AM} = (\theta^{Am}, \theta_m^A)$. Gaugings triggered by θ^{Am} are called electric in the sense that only electric two-forms participate in the gauged theory, while magnetic gaugings obtained from θ_m^A also involve magnetic two-forms together with additional three-form tensor fields $C_{\mu\nu\rho,A}$. In this case, the magnetic two-forms $B_{\mu\nu}^m$ can appear in the gauged Lagrangian by coupling to the three-form potentials via the topological term. The electric and magnetic two-forms are combined into a vector representation of the full global symmetry group $SO(5,5)$ denoted by $B_{\mu\nu,M} = (B_{\mu\nu,m}, B_{\mu\nu}^m)$.

The gauge covariance of the field strength tensors for all these fields requires the presence of higher-degree form fields leading to hierarchies of non-Abelian vector and tensor fields of various ranks. In particular, the covariant field strengths of vector and two-form fields are defined by

$$\mathcal{H}_{\mu\nu}{}^A = F_{\mu\nu}{}^A - \sqrt{2}g\theta^{AM}B_{\mu\nu,M}, \quad (10)$$

$$\begin{aligned} \mathcal{H}_{\mu\nu\rho,M} &= 3D_{[\mu}B_{\nu\rho],M} + 3\sqrt{2}(\Gamma_M)_{AB}A_{[\mu}{}^A \\ &\times \left(\partial_{\nu}A_{\rho]}{}^B + \frac{1}{3}gX_{[CD]}{}^BA_{\nu}{}^CA_{\rho]}{}^D \right) \\ &- \sqrt{2}g\eta_{MN}\theta^{AN}C_{\mu\nu\rho,A}, \end{aligned} \quad (11)$$

with the usual non-Abelian gauge field strengths,

$$F_{\mu\nu}{}^A = 2\partial_{[\mu}A_{\nu]}{}^A + gX_{[BC]}{}^AA_{\mu}{}^BA_{\nu}{}^C. \quad (12)$$

These field strengths satisfy the following Bianchi identities:

$$D_{[\mu}\mathcal{H}_{\nu\rho]}{}^A = -\frac{\sqrt{2}}{3}g\theta^{AM}\mathcal{H}_{\mu\nu\rho,M}, \quad (13)$$

$$\begin{aligned} D_{[\mu}\mathcal{H}_{\nu\rho\lambda],M} &= \frac{3}{2\sqrt{2}}(\Gamma_M)_{AB}\mathcal{H}_{[\mu\nu}{}^A\mathcal{H}_{\rho\lambda]}{}^B \\ &- \frac{1}{2\sqrt{2}}g\eta_{MN}\theta^{AN}\mathcal{H}_{\mu\nu\rho\lambda,A}, \end{aligned} \quad (14)$$

with the four-form field strengths defined by

$$\begin{aligned} \mathcal{H}_{\mu\nu\rho\lambda,A} &= 4D_{[\mu}C_{\nu\rho\lambda],A} - (\Gamma^M)_{AB}\left[6\sqrt{2}B_{\mu\nu,M}\mathcal{H}_{\rho\lambda}{}^B \right. \\ &+ 6g\theta^{BN}B_{[\mu\nu,M}B_{\rho\lambda],N} + 8(\Gamma_M)_{CD}A_{[\mu}{}^BA_{\nu}{}^CA_{\rho]}{}^D \\ &\left. + 2(\Gamma_M)_{CF}X_{DE}{}^FA_{[\mu}{}^BA_{\nu}{}^CA_{\rho]}{}^DA_{\lambda]}{}^E\right]. \end{aligned} \quad (15)$$

In addition, all of the aforementioned field strength tensors are subject to the following duality relations:

$$0 = g\theta^A{}_m\left(\mathcal{H}_{\mu\nu\rho}{}^m - \frac{1}{6}K^{mn}\epsilon_{\mu\nu\rho\lambda\sigma\tau}\mathcal{H}^{\lambda\sigma\tau}{}_n\right), \quad (16)$$

$$0 = g\theta^A{}_m\left(\mathcal{H}_{\mu\nu\rho\lambda,A} + \frac{1}{2}M_{AB}\epsilon_{\mu\nu\rho\lambda\sigma\tau}\mathcal{H}^{\sigma\tau,B}\right), \quad (17)$$

where K^{mn} and M_{AB} are scalar matrices written in terms of the scalar coset representative.

The matrix M_{AB} is defined by

$$M_{AB} = \Omega_{\alpha\beta}\Omega_{\dot{\alpha}\dot{\beta}}V_A{}^{\alpha\dot{\alpha}}V_B{}^{\beta\dot{\beta}}, \quad (18)$$

with $\Omega_{\alpha\beta}$ and $\Omega_{\dot{\alpha}\dot{\beta}}$ being the two $USp(4) \sim SO(5)$ symplectic forms used for lowering indices α and $\dot{\alpha}$,

respectively. Raising of indices α and $\dot{\alpha}$ is implemented by using $\Omega^{\alpha\beta} = (\Omega_{\alpha\beta})^*$ and $\Omega^{\dot{\alpha}\dot{\beta}} = (\Omega_{\dot{\alpha}\dot{\beta}})^*$. On the other hand, the matrix K^{mn} is defined in terms of the coset representative in vector representation of $SO(5,5)$ of the form

$$\mathcal{V}_M{}^{\underline{A}} = \begin{pmatrix} \mathcal{V}_m{}^a & \mathcal{V}_m{}^{\dot{a}} \\ \mathcal{V}^{ma} & \mathcal{V}^{m\dot{a}} \end{pmatrix}, \quad (19)$$

with $\underline{A} = (a, \dot{a})$. This is related to the coset representative in the chiral spinor representation $V_A{}^{\alpha\dot{\alpha}}$ by the following relations:

$$\mathcal{V}_M{}^a = \frac{1}{16}V^{A\alpha\dot{\alpha}}(\Gamma_M)_{AB}(\gamma^a)_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}}V^B{}_{\beta\dot{\beta}}, \quad (20)$$

$$\mathcal{V}_M{}^{\dot{a}} = -\frac{1}{16}V^{A\alpha\dot{\alpha}}(\Gamma_M)_{AB}(\gamma^{\dot{a}})_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}}V^B{}_{\beta\dot{\beta}}, \quad (21)$$

in which $V^{A\alpha\dot{\alpha}} = \Omega^{\alpha\beta}\Omega^{\dot{\alpha}\dot{\beta}}V^A{}_{\beta\dot{\beta}}$ and $V^A{}_{\alpha\dot{\alpha}}$ are inverse elements of the coset representative $V_A{}^{\alpha\dot{\alpha}}$. In these relations, $(\Gamma_M)_{AB}$ and $(\Gamma_{\underline{A}})_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}} = ((\gamma_a)_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}}, (\gamma_{\dot{a}})_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}})$ are, respectively, $SO(5,5)$ gamma matrices in nondiagonal η_{MN} and diagonal $\eta_{\underline{A}\underline{B}}$ bases, see Appendix A of [29] for more detail.

With all these, the matrix K^{mn} is defined by

$$K^{mn} = \mathcal{V}^{ma}(V_n{}^a)^{-1}P_+ - \mathcal{V}^{m\dot{a}}(V_n{}^{\dot{a}})^{-1}P_-, \quad (22)$$

in which $P_{\pm}$ are projections on the self- and antiself-dual parts given by

$$P_{\pm} = \frac{1}{2}(1 \pm j). \quad (23)$$

The operation j implements the duality transformation as

$$jH_{\mu\nu\rho} = \tilde{H}_{\mu\nu\rho} = \frac{1}{6}\epsilon\epsilon_{\mu\nu\rho\lambda\sigma\kappa}H^{\lambda\sigma\kappa}. \quad (24)$$

The bosonic Lagrangian of the general six-dimensional $N = (2, 2)$ gauged supergravity takes the form

$$\begin{aligned} e^{-1}\mathcal{L} &= \frac{1}{4}R - \frac{1}{16}P_{\mu}{}^{a\dot{a}}P^{\mu}{}_{a\dot{a}} - \frac{1}{4}M_{AB}\mathcal{H}_{\mu\nu}{}^A\mathcal{H}^{\mu\nu,B} \\ &- \frac{1}{12}K^{mn}\mathcal{H}_{\mu\nu\rho,m}\mathcal{H}^{\mu\nu\rho}{}_n - \mathbf{V} + e^{-1}\mathcal{L}_{\text{top}}, \end{aligned} \quad (25)$$

where $\mathcal{L}_{\text{top}}$ is the topological term given in [28]. The kinetic term for the scalar fields is written in terms of the scalar vielbein,

$$P_{\mu}{}^{a\dot{a}} = \frac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^{\dot{a}})_{\dot{\alpha}\dot{\beta}}V^{A\alpha\dot{\alpha}}D_{\mu}V_A{}^{\beta\dot{\beta}}, \quad (26)$$

where $(\gamma^a)_{\alpha\beta} = \frac{1}{4}\Omega_{\beta\gamma}(\gamma^a)_{\alpha\dot{\alpha}}^{\gamma\dot{\alpha}}$, and $(\gamma^{\dot{a}})_{\dot{\alpha}\dot{\beta}} = \frac{1}{4}\Omega_{\dot{\beta}\dot{\gamma}}(\gamma^{\dot{a}})_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}}$. The scalar potential is given in terms of the embedding tensor and the coset representatives by

$$\mathbf{V} = \frac{g^2}{2}\theta^{AM}\theta^{BN}\mathcal{V}_M{}^a\mathcal{V}_N{}^b[V_A{}^{\alpha\dot{\alpha}}(\gamma_a)_{\alpha}{}^{\beta}(\gamma_b)_{\beta}{}^{\gamma}V_{B\gamma\dot{\alpha}}]. \quad (27)$$

For all solutions given in the present paper, the three-form field strengths identically vanish. For simplicity, we accordingly set $\mathcal{H}_{\mu\nu\rho,M} = 0$ from now on. It is straightforward to verify that this is indeed a consistent truncation in all cases considered here.

With this, supersymmetry transformations of the fermionic fields read

$$\begin{aligned} \delta\psi_{+\mu\alpha} &= D_{\mu}\epsilon_{+\alpha} + \frac{g}{4}\hat{\gamma}_{\mu}T_{\alpha}{}^{\dot{\beta}}\epsilon_{-\dot{\beta}} \\ &\quad + \frac{1}{8}(\hat{\gamma}_{\mu}{}^{\nu\rho} - 6\delta_{\mu}^{\nu}\hat{\gamma}^{\rho})\mathcal{H}_{\nu\rho}{}^A\Omega_{\alpha\dot{\beta}}V_A{}^{\beta\dot{\beta}}\epsilon_{-\dot{\beta}}, \end{aligned} \quad (28)$$

$$\begin{aligned} \delta\psi_{-\mu\dot{\alpha}} &= D_{\mu}\epsilon_{-\dot{\alpha}} - \frac{g}{4}\hat{\gamma}_{\mu}T_{\dot{\alpha}}{}^{\beta}\epsilon_{+\beta} \\ &\quad + \frac{1}{8}(\hat{\gamma}_{\mu}{}^{\nu\rho} - 6\delta_{\mu}^{\nu}\hat{\gamma}^{\rho})\mathcal{H}_{\nu\rho}{}^A\Omega_{\dot{\alpha}\beta}V_A{}^{\beta\dot{\beta}}\epsilon_{+\beta}, \end{aligned} \quad (29)$$

$$\begin{aligned} \delta\chi_{+a\dot{\alpha}} &= \frac{1}{4}P^{\mu}{}_{a\dot{a}}\hat{\gamma}_{\mu}(\gamma^{\dot{a}})_{\dot{\alpha}}{}^{\dot{\beta}}\epsilon_{-\dot{\beta}} + 2g(T_a)_{\dot{\alpha}}{}^{\beta}\epsilon_{+\beta} \\ &\quad - \frac{g}{2}T_{\dot{\alpha}}{}^{\beta}(\gamma_a)_{\beta}{}^{\dot{\gamma}}\epsilon_{+\dot{\gamma}} + \frac{1}{4}\mathcal{H}_{\mu\nu}{}^A\hat{\gamma}^{\mu\nu}\Omega_{\dot{\alpha}\beta}V_A{}^{\alpha\dot{\beta}}(\gamma_a)_{\alpha}{}^{\beta}\epsilon_{+\beta}, \end{aligned} \quad (30)$$

$$\begin{aligned} \delta\chi_{-\dot{a}\alpha} &= \frac{1}{4}P^{\mu}{}_{\dot{a}\alpha}\hat{\gamma}_{\mu}(\gamma^{\alpha})_{\alpha}{}^{\beta}\epsilon_{+\beta} + 2g(T_{\dot{a}})_{\alpha}{}^{\dot{\beta}}\epsilon_{-\dot{\beta}} \\ &\quad + \frac{g}{2}T_{\alpha}{}^{\dot{\beta}}(\gamma_{\dot{a}})_{\dot{\beta}}{}^{\dot{\gamma}}\epsilon_{-\dot{\gamma}} + \frac{1}{4}\mathcal{H}_{\mu\nu}{}^A\hat{\gamma}^{\mu\nu}\Omega_{\alpha\dot{\beta}}V_A{}^{\beta\dot{\alpha}}(\gamma_{\dot{a}})_{\dot{\alpha}}{}^{\dot{\beta}}\epsilon_{-\dot{\beta}}. \end{aligned} \quad (31)$$

The covariant derivatives of supersymmetry parameters, $\epsilon_{+\alpha}$ and $\epsilon_{-\dot{\alpha}}$, are

$$D_{\mu}\epsilon_{+\alpha} = \partial_{\mu}\epsilon_{+\alpha} + \frac{1}{4}\omega_{\mu}{}^{\nu\rho}\hat{\gamma}_{\nu\rho}\epsilon_{+\alpha} + \frac{1}{4}Q_{\mu}{}^{ab}(\gamma_{ab})_{\alpha}{}^{\beta}\epsilon_{+\beta}, \quad (32)$$

$$D_{\mu}\epsilon_{-\dot{\alpha}} = \partial_{\mu}\epsilon_{-\dot{\alpha}} + \frac{1}{4}\omega_{\mu}{}^{\nu\rho}\hat{\gamma}_{\nu\rho}\epsilon_{-\dot{\alpha}} + \frac{1}{4}Q_{\mu}{}^{\dot{a}b}(\gamma_{\dot{a}b})_{\dot{\alpha}}{}^{\dot{\beta}}\epsilon_{-\dot{\beta}}, \quad (33)$$

with $\hat{\gamma}_{\mu} = e_{\mu}^{\hat{\mu}}\hat{\gamma}_{\hat{\mu}}$. Matrices $\hat{\gamma}_{\hat{\mu}}$ are spacetime gamma matrices, see the convention in Appendix B of [29]. For simplicity, we suppress all spacetime spinor indices. The $SO(5) \times SO(5)$ composite connections $Q_{\mu}{}^{ab}$ and $Q_{\mu}{}^{\dot{a}b}$ are given by

$$Q_{\mu}{}^{ab} = \frac{1}{8}(\gamma^{ab})_{\alpha\beta}\Omega_{\dot{\alpha}\dot{\beta}}V^{A\alpha\dot{\alpha}}D_{\mu}V_A{}^{\beta\dot{\beta}}, \quad (34)$$

$$Q_{\mu}{}^{\dot{a}b} = \frac{1}{8}\Omega_{\alpha\beta}(\gamma^{\dot{a}b})_{\dot{\alpha}\dot{\beta}}V^{A\alpha\dot{\alpha}}D_{\mu}V_A{}^{\beta\dot{\beta}}. \quad (35)$$

The T tensors in the fermionic supersymmetry transformations are defined as

$$(T^a)^{\alpha\dot{\alpha}} = \mathcal{V}_M{}^a\theta^{AM}V_A{}^{\alpha\dot{\alpha}}, \quad (T^{\dot{a}})^{\alpha\dot{\alpha}} = -\mathcal{V}_M{}^{\dot{a}}\theta^{AM}V_A{}^{\alpha\dot{\alpha}}, \quad (36)$$

with

$$T^{\alpha\dot{\alpha}} \equiv (T^a)^{\beta\dot{\beta}}(\gamma_a)_{\beta}{}^{\alpha} = -(T^{\dot{a}})^{\alpha\dot{\beta}}(\gamma_{\dot{a}})_{\dot{\beta}}{}^{\dot{\alpha}}. \quad (37)$$

In terms of the T tensors, the scalar potential is given by

$$\mathbf{V} = g^2 \left[(T^a)^{\alpha\dot{\alpha}}(T_a)_{\alpha\dot{\alpha}} - \frac{1}{2}T^{\alpha\dot{\alpha}}T_{\alpha\dot{\alpha}} \right]. \quad (38)$$

B. Gauging $CSO(p, q, 5-p-q)$ and $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ groups

In this paper, we are only interested in $CSO(p, q, 5-p-q)$ and $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ gauge groups that can be embedded in $GL(5) \subset SO(5, 5)$. The embedding tensor θ^{AM} in $\mathbf{144}_c$ representation of $SO(5, 5)$ decomposes under $GL(5)$ as

$$\mathbf{144}_c \rightarrow \bar{\mathbf{5}}^{+3} \oplus \mathbf{5}^{+7} \oplus \mathbf{10}^{-1} \oplus \mathbf{15}^{-1} \oplus \mathbf{24}^{-5} \oplus \overline{\mathbf{40}}^{-1} \oplus \overline{\mathbf{45}}^{+3}. \quad (39)$$

The $CSO(p, q, 5-p-q)$ gauge group with $p+q \leq 5$ corresponds to the embedding tensor in $\mathbf{15}^{-1}$ representation, while the $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ gauge group with $p+q \leq 4$ is obtained from the embedding tensor in $\overline{\mathbf{40}}^{-1}$ representation. Both of these gauge groups arise from a dimensional reduction on S^1 of the maximal gauged supergravity in seven dimensions with $CSO(p, q, 5-p-q)$ and $CSO(p, q, 4-p-q)$ gauge groups constructed in [33]. Moreover, as previously mentioned, the six-dimensional gauged supergravity with these two gauge groups can also be embedded in type IIA theory via consistent truncations on $H^{p,q} \times \mathbb{R}^{5-p-q}$ and $H^{p,q} \times \mathbb{R}^{4-p-q} \times S^1$, respectively.

Decomposing the $SO(5, 5)$ spinor representation under $GL(5)$ as $\mathbf{16}_s \rightarrow \bar{\mathbf{5}}^{+3} \oplus \mathbf{10}^{-1} \oplus \mathbf{1}^{-5}$, we find that the gauge generators can be written as

$$X_A = \mathbb{T}_{Am}X^m + \mathbb{T}_A^{mn}X_{mn} + \mathbb{T}_{A*}X_*. \quad (40)$$

Generators X^m , X_{mn} , and X_* correspond, respectively, to $\bar{\mathbf{5}}^{+3}$, $\mathbf{10}^{-1}$, and $\mathbf{1}^{-5}$ representations of $GL(5)$. The decomposition matrices $\mathbb{T}_{Am}$, $\mathbb{T}_A^{mn}$, and $\mathbb{T}_{A*}$ are given in Appendix A. Similarly, the vector fields in $\mathbf{16}_c$ representation decompose as

$$A_\mu^A = \mathbb{T}^{Am} A_{\mu,m} + \mathbb{T}_{mn}^A A_\mu^{mn} + \mathbb{T}_*^A A_\mu^*, \quad (41)$$

with matrices $\mathbb{T}^{Am}$, $\mathbb{T}_{mn}^A$, and $\mathbb{T}_*^A$ being inverses of $\mathbb{T}_{Am}$, $\mathbb{T}_A^{mn}$, and $\mathbb{T}_{A*}$ given by complex conjugations, $\mathbb{T}^A = (\mathbb{T}_A)^{-1} = (\mathbb{T}_A)^*$. Accordingly, the gauge covariant derivative can be written as

$$D_\mu = \nabla_\mu - g A_{\mu,m} X^m - g A_\mu^{mn} X_{mn} - g A_\mu^* X_*. \quad (42)$$

To identify relevant components of the embedding tensor, we recall the decomposition of $SO(5,5)$ vector representation $\mathbf{10} \rightarrow \mathbf{5}^{+2} \oplus \mathbf{\bar{5}}^{-2}$ under $GL(5)$. We can then write the embedding tensor as $\theta^{AM} = (\theta^{Am}, \theta^A_m)$ with the two components θ^{Am} and θ^A_m containing the following irreducible $GL(5)$ representations:

$$\theta^{Am}: \mathbf{\bar{5}}^{+3} \oplus \mathbf{10}^{-1} \oplus \mathbf{24}^{-5} \oplus \mathbf{\bar{40}}^{-1}, \quad (43)$$

$$\theta^A_m: \mathbf{\bar{5}}^{+3} \oplus \mathbf{5}^{+7} \oplus \mathbf{10}^{-1} \oplus \mathbf{15}^{-1} \oplus \mathbf{\bar{45}}^{+3}. \quad (44)$$

Gaugings in $\mathbf{15}^{-1}$ representation can be parametrized by a symmetric tensor $Y_{mn} = Y_{(mn)}$ leading to the nonvanishing components of the embedding tensor of the form

$$\theta^{Am} = 0 \quad \text{and} \quad \theta^A_m = \mathbb{T}^{An} Y_{mn}. \quad (45)$$

On the other hand, gaugings from $\mathbf{\bar{40}}^{-1}$ representation are characterized by a tensor $U^{np,m} = U^{[np],m}$ satisfying $U^{[np,m]} = 0$. This leads to the embedding tensor of the form

$$\theta^{Am} = \mathbb{T}_{np}^A U^{np,m} \quad \text{and} \quad \theta^A_m = 0. \quad (46)$$

We also point out that these gaugings are, respectively, purely magnetic and purely electric, see [29] for more detail.

To construct an explicit form of the scalar coset representative, we first consider the decomposition of 25 non-compact generators of $SO(5,5)$ under $GL(5)$ as

$$25 \rightarrow \underbrace{\mathbf{1} + \mathbf{14}}_{\hat{t}_{ab}} + \underbrace{\mathbf{10}}_{s_{mn}}, \quad (47)$$

$\hat{t}_{ab}$ denote noncompact generators of $GL(5)$ defined by

$$\hat{t}_{ab} = \frac{1}{2} (\mathbb{M}_a^M \mathbb{M}_b^N + \mathbb{M}_b^M \mathbb{M}_a^N) t_{MN}, \quad (48)$$

where $\mathbb{M}_A^M = (\mathbb{M}_a^M, \mathbb{M}_{\dot{a}}^M)$ is a transformation matrix whose explicit form is given in (A17).

In accordance with (47), we denote all the 25 scalar fields as

$$\Phi^I = \{\varphi, \phi_1, \dots, \phi_{14}, \varsigma_1, \dots, \varsigma_{10}\}, \quad (49)$$

with $I = 1, \dots, 25$. The scalar φ is the dilaton corresponding to the $SO(1,1)$ factor in $GL(5) \sim SL(5) \times SO(1,1)$ generated by

$$d = \hat{t}_{11} + \hat{t}_{22} + \hat{t}_{33} + \hat{t}_{44} + \hat{t}_{55}. \quad (50)$$

On the other hand, 14 scalars $\{\phi_1, \dots, \phi_{14}\}$ corresponding to generators $\hat{t}_{ab}$ parametrize the $SL(5)/SO(5)$ submanifold of $SO(5,5)/SO(5) \times SO(5)$. Finally, the remaining ten scalars $\{\varsigma_1, \dots, \varsigma_{10}\}$ correspond to the shift generators $s_{mn} = t_{mn}$ and are called shift scalars. From a higher-dimensional point of view, these scalars arise from the S^1 reduction of the vector fields in the maximal seven-dimensional gauged supergravity.

III. SUPERSYMMETRIC QUANTUM MECHANICS FROM $CSO(p, q, 5-p-q)$ GAUGED SUPERGRAVITY

We begin the study of supersymmetric solutions describing D4-branes wrapped on a four-manifold $\mathcal{M}_4$ from the maximal gauged supergravity with $CSO(p, q, 5-p-q)$ gauge group. From the embedding tensor given in (45), we can choose a basis in which the symmetric tensor Y_{mn} takes the form

$$Y_{mn} = \text{diag}(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q, \underbrace{0, \dots, 0}_r), \quad (51)$$

for $p + q + r = 5$. In this case, the only nonvanishing gauge generators are given by

$$X_{mn} = \mathbb{T}_{mn}^A X_A = 2Y_{p[m} t_{n]}, \quad (52)$$

with t_n^m being $GL(5)$ generators. Therefore, only ten gauge fields $A_\mu^{mn} = A_\mu^{[mn]}$ appear in the gauge covariant derivatives

$$D_\mu = \nabla_\mu - g A_\mu^{mn} X_{mn}. \quad (53)$$

The gauge generators satisfy the relations

$$[X_{mn}, X_{pq}] = (X_{mn})_{pq}{}^{rs} X_{rs}, \quad (54)$$

with $(X_{mn})_{pq}{}^{rs} = 2(X_{mn})_{[p}{}^{[r} \delta_{q]}^s]$. Therefore, the corresponding gauge group is determined to be

$$G_0 = CSO(p, q, r) = SO(p, q) \ltimes \mathbb{R}^{(p+q) \cdot r}. \quad (55)$$

Recall that gaugings in $\mathbf{15}^{-1}$ representation are magnetic, and we find that, in general, the two- and three-form fields $B_{\mu\nu}^m$ and $C_{\mu\nu\rho,A}$ can appear in the Lagrangian. With $\theta^{Am} = 0$, the two-form field strengths take the form

$$\mathcal{H}_{\mu\nu}^A = F_{\mu\nu}^A - \sqrt{2}g\mathbb{T}^{An}Y_{nm}B_{\mu\nu}^m, \quad (56)$$

which can be decomposed as

$$\begin{aligned} \mathcal{H}_{\mu\nu,m} &= \mathbb{T}_{Am}\mathcal{H}_{\mu\nu}^A = -\sqrt{2}gY_{mn}B_{\mu\nu}^n, \\ \mathcal{H}_{\mu\nu}^{mn} &= \mathbb{T}_A^{mn}\mathcal{H}_{\mu\nu}^A = F_{\mu\nu}^{mn}. \end{aligned} \quad (57)$$

$F_{\mu\nu}^{mn}$ denote the usual non-Abelian gauge field strengths given by

$$F_{\mu\nu}^{mn} = \mathbb{T}_A^{mn}F_{\mu\nu}^A = 2\partial_{[\mu}A_{\nu]}^{mn} + g(X_{pq})_{rs}^{mn}A_{\mu}^{pq}A_{\nu}^{rs}. \quad (58)$$

A. D4-branes wrapped on a Riemannian four-manifold

We first consider the case of supersymmetric solutions describing D4-branes wrapped on a Riemannian four-manifold Σ_k^4 with constant curvature. The curvature of Σ_k^4 is characterized by a constant k with values $k = 1, 0, -1$ corresponding to a four-dimensional sphere S^4 , a flat space $\mathbb{R}^4$, or a hyperbolic space H^4 , respectively. Although there are many possibilities to perform topological twists in this case, we restrict to the case of Σ_k^4 being a constant curvature manifold for simplicity and also for the sake of comparison with a similar work in seven dimensions studied in [38].

The metric *Ansatz* for the six-dimensional spacetime is given by

$$ds_6^2 = -e^{2U(r)}dt^2 + dr^2 + e^{2V(r)}ds_{\Sigma_k^4}^2. \quad (59)$$

$ds_{\Sigma_k^4}^2$ is the metric on Σ_k^4 with the explicit form written as

$$ds_{\Sigma_k^4}^2 = d\chi^2 + f_k(\chi)^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\zeta^2)], \quad (60)$$

for $\chi, \psi, \theta \in [0, \frac{\pi}{2}]$, $\zeta \in [0, 2\pi]$. The function $f_k(\chi)$ is defined by

$$f_k(\chi) = \begin{cases} \sin\chi, & \text{for } k = +1 \\ \chi, & \text{for } k = 0 \\ \sinh\chi, & \text{for } k = -1 \end{cases}. \quad (61)$$

With the following choice of vielbein:

$$\begin{aligned} e^{\hat{t}} &= e^U dt, & e^{\hat{r}} &= dr, & e^{\hat{\chi}} &= e^V d\chi, & e^{\hat{\psi}} &= e^V f_k(\chi) d\psi, \\ e^{\hat{\theta}} &= e^V f_k(\chi) \sin\psi d\theta, & e^{\hat{\zeta}} &= e^V f_k(\chi) \sin\psi \sin\theta d\zeta, \end{aligned} \quad (62)$$

we obtain the nonvanishing components of the spin connection given by

$$\omega_{\hat{t}}^{\hat{r}} = U', \quad \omega_{\hat{r}}^{\hat{t}} = V'\delta_{\hat{t}}^{\hat{r}},$$

$$\omega_{\hat{\psi}}^{\hat{\chi}} = \omega_{\hat{\theta}}^{\hat{\chi}} = \omega_{\hat{\zeta}}^{\hat{\chi}} = e^{-V} \frac{f'_k(\chi)}{f_k(\chi)},$$

$$\omega_{\hat{\theta}}^{\hat{\psi}} = \omega_{\hat{\zeta}}^{\hat{\psi}} = e^{-V} \frac{\cot\psi}{f_k(\chi)}, \quad \omega_{\hat{\zeta}}^{\hat{\theta}} = e^{-V} \frac{\cot\theta}{f_k(\chi) \sin\psi}. \quad (63)$$

We have split the six-dimensional coordinates as $x^\mu = (t, r, x^i)$, $i = \chi, \psi, \theta, \zeta$, with indices $\hat{i}, \hat{j}, \dots = \hat{\chi}, \hat{\psi}, \hat{\theta}, \hat{\zeta}$ being flat indices on Σ_k^4 . The r derivatives will generally be denoted by $'$ with an exception that a $'$ on any function with an explicit argument refers to the derivative of the function with respect to that argument as in $f'_k(\chi) = \frac{df_k(\chi)}{d\chi}$.

To preserve some amount of supersymmetry, we perform a topological twist by turning on some gauge fields to cancel the spin connection $\omega_{\hat{t}}^{\hat{j}\hat{k}}$ on Σ_k^4 . In the present case, we achieve this by turning on $SO(4)$ gauge fields as follows:

$$\begin{aligned} A_{\hat{\psi}}^{\hat{\chi}} &= A_{\hat{\theta}}^{\hat{\chi}} = A_{\hat{\zeta}}^{\hat{\chi}} = e^{-V} \frac{p f'_k(\chi)}{4k f_k(\chi)}, \\ A_{\hat{\theta}}^{\hat{\psi}} &= A_{\hat{\zeta}}^{\hat{\psi}} = e^{-V} \frac{p \cot\psi}{4k f_k(\chi)}, \\ A_{\hat{\zeta}}^{\hat{\theta}} &= e^{-V} \frac{p \cot\theta}{4k f_k(\chi) \sin\psi}, \end{aligned} \quad (64)$$

where p is a constant magnetic charge. In order to perform this topological twist, we consider gauged supergravity with the gauge group containing an $SO(4)$ subgroup. These gauge groups are given by $SO(5)$, $SO(4, 1)$, and $CSO(4, 0, 1)$ which can be characterized by a tensor Y_{mn} in the embedding tensor of the form

$$Y_{mn} = \text{diag}(1, 1, 1, 1, \lambda), \quad (65)$$

for $\lambda = +1, -1, 0$. Among the 25 scalar fields, there are two $SO(4)$ singlet scalars parametrized by the coset representative,

$$V = e^{\varphi \mathbf{d} + \phi \mathcal{Y}}. \quad (66)$$

The first scalar is the dilaton $\varphi(r)$ associated with the $SO(1, 1)$ generator $\mathbf{d}$, and the other one $\phi(r)$ corresponds to the noncompact generator,

$$\mathcal{Y} = \hat{\mathbf{t}}_{11} + \hat{\mathbf{t}}_{22} + \hat{\mathbf{t}}_{33} + \hat{\mathbf{t}}_{44} - 4\hat{\mathbf{t}}_{55}. \quad (67)$$

The topological twist can be achieved by imposing the twist condition

$$gp = k, \quad (68)$$

together with the following projection conditions:

$$\begin{aligned}\hat{\gamma}_{\hat{\psi}\hat{\chi}}\epsilon_{+\alpha} &= (\gamma_{12})_{\alpha}^{\beta}\epsilon_{+\beta}, & \hat{\gamma}_{\hat{\theta}\hat{\psi}}\epsilon_{+\alpha} &= (\gamma_{23})_{\alpha}^{\beta}\epsilon_{+\beta}, \\ \hat{\gamma}_{\hat{\zeta}\hat{\theta}}\epsilon_{+\alpha} &= (\gamma_{34})_{\alpha}^{\beta}\epsilon_{+\beta},\end{aligned}\quad (69)$$

$$\begin{aligned}\text{and } \hat{\gamma}_{\hat{\psi}\hat{\chi}}\epsilon_{-\alpha} &= (\gamma_{1\dot{2}})_{\alpha}^{\dot{\beta}}\epsilon_{-\dot{\beta}}, & \hat{\gamma}_{\hat{\theta}\hat{\psi}}\epsilon_{-\alpha} &= (\gamma_{2\dot{3}})_{\alpha}^{\dot{\beta}}\epsilon_{-\dot{\beta}}, \\ \hat{\gamma}_{\hat{\zeta}\hat{\theta}}\epsilon_{-\alpha} &= (\gamma_{3\dot{4}})_{\alpha}^{\dot{\beta}}\epsilon_{-\dot{\beta}}.\end{aligned}\quad (70)$$

To obtain the corresponding Bogomol'nyi-Prasad-Sommerfield (BPS) equations, we need to further impose an additional projector of the form

$$\hat{\gamma}_r\epsilon_{\pm} = \epsilon_{\mp}, \quad (71)$$

due to the radial dependence of scalars. We also take the supersymmetry parameters to depend only on r . With all these projectors, the unbroken supersymmetry has two supercharges. Each chirality of $\epsilon_{\pm}$ consists of two unbroken supercharges due to the projectors (69) and (70), and the two chiralities are related by the $\hat{\gamma}_r$ projector (71).

The nonvanishing components of the $SO(4)$ gauge field strengths are given by

$$\mathcal{H}_{\hat{i}\hat{j}}^{\tilde{m}\tilde{n}} = F_{\hat{i}\hat{j}}^{\tilde{m}\tilde{n}} = -\frac{1}{2}e^{-2V}p\delta_{[\hat{i}}^{\tilde{m}}\delta_{\hat{j}]}^{\tilde{n}}, \quad (72)$$

in which we have split the $GL(5)$ index as $m = (\tilde{m}, 5)$ with $\tilde{m}, \tilde{n} = 1, 2, 3, 4$ and $\hat{i}, \hat{j} = \{1, 2, 3, 4\} = \{\hat{\chi}, \hat{\psi}, \hat{\theta}, \hat{\zeta}\}$. These identically satisfy the Bianchi identity (13) and the two-/four-form duality (17). However, the above gauge field strengths also lead to a nonvanishing term in the Bianchi identity given in (14). This term takes the form

$$(\Gamma_m)_{AB}\mathbb{T}_{np}^A\mathbb{T}_{qr}^B\mathcal{H}_{[\hat{i}\hat{j}}^{np}\mathcal{H}_{\hat{k}\hat{l}]}^{qr} = \frac{p^2}{4\sqrt{2}}e^{-4V}\delta_m^5\epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}}. \quad (73)$$

Therefore, to satisfy the Bianchi identity (14), we turn on a magnetic two-form potential of the form

$$B_{\hat{r}\hat{5}} = -\frac{3p^2}{8g^2}e^{-2(2V+3\varphi+8\phi)}, \quad (74)$$

giving rise to the following two-form field strength:

$$\mathcal{H}_{\hat{r}\hat{5}} = \frac{3p^2}{4\sqrt{2}g}e^{-2(2V+3\varphi+8\phi)}. \quad (75)$$

However, with this new contribution in $\mathcal{H}_{\mu\nu,5}$, the two-/four-form duality (17) is no longer satisfied. To remedy this, we turn on the four-form field strength of the form

$$\begin{aligned}\theta^A{}_m\mathcal{H}_{\hat{i}\hat{j}\hat{k}\hat{l},A} &= -\theta^A{}_mM_{AB}\mathbb{T}^{Bn}\epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}}\mathcal{H}^{\hat{i}\hat{r}}{}_n \\ &= \frac{3\lambda^2p^2}{4\sqrt{2}g}e^{-4V}\delta_m^5\epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}}.\end{aligned}\quad (76)$$

With all these, all the Bianchi identities as well as the field equations for vector and tensor fields are satisfied.

It should be pointed out that for the $CSO(4, 0, 1)$ gauge group with $\lambda = 0$, the four-form field strength in (76) vanishes identically. In this case, the problematic term in (73) cannot be canceled and must vanish implying $p = 0$. This leads to untwisted solutions with all the gauge fields vanishing and corresponds to the flat domain wall solutions studied in [29]. Accordingly, we do not consider the nonsemisimple gauge group $CSO(4, 0, 1)$ in the following analysis.

Substituting the above *Ansatz* into the conditions $\delta\psi_{+\mu\alpha} = 0$, $\delta\psi_{-\mu\dot{\alpha}} = 0$, $\delta\chi_{+a\dot{\alpha}} = 0$, and $\delta\chi_{-\dot{a}\alpha} = 0$, and imposing the projectors (69)–(71), we find the following BPS equations:

$$U' = \frac{g}{4\sqrt{2}}e^{\varphi-4\phi}(4 + \lambda e^{20\phi}) - \frac{3p}{4\sqrt{2}}e^{-2V-\varphi+4\phi} + \frac{9\lambda p^2}{16\sqrt{2}g}e^{-4V-3\varphi-8\phi}, \quad (77)$$

$$V' = \frac{g}{4\sqrt{2}}e^{\varphi-4\phi}(4 + \lambda e^{20\phi}) + \frac{3p}{4\sqrt{2}}e^{-2V-\varphi+4\phi} - \frac{3\lambda p^2}{16\sqrt{2}g}e^{-4V-3\varphi-8\phi}, \quad (78)$$

$$\varphi' = -\frac{g}{20\sqrt{2}}e^{\varphi-4\phi}(4 + \lambda e^{20\phi}) + \frac{3p}{20\sqrt{2}}e^{-2V-\varphi+4\phi} - \frac{9\lambda p^2}{80\sqrt{2}g}e^{-4V-3\varphi-8\phi}, \quad (79)$$

$$\phi' = \frac{g}{5\sqrt{2}}e^{\varphi-4\phi}(1 - \lambda e^{20\phi}) - \frac{3p}{20\sqrt{2}}e^{-2V-\varphi+4\phi} - \frac{3\lambda p^2}{40\sqrt{2}g}e^{-4V-3\varphi-8\phi}, \quad (80)$$

together with a simple form of the Killing spinors,

$$\epsilon_{\pm} = e^{\frac{U(r)}{2}}\epsilon_{\pm}^0. \quad (81)$$

$\epsilon_{\pm}^0$ are constant symplectic-Majorana–Weyl spinors satisfying the projectors (69)–(71). It can also be verified that

these equations satisfy all bosonic field equations derived from the Lagrangian (25). We also note that these equations do not admit a fixed point solution of the form $\text{AdS}_2 \times \Sigma_k^4$ at which $\varphi' = \phi' = V' = 0$ and $U' = \text{const}$.

The resulting BPS equations cannot be solved analytically, so we look for numerical solutions. We are interested

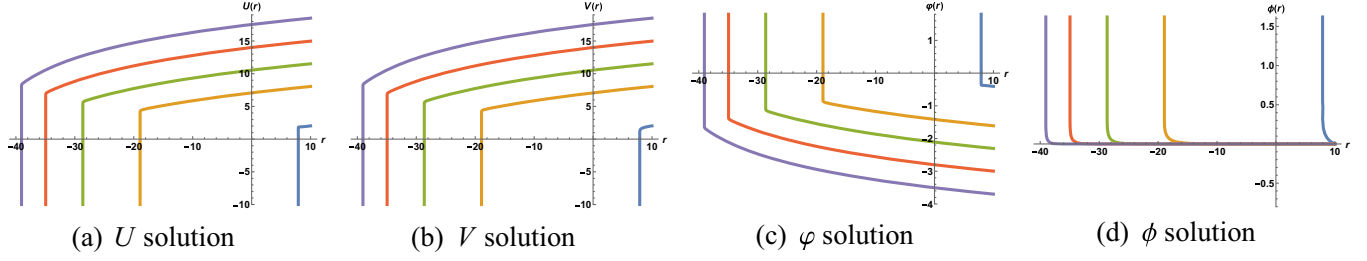


FIG. 1. Interpolating solutions between the locally $SO(5)$ flat domain wall as $r \rightarrow +\infty$ and $t \times S^4$ -sliced curved domain walls for the $SO(4)$ twist in the $SO(5)$ gauge group. The blue, orange, green, red, and purple curves refer to $g = 0.15, 0.50, 1, 2$, and 4.04 , respectively.

in solutions interpolating between asymptotically locally flat domain walls studied in [29] and another domain wall with the world-volume given by $t \times \Sigma_k^4$. According to the DW/QFT correspondence, we expect these solutions to be holographically dual to supersymmetric quantum mechanics preserving two supercharges in the IR. This quantum mechanics is obtained from twisted compactification on a Riemannian four-manifold of the maximal SYM theory in five dimensions dual to the flat domain wall solutions of [29].

The first class of solutions is given by solutions that are asymptotically $SO(5)$ symmetric domain wall of the form

$$U \sim V \sim 5 \ln gr, \quad \phi \sim -\ln gr, \quad \phi \sim \frac{1}{g^{16} r^{16}}. \quad (82)$$

We also note that the $SO(4)$ singlet scalar $\phi \rightarrow 0$ as $r \rightarrow \infty$, while the dilaton φ is invariant under $SO(5)$. This asymptotic geometry can only exist in the $SO(5)$ gauge group. Examples of numerical solutions interpolating between this locally $SO(5)$ flat domain wall and $t \times \Sigma_k^4$ -sliced curved domain walls are given in Figs. 1 and 2 for $k = 1$ and $k = -1$ with various values of the gauge coupling constant g . For the solutions with $t \times \mathbb{R}^4$ slices and $k = 0$, the domain walls are entirely flat since the twist condition implies $p = 0$. Accordingly, we have omitted these solutions.

As usually the case for holographic solutions dual to nonconformal field theories, all solutions flow to singular geometries in the IR. To determine whether these

singularities lead to any physical interpretations, we can uplift the solutions to type IIA theory and look for the behavior of the ten-dimensional metric component $\hat{g}_{00}$. We have put a hat on the fields in ten dimensions. Using the formulas given in Appendix B, we find that for the case of the $SO(5)$ gauge group, the (00)-component of type IIA metric takes the form

$$\hat{g}_{00} = e^{2U+\varphi} \Delta^{3/8}. \quad (83)$$

The warp factor Δ is given by

$$\Delta = e^{-4\phi} [(\mu^1)^2 + (\mu^2)^2 + (\mu^3)^2 + (\mu^4)^2] + \lambda^2 e^{8\phi} (\mu^5)^2, \quad (84)$$

in which μ^a , $a = 1, 2, \dots, 5$, are coordinates on S^4 satisfying $\mu^a \mu^a = 1$.

According to the criterion of [18], $\hat{g}_{00}$ should not diverge when approaching the singularity in order for the IR singularity to be physically acceptable. For all the solutions shown in Figs. 1 and 2, we give the behavior of $\hat{g}_{00}$ in Fig. 3. From this figure, we readily see that all the solutions with $t \times H^4$ slices are uplifted to type IIA theory solutions with physically acceptable singularities since $\hat{g}_{00} \rightarrow 0$ near the singularities. According to the DW/QFT correspondence, these solutions give holographic descriptions of twisted compactifications of the maximal five-dimensional SYM theory on H^4 leading to supersymmetric quantum mechanics preserving two supercharges. Moreover, the uplifted type IIA solutions can be interpreted as D4-branes

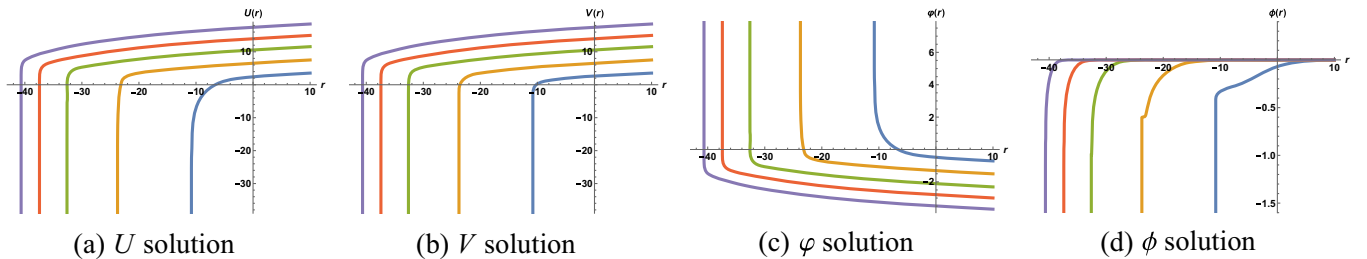


FIG. 2. Interpolating solutions between the locally $SO(5)$ flat domain wall as $r \rightarrow +\infty$ and $t \times H^4$ -sliced curved domain walls for the $SO(4)$ twist in the $SO(5)$ gauge group. The blue, orange, green, red, and purple curves refer to $g = 0.20, 0.45, 1, 2$, and 4 , respectively.

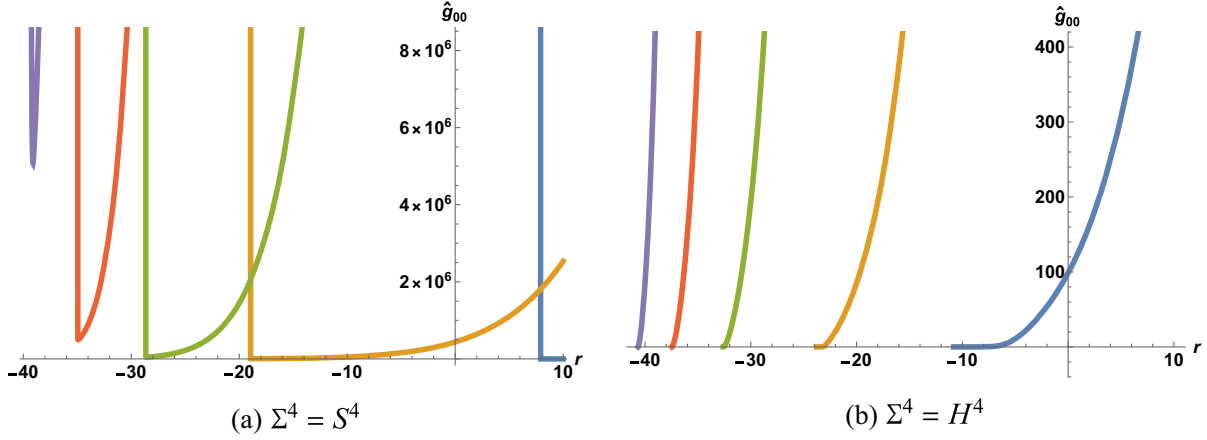


FIG. 3. The behavior of $\hat{g}_{00}$ along the RG flows given in Figs. 1 and 2, respectively, for $\Sigma_1^4 = S^4$ and $\Sigma_{-1}^4 = H^4$ in the $SO(5)$ gauge group.

wrapped on H^4 . On the other hand, all solutions with $t \times S^4$ slices lead to unphysical type IIA solutions.

We now move to another class of solutions with asymptotic behaviors given by a locally flat domain wall with $SO(4)$ symmetry of the form

$$U \sim V \sim -\frac{1}{5} \ln \left[-\frac{5g\rho}{\sqrt{2}} \right], \quad \varphi \sim -\phi \sim \frac{1}{25} \ln \left[-\frac{5g\rho}{\sqrt{2}} \right]. \quad (85)$$

The new radial coordinate ρ is defined by $\frac{d\rho}{dr} = e^{-6V}$. We choose the gauge coupling constant $g < 0$ in order to identify the UV limit with $\rho \rightarrow +\infty$. It turns out that, in this case, all of the numerical solutions are uplifted to type IIA solutions with unphysical IR singularities since $\hat{g}_{00}$ diverges near the singularities. Accordingly, we do not present these solutions here.

For convenience and the sake of comparison with other cases to be given in subsequent analyses, we summarize physical acceptabilities of the IR singularities according to the criterion of [18] in Table I. Note also that solutions with $t \times \mathbb{R}^4$ slices are not included in the table since these are untwisted flat domain walls with $p = k = 0$ along the entire flow solutions. In this table and subsequent ones, we use the $\times$ symbol to mark the cases in which the criterion of

[18] is violated. For the solutions with physically acceptable singularities, the possible values of the independent parameter, in this case g , are also given.

B. D4-branes wrapped on a product of two Riemann surfaces

We now carry out a similar analysis for supersymmetric solutions describing D4-branes wrapped on a product of two Riemann surfaces $\Sigma_{k_1}^2 \times \Sigma_{k_2}^2$. In this case, the *Ansatz* for the metric takes the form of

$$ds_6^2 = -e^{2U(r)} dt^2 + dr^2 + e^{2V_1(r)} ds_{\Sigma_{k_1}^2}^2 + e^{2V_2(r)} ds_{\Sigma_{k_2}^2}^2, \quad (86)$$

in which the metrics on $\Sigma_{k_1}^2$ and $\Sigma_{k_2}^2$ are given by

$$ds_{\Sigma_{k_i}^2}^2 = d\theta_i^2 + f_{k_i}(\theta_i)^2 d\zeta_i^2, \quad i = 1, 2. \quad (87)$$

The function $f_{k_i}(\theta_i)$ is defined as in (61) with $k_i = 0, \pm 1$ characterizing the constant curvature of $\Sigma_{k_i}^2$. We split the six-dimensional coordinates as $x^\mu = (t, r, x^{i_1}, x^{i_2})$ with $x^{i_1} = (x^{\theta_1}, x^{\zeta_1})$ and $x^{i_2} = (x^{\theta_2}, x^{\zeta_2})$.

Using the following choice of the vielbein:

$$\begin{aligned} e^{\hat{t}} &= e^U dt, & e^{\hat{r}} &= dr, & e^{\hat{\theta}_1} &= e^{V_1} d\theta_1, & e^{\hat{\theta}_2} &= e^{V_2} d\theta_2, \\ e^{\hat{\zeta}_1} &= e^{V_1} f_{k_1}(\theta_1) d\zeta_1, & e^{\hat{\zeta}_2} &= e^{V_2} f_{k_2}(\theta_2) d\zeta_2, \end{aligned} \quad (88)$$

we obtain all nonvanishing components of the spin connection as follows:

$$\begin{aligned} \omega_{\hat{t}\hat{r}}^{\hat{t}} &= U', & \omega_{\hat{t}\hat{\theta}_1}^{\hat{t}} &= V_1' \delta_{\hat{t}\hat{\theta}_1}^{\hat{t}}, & \omega_{\hat{t}\hat{\theta}_2}^{\hat{t}} &= V_2' \delta_{\hat{t}\hat{\theta}_2}^{\hat{t}}, \\ \omega_{\hat{\zeta}_1\hat{\theta}_1}^{\hat{\zeta}_1} &= e^{-V_1} \frac{f_{k_1}'(\theta_1)}{f_{k_1}(\theta_1)}, & \omega_{\hat{\zeta}_2\hat{\theta}_2}^{\hat{\zeta}_2} &= e^{-V_2} \frac{f_{k_2}'(\theta_2)}{f_{k_2}(\theta_2)}, \end{aligned} \quad (89)$$

TABLE I. Summary of satisfaction of the physical criterion given in [18] for the IR singularities of the curved domain wall solutions obtained from $SO(4)$ twist on a Riemannian four-manifold in $SO(5)$ and $SO(4, 1)$ gauge groups.

UV flat domain wall	Gauge group	Riemannian four-manifold	Physical IR singularity
$SO(5)$	$SO(5)$	S^4	$\times$
		H^4	any g
$SO(4)$	$SO(5)$	S^4, H^4	$\times$
	$SO(4, 1)$	S^4, H^4	$\times$

with $\hat{i}_1 = \hat{\theta}_1, \hat{\zeta}_1$ and $\hat{i}_2 = \hat{\theta}_2, \hat{\zeta}_2$ being flat indices on $\Sigma_{k_1}^2$ and $\Sigma_{k_2}^2$, respectively.

To perform the topological twist, we turn on $SO(2) \times SO(2)$ gauge fields corresponding to X_{12} and X_{34} gauge generators of the form

$$\begin{aligned} A_{\hat{\zeta}_1}^{12} &= e^{-V_1} \frac{p_{11} f'_{k_1}(\theta_1)}{4k_1 f_{k_1}(\theta_1)}, & A_{\hat{\zeta}_2}^{12} &= e^{-V_2} \frac{p_{12} f'_{k_2}(\theta_2)}{4k_2 f_{k_2}(\theta_2)}, \\ A_{\hat{\zeta}_1}^{34} &= e^{-V_1} \frac{p_{21} f'_{k_1}(\theta_1)}{4k_1 f_{k_1}(\theta_1)}, & A_{\hat{\zeta}_2}^{34} &= e^{-V_2} \frac{p_{22} f'_{k_2}(\theta_2)}{4k_2 f_{k_2}(\theta_2)}. \end{aligned} \quad (90)$$

The resulting two-form field strengths have the following nonvanishing components:

$$\begin{aligned} \mathcal{H}_{\hat{\theta}_1 \hat{\zeta}_1}^{12} &= -\frac{1}{4} p_{11} e^{-2V_1}, & \mathcal{H}_{\hat{\theta}_2 \hat{\zeta}_2}^{12} &= -\frac{1}{4} p_{12} e^{-2V_2}, \\ \mathcal{H}_{\hat{\theta}_1 \hat{\zeta}_1}^{34} &= -\frac{1}{4} p_{21} e^{-2V_1}, & \mathcal{H}_{\hat{\theta}_2 \hat{\zeta}_2}^{34} &= -\frac{1}{4} p_{22} e^{-2V_2}. \end{aligned} \quad (91)$$

As in the previous case, we consider solutions in all gauge groups with an $SO(2) \times SO(2)$ subgroup. The embedding tensor for these gauge groups are characterized by the symmetric tensor,

$$Y_{mn} = \text{diag}(1, 1, \kappa, \kappa, \lambda), \quad (92)$$

for $\lambda = 0, \pm 1$ and $\kappa = \pm 1$. All possible gauge groups are then given explicitly by $SO(5)$ ($\kappa = \lambda = 1$), $SO(4, 1)$ ($\kappa = -\lambda = 1$), $SO(3, 2)$ ($\kappa = -\lambda = -1$), $CSO(4, 0, 1)$ ($\kappa = 1, \lambda = 0$), and $CSO(2, 2, 1)$ ($\kappa = -1, \lambda = 0$).

Under $SO(2) \times SO(2)$ generated by X_{12} and X_{34} , there are five singlet scalars. One of them is the dilaton $\varphi(r)$ corresponding to the $SO(1, 1)$ generator d . The remaining four are denoted by $\phi_1(r)$, $\phi_2(r)$, $\varsigma_1(r)$, and $\varsigma_2(r)$ and correspond, respectively, to the following noncompact generators:

$$\begin{aligned} \mathcal{Y}_1 &= \hat{t}_{11} + \hat{t}_{22} - 2\hat{t}_{55}, & \mathcal{Y}_2 &= \hat{t}_{33} + \hat{t}_{44} - 2\hat{t}_{55}, \\ \mathcal{Y}_3 &= s_{12}, & \mathcal{Y}_4 &= s_{34}. \end{aligned} \quad (93)$$

The coset representative is then given by

$$V = e^{\varphi d + \phi_1 \mathcal{Y}_1 + \phi_2 \mathcal{Y}_2 + \varsigma_1 \mathcal{Y}_3 + \varsigma_2 \mathcal{Y}_4}. \quad (94)$$

As in the solutions with a $SO(2) \times SO(2)$ twist on a single Riemann surface considered in [32], the $SO(2) \times SO(2)$ gauge field strengths lead to the following nonvanishing terms in the two-/four-form duality, given in (17),

$$\begin{aligned} g\theta^A{}_m M_{AB} \mathbb{T}_{np}^B \mathcal{H}_{\hat{\theta}_1 \hat{\zeta}_1}^{np} &= -\frac{g\lambda}{\sqrt{2}} e^{-2(V_1 - \varphi - 2\phi_1 - 2\phi_2)} \\ &\quad \times (p_{21} e^{4\phi_2} \varsigma_1 + p_{11} e^{4\phi_1} \varsigma_2) \delta_m^5, \\ g\theta^A{}_m M_{AB} \mathbb{T}_{np}^B \mathcal{H}_{\hat{\theta}_2 \hat{\zeta}_2}^{np} &= -\frac{g\lambda}{\sqrt{2}} e^{-2(V_2 - \varphi - 2\phi_1 - 2\phi_2)} \\ &\quad \times (p_{22} e^{4\phi_2} \varsigma_1 + p_{12} e^{4\phi_1} \varsigma_2) \delta_m^5. \end{aligned} \quad (95)$$

Accordingly, we would need to turn on the following components of the four-form fields $\mathcal{H}_{tr\theta_1\zeta_1,A}$ and $\mathcal{H}_{tr\theta_2\zeta_2,A}$. In addition, this leads to nonvanishing components $\mathcal{H}_{tr}^{mn}$ and algebraic constraints on the magnetic charges and the warp factors V_1 and V_2 . To avoid these complications as well as unnecessary restrictions on the solutions, we set the two terms in (95) to zero. As in [32], there are two possibilities, namely, vanishing shift scalars $\varsigma_1 = \varsigma_2 = 0$ or imposing some additional constraints on the magnetic charges with nonvanishing shift scalars.

1. Solutions with $\varsigma_1 = \varsigma_2 = 0$

Although setting $\varsigma_1 = \varsigma_2 = 0$ solves the two-/four-form duality (17), there is an additional term in the Bianchi identity (14) of the form

$$\begin{aligned} (\Gamma_m)_{AB} \mathbb{T}_{np}^A \mathbb{T}_{qr}^B \mathcal{H}_{\hat{\theta}_1 \hat{\zeta}_1}^{np} \mathcal{H}_{\hat{\theta}_2 \hat{\zeta}_2}^{qr} \\ = \frac{1}{12\sqrt{2}} (p_{11} p_{22} + p_{12} p_{21}) e^{-2V_1 - 2V_2} \delta_m^5. \end{aligned} \quad (96)$$

This can be canceled by turning on the following magnetic two-form potential,

$$B_{\hat{r}\hat{s}}^5 = -\frac{1}{8g^2} (p_{11} p_{22} + p_{12} p_{21}) e^{-2(V_1 + V_2 + 3\varphi + 4\phi_1 + 4\phi_2)}, \quad (97)$$

which leads to another contribution to the two-form field strengths as

$$\mathcal{H}_{\hat{r}\hat{s}}^5 = \frac{\lambda}{4\sqrt{2}g} (p_{11} p_{22} + p_{12} p_{21}) e^{-2(V_1 + V_2 + 3\varphi + 4\phi_1 + 4\phi_2)}. \quad (98)$$

This, in turn, gives rise to, via the Bianchi identity (14) and the two-/four-form duality (17), the four-form field strength,

$$\begin{aligned} \theta^A{}_m \mathcal{H}_{\hat{\theta}_1 \hat{\zeta}_1 \hat{\theta}_2 \hat{\zeta}_2, A} &= -\theta^A{}_m M_{AB} \mathbb{T}_{np}^B \mathcal{H}_{\hat{r}\hat{s}}^n \\ &= \frac{\lambda^2}{4\sqrt{2}g} (p_{11} p_{22} + p_{12} p_{21}) e^{-2V_1 - 2V_2} \delta_m^5. \end{aligned} \quad (99)$$

For gauge groups with $\lambda = \pm 1$, the two terms in (96) and (99) cancel each other recovering the Bianchi identity (14).

On the other hand, for $CSO(4, 0, 1)$ and $CSO(2, 2, 1)$ gauge groups with $\lambda = 0$, the problematic term (96) within the Bianchi identity (14) cannot be canceled since the corresponding four-form field strengths do not appear. To satisfy the Bianchi identity (14) in this case, we further

impose the following condition between the magnetic charges:

$$p_{11}p_{22} + p_{12}p_{21} = 0. \quad (100)$$

Implementing the relevant topological twist by imposing the projection condition $\hat{\gamma}_r \epsilon_{\pm} = \epsilon_{\mp}$ as well as

$$\hat{\gamma}_{\hat{\zeta}_1 \hat{\theta}_1} \epsilon_{+\alpha} = \hat{\gamma}_{\hat{\zeta}_2 \hat{\theta}_2} \epsilon_{+\alpha} = (\gamma_{12})_{\alpha}^{\beta} \epsilon_{+\beta} = (\gamma_{34})_{\alpha}^{\beta} \epsilon_{+\beta} \quad (101)$$

$$\text{and } \hat{\gamma}_{\hat{\zeta}_1 \hat{\theta}_1} \epsilon_{-\alpha} = \hat{\gamma}_{\hat{\zeta}_2 \hat{\theta}_2} \epsilon_{-\alpha} = (\gamma_{1\dot{2}})_{\alpha}^{\dot{\beta}} \epsilon_{-\dot{\beta}} = (\gamma_{3\dot{4}})_{\alpha}^{\dot{\beta}} \epsilon_{-\dot{\beta}}, \quad (102)$$

together with the twist conditions

$$g(p_{11} + \kappa p_{21}) = k_1 \quad \text{and} \quad g(p_{12} + \kappa p_{22}) = k_2, \quad (103)$$

we find the following set of BPS equations:

$$U' = \frac{ge^{\varphi}}{4\sqrt{2}} (2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) + \frac{3\lambda(p_{11}p_{22} + p_{12}p_{21})}{16\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} - \frac{e^{-\varphi}}{8\sqrt{2}} \left[e^{-2V_1}(p_{11}e^{4\phi_1} + p_{21}e^{4\phi_2}) + e^{-2V_2}(p_{12}e^{4\phi_1} + p_{22}e^{4\phi_2}) \right], \quad (104)$$

$$V'_1 = \frac{ge^{\varphi}}{4\sqrt{2}} (2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) - \frac{\lambda(p_{11}p_{22} + p_{12}p_{21})}{16\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} + \frac{e^{-\varphi}}{8\sqrt{2}} \left[3e^{-2V_1}(p_{11}e^{4\phi_1} + p_{21}e^{4\phi_2}) - e^{-2V_2}(p_{12}e^{4\phi_1} + p_{22}e^{4\phi_2}) \right], \quad (105)$$

$$V'_2 = \frac{ge^{\varphi}}{4\sqrt{2}} (2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) - \frac{\lambda(p_{11}p_{22} + p_{12}p_{21})}{16\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} - \frac{e^{-\varphi}}{8\sqrt{2}} \left[e^{-2V_1}(p_{11}e^{4\phi_1} + p_{21}e^{4\phi_2}) - 3e^{-2V_2}(p_{12}e^{4\phi_1} + p_{22}e^{4\phi_2}) \right], \quad (106)$$

$$\varphi' = -\frac{ge^{\varphi}}{20\sqrt{2}} (2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) - \frac{3\lambda(p_{11}p_{22} + p_{12}p_{21})}{80\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} + \frac{e^{-\varphi}}{40\sqrt{2}} \left[e^{-2V_1}(p_{11}e^{4\phi_1} + p_{21}e^{4\phi_2}) + e^{-2V_2}(p_{12}e^{4\phi_1} + p_{22}e^{4\phi_2}) \right], \quad (107)$$

$$\phi'_1 = \frac{ge^{\varphi}}{5\sqrt{2}} (3e^{-4\phi_1} - 2\kappa e^{-4\phi_2} - \lambda e^{8(\phi_1+\phi_2)}) - \frac{\lambda(p_{11}p_{22} + p_{12}p_{21})}{40\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} - \frac{e^{-\varphi}}{20\sqrt{2}} \left[e^{-2V_1}(3p_{11}e^{4\phi_1} - 2p_{21}e^{4\phi_2}) + e^{-2V_2}(3p_{12}e^{4\phi_1} - 2p_{22}e^{4\phi_2}) \right], \quad (108)$$

$$\phi'_2 = \frac{ge^{\varphi}}{5\sqrt{2}} (3\kappa e^{-4\phi_2} - 2e^{-4\phi_1} - \lambda e^{8(\phi_1+\phi_2)}) - \frac{\lambda(p_{11}p_{22} + p_{12}p_{21})}{40\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} + \frac{e^{-\varphi}}{20\sqrt{2}} \left[e^{-2V_1}(2p_{11}e^{4\phi_1} - 3p_{21}e^{4\phi_2}) + e^{-2V_2}(2p_{12}e^{4\phi_1} - 3p_{22}e^{4\phi_2}) \right], \quad (109)$$

with the Killing spinors given by (81). As in the previous case, we numerically solve these equations for solutions interpolating between locally asymptotically flat domain walls and $t \times \Sigma_{k_1}^2 \times \Sigma_{k_2}^2$ -sliced domain walls. We consider the solutions from gauge groups with $\lambda \neq 0$ and $\lambda = 0$ separately.

Solutions from $SO(5)$, $SO(4,1)$, and $SO(3,2)$ gauge groups. We first consider semisimple gauge groups $SO(5)$, $SO(4,1)$, and $SO(3,2)$ with $\lambda \neq 0$. We begin with solutions from $SO(5)$ gauge group that, as $\rho \rightarrow \infty$, are asymptotically flat domain walls with $SO(5)$ symmetry given by

$$U \sim V_1 \sim V_2 \sim \frac{5g\lambda\rho}{4\sqrt{2}}, \quad \varphi \sim -\frac{g\lambda\rho}{4\sqrt{2}}, \quad \phi_1 \sim \phi_2 \sim 0, \quad (110)$$

for the new radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{\varphi+8(\phi_1+\phi_2)}$.

Before giving examples of numerical solutions, we note that the solutions with $t \times \Sigma_{k_2}^2 \times \Sigma_{k_1}^2$ slices are equivalent to the solutions with $t \times \Sigma_{k_1}^2 \times \Sigma_{k_2}^2$ slices with all subscripts 1 and 2 interchanged. For the completely flat case with $k_1 = 0$ and $k_2 = 0$, the topological twist is not needed since there is no spin connection on $\mathbb{R}^2 \times \mathbb{R}^2$ to be canceled. As a result, these solutions preserve eight supercharges due to the $\hat{\gamma}_r$ projector and the last projector in (101) and (102). Although the solutions take the form of a flat $t \times \mathbb{R}^2 \times \mathbb{R}^2$ -sliced domain wall, the gauge fields are nonvanishing with the magnetic charges related by $p_{11} = -\kappa p_{21}$ and $p_{12} = -\kappa p_{22}$. Therefore, we also give numerical solutions in this case. For solutions with an $\mathbb{R}^2$ factor corresponding to either $k_1 = 0$ or $k_2 = 0$, the unbroken supersymmetry consists of four supercharges due to the two projectors on $\epsilon_{\pm}$ and the $\hat{\gamma}_r$ projector. Finally, for the case of $k_1 \neq 0$ and $k_2 \neq 0$, both ϵ_{+} and ϵ_{-} are subject to three independent projectors together with the $\hat{\gamma}_r$ projector relating the two chiralities, so the corresponding solutions preserve only two supercharges.

To characterize the solutions, it is also useful to define the following two parameters:

$$z_1 = g(p_{11} - \kappa p_{21}) \quad \text{and} \quad z_2 = g(p_{12} - \kappa p_{22}). \quad (111)$$

According to the twist conditions given in (103), the solutions with $z_1 = z_2 = 0$ are special in the sense that $p_{11} = \kappa p_{21} = \frac{k_1}{2g}$ and $p_{12} = \kappa p_{22} = \frac{k_2}{2g}$. In particular, the solution with $z_1 = z_2 = 0$ and $k_1 = k_2 = 0$ corresponds to the untwisted flat domain wall solution preserving 16 supercharges. Examples of numerical solutions asymptotic to the locally flat domain wall (110) with $SO(5)$ symmetry are given in Figs. 4–9. In these figures, we have chosen $g = 0.5$ and also given the behavior of the type IIA metric component $\hat{g}_{00}$.

From these figures, we readily see that only the solutions with $t \times H^2 \times H^2$ slices in Fig. 7 and solutions with $t \times H^2 \times \mathbb{R}^2$ slices in Fig. 8 are uplifted to type IIA solutions with physically acceptable IR singularities for specific values of z_1 and z_2 . We further examine these cases and find regions in the parameter space (z_1, z_2) for the IR singularities to be physically acceptable. The result is shown in Fig. 10, in which the physical regions (represented by the yellow areas) appear along the line $z_1 + z_2 \approx 0$ for small z_1 and z_2 in all cases. We also point out that these regions expand as g increases.

We now turn to solutions with an asymptotic behavior given by $SO(2) \times SO(2)$ symmetric flat domain wall of the form

$$U \sim V_1 \sim V_2 \sim \frac{g\lambda\rho}{4\sqrt{2}}, \quad \varphi \sim -\frac{g\lambda\rho}{20\sqrt{2}}, \quad \phi_1 \sim \phi_2 \sim -\frac{g\lambda\rho}{5\sqrt{2}}. \quad (112)$$

We have chosen $g\lambda < 0$ in order to identify the UV limit with $\rho \rightarrow +\infty$. From the numerical analysis, it turns out that all of the solutions in $SO(5)$ gauge group are uplifted to type IIA solutions with unphysical IR singularities. For

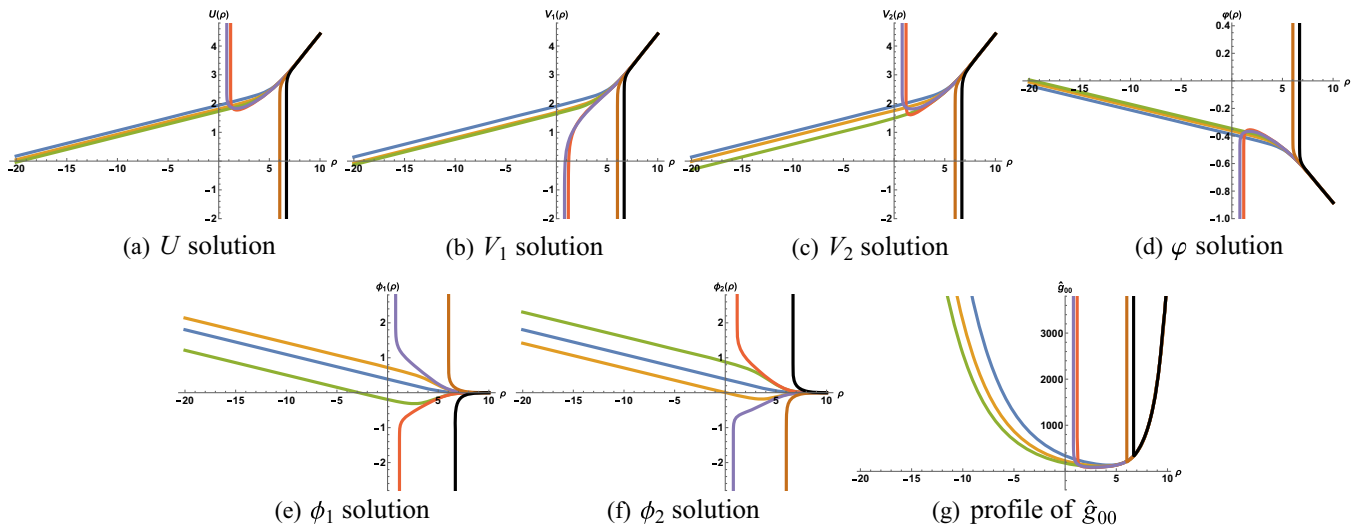


FIG. 4. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times S^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (1.7, 0), (0, -1.9), (-2, 0), (3, -1), (4, 3), (0, -16)$.

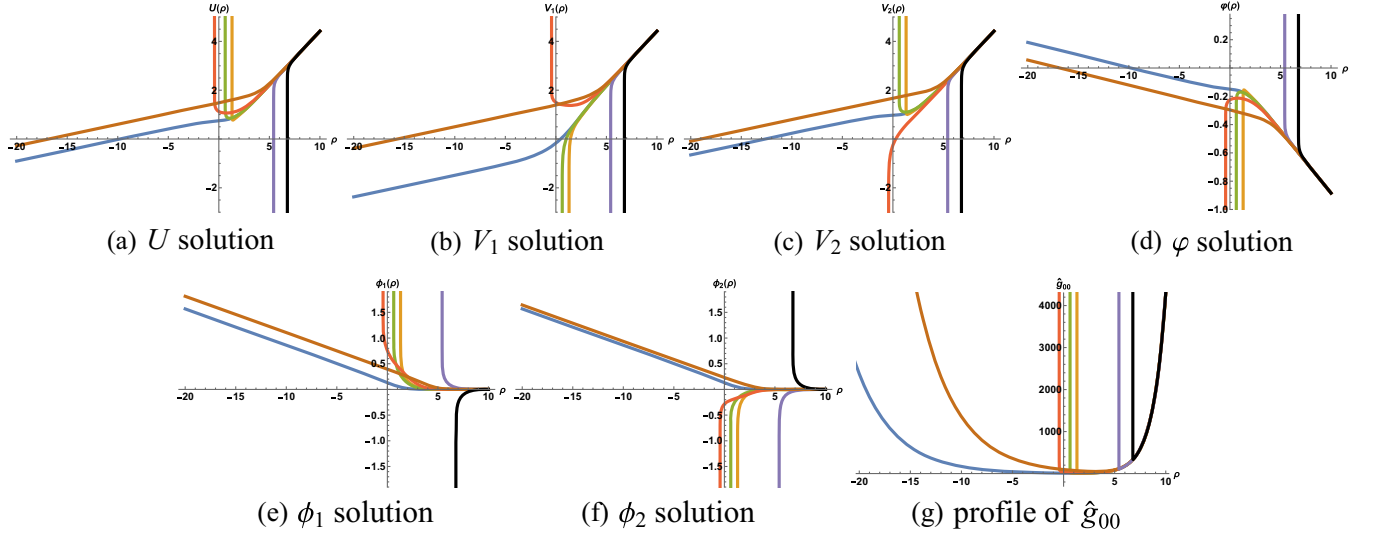


FIG. 5. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (0.01, 0), (0.50, -0.50), (-2, 2), (3, -1), (8, -8), (0, -16)$.

$SO(4, 1)$ gauge group with $g = 1$, only solutions with $t \times H^2 \times \mathbb{R}^2$ and $t \times \mathbb{R}^2 \times \mathbb{R}^2$ slices admit physical IR singularities for $-1 \leq z_1 \leq 1$ and $z_2 = 0$, and $z_1 = z_2 = 0$, respectively. The former corresponds to the $Mkw_3 \times H^2$ -sliced curved domain walls obtained from $SO(2) \times SO(2)$ twist on a Riemann surface previously studied in [32], while the latter, with $z_1 = z_2 = k_1 = k_2 = 0$, is the $SO(2) \times SO(2)$ symmetric flat domain wall. Accordingly, we do not give all of these solutions and those with unphysical IR singularities here.

For $SO(3, 2)$ gauge group, examples of numerical solutions with $g = -1$ are given in Figs. 11–16. In these

figures, we have shown the regions in the parameter space (z_1, z_2) where physical IR singularities exist in each case. Furthermore, we have also located solutions with physical IR singularities in the yellow regions by dots with the colors corresponding to the numerical curves of these solutions. As in the previous case, $\times$ marks that lie outside the yellow regions represent solutions with unphysical IR singularities. We end this section by giving a summary of all the solutions from $SO(5)$, $SO(4, 1)$ and $SO(3, 2)$ gauge groups in Table II.

Solutions from $CSO(4, 0, 1)$ and $CSO(2, 2, 1)$ gauge groups: We now consider the case of nonsemisimple gauge

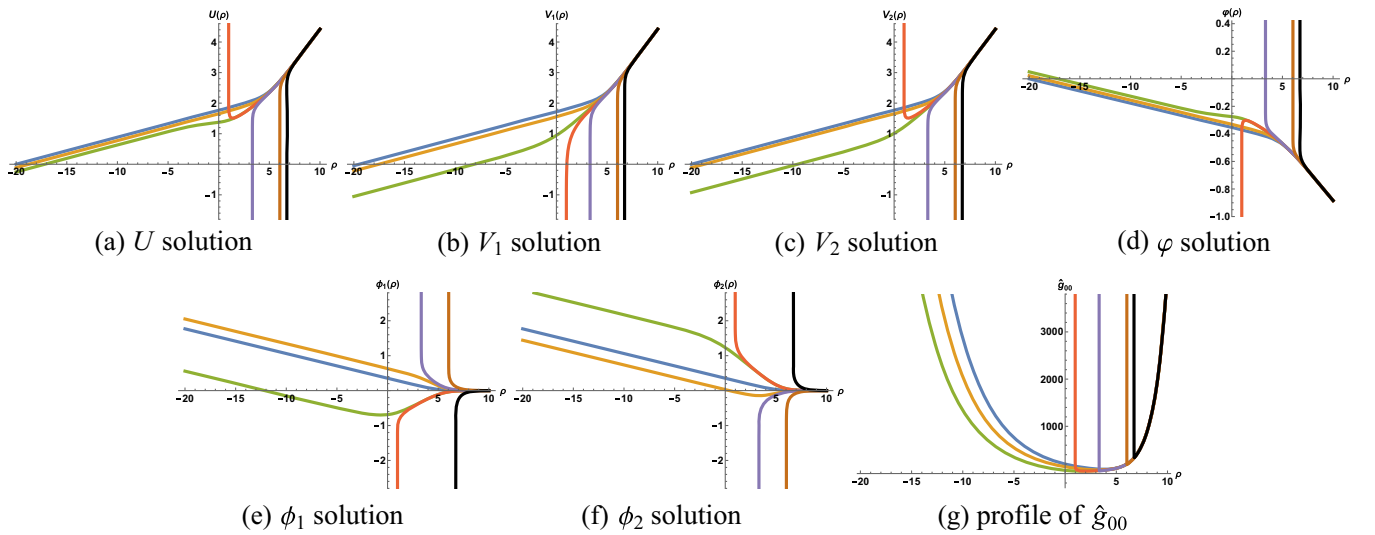


FIG. 6. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (0.8, 0), (0, -1), (-1, 0), (0.5, 0.6), (3, 3), (0, -16)$.

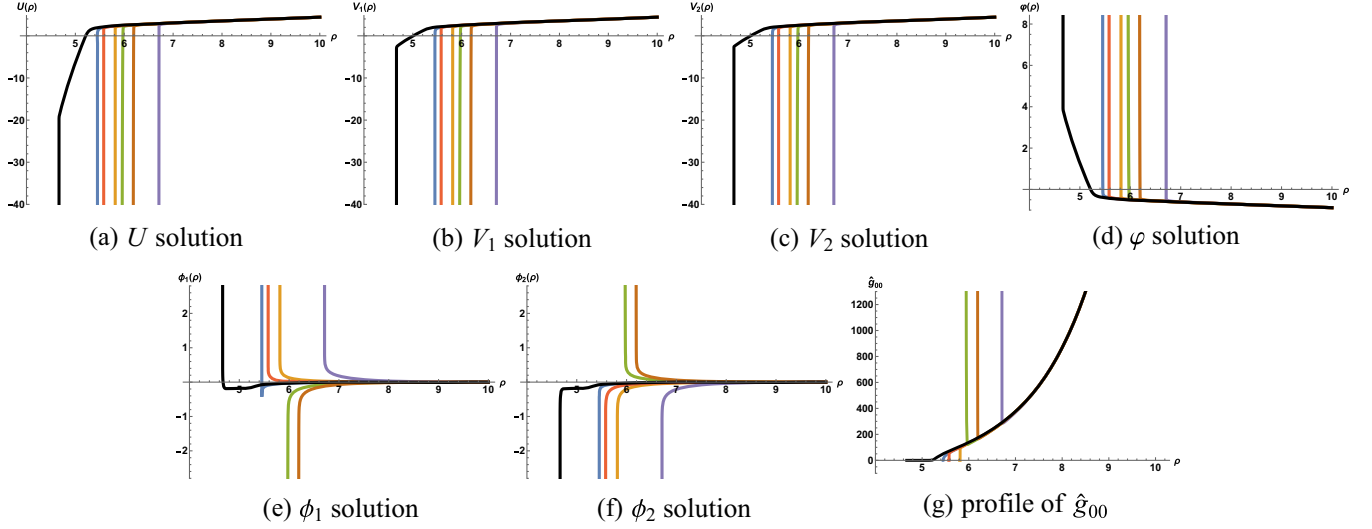


FIG. 7. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (1.5, 0), (0, -2.5), (-3, 3.5), (7, 5), (-9, 4.5), (8, -8)$.

groups with $\lambda = 0$. In this case, we define a new radial coordinate ρ by the relation $\frac{d\rho}{dr} = e^{-\frac{5}{2}\phi_1 - \frac{15}{2}\phi_2}$. As $\rho \rightarrow +\infty$, we require that the solutions are asymptotic to the $SO(2) \times SO(2)$ symmetric flat domain wall,

$$\begin{aligned} U \sim V_1 \sim V_2 &\sim \frac{3}{4} \ln \rho - \frac{5}{4} \ln(-g\kappa\rho), \\ \varphi &\sim \frac{1}{4} \ln(-g\kappa\rho) - \frac{3}{20} \ln \rho, \\ \phi_1 &\sim \frac{9}{10} \ln \rho - \frac{1}{2} \ln(-g\kappa\rho), \quad \phi_2 \sim -\frac{3}{5} \ln \rho. \end{aligned} \quad (113)$$

All numerical solutions in $CSO(4, 0, 1)$ gauge group lead to unphysical IR singularities, so we omit these solutions here.

Examples of numerical solutions from $CSO(2, 2, 1)$ gauge group are given in Figs. 17–22. We also note that, in this case, the parameters z_1 and z_2 are not independent from each other due to the constraint $p_{11}p_{22} + p_{12}p_{21} = 0$ required by the Bianchi identity (14) for $\lambda = 0$. This leads to the condition

$$z_1 z_2 = k_1 k_2. \quad (114)$$

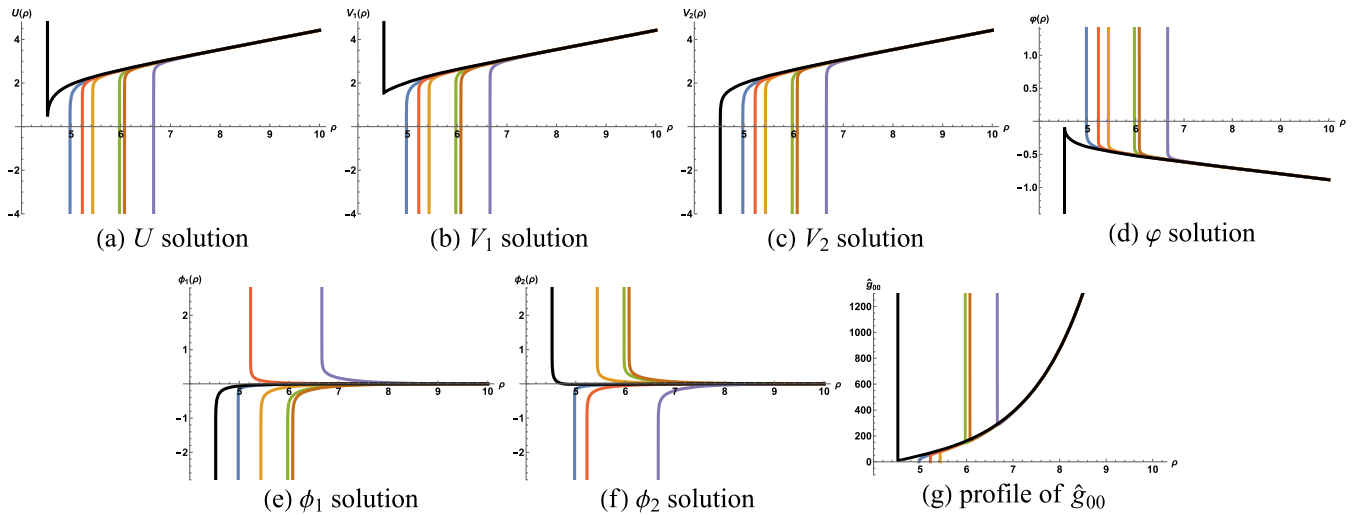


FIG. 8. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (0.5, -1.5), (-1.5, -2), (-2.5, 3), (7, 5), (-9, 4.5), (8, -8)$.

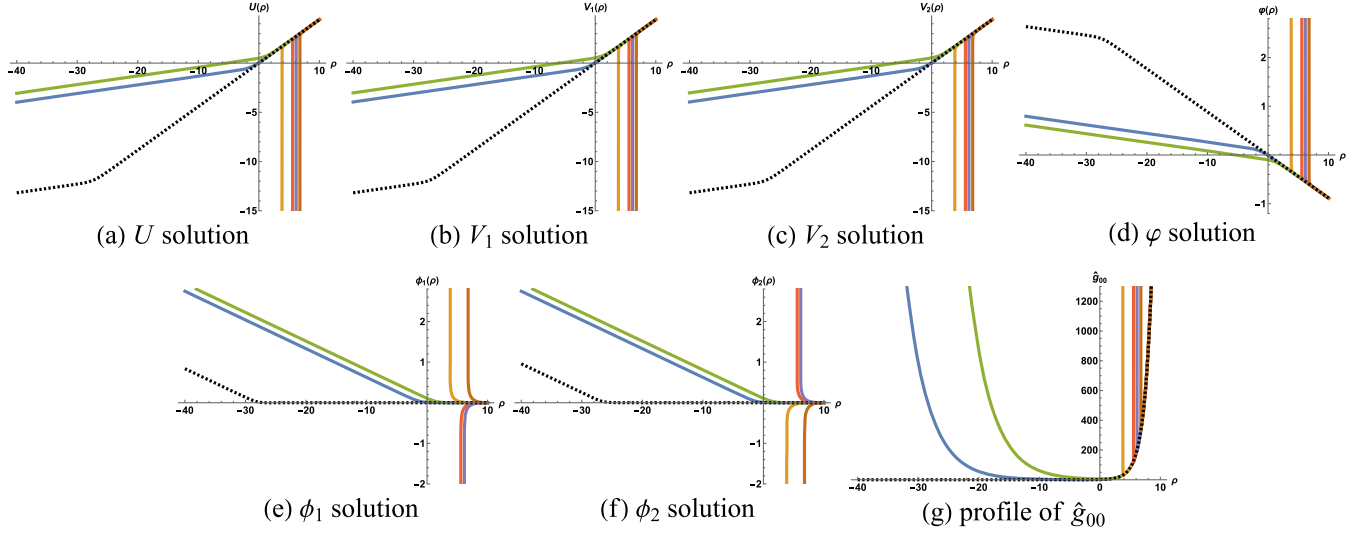


FIG. 9. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times \mathbb{R}^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group. The blue, orange, green, red, purple, and brown curves refer to $(z_1, z_2) = (0.1, -0.1), (0.2, 0), (0.8, -0.8), (-1, -1.5), (3, -3), (0, 16)$. The dashed curve is the $SO(2) \times SO(2)$ flat domain wall with $z_1 = z_2 = 0$.

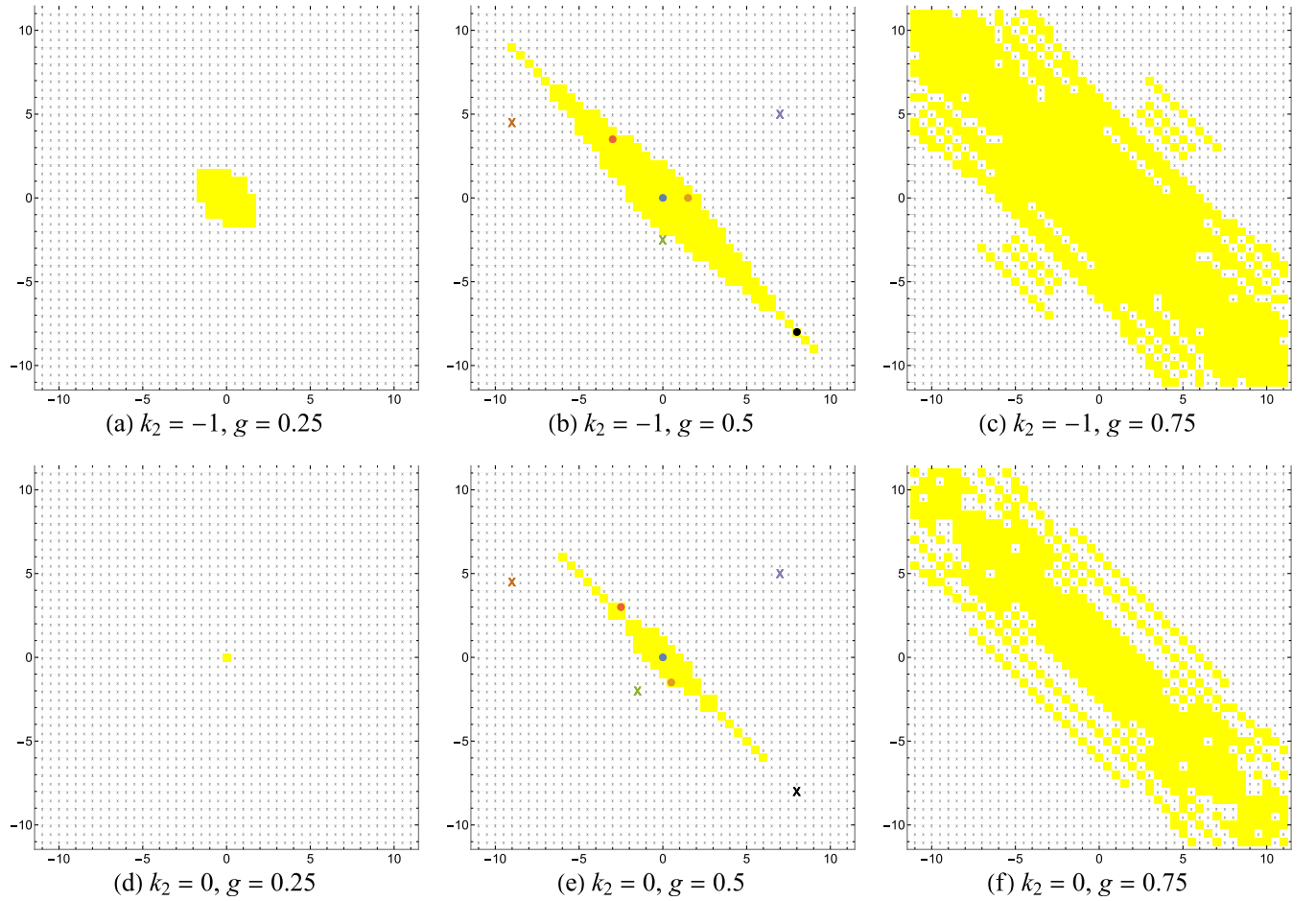


FIG. 10. Regions (yellow) in the parameter space (z_1, z_2) where the physical IR singularities exist from $t \times H^2 \times H^2$ - and $t \times H^2 \times \mathbb{R}^2$ -sliced curved domain walls with $k_1 = -1$ and $g = 0.25, 0.50, 0.75$. In the middle panels, we also show the colored dots corresponding to numerical solutions with physical IR singularities given in Figs. 7 and 8, while the $\times$ symbol refer to solutions with $\hat{g}_{00} \rightarrow +\infty$ near the IR singularities.

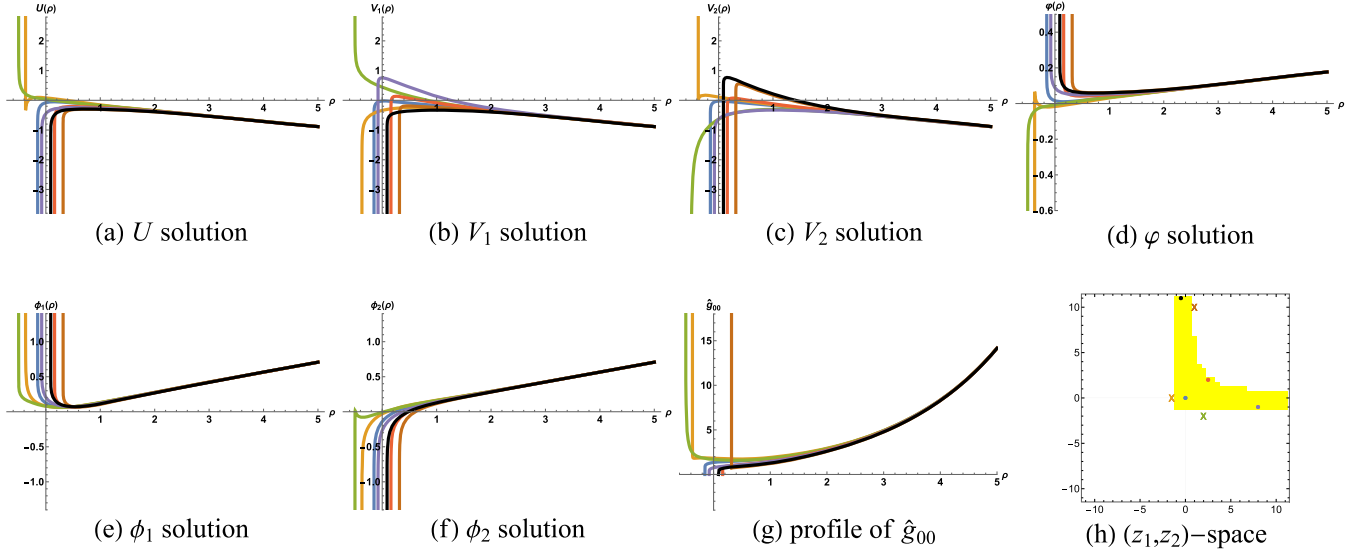


FIG. 11. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times S^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3, 2)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (-1.5, 0), (2, -2), (2.5, 2), (8, -1), (1, 10), (-0.5, 11)$. The yellow region in (z_1, z_2) space shows the area of physical IR singularities.

In particular, solutions with $z_1 = 0$ for $k_1 k_2 \neq 0$ do not exist. In the figures, we have set $g = -\kappa = 1$ and given the solutions for different values of z_1 . We collect all possible values of z_1 giving rise to type IIA solutions with physical IR singularities in Table III.

2. Solutions with nonvanishing ς_1 and ς_2

We now repeat a similar analysis for solutions with nonvanishing shift scalars. In order to keep ς_1 and ς_2

nonvanishing, we need to impose the following conditions on the magnetic charges:

$$p_{21} = \kappa p_{11} \quad \text{and} \quad p_{22} = \kappa p_{12}, \quad (115)$$

together with an algebraic constraint

$$\kappa e^{4\phi_2} \varsigma_1 + e^{4\phi_1} \varsigma_2 = 0. \quad (116)$$

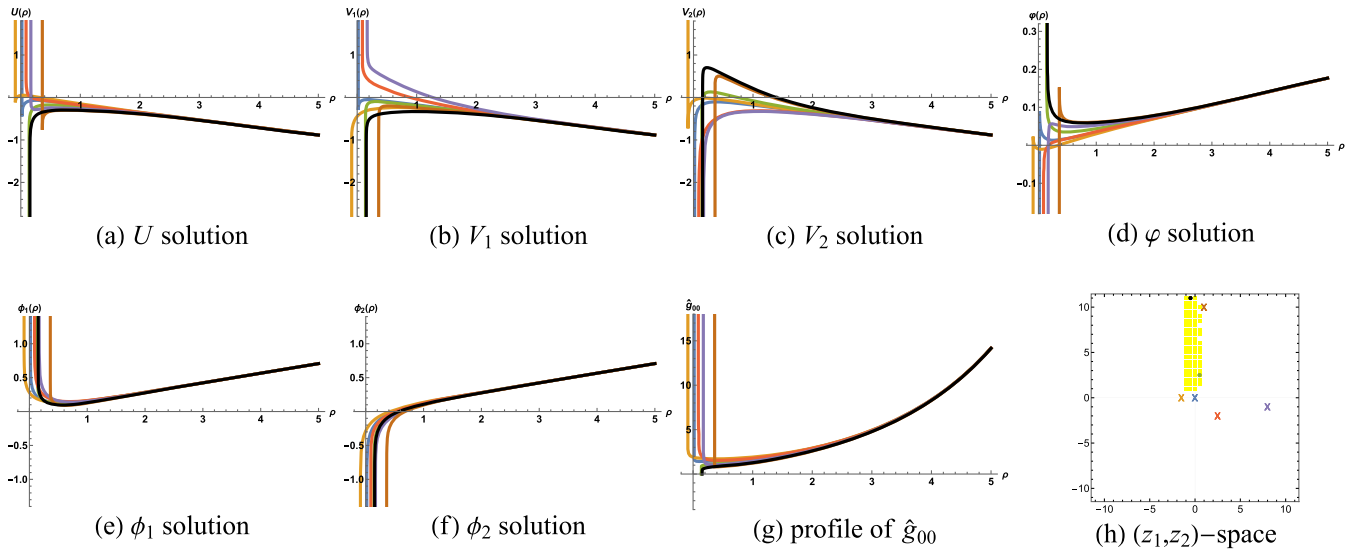


FIG. 12. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3, 2)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (-1.5, 0), (2, -2), (2.5, 2), (8, -1), (1, 10), (-0.5, 11)$. The yellow region in (z_1, z_2) space shows the area of physical IR singularities.

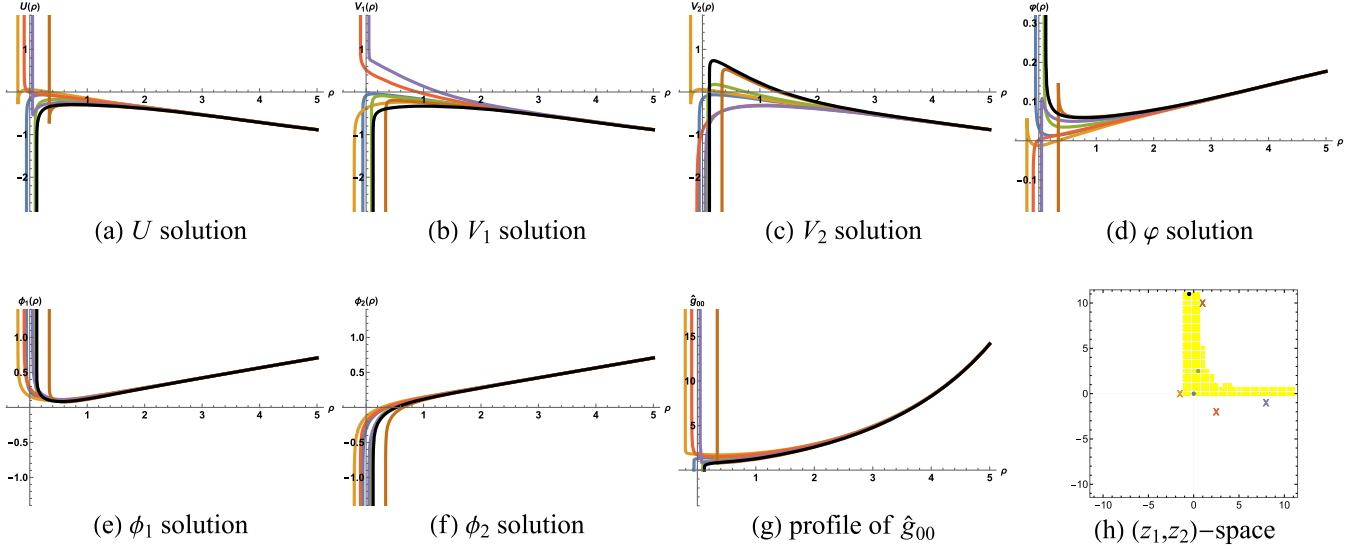


FIG. 13. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (-1.5, 0), (2, -2), (2.5, 2), (8, -1), (1, 10), (-0.5, 11)$. The yellow region in (z_1, z_2) space shows the area of physical IR singularities.

With all these and the magnetic two-form potential (97) as well as the twist conditions

$$2gp_{11} = k_1 \quad \text{and} \quad 2gp_{12} = k_2, \quad (117)$$

we find the following BPS equations:

$$U' = \frac{ge^\varphi}{4\sqrt{2}}(2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1 + \phi_2)}) + \frac{3\kappa\lambda p_{11}p_{12}}{8\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} - \frac{e^{-\varphi}}{8\sqrt{2}}(p_{11}e^{-2V_1} + p_{12}e^{-2V_2})(e^{4\phi_1} + \kappa e^{4\phi_2}), \quad (118)$$

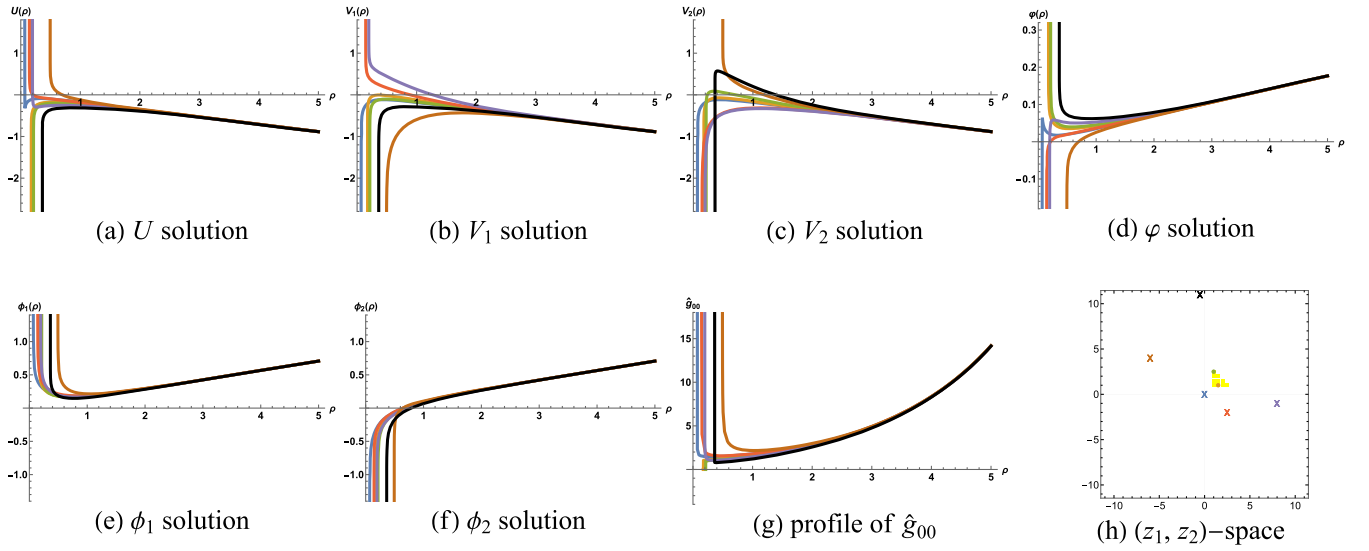


FIG. 14. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (-1.5, 0), (2, -2), (2.5, 2), (8, -1), (1, 10), (-0.5, 11)$. The yellow region in (z_1, z_2) space shows the area of physical IR singularities.

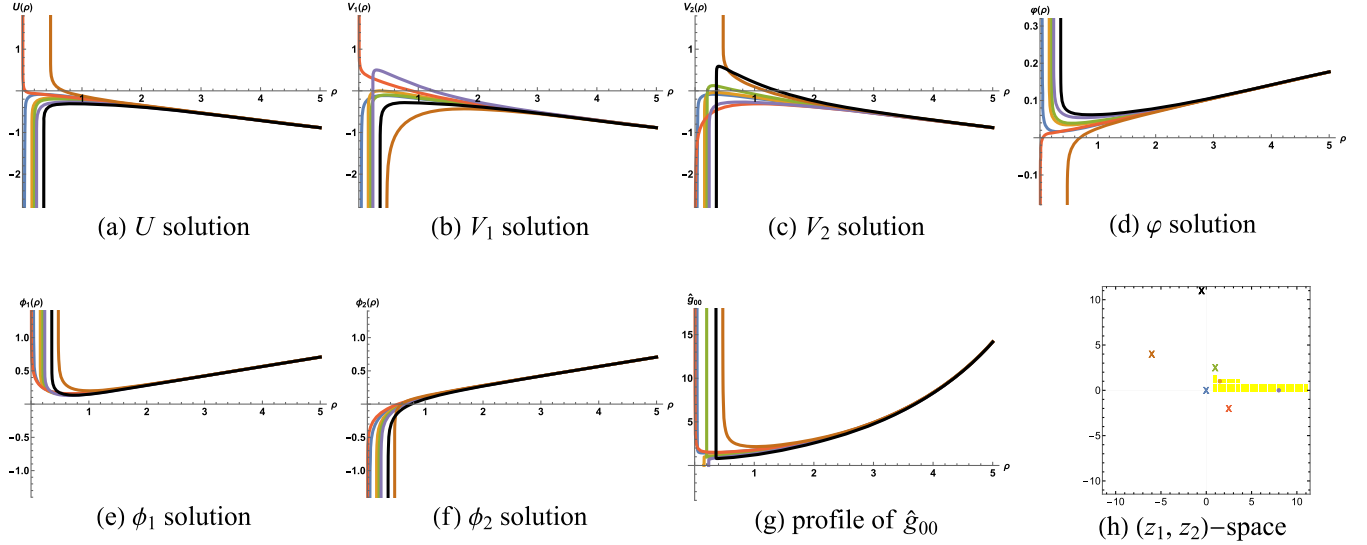


FIG. 15. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $(z_1, z_2) = (0, 0), (-1.5, 0), (2, -2), (2.5, 2), (8, -1), (1, 10), (-0.5, 11)$. The yellow region in (z_1, z_2) space shows the area of physical IR singularities.

$$V_1' = \frac{ge^\varphi}{4\sqrt{2}}(2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) - \frac{\kappa\lambda p_{11}p_{12}}{8\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} + \frac{e^{-\varphi}}{8\sqrt{2}}(3p_{11}e^{-2V_1} - p_{12}e^{-2V_2})(e^{4\phi_1} + \kappa e^{4\phi_2}), \quad (119)$$

$$V_2' = \frac{ge^\varphi}{4\sqrt{2}}(2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) - \frac{\kappa\lambda p_{11}p_{12}}{8\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} - \frac{e^{-\varphi}}{8\sqrt{2}}(p_{11}e^{-2V_1} - 3p_{12}e^{-2V_2})(e^{4\phi_1} + \kappa e^{4\phi_2}), \quad (120)$$

$$\begin{aligned} \varphi' = & -\frac{ge^\varphi}{20\sqrt{2}}(2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) - \frac{3\kappa\lambda p_{11}p_{12}}{40\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} \\ & + \frac{e^{-\varphi}}{40\sqrt{2}}(p_{11}e^{-2V_1} + p_{12}e^{-2V_2})(e^{4\phi_1} + \kappa e^{4\phi_2}), \end{aligned} \quad (121)$$

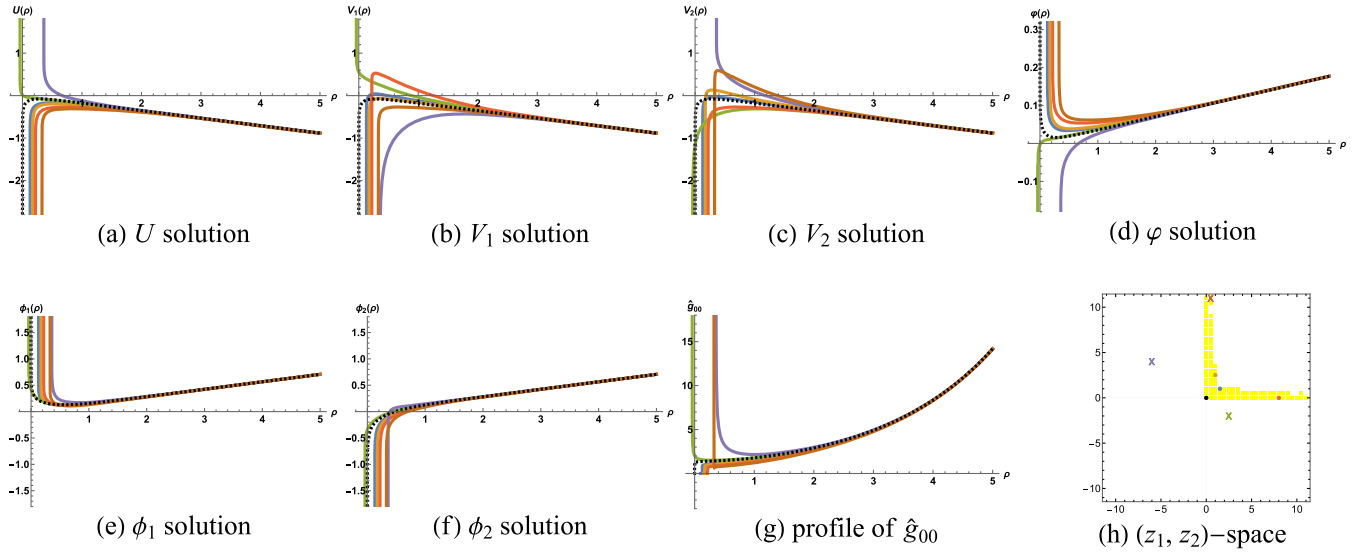


FIG. 16. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times \mathbb{R}^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, and brown curves refer to $(z_1, z_2) = (-1.5, 0), (2, -2), (2.5, 2), (8, -1), (1, 10), (-0.5, 11)$. The dashed curve is the $SO(2) \times SO(2)$ flat domain wall with $z_1 = z_2 = 0$. The yellow region in (z_1, z_2) space shows the area of physical IR singularities.

TABLE II. Summary of satisfaction of the criterion [18] for the IR singularities of the solutions with $t \times \Sigma_{k_1}^2 \times \Sigma_{k_2}^2$ slices obtained from $SO(2) \times SO(2)$ twist with $\varsigma_1 = \varsigma_2 = 0$ in $SO(5)$, $SO(4, 1)$, and $SO(3, 2)$ gauge groups.

UV flat domain wall	Gauge group	Two Riemann surfaces	Physical IR singularity
$SO(5)$	$SO(5)$	$S^2 \times \Sigma^2$	$\times$
		$H^2 \times H^2$	Figs. 10(a)–10(c)
		$H^2 \times \mathbb{R}^2$	Figs. 10(d)–10(f)
		$\mathbb{R}^2 \times \mathbb{R}^2$	$z_1 = z_2 = 0$
$SO(2) \times SO(2)$	$SO(5)$	$\Sigma^2 \times \Sigma^2$	$\times$
	$SO(4, 1)$	$S^2 \times \Sigma^2, H^2 \times H^2$	$\times$
		$H^2 \times \mathbb{R}^2$	$-1 \leq z_1 \leq 1, z_2 = 0$
		$\mathbb{R}^2 \times \mathbb{R}^2$	$z_1 = z_2 = 0$
	$SO(3, 2)$	$S^2 \times S^2$	Fig. 11(h)
		$S^2 \times H^2$	Fig. 12(h)
		$S^2 \times \mathbb{R}^2$	Fig. 13(h)
		$H^2 \times H^2$	Fig. 14(h)
		$H^2 \times \mathbb{R}^2$	Fig. 15(h)
		$\mathbb{R}^2 \times \mathbb{R}^2$	Fig. 16(h)

$$\begin{aligned} \phi'_1 = & \frac{ge^\varphi}{5\sqrt{2}}(3e^{-4\phi_1} - 2\kappa e^{-4\phi_2} - \lambda e^{8(\phi_1+\phi_2)}) - \frac{\kappa\lambda p_{11}p_{12}}{20\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} \\ & - \frac{e^{-\varphi}}{20\sqrt{2}}(p_{11}e^{-2V_1} + p_{12}e^{-2V_2})(3e^{4\phi_1} - 2\kappa e^{4\phi_2}), \end{aligned} \quad (122)$$

$$\begin{aligned} \phi'_2 = & \frac{ge^\varphi}{5\sqrt{2}}(3\kappa e^{-4\phi_2} - 2e^{-4\phi_1} - \lambda e^{8(\phi_1+\phi_2)}) - \frac{\kappa\lambda p_{11}p_{12}}{20\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} \\ & + \frac{e^{-\varphi}}{20\sqrt{2}}(p_{11}e^{-2V_1} + p_{12}e^{-2V_2})(2e^{4\phi_1} - 3\kappa e^{4\phi_2}), \end{aligned} \quad (123)$$

$$\begin{aligned} \varsigma'_1 = & \frac{ge^\varphi}{\sqrt{2}}(2e^{-4\phi_1} - 2\kappa e^{-4\phi_2} - \lambda e^{8(\phi_1+\phi_2)})\varsigma_1 - \frac{\kappa\lambda p_{11}p_{12}\varsigma_1}{2\sqrt{2}ge^{2V_1+2V_2+3\varphi+4\phi_1+4\phi_2}} \\ & - \frac{e^{-\varphi}}{2\sqrt{2}}(p_{11}e^{-2V_1} + p_{12}e^{-2V_2})(e^{4\phi_1} - \kappa e^{4\phi_2})\varsigma_1. \end{aligned} \quad (124)$$

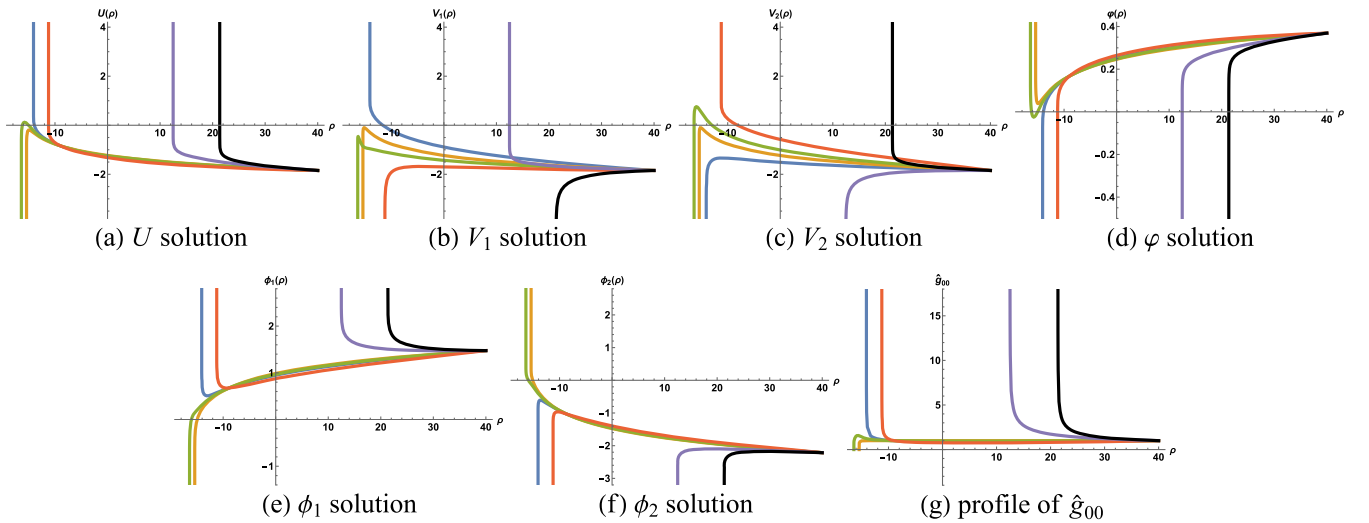


FIG. 17. Numerical solutions interpolating between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times S^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $CSO(2, 2, 1)$ gauge group. The blue, orange, green, red, purple, and black curves, respectively, refer to $z_1 = -2.80, -1, -0.50, -0.15, 0.50, 4$.

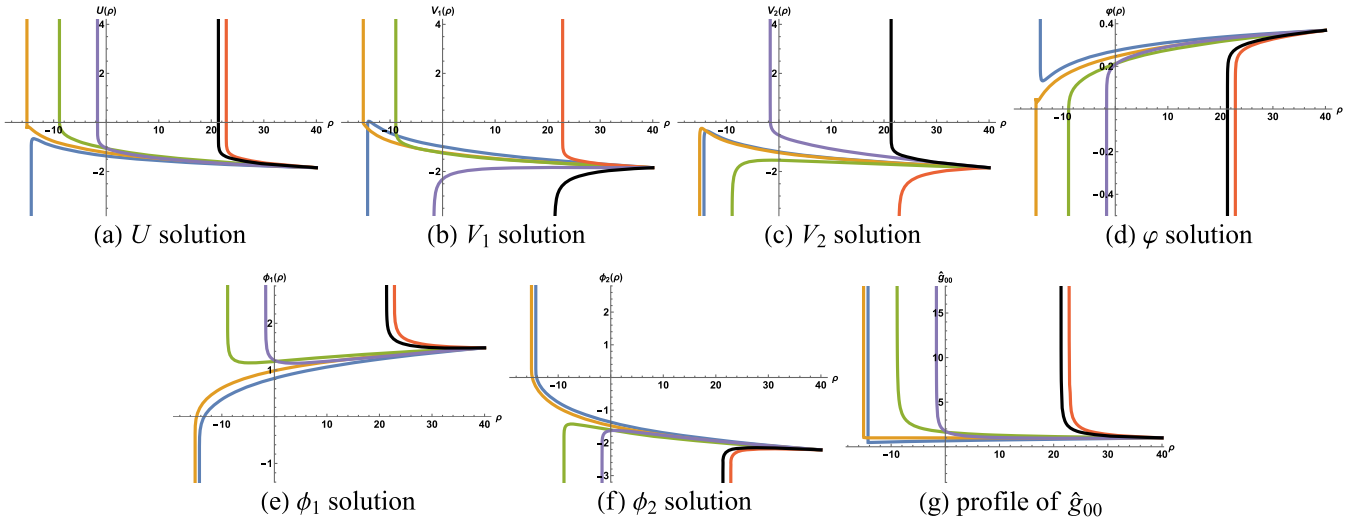


FIG. 18. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $CSO(2,2,1)$ gauge group. The blue, orange, green, red, purple, and black curves, respectively, refer to $z_1 = -2.80, -1, -0.50, -0.15, 0.50, 4$.

It should be pointed out that these are the same as those given in (104) to (109) subject to the conditions (115) together with an additional equation for ς_1 .

For nonvanishing ς_1 and ς_2 , the Killing spinors generally take the form

$$\epsilon_+ = e^{\frac{1}{2}(U+W\gamma_{12})}\epsilon_+^0 \quad \text{and} \quad \epsilon_- = e^{\frac{1}{2}(U+W\gamma_{12})}\epsilon_-^0, \quad (125)$$

with the r -dependent function W determined by the composite connection via

$$\begin{aligned} W' &= Q_r^{12} + Q_r^{34} \\ &= \varsigma_1' + \varsigma_2' - 4\varphi'(\varsigma_1 + \varsigma_2) - 4\varsigma_1\phi_1' - 4\varsigma_2\phi_2'. \end{aligned} \quad (126)$$

However, it turns out that the BPS equations given above imply that $W' = 0$. Without loss of generality, we can choose the constant W to be zero leading to the Killing spinors of the form (81).

For $SO(5)$, $SO(4,1)$, and $SO(3,2)$ gauge groups with $\lambda \neq 0$, these BPS equations are compatible with all the bosonic field equations. On the other hand, for nonsemi-simple gauge groups $CSO(4,0,1)$ and $CSO(2,2,1)$ with $\lambda = 0$, the four-form field strengths given in (99) do not

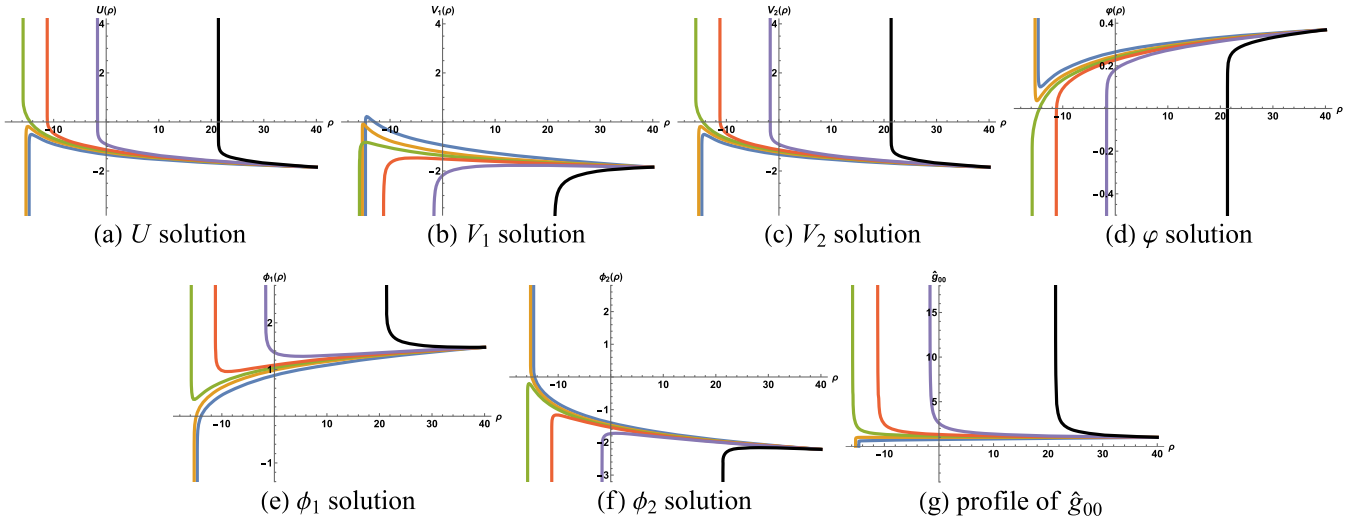


FIG. 19. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $CSO(2,2,1)$ gauge group. The blue, orange, green, red, purple, and black curves, respectively, refer to $z_1 = -2.80, -1, -0.50, -0.15, 0.50, 4$.

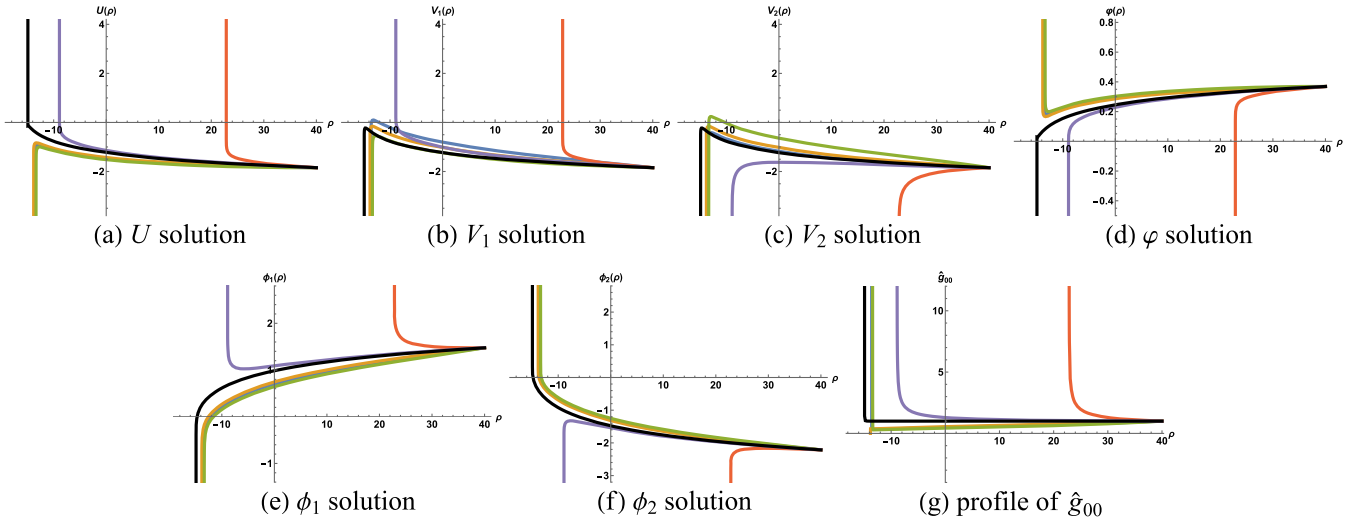


FIG. 20. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $CSO(2, 2, 1)$ gauge group. The blue, orange, green, red, purple, and black curves, respectively, refer to $z_1 = -3.50, -1, -0.15, 0.15, 0.50, 1$.

contribute to the Bianchi identity (14). Accordingly, the contribution of $SO(2) \times SO(2)$ gauge field strengths to the Bianchi identity (14) must vanish identically. This leads to the following condition:

$$(\Gamma_m)_{AB} \mathbb{T}_{np}^A \mathbb{T}_{qr}^B \mathcal{H}_{\hat{\theta}_1 \hat{\zeta}_1}^{np} \mathcal{H}_{\hat{\theta}_2 \hat{\zeta}_2}^{qr} = \frac{\kappa}{6\sqrt{2}} p_{11} p_{12} e^{-2V_1 - 2V_2} \delta_m^5 = 0. \quad (127)$$

However, this implies that either $p_{11} = 0$ or $p_{12} = 0$ incompatible with the twist conditions for $k_1 \neq 0$ and $k_2 \neq 0$. Accordingly, we do not consider solutions from

nonsemisimple gauge groups $CSO(4, 0, 1)$ and $CSO(2, 2, 1)$ in the following analysis.

Similar to the previous case, we look for numerical solutions with asymptotic behaviors given by flat domain walls. For the $SO(5)$ gauge group, all solutions asymptotic to $SO(5)$ symmetric domain wall in (110) have $\varsigma_1 = 0$ and $\varsigma_2 = 0$ along the entire flow solutions. Moreover, imposing the conditions (115) and the twist conditions (117) is equivalent to setting $z_1 = z_2 = 0$ in the previous case. Therefore, the solutions are similar to those already found in the previous section. Accordingly, we do not give the numerical solutions for this case to avoid repetition.

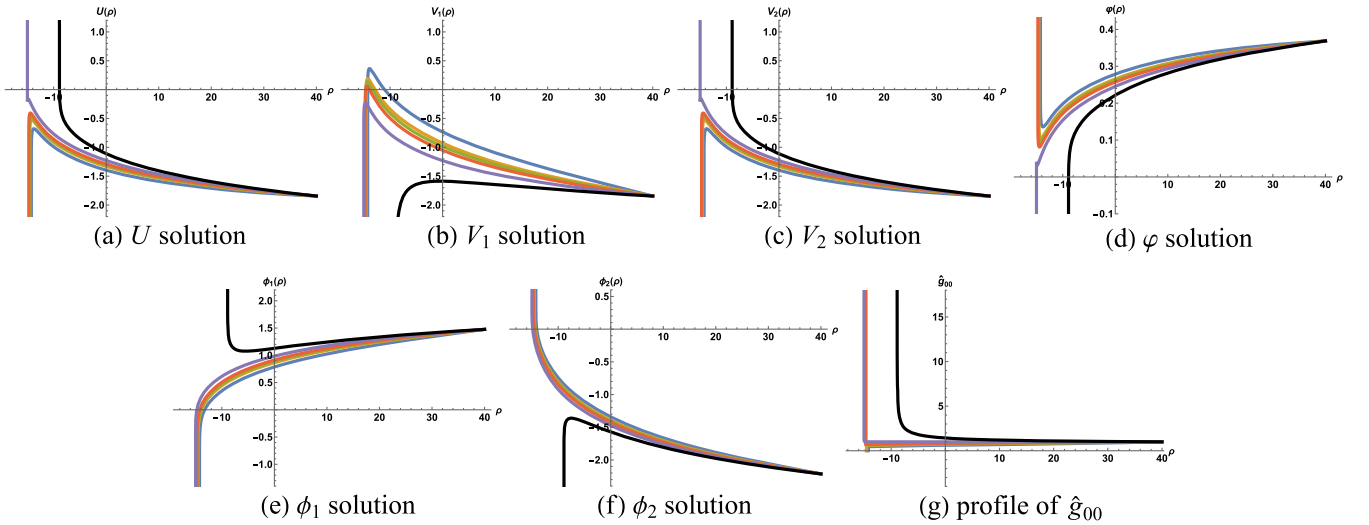


FIG. 21. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $CSO(2, 2, 1)$ gauge group. The blue, orange, green, red, purple, and black curves, respectively, refer to $z_1 = -3.50, -1, -0.50, 0.01, 1, 2$.

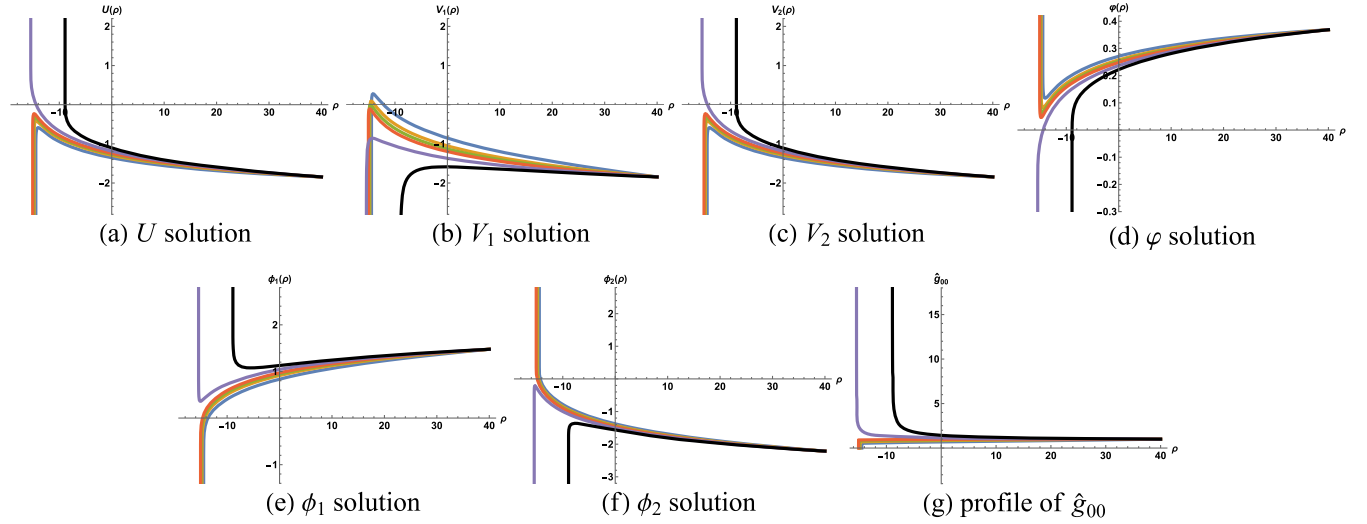


FIG. 22. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times \mathbb{R}^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $CSO(2,2,1)$ gauge group. The blue, orange, green, red, purple, and black curves, respectively, refer to $z_1 = -2.80, -1, -0.50, -0.15, 0.50, 1$.

We now consider solutions with the asymptotic behavior given by $SO(2) \times SO(2)$ symmetric flat domain wall of the form

$$\begin{aligned} U \sim V_1 \sim V_2 \sim \frac{g\lambda\rho}{4\sqrt{2}}, \quad \varphi \sim -\frac{g\lambda\rho}{20\sqrt{2}}, \\ \phi_1 \sim \phi_2 \sim -\frac{g\lambda\rho}{5\sqrt{2}}, \quad \varsigma_1 \sim e^{-\frac{g\lambda\rho}{\sqrt{2}}}, \end{aligned} \quad (128)$$

with the new radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{\varphi+8(\phi_1+\phi_2)}$. We choose $g\lambda < 0$ in order to identify the UV limit with $\rho \rightarrow +\infty$.

As in the previous case with $\varsigma_1 = \varsigma_2 = 0$, all of the solutions in $SO(5)$ gauge group are uplifted to type IIA solutions with unphysical IR singularities, so we also neglect these solutions. For $SO(4,1)$ and $SO(3,2)$ gauge groups, examples of numerical solutions are given in Figs. 23–32 for different values of g in Appendix C 1 a.

TABLE III. Summary of satisfaction of the criterion [18] for the IR singularities of the solutions interpolating between the locally $SO(2) \times SO(2)$ flat domain wall and $t \times \Sigma_{k_1}^2 \times \Sigma_{k_2}^2$ -sliced curved domain walls obtained from $SO(2) \times SO(2)$ twist with $\varsigma_1 = \varsigma_2 = 0$ in $CSO(4,0,1)$ and $CSO(2,2,1)$ gauge groups.

Gauge group	Two Riemann surfaces	Physical IR singularity
$CSO(4,0,1)$	$\Sigma^2 \times \Sigma^2$	$\times$
$CSO(2,2,1)$	$S^2 \times S^2$	$-1 \leq z_1 \leq -0.5$
	$S^2 \times H^2$	$\times$
	$S^2 \times \mathbb{R}^2, H^2 \times \mathbb{R}^2$	$z_1 \leq -1$
	$H^2 \times H^2$	$z_1 = -1$
	$\mathbb{R}^2 \times \mathbb{R}^2$	$z_1 \leq -0.15$

We also recall that, in this case, all the four magnetic charges have been fixed in terms of the gauge coupling constant by the four conditions given in (115) and (117). In addition, we have not given $t \times \mathbb{R}^2 \times \mathbb{R}^2$ -sliced solutions since, in these solutions, we have $k_1 = k_2 = 0$ and $p_{11} = p_{12} = p_{21} = p_{22} = 0$. Therefore, the solutions are essentially flat domain walls. From the figures, we find that all solutions from $SO(3,2)$ gauge group lead to physical IR singularities for any values of k_1 and k_2 . On the other hand, for the $SO(4,1)$ gauge group, only solutions with an $\mathbb{R}^2$ factor, $k_1 = 0$ or $k_2 = 0$, admit physical IR singularities. We collect all possible values of the gauge coupling constant g giving rise to type IIA solutions with physical IR singularities in Table IV.

C. D4-branes wrapped on a Kahler four-cycle

In this section, we perform a similar analysis for D4-branes wrapped on a Kahler four-cycle K_k^4 . As in the

TABLE IV. Summary of satisfaction of the criterion [18] for the IR singularities of the solutions interpolating between the locally $SO(2) \times SO(2)$ flat domain wall and $t \times \Sigma_{k_1}^2 \times \Sigma_{k_2}^2$ -sliced curved domain walls with shift scalars in $SO(5)$, $SO(4,1)$, and $SO(3,2)$ gauge groups.

Gauge group	Two Riemann surfaces	Physical IR singularity
$SO(5)$	$\Sigma^2 \times \Sigma^2$	$\times$
$SO(4,1)$	$S^2 \times S^2, S^2 \times H^2, H^2 \times H^2$	$\times$
	$S^2 \times \mathbb{R}^2$	$g \geq 2$
	$H^2 \times \mathbb{R}^2$	any g
$SO(3,2)$	$S^2 \times S^2, S^2 \times \mathbb{R}^2$	$g \leq -0.6$
	$S^2 \times H^2, H^2 \times H^2, H^2 \times \mathbb{R}^2$	$g \leq -2$

previous cases, the constant k characterizes the constant curvature of K_k^4 with $k = 1, 0, -1$ corresponding to a two-dimensional complex projective space CP^2 , a four-dimensional flat space $\mathbb{R}^4$, and a two-dimensional complex hyperbolic space CH^2 , respectively. The manifold K_k^4 has a $U(2) \sim U(1) \times SU(2)$ spin connection. Therefore, we can perform a twist by turning on $SO(2) \sim U(1)$ or $SO(3) \sim SU(2)$ gauge fields to cancel the $U(1)$ or $SU(2)$ parts of the spin connection.

A general *Ansatz* for the six-dimensional metric takes the form of

$$ds_6^2 = -e^{2U(r)} dt^2 + dr^2 + e^{2V(r)} ds_{K_k^4}^2, \quad (129)$$

in which the explicit form of the metric on a Kahler four-cycle is specified in each case.

1. Solutions with $SO(3)$ twist

We begin with a twist along the $SU(2) \sim SO(3)$ part of the spin connection and choose the following form of the metric on a Kahler four-cycle:

$$ds_{K_k^4}^2 = d\psi^2 + f_k(\psi)^2(\tau_1^2 + \tau_2^2 + \tau_3^2), \quad (130)$$

with $\psi \in [0, \frac{\pi}{2}]$. $f_k(\psi)$ is a function defined as in (61), while τ_i , $i = 1, 2, 3$, are $SU(2)$ left-invariant one-forms satisfying

$$d\tau_i = \varepsilon_{ijl} \tau_j \wedge \tau_l. \quad (131)$$

The explicit forms of τ_i can be chosen to be

$$\begin{aligned} \tau_1 &= \frac{1}{2}(-\sin \vartheta d\theta + \cos \vartheta \sin \theta d\zeta), \\ \tau_2 &= \frac{1}{2}(\cos \vartheta d\theta + \sin \vartheta \sin \theta d\zeta), \\ \tau_3 &= \frac{1}{2}(d\vartheta + \cos \theta d\zeta), \end{aligned} \quad (132)$$

with the ranges of the angular coordinates given by $\theta \in [0, \pi]$, $\zeta \in [0, 2\pi]$, and $\vartheta \in [0, 4\pi]$.

Using the following choice of vielbein:

$$\begin{aligned} e^{\hat{t}} &= e^U dt, & e^{\hat{r}} &= dr, & e^{\hat{\psi}} &= e^V d\psi, \\ e^{\hat{1}} &= e^V f_k(\psi) \tau_1, & e^{\hat{2}} &= e^V f_k(\psi) \tau_2, & e^{\hat{3}} &= e^V f_k(\psi) \tau_3, \end{aligned} \quad (133)$$

we find the following nonvanishing components of the spin connection,

$$\begin{aligned} \omega_{\hat{t}\hat{r}} &= U', & \omega_{\hat{t}\hat{\psi}} &= V'\delta_{\hat{t}\hat{\psi}}, \\ \omega_{\hat{1}\hat{\psi}} &= e^{-V} \frac{f'_k(\psi)}{f_k(\psi)} \delta_{\hat{1}\hat{\psi}}, & \omega_{\hat{1}\hat{2}} &= e^{-V} \frac{1}{f_k(\psi)} \varepsilon_{\hat{1}\hat{2}}^{\hat{3}\hat{4}}. \end{aligned} \quad (134)$$

We have split the six-dimensional coordinates as $x^\mu = (t, r, x^I)$, $I = (\psi, i)$ for $i = 1, 2, 3$, and used $\hat{I} = \{\hat{\psi}, \hat{i}\}$ as a flat index on K_k^4 with $\hat{i}, \hat{j}, \hat{l} = \hat{1}, \hat{2}, \hat{3}$. The $SU(2)$ spin connections on K_k^4 are given by $\omega_{\hat{i}\hat{\psi}}^{\hat{j}\hat{\psi}}$ and $\omega_{\hat{i}\hat{j}}^{\hat{l}\hat{l}}$. To perform a topological twist, we turn on $SO(3)$ gauge fields with the following *Ansatz*:

$$A_{\hat{1}}^{23} = A_{\hat{2}}^{31} = A_{\hat{3}}^{12} = e^{-V} \frac{p}{4k} \frac{[f'_k(\psi) - 1]}{f_k(\psi)}. \quad (135)$$

We then consider all gauge groups containing an $SO(3)$ subgroup. By choosing the $SO(3)$ gauge generators to be X_{12} , X_{13} , and X_{23} , we can describe all possible gauge groups using the embedding tensor characterized by the following symmetric tensor:

$$Y_{mn} = \text{diag}(1, 1, 1, \kappa, \lambda). \quad (136)$$

There are six possible gauge groups given by $SO(5)$ ($\kappa = \lambda = 1$), $SO(4, 1)$ ($\kappa = -\lambda = 1$), $SO(3, 2)$ ($\kappa = \lambda = -1$), $CSO(4, 0, 1)$ ($\kappa = 1, \lambda = 0$), $CSO(3, 1, 1)$ ($\kappa = -1, \lambda = 0$), and $CSO(3, 0, 2)$ ($\kappa = \lambda = 0$).

In addition to the dilaton, there are four $SO(3)$ singlet scalars corresponding to the following noncompact generators:

$$\begin{aligned} \bar{\mathcal{Y}}_1 &= 2\hat{\mathbf{t}}_{11} + 2\hat{\mathbf{t}}_{22} + 2\hat{\mathbf{t}}_{33} - 3\hat{\mathbf{t}}_{44} - 3\hat{\mathbf{t}}_{55}, \\ \bar{\mathcal{Y}}_2 &= \hat{\mathbf{t}}_{45}, & \bar{\mathcal{Y}}_3 &= \hat{\mathbf{t}}_{44} - \hat{\mathbf{t}}_{55}, & \bar{\mathcal{Y}}_4 &= s_{45}, \end{aligned} \quad (137)$$

leading to the coset representative of the form

$$V = e^{\varphi d + \phi_1 \bar{\mathcal{Y}}_1 + \phi_2 \bar{\mathcal{Y}}_2 + \phi_3 \bar{\mathcal{Y}}_3 + \varsigma \bar{\mathcal{Y}}_4}. \quad (138)$$

With $B_{\mu\nu, m} = 0$, the only nonvanishing components of the modified gauge field strengths are given by

$$\begin{aligned} \mathcal{H}_{\hat{\psi}\hat{1}}^{23} &= \mathcal{H}_{\hat{2}\hat{3}}^{23} = \mathcal{H}_{\hat{\psi}\hat{2}}^{31} = \mathcal{H}_{\hat{3}\hat{1}}^{31} \\ &= \mathcal{H}_{\hat{\psi}\hat{3}}^{12} = \mathcal{H}_{\hat{1}\hat{2}}^{12} = -\frac{e^{-2V} p}{4}. \end{aligned} \quad (139)$$

These satisfy all the Bianchi identities without any further conditions. However, there is a nonvanishing term in the two-/four-form duality given in (17) of the form

$$\mathbb{T}^{Am} M_{AB} \mathbb{T}_{np}^B \mathcal{H}_{\hat{\psi}\hat{i}}^{np} = -\frac{1}{\sqrt{2}} e^{-2(V-\varphi-2\phi_1)} p \varsigma \delta_{\hat{i}}^m, \quad (140)$$

$$\mathbb{T}^{Am} M_{AB} \mathbb{T}_{np}^B \mathcal{H}_{\hat{i}\hat{j}}^{np} = -\frac{1}{\sqrt{2}} e^{-2(V-\varphi-2\phi_1)} p \varsigma \varepsilon_{\hat{i}\hat{j}\hat{k}}^{\hat{l}} \delta_{\hat{k}}^m. \quad (141)$$

In order to keep $p \neq 0$, we set $\varsigma = 0$ such that the two-/four-form duality is identically satisfied.

We now impose the twist condition $gp = k$ and the projectors $\hat{\gamma}_r \epsilon_{\pm} = \epsilon_{\mp}$ as well as

$$\hat{\gamma}_{\hat{1}\hat{2}}\epsilon_{+\alpha} = -(\gamma_{12})_{\alpha}^{\beta}\epsilon_{+\beta}, \quad \hat{\gamma}_{\hat{2}\hat{3}}\epsilon_{+\alpha} = -(\gamma_{23})_{\alpha}^{\beta}\epsilon_{+\beta}, \quad \hat{\gamma}_{\hat{\psi}\hat{1}}\epsilon_{+\alpha} = \hat{\gamma}_{\hat{2}\hat{3}}\epsilon_{+\alpha}, \quad (142)$$

together with

$$\hat{\gamma}_{\hat{1}\hat{2}}\epsilon_{-\dot{\alpha}} = -(\gamma_{12})_{\dot{\alpha}}^{\dot{\beta}}\epsilon_{-\dot{\beta}}, \quad \hat{\gamma}_{\hat{2}\hat{3}}\epsilon_{-\dot{\alpha}} = -(\gamma_{23})_{\dot{\alpha}}^{\dot{\beta}}\epsilon_{-\dot{\beta}}, \quad \hat{\gamma}_{\hat{\psi}\hat{1}}\epsilon_{-\dot{\alpha}} = \hat{\gamma}_{\hat{2}\hat{3}}\epsilon_{-\dot{\alpha}}. \quad (143)$$

The Killing spinors take the form

$$\epsilon_{+}(r) = e^{\frac{1}{2}[U(r)+W(r)\gamma_{45}]} \epsilon_{+}^0 \quad \text{and} \quad \epsilon_{-}(r) = e^{\frac{1}{2}[U(r)-W(r)\gamma_{45}]} \epsilon_{-}^0, \quad (144)$$

for an r -dependent function $W(r)$. With all these, the resulting BPS equations read

$$U' = \frac{ge^{\varphi-8\phi_1}}{4\sqrt{2}} \left[3 + e^{20\phi_1} ((\kappa + \lambda) \cosh 2\phi_2 \cosh 4\phi_3 - (\kappa - \lambda) \sinh 4\phi_3) \right] - \frac{3p}{4\sqrt{2}} e^{-2V-\varphi+8\phi_1}, \quad (145)$$

$$V' = \frac{ge^{\varphi-8\phi_1}}{4\sqrt{2}} \left[3 + e^{20\phi_1} ((\kappa + \lambda) \cosh 2\phi_2 \cosh 4\phi_3 - (\kappa - \lambda) \sinh 4\phi_3) \right] + \frac{3p}{4\sqrt{2}} e^{-2V-\varphi+8\phi_1}, \quad (146)$$

$$\varphi' = -\frac{ge^{\varphi-8\phi_1}}{20\sqrt{2}} \left[3 + e^{20\phi_1} ((\kappa + \lambda) \cosh 2\phi_2 \cosh 4\phi_3 - (\kappa - \lambda) \sinh 4\phi_3) \right] + \frac{3p}{20\sqrt{2}} e^{-2V-\varphi+8\phi_1}, \quad (147)$$

$$\phi_1' = \frac{ge^{\varphi-8\phi_1}}{10\sqrt{2}} \left[2 - e^{20\phi_1} ((\kappa + \lambda) \cosh 2\phi_2 \cosh 4\phi_3 - (\kappa - \lambda) \sinh 4\phi_3) \right] - \frac{p}{5\sqrt{2}} e^{-2V-\varphi+8\phi_1}, \quad (148)$$

$$\phi_2' = -\frac{ge^{\varphi+12\phi_1}}{\sqrt{2}} (\kappa + \lambda) \sinh 2\phi_2 \operatorname{sech} 4\phi_3, \quad (149)$$

$$\phi_3' = \frac{ge^{\varphi+12\phi_1}}{2\sqrt{2}} ((\kappa - \lambda) \cosh 4\phi_3 - (\kappa + \lambda) \cosh 2\phi_2 \sinh 4\phi_3), \quad (150)$$

together with

$$W' = -\frac{ge^{\varphi+12\phi_1}}{2\sqrt{2}} (\kappa + \lambda) \sinh 2\phi_2 \tanh 4\phi_3. \quad (151)$$

It can be verified that these BPS equations imply all the second-ordered field equations. We also note that for $k = 0$ the twist condition implies $p = 0$. This leads to untwisted solutions of standard flat domain walls preserving 16 supercharges. As in the previous cases, we do not further consider this type of solutions. On the other hand, for $k = \pm 1$, the above projection conditions imply that the solutions preserve only two supercharges.

For the $CSO(3, 0, 2)$ gauge group with $\kappa = \lambda = 0$, the BPS equations can be analytically solved. In this case, the BPS equations lead to $\phi_2' = \phi_3' = W' = 0$, and we can choose $\phi_2 = \phi_3 = W = 0$ without loss of generality. The other BPS equations give $U' = -5\varphi'$ and $\phi_1' = -\frac{4}{3}\varphi'$ with the solutions given by

$$U = -5\varphi \quad \text{and} \quad \phi_1 = C_1 - \frac{4}{3}\varphi, \quad (152)$$

for an integration constant C_1 . We have neglected an additive integration constant for U which can be absorbed by rescaling the time coordinate.

Taking a linear combination $V' + 5\varphi'$ and changing to a new radial coordinate ρ given by $\frac{d\rho}{dr} = e^{4\varphi+8\phi_1-V}$, we find

$$V = \ln \left[\frac{3p\rho}{2\sqrt{2}} + C \right] - 5\varphi. \quad (153)$$

The integration constant C can also be set to zero by shifting the coordinate ρ . With all these results and $C = 0$, the equation for φ' gives

$$\varphi = \frac{6}{5}C_1 - \frac{3}{40} \ln \left[\frac{9(p\rho^{10/3} + C_0)}{20g\rho^{4/3}} \right], \quad (154)$$

in which C_0 is another integration constant.

In the UV limit as $\rho \rightarrow +\infty$, we find the following asymptotic behavior:

$$\begin{aligned} U &\sim \frac{3}{4} \ln \rho, & V &\sim \frac{7}{4} \ln \rho, \\ \varphi &\sim -\frac{3}{20} \ln \rho, & \phi_1 &\sim \frac{1}{5} \ln \rho, \end{aligned} \quad (155)$$

which leads to the six-dimensional metric of the form

$$ds^2 = \rho^{\frac{3}{2}}(-dt^2 + d\rho^2 + \rho^2 ds_{K_k^4}^2). \quad (156)$$

As $\rho \rightarrow 0$ and $C_0 \neq 0$, the solution becomes

$$\begin{aligned} U &\sim -\frac{1}{2} \ln \rho, & V &\sim \frac{1}{2} \ln \rho, \\ \varphi &\sim \frac{1}{10} \ln \rho, & \phi_1 &\sim -\frac{2}{15} \ln \rho, \end{aligned} \quad (157)$$

leading to the six-dimensional metric,

$$ds^2 = -\rho^{-1} dt^2 + \rho^{\frac{7}{2}} d\rho^2 + \rho ds_{K_k^4}^2. \quad (158)$$

By uplifting the solution to type IIA theory, we can find the behavior of the ten-dimensional metric component $\hat{g}_{00}$ near this IR singularity as

$$\hat{g}_{00} = e^{2U+\varphi-3\phi_1} \sim \frac{1}{\sqrt{\rho}} \rightarrow +\infty, \quad (159)$$

which implies that the singularity is unphysical.

For $C_0 = 0$, we find the same asymptotic behavior (155) as $\rho \rightarrow 0$. The six-dimensional metric is also given by (156), and the behavior of $\hat{g}_{00}$ near this IR singularity is

$$\hat{g}_{00} = e^{2U+\varphi-3\phi_1} \sim \rho^{3/4} \rightarrow 0. \quad (160)$$

Accordingly, in this case, the singularity is physical, and the solution describes D4-branes wrapped on K_k^4 giving rise to $N = 2$ supersymmetric quantum mechanics in the IR.

In other gauge groups, we need to solve the BPS equations numerically. We begin with solutions that are asymptotic to a locally flat domain walls with $SO(5)$ symmetry given by

$$U \sim V \sim \frac{5g\rho}{4\sqrt{2}}, \quad \varphi \sim -\frac{g\rho}{4\sqrt{2}}, \quad \phi_1 \sim \phi_2 \sim \phi_3 \sim 0, \quad (161)$$

with the radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{\varphi-8\phi_1}$. Examples of numerical solutions interpolating between this asymptotic behavior and $t \times K_k^4$ -sliced curved domain walls in $SO(5)$ gauge group for different values of g are given in Figs. 33 and 34 in Appendix C 2 a. In all of these solutions, we find that ϕ_2 and ϕ_3 vanish identically along the entire flows, so we have neglected the corresponding numerical solutions for ϕ_2 and ϕ_3 for convenience. From the behavior of the ten-dimensional metric component $\hat{g}_{00}$, we find that all $t \times CH^2$ -sliced solutions are uplifted to type IIA solutions with physically acceptable IR singularities. On the other hand, the $t \times CP^2$ -sliced solutions admit physical IR singularities only for $g \geq 0.3$.

Another class of solutions consists of solutions that are asymptotically locally $SO(3)$ symmetric flat domain wall of the form

$$\begin{aligned} U \sim V &\sim \frac{3g\rho}{4\sqrt{2}}, & \varphi &\sim -\frac{3g\rho}{20\sqrt{2}}, \\ \phi_1 &\sim \frac{g\rho}{5\sqrt{2}}, & \phi_2 \sim \phi_3 &\sim 0. \end{aligned} \quad (162)$$

Choosing $g < 0$ in order to identify the UV limit with $\rho \rightarrow +\infty$, we find a number of numerical solutions interpolating between this locally $SO(3)$ symmetric flat domain wall and singular geometries in the IR. It turns out that all solutions in $SO(5)$, $SO(4,1)$, $CSO(4,0,1)$, and $CSO(3,1,1)$ gauge groups are uplifted to type IIA solutions with unphysical IR singularities. We do not give these solutions here. In the case of $SO(3,2)$ gauge group, examples of numerical solutions are shown in Figs. 35 and 36 in Appendix C 2 a. We also note that $\phi_2 = \phi_3 = 0$ in all of the solutions, so we have omitted the numerical solutions for ϕ_2 and ϕ_3 in the figures. From the figures, we see that all $t \times CP^2$ -sliced solutions are uplifted to ten dimensions with physical IR singularities for any values of g . On the other hand, the $t \times CH^2$ -sliced solutions admit physical IR singularities only for $g < -2$. All of these $t \times K_k^4$ -sliced curved domain walls obtained from $SO(3)$ twist are summarized in Table V. We also note that these results are very similar to solutions with an $SO(3)$ twist on a Riemannian three-manifold studied recently in [32].

2. Solutions with $SU(2)$ twist

We now move to the topological twist obtained by canceling the $SU(2)$ part of the spin connection with the self-dual $SU(2)_+$ subgroup of $SU(2)_+ \times SU(2)_- \sim SO(4) \subset SO(5)$ generated by $X_{12} + X_{34}$, $X_{13} + X_{24}$, and $X_{23} + X_{14}$. We then turn on the following gauge fields:

TABLE V. Summary of satisfaction of the criterion [18] for the IR singularities of the $t \times K_k^4$ -sliced curved domain walls obtained from $SO(3)$ twist in $SO(5)$, $SO(4, 1)$, $SO(3, 2)$, $CSO(4, 0, 1)$, $CSO(3, 1, 1)$, and $CSO(3, 0, 2)$ gauge groups.

UV flat domain wall	Gauge group	Kahler four-cycle	Physical IR singularity
$SO(5)$	$SO(5)$	CP^2 CH^2	$g \geq 0.3$ Any g
$SO(3)$	$SO(5)$	CP^2, CH^2	$\times$
	$SO(4, 1)$	CP^2, CH^2	$\times$
	$SO(3, 2)$	CP^2	Any g
		CH^2	$g \leq -2$
	$CSO(4, 0, 1)$	CP^2, CH^2	$\times$
	$CSO(3, 1, 1)$	CP^2, CH^2	$\times$
	$CSO(3, 0, 2)$ (analytic)	CP^2, CH^2	$C_0 = 0$

$$A_1^{23} = A_2^{31} = A_3^{12} = e^{-V} \frac{P}{8k} \frac{[f'_k(\psi) - 1]}{f_k(\psi)} \quad (163)$$

$$\text{and } A_1^{14} = A_2^{24} = A_3^{34} = e^{-V} \frac{P}{8k} \frac{[f'_k(\psi) - 1]}{f_k(\psi)}. \quad (164)$$

The gauge fields can also be written compactly as

$$A_i^i = \frac{1}{2} \varepsilon^{ijl} A_i^{jl} + A_i^{i4} = e^{-V} \frac{P}{4k} \frac{[f'_k(\psi) - 1]}{f_k(\psi)} \delta_i^i. \quad (165)$$

In this case, we perform the twist by imposing the twist condition $gp = k$ and the projections given in (142) and (143) as well as two additional projectors

$$\begin{aligned} (\gamma_{12})_\alpha^\beta \epsilon_{+\beta} &= (\gamma_{34})_\alpha^\beta \epsilon_{+\beta} \\ \text{and } (\gamma_{12})_{\dot{\alpha}}^{\dot{\beta}} \epsilon_{-\dot{\beta}} &= (\gamma_{34})_{\dot{\alpha}}^{\dot{\beta}} \epsilon_{-\dot{\beta}}. \end{aligned} \quad (166)$$

Accordingly, the resulting solutions preserve only one supercharges.

In addition, we also need to turn on the magnetic two-form potential

$$B_{\hat{t}\hat{r}}^5 = -\frac{3p^2}{16g^2} e^{-2(2V+3\varphi+8\phi)}. \quad (167)$$

Recall that the vector representation of $SO(4)$ is identified with $(2, 2)$ representation of $SU(2)_+ \times SU(2)_-$, we see that the $SU(2)_+$ is embedded in $SO(4)$ in such a way that the $SU(2)_+$ singlets are essentially $SO(4)$ singlets. Accordingly, we need to consider gauge groups with an $SO(4)$ subgroup and singlet scalars under $SO(4)$. These gauge groups are $SO(5)$, $SO(4, 1)$, and $CSO(4, 0, 1)$ with the embedding tensor given by (65). As in the case of $SO(4)$ twist, we consider only $SO(5)$ and $SO(4, 1)$ gauge groups with $\lambda \neq 0$ in order to satisfy the Bianchi identities and the two-/four-form duality.

Using the $SO(4)$ invariant coset representative (66) and the $\hat{\gamma}_r$ projection (71), we find the following BPS equations:

$$U' = \frac{g}{4\sqrt{2}} e^{\varphi-4\phi} (4 + \lambda e^{20\phi}) - \frac{3p}{4\sqrt{2}} e^{-2V-\varphi+4\phi} + \frac{9\lambda p^2}{32\sqrt{2}g} e^{-4V-3\varphi-8\phi}, \quad (168)$$

$$V' = \frac{g}{4\sqrt{2}} e^{\varphi-4\phi} (4 + \lambda e^{20\phi}) + \frac{3p}{4\sqrt{2}} e^{-2V-\varphi+4\phi} - \frac{3\lambda p^2}{32\sqrt{2}g} e^{-4V-3\varphi-8\phi}, \quad (169)$$

$$\varphi' = -\frac{g}{20\sqrt{2}} e^{\varphi-4\phi} (4 + \lambda e^{20\phi}) + \frac{3p}{20\sqrt{2}} e^{-2V-\varphi+4\phi} - \frac{9\lambda p^2}{160\sqrt{2}g} e^{-4V-3\varphi-8\phi}, \quad (170)$$

$$\phi' = \frac{g}{5\sqrt{2}} e^{\varphi-4\phi} (1 - \lambda e^{20\phi}) - \frac{3p}{20\sqrt{2}} e^{-2V-\varphi+4\phi} - \frac{3\lambda p^2}{80\sqrt{2}g} e^{-4V-3\varphi-8\phi}. \quad (171)$$

We then look for numerical solutions asymptotic to locally flat domain walls with $SO(5)$ and $SO(4)$ symmetries given in (82) and (85), respectively. We first note that the resulting BPS equations take a very similar form as those in Sec. III A. The corresponding numerical solutions are also very similar to those obtained from the $SO(4)$ twist on a Riemannian four-manifold in Sec. III A. We do not present these solutions here to avoid repetition but simply give a summary of the physical IR singularities in Table VI.

3. Solutions with $SO(2) \times SO(2)$ twist

Another possibility to perform a topological twist on a Kahler four-cycle is achieved by canceling the $U(1)$ part of the spin connection. In this case, it is convenient to write the metric on K_k^4 as

TABLE VI. Summary of satisfaction of the criterion [18] for the IR singularities of the curved domain wall solutions obtained from $SU(2)$ twist on a Kahler four-cycle in $SO(5)$ and $SO(4, 1)$ gauge groups.

UV flat domain wall	Gauge group	Kahler four-cycle	Physical IR singularity
$SO(5)$	$SO(5)$	CP^2	$\times$
		CH^2	Any g
$SO(4)$	$SO(5)$	CP^2, CH^2	$\times$
	$SO(4, 1)$	CP^2, CH^2	$\times$

$$ds_{K_k^4}^2 = \frac{d\psi^2}{(k\psi^2 + 1)^2} + \frac{\psi^2 \tau_3^2}{(k\psi^2 + 1)^2} + \frac{\psi^2}{(k\psi^2 + 1)} (\tau_1^2 + \tau_2^2), \quad (172)$$

with τ_i being the $SU(2)$ left-invariant one-forms given in (132). The six-dimensional metric is still given by (129) with $ds_{K_k^4}^2$ given by (172).

With the vielbein of the form

$$e^{\hat{t}} = e^U dt, \quad e^{\hat{r}} = dr, \quad e^{\hat{\psi}} = \frac{e^V d\psi}{(k\psi^2 + 1)},$$

$$e^{\hat{1}} = \frac{e^V \psi}{\sqrt{k\psi^2 + 1}} \tau_1, \quad e^{\hat{2}} = \frac{e^V \psi}{\sqrt{k\psi^2 + 1}} \tau_2, \quad e^{\hat{3}} = \frac{e^V \psi}{(k\psi^2 + 1)} \tau_3, \quad (173)$$

all nonvanishing components of the spin connection are given by

$$\omega_{\hat{t}\hat{r}} = U', \quad \omega_{\hat{t}\hat{\psi}} = V' \delta_{\hat{t}}^{\hat{\psi}}, \quad \hat{I}, \hat{J} = \hat{\psi}, \hat{1}, \hat{2}, \hat{3},$$

$$\omega_{\hat{1}\hat{\psi}} = \omega_{\hat{1}\hat{2}} = \omega_{\hat{2}\hat{1}} = \omega_{\hat{2}\hat{\psi}} = \frac{e^{-V}}{\psi},$$

$$\omega_{\hat{3}\hat{\psi}} = e^{-V} \frac{(1 - k\psi^2)}{\psi}, \quad \omega_{\hat{3}\hat{1}} = e^{-V} \frac{(2k\psi^2 + 1)}{\psi}. \quad (174)$$

Imposing the projector

$$\hat{\gamma}_{\hat{\psi}} \hat{\epsilon}_{\pm} = \hat{\gamma}_{\hat{2}\hat{3}} \epsilon_{\pm}, \quad (175)$$

we find that $\omega_{\hat{1}\hat{\psi}}$ and $\omega_{\hat{1}\hat{2}}$ as well as $\omega_{\hat{2}\hat{1}}$ and $\omega_{\hat{2}\hat{\psi}}$ cancel each other in the covariant derivatives of the supersymmetry parameters. The remaining $U(1)$ part takes the form of

$$\omega_{\hat{3}\hat{\psi}} - \omega_{\hat{3}\hat{1}} = e^{-V} k\psi. \quad (176)$$

We perform a twist to cancel this by turning on $SO(2) \times SO(2)$ gauge fields,

$$A_3^{12} = \frac{3p_1}{4} e^{-V} \psi \quad \text{and} \quad A_3^{34} = \frac{3p_2}{4} e^{-V} \psi. \quad (177)$$

We also need to impose the following projection conditions on the Killing spinors:

$$\hat{\gamma}_{\hat{1}\hat{2}} \epsilon_{+\alpha} = (\gamma_{12})_{\alpha}^{\beta} \epsilon_{+\beta} = (\gamma_{34})_{\alpha}^{\beta} \epsilon_{+\beta} \quad (178)$$

and

$$\hat{\gamma}_{\hat{1}\hat{2}} \epsilon_{-\alpha} = (\gamma_{12})_{\alpha}^{\beta} \epsilon_{-\beta} = (\gamma_{34})_{\alpha}^{\beta} \epsilon_{-\beta} \quad (179)$$

together with the twist condition

$$g(p_1 + \kappa p_2) = k. \quad (180)$$

Including the $\hat{\gamma}_r$ projector in (71), the solutions preserve two supercharges.

As in the case of $SO(2) \times SO(2)$ twist on two Riemann surfaces, the embedding tensor and scalar coset representative are given, respectively, by (92) and (94). The two-form gauge field strengths take the form

$$\mathcal{H}_{\hat{1}\hat{2}}^{12} = \mathcal{H}_{\hat{\psi}\hat{3}}^{12} = \frac{3p_1}{2} e^{-2V}$$

$$\text{and} \quad \mathcal{H}_{\hat{1}\hat{2}}^{34} = \mathcal{H}_{\hat{\psi}\hat{3}}^{34} = \frac{3p_2}{2} e^{-2V}. \quad (181)$$

In order to satisfy the Bianchi identity (14), we also need to turn on the magnetic two-form potential of the form

$$B_{\hat{t}\hat{r}}^5 = -\frac{9}{g^2} p_1 p_2 e^{-2(2V+3\varphi+4\phi_1+4\phi_2)}. \quad (182)$$

As in the previous cases, for the gauge groups with $\lambda = 0$, the Bianchi identity implies either $p_1 = 0$ or $p_2 = 0$. Accordingly, to keep both of the $SO(2)$ gauge fields nonvanishing, we consider only gauge groups with $\lambda \neq 0$. Furthermore, consistency of the BPS equations also requires either $\varsigma_1 = \varsigma_2 = 0$ or $p_2 = p_1/\kappa$ with $\varsigma_1 \neq 0$ and $\varsigma_2 \neq 0$. We consider these two possibilities separately.

Solutions with $\varsigma_1 = \varsigma_2 = 0$. For $\varsigma_1 = \varsigma_2 = 0$, the corresponding BPS equations read

$$U' = \frac{ge^{\varphi}}{4\sqrt{2}} (2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) - \frac{3e^{-2V-\varphi}}{2\sqrt{2}} (p_1 e^{4\phi_1} + p_2 e^{4\phi_2}) + \frac{27\lambda}{2\sqrt{2}g} p_1 p_2 e^{-4V-3\varphi-4\phi_1+4\phi_2}, \quad (183)$$

$$V' = \frac{ge^{\varphi}}{4\sqrt{2}} (2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) + \frac{3e^{-2V-\varphi}}{2\sqrt{2}} (p_1 e^{4\phi_1} + p_2 e^{4\phi_2}) - \frac{9\lambda}{2\sqrt{2}g} p_1 p_2 e^{-4V-3\varphi-4\phi_1+4\phi_2}, \quad (184)$$

$$\varphi' = -\frac{ge^{\varphi}}{20\sqrt{2}} (2e^{-4\phi_1} + 2\kappa e^{-4\phi_2} + \lambda e^{8(\phi_1+\phi_2)}) + \frac{3e^{-2V-\varphi}}{10\sqrt{2}} (p_1 e^{4\phi_1} + p_2 e^{4\phi_2}) - \frac{27\lambda}{10\sqrt{2}g} p_1 p_2 e^{-4V-3\varphi-4\phi_1+4\phi_2}, \quad (185)$$

$$\phi'_1 = \frac{ge^\varphi}{5\sqrt{2}}(3e^{-4\phi_1} - 2\kappa e^{-4\phi_2} - \lambda e^{8(\phi_1+\phi_2)}) - \frac{3e^{-2V-\varphi}}{5\sqrt{2}}(3p_1 e^{4\phi_1} - 2p_2 e^{4\phi_2}) - \frac{9\lambda}{5\sqrt{2}g}p_1 p_2 e^{-4V-3\varphi-4\phi_1+4\phi_2}, \quad (186)$$

$$\phi'_2 = \frac{ge^\varphi}{5\sqrt{2}}(3\kappa e^{-4\phi_2} - 2e^{-4\phi_1} - \lambda e^{8(\phi_1+\phi_2)}) + \frac{3e^{-2V-\varphi}}{5\sqrt{2}}(2p_1 e^{4\phi_1} - 3p_2 e^{4\phi_2}) - \frac{9\lambda}{5\sqrt{2}g}p_1 p_2 e^{-4V-3\varphi-4\phi_1+4\phi_2}. \quad (187)$$

We begin with solutions that are asymptotic to a locally $SO(5)$ symmetric flat domain wall with the following asymptotic behavior, as $\rho \rightarrow +\infty$:

$$U \sim V \sim \frac{5g\lambda\rho}{4\sqrt{2}}, \quad \varphi \sim -\frac{g\lambda\rho}{4\sqrt{2}}, \quad \phi_1 \sim \phi_2 \sim 0. \quad (188)$$

The new radial coordinate ρ is defined by $\frac{d\rho}{dr} = e^{\varphi+8(\phi_1+\phi_2)}$. Examples of numerical solutions from $SO(5)$ gauge group are given in Figs. 37–39 in Appendix C 2 b. In these figures, we have set $g = 1$ and different values of p_1 . All $t \times CP^2$ -sliced domain walls are uplifted to type IIA theory with unphysical IR singularities. On the other hand, for $t \times CH^2$ - or $t \times \mathbb{R}^4$ -sliced domain walls, all solutions lead to physical solutions in type IIA theory. We also point out that the solutions with $t \times \mathbb{R}^4$ slices do not need a topological twists and preserve eight supercharges due to the projectors (71) and (175). However, the $SO(2) \times SO(2)$ gauge fields are nonvanishing in these solutions. For the sake of comparison, we also give the flat domain wall solutions with $p_1 = p_2 = k = 0$ represented by the dashed line in Fig. 39.

Another class of solutions consists of solutions with the asymptotic geometry given by an $SO(2) \times SO(2)$ symmetric flat domain wall of the form

$$U \sim V \sim \frac{g\lambda\rho}{4\sqrt{2}}, \quad \varphi \sim -\frac{g\lambda\rho}{20\sqrt{2}}, \quad \phi_1 \sim \phi_2 \sim -\frac{g\lambda\rho}{5\sqrt{2}}. \quad (189)$$

We choose $g\lambda < 0$ in order to identify the UV limit with $\rho \rightarrow +\infty$. For the $SO(5)$ gauge group, all solutions are uplifted to type IIA theory with unphysical IR singularities. For $SO(4,1)$ and $SO(3,2)$ gauge groups, examples of numerical solutions are shown in Figs. 40–45 in Appendix C 2 b. From these figures, we see that only the solutions with specific values of the magnetic charge p_1 can be uplifted to ten-dimensional solutions with physical IR singularities. The conditions for obtaining solutions with physical IR singularities are summarized in Table VII.

Solutions with ς_1 and ς_2 scalars. To include nonvanishing shift scalars ς_1 and ς_2 in the BPS equations, we have to impose the condition $p_2 = p_1/\kappa$ and an algebraic condition,

$$\kappa e^{4\phi_2}\varsigma_1 + e^{4\phi_1}\varsigma_2 = 0. \quad (190)$$

The resulting BPS equations are the same as those given in (183) to (187) with $p_2 = p_1/\kappa$ together with an additional equation,

$$\begin{aligned} \varsigma'_1 = & \frac{ge^\varphi}{\sqrt{2}}(2e^{-4\phi_1} - 2\kappa e^{-4\phi_2} - \lambda e^{8(\phi_1+\phi_2)})\varsigma_1 \\ & - \frac{9\sqrt{2}\lambda p_1^2 \varsigma_1}{g\kappa} e^{-4V-3\varphi-4\phi_1-4\phi_2} \\ & - 3\sqrt{2}p_1 e^{-2V-\varphi}(e^{4\phi_1} - \kappa e^{4\phi_2})\varsigma_1. \end{aligned} \quad (191)$$

It turns out that all solutions asymptotic to an $SO(5)$ symmetric flat domain wall from $SO(5)$ gauge group have $\varsigma_1 = \varsigma_2 = 0$ along the entire solutions. Since these solutions are qualitatively the same as those considered in the previous case, we do not give the corresponding numerical solutions here.

We now move to solutions with the asymptotic geometry given by an $SO(2) \times SO(2)$ symmetric flat domain wall of the form

TABLE VII. Summary of satisfaction of the criterion [18] for the IR singularities of the solutions with $t \times K_k^4$ slices obtained from $SO(2) \times SO(2)$ twist with $\varsigma_1 = \varsigma_2 = 0$ in $SO(5)$, $SO(4,1)$, and $SO(3,2)$ gauge groups.

UV flat domain wall	Gauge group	Kahler four-cycle	Physical IR singularity
$SO(5)$	$SO(5)$	CP^2	$\times$
		$CH^2, \mathbb{R}^4$	Any p_1
$SO(2) \times SO(2)$	$SO(5)$	K^4	$\times$
		CP^2	$\times$
		CH^2	$-1 \leq p_1 \leq 0$
	$SO(4,1)$	$\mathbb{R}^4$	$p_1 = 0$
		CP^2	$-0.8 \leq p_1 \leq -1.1$
		$CH^2, \mathbb{R}^4$	$p_1 = 0$

TABLE VIII. Summary of satisfaction of the criterion [18] for the IR singularities of the solutions interpolating between the locally $SO(2) \times SO(2)$ flat domain wall and $t \times K_k^4$ -sliced curved domain walls with shift scalars in $SO(5)$, $SO(4, 1)$, and $SO(3, 2)$ gauge groups.

Gauge group	Kahler four-cycles	Physical IR singularity
$SO(5)$	CP^2, CH^2	$\times$
$SO(4, 1)$	CP^2, CH^2	$\times$
$SO(3, 2)$	CP^2	$g \leq -1.5$
	CH^2	$\times$

$$U \sim V \sim \frac{g\lambda\rho}{4\sqrt{2}}, \quad \varphi \sim -\frac{g\lambda\rho}{20\sqrt{2}},$$

$$\phi_1 \sim \phi_2 \sim -\frac{g\lambda\rho}{5\sqrt{2}}, \quad \varsigma_1 \sim e^{-\frac{g\lambda\rho}{\sqrt{2}}}, \quad (192)$$

with the new radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{\varphi+8(\phi_1+\phi_2)}$. By choosing $g\lambda < 0$, we can identify the UV limit with $\rho \rightarrow +\infty$. From the numerical analysis, we find that all solutions from $SO(5)$ and $SO(4, 1)$ gauge groups are uplifted to ten-dimensional solutions with unphysical IR singularities. Accordingly, we omit all of these solutions here. For the $SO(3, 2)$ gauge group, examples of numerical solutions for different values of g are given in Figs. 46 and 47 in Appendix C 2 b. We have omitted the $t \times \mathbb{R}^4$ -sliced solutions with $k = 0$ and $p_2 = p_1 = 0$ since, in this case, the solutions are essentially flat domain walls. From these figures, we find that only $t \times CP^2$ -sliced domain walls with $g \leq -1.5$ (an example of these solutions is given by the purple line in Fig. 46) are uplifted to solutions of type IIA theory with physical IR singularities. All of the solutions are summarized in Table VIII.

IV. SUPERSYMMETRIC QUANTUM MECHANICS FROM $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ GAUGED SUPERGRAVITY

In this section, we repeat the same analysis for solutions describing D4-branes wrapped on a four-manifold from six-dimensional gauged supergravity with $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$. This gauge group arises from the embedding tensor in $\overline{40}^{-1}$ representation of $GL(5)$. The explicit form of the embedding tensor is given by

$$\theta^{Am} = \mathbb{T}_{np}^A U^{np,m} \quad \text{and} \quad \theta^A_m = 0. \quad (193)$$

We also recall that the tensor $U^{mn,p}$ satisfies the conditions $U^{mn,p} = U^{[mn],p}$ and $U^{[mn,p]} = 0$. The quadratic constraint can be solved by taking

$$U^{mn,p} = v^{[m} w^{n]p}, \quad (194)$$

in which v^m is a $GL(5)$ vector, and w^{mn} is a symmetric tensor, $w^{mn} = w^{(mn)}$.

Using $SL(5)$ symmetry, we can choose the vector $v^m = \delta_5^m$ and split the $SL(5)$ index $m = (i, 5)$ with $i = 1, \dots, 4$. In addition, we also restrict the discussion to cases with $w^{i5} = w^{55} = 0$. With a proper $SL(4) \subset SL(5)$ transformation that leaves $v^m = \delta_5^m$ invariant, we can write w^{ij} as

$$w^{ij} = \text{diag}(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q, \underbrace{0, \dots, 0}_r), \quad (195)$$

with $p + q + r = 4$. The nonvanishing gauge generators are given by

$$X_{ij} = \frac{1}{\sqrt{2}} \varepsilon_{ijl_1 l_2} w^{l_1 l_3} t^{l_2}_{l_3} \quad \text{and} \quad X^i = w^{ij} s_{5j}. \quad (196)$$

These generators satisfy the following Lie algebra:

$$[X^i, X^j] = 0, \quad [X_{ij}, X^l] = (X_{ij})_l{}^{l'} X^{l'},$$

$$[X_{i_1 i_2}, X_{i_3 i_4}] = (X_{i_1 i_2})_{i_3 i_4}{}^{i_5 i_6} X_{i_5 i_6}, \quad (197)$$

with $(X_{i_1 i_2})_{i_3 i_4}{}^{i_5 i_6} = 2(X_{i_1 i_2})_{[i_3}{}^{[i_5} \delta_{i_4]}^{i_6]}$. The gauge group is then given by

$$G_0 = CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$$

$$= SO(p, q) \ltimes (\mathbb{R}^{(p+q)(4-p-q)} \times \mathbb{R}^4). \quad (198)$$

The $CSO(p, q, 4-p-q) \subset SL(4) \subset SL(5)$ factor and the four-dimensional translation group $\mathbb{R}^4$ are, respectively, generated by X_{ij} and X^i .

In this case, it is convenient to describe scalar fields in the $SL(4)$ basis. In accordance with the branching $SL(5) \rightarrow SL(4) \times \mathbb{R}^+$, we split the $SO(5) \times SO(5)$ vector indices as $a = (i, 5)$ and $\dot{a} = (\dot{i}, \dot{5})$. The $SL(5)$ noncompact generators then decompose as

$$\hat{t}_{ab} \rightarrow (\hat{t}_{ij}, \hat{t}_{i\dot{5}}, \hat{t}_{\dot{5}\dot{5}}). \quad (199)$$

The $\mathbb{R}^+ \sim SO(1, 1)$ generator is given by

$$\tilde{\mathcal{Y}}_0 = \hat{t}_{11} + \hat{t}_{22} + \hat{t}_{33} + \hat{t}_{44} - 4\hat{t}_{5\dot{5}}, \quad (200)$$

while the noncompact generators of $SL(4)$ can be written as

$$\bar{t}_{ij} = \hat{t}_{ij} + \frac{1}{4} \hat{t}_{5\dot{5}} \delta_{ij}, \quad (201)$$

with $\hat{t}_{11} + \hat{t}_{22} + \hat{t}_{33} + \hat{t}_{44} = -\hat{t}_{5\dot{5}}$ due to the tracelessness of $SL(5)$ generators. The nine scalar fields corresponding to $\bar{t}_{ij}$ generators parametrize an $SL(4)/SO(4) \subset SL(5)/SO(5)$ coset. The remaining four scalars associated with $\hat{t}_{i\dot{5}}$ are nilpotent scalars and are denoted by b_i .

In addition to the dilaton associated with generator $\mathbf{d}$, there are also ten axions corresponding to the antisymmetric shift generators s_{mn} as in the previous section.

The gauge generators couple to ten gauge fields denoted by $A_\mu^{ij} = A_\mu^{[ij]}$ and $A_{\mu,i}$ in the gauge covariant derivatives,

$$D_\mu = \nabla_\mu - gA_{\mu,i}X^i - gA_\mu^{ij}X_{ij}. \quad (202)$$

We perform topological twists by turning on gauge fields A_μ^{ij} associated with compact residual symmetries that are subgroups of the $CSO(p, q, 4 - p - q)$ factor and truncate out all the gauge fields $A_{\mu,i}$. Moreover, since gaugings in $\mathbf{40}^{-1}$ representation are purely electric, the three-forms $C_{\mu\nu\rho,A}$ and the magnetic two-forms $B_{\mu\nu}^m$ do not appear in the Lagrangian.

As in the previous section, the gauge fields arising from the topological twist on a four-manifold usually lead to a nonvanishing term in the Bianchi identity (14) of the form $(\Gamma_m)_{AB} \mathbb{T}_{i_1 i_2}^A \mathbb{T}_{i_3 i_4}^B \mathcal{H}_{[\mu\nu}^{i_1 i_2} \mathcal{H}_{\rho\sigma]}^{i_3 i_4}$. However, unlike the case of $CSO(p, q, 5 - p - q)$ gauge group, there are no magnetic two-form fields $B_{\mu\nu}^m$ to cancel this term in such a way that the Bianchi identity is satisfied. On the other hand, the Bianchi identity could be recovered by turning on the electric two-form fields $B_{\mu\nu,m}$ to make

$$\mathcal{D}_{[\mu} \mathcal{H}_{\nu\rho\sigma],m} = \frac{3}{2\sqrt{2}} (\Gamma_m)_{AB} \mathbb{T}_{i_1 i_2}^A \mathbb{T}_{i_3 i_4}^B \mathcal{H}_{[\mu\nu}^{i_1 i_2} \mathcal{H}_{\rho\sigma]}^{i_3 i_4}. \quad (203)$$

However, we are not able to find a consistent set of BPS equations with nonvanishing three-form field strengths. Therefore, in this section, we set all the tensor fields to zero, namely,

$$B_{\mu\nu,m} = 0, \quad B_{\mu\nu}^m = 0, \quad C_{\mu\nu\rho,A} = 0, \quad (204)$$

and solve the problematic Bianchi identity by requiring the term $(\Gamma_m)_{AB} \mathbb{T}_{i_1 i_2}^A \mathbb{T}_{i_3 i_4}^B \mathcal{H}_{[\mu\nu}^{i_1 i_2} \mathcal{H}_{\rho\sigma]}^{i_3 i_4}$ to vanish identically. This condition is rather restrictive. In particular, a topological twist on a Riemannian four-manifold Σ_k^4 by turning on $SO(4)$ gauge fields is not possible. In the following analysis, we then consider topological twists only on four-dimensional manifolds given by a product of two Riemann surfaces $\Sigma_{k_1}^2 \times \Sigma_{k_2}^2$ and a Kahler four-cycle K_k^4 in this section.

A. D4-branes wrapped on a product of two Riemann surfaces

For solutions describing D4-branes wrapped on a product of two Riemann surfaces, the *Ansatz* for the metric is still given in (86). To cancel the spin connection on $\Sigma_{k_1}^2 \times \Sigma_{k_2}^2$, we turn on the following $SO(2) \times SO(2)$ gauge fields:

$$\begin{aligned} A_{\hat{\xi}_1}^{12} &= -\frac{e^{-V_1} p_{21} f'_{k_1}(\theta_1)}{2\sqrt{2}k_1 f_{k_1}(\theta_1)}, & A_{\hat{\xi}_2}^{12} &= -\frac{e^{-V_2} p_{22} f'_{k_2}(\theta_2)}{2\sqrt{2}k_2 f_{k_2}(\theta_2)}, \\ A_{\hat{\xi}_1}^{34} &= -\frac{e^{-V_1} p_{11} f'_{k_1}(\theta_1)}{2\sqrt{2}k_1 f_{k_1}(\theta_1)}, & A_{\hat{\xi}_2}^{34} &= -\frac{e^{-V_2} p_{12} f'_{k_2}(\theta_2)}{2\sqrt{2}k_2 f_{k_2}(\theta_2)}, \end{aligned} \quad (205)$$

with the corresponding two-form field strengths,

$$\begin{aligned} \mathcal{H}_{\hat{\theta}_1 \hat{\xi}_1}^{12} &= \frac{e^{-2V_1} p_{21}}{2\sqrt{2}}, & \mathcal{H}_{\hat{\theta}_2 \hat{\xi}_2}^{12} &= \frac{e^{-2V_2} p_{22}}{2\sqrt{2}}, \\ \mathcal{H}_{\hat{\theta}_1 \hat{\xi}_1}^{34} &= \frac{e^{-2V_1} p_{11}}{2\sqrt{2}}, & \mathcal{H}_{\hat{\theta}_2 \hat{\xi}_2}^{34} &= \frac{e^{-2V_2} p_{12}}{2\sqrt{2}}. \end{aligned} \quad (206)$$

These lead to a nonvanishing term in the Bianchi identity (14) of the form

$$\begin{aligned} &(\Gamma_m)_{AB} \mathbb{T}_{i_1 i_2}^A \mathbb{T}_{i_3 i_4}^B \mathcal{H}_{\hat{\theta}_1 \hat{\xi}_1}^{i_1 i_2} \mathcal{H}_{\hat{\theta}_2 \hat{\xi}_2}^{i_3 i_4} \\ &= \frac{1}{6\sqrt{2}} (p_{11} p_{22} + p_{12} p_{21}) e^{-2V_1 - 2V_2} \delta_m^5. \end{aligned} \quad (207)$$

As previously mentioned, in order for the Bianchi identity to hold, we require that the four magnetic charges satisfy the following relation:

$$p_{11} p_{22} + p_{12} p_{21} = 0. \quad (208)$$

There are two gauge groups that contain an $SO(2) \times SO(2)$ subgroup. These are $SO(4) \ltimes \mathbb{R}^4$ and $SO(2, 2) \ltimes \mathbb{R}^4$ obtained from the embedding tensor with

$$w^{ij} = \text{diag}(1, 1, \kappa, \kappa), \quad (209)$$

for $\kappa = 1, -1$, respectively. Under $SO(2) \times SO(2)$ residual symmetry generated by X_{12} and X_{34} , there are five singlet scalars with the coset representative of the form

$$V = e^{\varphi \mathbf{d} + \phi_0 \hat{Y}_0 + \phi \hat{Y}_1 + \varsigma_1 \hat{Y}_2 + \varsigma_2 \hat{Y}_3}. \quad (210)$$

$\varphi(r)$ is the dilaton corresponding to the $SO(1, 1)$ generator $\mathbf{d}$ while $\phi_0(r)$ is another dilatonic scalar associated with the $SO(1, 1)$ factor in $SL(5) \rightarrow SL(4) \times SO(1, 1)$ branching. The remaining three scalars correspond to the following $SO(5, 5)$ noncompact generators:

$$\hat{Y}_1 = \bar{t}_{11} + \bar{t}_{22} - \bar{t}_{33} - \bar{t}_{44}, \quad \hat{Y}_2 = s_{12}, \quad \hat{Y}_3 = s_{34}. \quad (211)$$

We now perform an $SO(2) \times SO(2)$ twist by imposing the twist condition (103) together with the projection conditions (101), (102) and

$$\hat{\gamma}_r \epsilon_\pm = \gamma^5 \epsilon_\mp \quad (212)$$

on the Killing spinors (81). Similar to the $SO(2) \times SO(2)$ twist performed in $CSO(p, q, 5 - p - q)$ gauge group,

consistency requires either vanishing shift scalars or additional conditions on the magnetic charges $p_{21} = \kappa p_{11}$ and $p_{22} = \kappa p_{12}$ with nonvanishing shift scalars. However, the latter together with the aforementioned algebraic constraint (208) leads to $p_{11}p_{12} = 0$ which is not compatible with the topological twist. Accordingly, we do not consider the solutions with nonvanishing ς_1 and ς_2 in the following analysis.

With $\varsigma_1 = \varsigma_2 = 0$, we find the following BPS equations:

$$U' = \frac{g}{4} e^{\varphi-4(\phi_0+\phi)} (e^{8\phi} + \kappa) - \frac{1}{8} e^{-\varphi+4\phi_0} [e^{-2V_1} (p_{11}e^{-4\phi} + p_{21}e^{4\phi}) + e^{-2V_2} (p_{12}e^{-4\phi} + p_{22}e^{4\phi})], \quad (213)$$

$$V_1' = \frac{g}{4} e^{\varphi-4(\phi_0+\phi)} (e^{8\phi} + \kappa) + \frac{1}{8} e^{-\varphi+4\phi_0} [3e^{-2V_1} (p_{11}e^{-4\phi} + p_{21}e^{4\phi}) - e^{-2V_2} (p_{12}e^{-4\phi} + p_{22}e^{4\phi})], \quad (214)$$

$$V_2' = \frac{g}{4} e^{\varphi-4(\phi_0+\phi)} (e^{8\phi} + \kappa) - \frac{1}{8} e^{-\varphi+4\phi_0} [e^{-2V_1} (p_{11}e^{-4\phi} + p_{21}e^{4\phi}) - 3e^{-2V_2} (p_{12}e^{-4\phi} + p_{22}e^{4\phi})], \quad (215)$$

$$\varphi' = -\frac{g}{20} e^{\varphi-4(\phi_0+\phi)} (e^{8\phi} + \kappa) + \frac{1}{40} e^{-\varphi+4\phi_0} [e^{-2V_1} (p_{11}e^{-4\phi} + p_{21}e^{4\phi}) + e^{-2V_2} (p_{12}e^{-4\phi} + p_{22}e^{4\phi})], \quad (216)$$

$$\phi_0' = \frac{g}{20} e^{\varphi-4(\phi_0+\phi)} (e^{8\phi} + \kappa) - \frac{1}{40} e^{-\varphi+4\phi_0} [e^{-2V_1} (p_{11}e^{-4\phi} + p_{21}e^{4\phi}) + e^{-2V_2} (p_{12}e^{-4\phi} + p_{22}e^{4\phi})], \quad (217)$$

$$\phi' = -\frac{g}{4} e^{\varphi-4(\phi_0+\phi)} (e^{8\phi} - \kappa) + \frac{1}{8} e^{-\varphi+4\phi_0} [e^{-2V_1} (p_{11}e^{-4\phi} - p_{21}e^{4\phi}) + e^{-2V_2} (p_{12}e^{-4\phi} - p_{22}e^{4\phi})]. \quad (218)$$

As in the previous section, we solve these equations numerically. For the $SO(4) \ltimes \mathbb{R}^4$ gauge group with $\kappa = 1$, we first look for solutions with an asymptotic behavior, as $r \rightarrow +\infty$, given by a locally $SO(4)$ symmetric flat domain wall,

$$U \sim V_1 \sim V_2 \sim \ln \left[\frac{gr}{2} \right], \quad \varphi \sim -\phi_0 \sim -\frac{1}{5} \ln \left[\frac{gr}{2} \right], \quad (219)$$

$$\phi \sim \frac{16}{g^4 r^4}.$$

This behavior implies that $\phi \rightarrow 0$ as $r \rightarrow +\infty$, so the $SO(4)$ symmetry is unbroken in this limit. In this case, the parameters z_1 and z_2 , defined in (111), are not independent due to the algebraic condition $p_{11}p_{22} + p_{12}p_{21} = 0$ or, equivalently, in terms of z_1 and z_2 , $z_1 z_2 = k_1 k_2$. From the behavior of the ten-dimensional metric component $\hat{g}_{00}$, we find that all solutions involving S^2 factors are uplifted to type IIA solutions with unphysical IR singularities. Accordingly, we have omitted the corresponding numerical solutions in these cases. For $t \times \mathbb{R}^2 \times \mathbb{R}^2$ -sliced domain walls, all solutions but for $z_1 = 0$ lead to unphysical solutions in type IIA theory. The solutions with $z_1 = 0$ corresponds to a flat domain wall with $p_{11} = p_{12} = p_{21} = p_{22} = k_1 = k_2 = 0$ due to the twist conditions, so all $t \times \mathbb{R}^2 \times \mathbb{R}^2$ -sliced solutions have been omitted as well.

Examples of numerical solutions for the remaining two cases are given in Figs. 48 and 49 in Appendix D 1 for

$g = 1$ and different values of z_1 . For $t \times H^2 \times H^2$ -sliced domain walls, the solutions with $z_1 \approx \pm 1$ lead to solutions in type IIA theory with physical IR singularities. On the other hand, solutions with $t \times H^2 \times \mathbb{R}^2$ slices admit physical IR singularities for $-1 \leq z_1 \leq 1$.

Another class of solutions consists of those with an asymptotic behavior taking the form of an $SO(2) \times SO(2)$ symmetric flat domain wall,

$$U \sim V_1 \sim V_2 \sim -\phi \sim -\ln \left[-\frac{g\rho}{4} \right] \quad (220)$$

$$\text{and } \varphi \sim -\phi_0 \sim \frac{1}{5} \ln \left[-\frac{g\rho}{4} \right],$$

with the radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{5\phi-5\phi_0}$. We also choose $g < 0$ in order to identify the UV limit with $\rho \rightarrow +\infty$. Solutions of these type can be found in both $SO(4) \ltimes \mathbb{R}^4$ and $SO(2, 2) \ltimes \mathbb{R}^4$ gauge groups. For the $SO(4) \ltimes \mathbb{R}^4$ gauge group, all solutions turn out to be unphysical upon uplifted to type IIA theory, so we do not present the corresponding numerical solutions here. For the $SO(2, 2) \ltimes \mathbb{R}^4$ gauge group, examples of numerical solutions are shown in Figs. 50–55 in Appendix D 1. From these figures, we see that many solutions can be uplifted to ten-dimensional solutions with physical IR singularities. We have summarized the conditions on z_1 in order for the IR singularities to be physically acceptable in Table IX.

TABLE IX. Summary of satisfaction of the criterion [18] for the IR singularities of the solutions with $t \times \Sigma_{k_1}^2 \times \Sigma_{k_2}^2$ slices obtained from $SO(2) \times SO(2)$ twist in $SO(4) \ltimes \mathbb{R}^4$ and $SO(2, 2) \ltimes \mathbb{R}^4$ gauge groups.

UV flat domain wall	Gauge group	Two Riemann surfaces	Physical IR singularity
$SO(4)$	$SO(4) \ltimes \mathbb{R}^4$	$S^2 \times \Sigma^2$ $H^2 \times H^2$ $H^2 \times \mathbb{R}^2$ $\mathbb{R}^2 \times \mathbb{R}^2$	$\times$ $z_1 \approx \pm 1$ $-1 \leq z_1 \leq 1$ $z_1 \approx 0$
$SO(2) \times SO(2)$	$SO(4) \ltimes \mathbb{R}^4$ $SO(2, 2) \ltimes \mathbb{R}^4$	$\Sigma^2 \times \Sigma^2$ $S^2 \times S^2$ $S^2 \times H^2$ $S^2 \times \mathbb{R}^2$ $H^2 \times H^2, H^2 \times \mathbb{R}^2, \mathbb{R}^2 \times \mathbb{R}^2$	$\times$ $0.48 \leq z_1 \leq 2$ $\times$ $z_1 \geq 0.54$ $z_1 > 0$

B. D4-branes wrapped on a Kahler four-cycle

We now consider D4-branes wrapped on a Kahler four-cycle. Unlike the previous case of the $CSO(p, q, 5 - p - q)$ gauge group, we cannot perform an $SU(2)$ twist on a Kahler four-cycle. As previously mentioned, in the absence of tensor fields, the Bianchi identity (14) requires the $SU(2)_\pm \subset SO(4)$ gauge fields to vanish.

1. Solutions with $SO(3)$ twist

We first consider an $SO(3)$ twist from $SO(4) \ltimes \mathbb{R}^4$, $SO(3, 1) \ltimes \mathbb{R}^4$, and $CSO(3, 0, 1) \ltimes \mathbb{R}^4$ gauge groups which can be collectively characterized by a symmetric tensor,

$$w^{ij} = \text{diag}(1, 1, 1, \kappa), \quad (221)$$

for $\kappa = 1, -1, 0$, respectively. The metric *Ansatz* is still given by (129) with the metric on K_k^4 given in (130). To cancel the $SU(2)$ part of the spin connection, we turn on $SO(3)$ gauge fields of the form

$$A_1^{14} = A_2^{24} = A_3^{34} = -\frac{e^{-V} p}{2\sqrt{2}k} \frac{[f'_k(\psi) - 1]}{f_k(\psi)}. \quad (222)$$

There are five $SO(3)$ singlet scalars consisting of the two dilatons and three additional scalars corresponding to the following noncompact generators:

$$\tilde{\mathcal{Y}}_1 = \tilde{t}_{11} + \tilde{t}_{22} + \tilde{t}_{33} - 3\tilde{t}_{44}, \quad \tilde{\mathcal{Y}}_2 = \hat{t}_{45}, \quad \tilde{\mathcal{Y}}_3 = s_{45}. \quad (223)$$

The coset representative then takes the form

$$V = e^{\varphi d + \phi_0 \tilde{\mathcal{Y}}_0 + \phi \tilde{\mathcal{Y}}_1 + b \tilde{\mathcal{Y}}_2 + \varsigma \tilde{\mathcal{Y}}_3}. \quad (224)$$

The gauge two-form field strengths are given by

$$\begin{aligned} \mathcal{H}_{\hat{\psi} \hat{1}}^{14} &= \mathcal{H}_{\hat{2} \hat{3}}^{14} = \mathcal{H}_{\hat{\psi} \hat{2}}^{24} = \mathcal{H}_{\hat{3} \hat{1}}^{24} \\ &= \mathcal{H}_{\hat{\psi} \hat{3}}^{34} = \mathcal{H}_{\hat{1} \hat{2}}^{34} = \frac{e^{-2V} p}{2\sqrt{2}}. \end{aligned} \quad (225)$$

We also note that these nonvanishing components lead to $\mathcal{H}^{ij} \wedge \mathcal{H}^{kl} = 0$ such that the Bianchi identities are satisfied identically. However, this is not the case for $SU(2)$ or $SO(4)$ twists.

We perform the $SO(3)$ twist by imposing a twist condition $gp = k$ together with the projections (142), (143), and (212) on the Killing spinors (81). However, it turns out that consistency of the BPS equations requires the axionic scalar b to vanish. With $b = 0$, the resulting BPS equations are given by

$$U' = \frac{g}{8} e^{\varphi - 4(\phi_0 + 3\phi)} (3e^{16\phi} + \kappa) - \frac{3p}{4} e^{-2V - \varphi + 4(\phi_0 - \phi)}, \quad (226)$$

$$V' = \frac{g}{8} e^{\varphi - 4(\phi_0 + 3\phi)} (3e^{16\phi} + \kappa) + \frac{3p}{4} e^{-2V - \varphi + 4(\phi_0 - \phi)}, \quad (227)$$

$$\varphi' = -\frac{g}{40} e^{\varphi - 4(\phi_0 + 3\phi)} (3e^{16\phi} + \kappa) + \frac{3p}{20} e^{-2V - \varphi + 4(\phi_0 - \phi)}, \quad (228)$$

$$\phi'_0 = \frac{g}{40} e^{\varphi - 4(\phi_0 + 3\phi)} (3e^{16\phi} + \kappa) - \frac{3p}{20} e^{-2V - \varphi + 4(\phi_0 - \phi)}, \quad (229)$$

$$\phi' = -\frac{g}{8} e^{\varphi - 4(\phi_0 + 3\phi)} (e^{16\phi} - \kappa) + \frac{p}{4} e^{-2V - \varphi + 4(\phi_0 - \phi)}, \quad (230)$$

$$\varsigma' = -g\kappa e^{\varphi - 4(\phi_0 + 3\phi)} \varsigma. \quad (231)$$

For $CSO(3, 0, 1) \ltimes \mathbb{R}^4$ gauge group with $\kappa = 0$, we can analytically solve the BPS equations. In this case, the BPS equations imply $\varsigma' = 0$. We choose the constant value of ς to be zero for simplicity. The other equations give the following solutions for U , φ , and ϕ_0 as functions of ϕ :

$$U = -3\phi, \quad \varphi = C + \frac{3}{5}\phi, \quad \phi_0 = C_0 - \frac{3}{5}\phi. \quad (232)$$

As usual, we have neglected an additive integration constant for the warp factor U .

We then consider a linear combination of the BPS equations of the form

$$V' + 3\phi' = \frac{3}{2} p e^{-2V-\varphi+4(\phi_0-\phi)}. \quad (233)$$

Using the results in (232) and changing to a new radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{-\phi}$, we find

$$V = \frac{1}{2} \ln [3p\rho e^{4C_0}] - 3\phi - \frac{C}{2}. \quad (234)$$

With all these results, we finally find the solution for $\phi(\rho)$ of the form

$$\phi = \frac{C_0}{2} - \frac{1}{8} \ln \left[\frac{3p}{5k} e^C \rho + \frac{C_1}{\rho^{2/3}} \right], \quad (235)$$

in which C_1 is another integration constant.

As $\rho \rightarrow +\infty$, the asymptotic behavior of the solution is given by

$$\begin{aligned} U &\sim \frac{3}{8} \ln \rho, & V &\sim \frac{7}{8} \ln \rho, \\ \varphi &\sim -\phi_0 \sim -\frac{3}{40} \ln \rho, & \phi &\sim -\frac{1}{8} \ln \rho. \end{aligned} \quad (236)$$

This leads to the six-dimensional metric of the form

$$ds^2 = \rho^{\frac{3}{4}} (-dt^2 + \rho^{-1} d\rho^2 + \rho ds_{K^4}^2). \quad (237)$$

As $\rho \rightarrow 0$ and $C_1 \neq 0$, we find

$$\begin{aligned} U &\sim -\frac{1}{4} \ln \rho, & V &\sim \frac{1}{4} \ln \rho, \\ \varphi &\sim -\phi_0 \sim \frac{1}{20} \ln \rho, & \phi &\sim \frac{1}{12} \ln \rho, \end{aligned} \quad (238)$$

with the six-dimensional metric given by

$$ds^2 = -\rho^{\frac{1}{2}} dt^2 + \rho^{\frac{1}{6}} d\rho^2 + \rho^{\frac{1}{2}} ds_{K^4}^2. \quad (239)$$

Upon uplifted to type IIA theory, the solution gives the behavior of the ten-dimensional metric component $\hat{g}_{00}$ near this IR singularity of the form

$$\hat{g}_{00} = e^{2U+2\varphi-\frac{3}{5}\phi_0-\phi} \sim \frac{1}{\rho^{107/240}} \rightarrow +\infty, \quad (240)$$

which implies that the singularity is unphysical.

On the other hand, for $C_1 = 0$ and $\rho \rightarrow 0$, we find the asymptotic behavior as given in (236). In this case, the behavior of $\hat{g}_{00}$ near this IR singularity is

$$\hat{g}_{00} = e^{2U+2\varphi-\frac{3}{5}\phi_0-\phi} \sim \rho^{107/160} \rightarrow 0, \quad (241)$$

indicating that the singularity is physical.

For other gauge groups, we look for numerical solutions to the BPS equations. For $SO(4) \ltimes \mathbb{R}^4$ gauge group with $\kappa = 1$, there are solutions with an asymptotic geometry, as $r \rightarrow +\infty$, given by a locally $SO(4)$ symmetric flat domain wall given by

$$\begin{aligned} U &\sim V \sim \ln \left[\frac{gr}{2} \right], & \varphi &\sim -\phi_0 \sim -\frac{1}{5} \ln \left[\frac{gr}{2} \right], \\ \phi &\sim \frac{16}{g^4 r^4}, & \varsigma &\sim \frac{1}{r^2}. \end{aligned} \quad (242)$$

Examples of numerical solutions for this type of solution for different values of g are shown in Figs. 56 and 57 in Appendix D 2 a. We find that all $t \times CH^2$ -sliced solutions admit physical IR singularities upon uplifted to type IIA theory. For $t \times CP^2$ -sliced domain walls, only solutions with $g \geq 1.5$ can lead to physical solutions in type IIA theory.

We then move to solutions with an asymptotic geometry given by an $SO(3)$ symmetric flat domain wall of the form

$$\begin{aligned} U &\sim V \sim \phi \sim -\ln \left[-\frac{g\kappa\rho}{8} \right], \\ \varphi &\sim -\phi_0 \sim \frac{1}{5} \ln \left[-\frac{g\kappa\rho}{8} \right], & \varsigma &\sim \rho^8, \end{aligned} \quad (243)$$

with the new radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{-V-13\phi}$. We choose $g\kappa < 0$ to identify the UV limit with $\rho \rightarrow +\infty$. For the $SO(4) \ltimes \mathbb{R}^4$ gauge group, all solutions are uplifted to ten-dimensional solutions with unphysical IR singularities, so we do not give the numerical solutions in this case. For the $SO(3, 1) \ltimes \mathbb{R}^4$ gauge group, examples of numerical solutions for different values of g are shown in Figs. 58 and 59 in Appendix D 2 a. In this case, all $t \times CH^2$ -sliced domain walls can be uplifted to type IIA solutions with physical IR singularities. For $t \times CP^2$ -sliced domain walls, only the solutions represented by the purple and brown curves in Fig. 58 lead to physical ten-dimensional solutions. All of the solutions in this case are summarized in Table X.

2. Solutions with $SO(2)$ twist

We end this paper by considering D4-branes wrapped on a Kahler four-cycle with an $SO(2)$ twist. To cancel the $U(1)$ part of the spin connection on K_k^4 with the metric given by (172), we turn on an $SO(2)$ gauge field of the form

TABLE X. Summary of satisfaction of the criterion [18] for the IR singularities of the $t \times K_k^4$ -sliced curved domain walls obtained from $SO(3)$ twist in $SO(4) \ltimes \mathbb{R}^4$, $SO(3, 1) \ltimes \mathbb{R}^4$, and $CSO(3, 0, 1) \ltimes \mathbb{R}^4$ gauge groups.

UV flat domain wall	Gauge group	Kahler four-cycle	Physical IR singularity
$SO(4)$	$SO(4) \ltimes \mathbb{R}^4$	CP^2 CH^2	$g \geq 1.5$ Any g
$SO(3)$	$SO(4) \ltimes \mathbb{R}^4$	CP^2, CH^2	$\times$
	$SO(3, 1) \ltimes \mathbb{R}^4$	CP^2 CH^2	$g \geq 2.56$ Any g
	$CSO(3, 0, 1) \ltimes \mathbb{R}^4$ (analytic)	CP^2, CH^2	$C_1 = 0$

$$A_3^{12} = -\frac{3e^{-V}p}{2\sqrt{2}}\psi. \quad (244)$$

All gauge groups with an $SO(2)$ subgroup are collectively characterized by the following component of the embedding tensor:

$$w^{ij} = \text{diag}(1, 1, \kappa, \lambda). \quad (245)$$

Apart from the two dilatons, there are additional nine $SO(2)$ singlet scalars. Three scalars are in the $SL(4)/SO(4)$ coset with noncompact generators,

$$\bar{\mathcal{Y}}_1 = \bar{t}_{11} + \bar{t}_{22} - \bar{t}_{33} - \bar{t}_{44}, \quad \bar{\mathcal{Y}}_2 = \bar{t}_{34}, \quad \bar{\mathcal{Y}}_3 = \bar{t}_{33} - \bar{t}_{44}. \quad (246)$$

The other two scalars correspond to two nilpotent generators,

$$\bar{\mathcal{Y}}_4 = \hat{t}_{35}, \quad \bar{\mathcal{Y}}_5 = \hat{t}_{45}, \quad (247)$$

and the remaining four scalars are associated with the following generators:

$$\bar{\mathcal{Y}}_6 = s_{12}, \quad \bar{\mathcal{Y}}_7 = s_{35}, \quad \bar{\mathcal{Y}}_8 = s_{45}, \quad \bar{\mathcal{Y}}_9 = s_{34}. \quad (248)$$

The analysis turns out to be highly complicated, so we perform a subtruncation by setting the shift and nilpotent scalars corresponding to s_{12} , $\hat{t}_{35}$, and $\hat{t}_{45}$ to zero. This is a consistent truncation and still gives interesting results. In addition, consistency of the BPS equations also requires the vanishing of the shift scalar corresponding to s_{35} . We then end up with seven scalars and the coset representative,

$$V = e^{\varphi d + \phi_0 \bar{\mathcal{Y}}_0 + \phi_1 \bar{\mathcal{Y}}_1 + \phi_2 \bar{\mathcal{Y}}_2 + \phi_3 \bar{\mathcal{Y}}_3 + \varsigma_1 \bar{\mathcal{Y}}_8 + \varsigma_2 \bar{\mathcal{Y}}_9}. \quad (249)$$

With a twist condition $gp = k$, and the projector (212), together with

$$\begin{aligned} \hat{\gamma}_{\hat{1}\hat{3}}\epsilon_{+\alpha} &= \hat{\gamma}_{\hat{1}\hat{2}}\epsilon_{+\alpha} = (\gamma_{12})_{\alpha}^{\beta}\epsilon_{+\beta} \\ \text{and } \hat{\gamma}_{\hat{1}\hat{3}}\epsilon_{-\alpha} &= \hat{\gamma}_{\hat{1}\hat{2}}\epsilon_{-\alpha} = (\gamma_{12})_{\alpha}^{\beta}\epsilon_{-\beta} \end{aligned} \quad (250)$$

on the Killing spinors

$$\epsilon_+ = e^{\frac{U(r)}{2} + W(r)\gamma_{34}}\epsilon_+^0 \quad \text{and} \quad \epsilon_- = e^{\frac{U(r)}{2} + W(r)\gamma_{34}}\epsilon_-^0, \quad (251)$$

the resulting BPS equations read

$$U' = \frac{g}{8}e^{\varphi-4(\phi_0+\phi_1)}[2e^{8\phi_1} + (\kappa + \lambda)\cosh 2\phi_2 \cosh 4\phi_3 + (\kappa - \lambda)\sinh 4\phi_3] - \frac{3p}{2}e^{-2V-\varphi+4(\phi_0-\phi_1)}, \quad (252)$$

$$V' = \frac{g}{8}e^{\varphi-4(\phi_0+\phi_1)}[2e^{8\phi_1} + (\kappa + \lambda)\cosh 2\phi_2 \cosh 4\phi_3 + (\kappa - \lambda)\sinh 4\phi_3] + \frac{3p}{2}e^{-2V-\varphi+4(\phi_0-\phi_1)}, \quad (253)$$

$$\varphi' = -\frac{g}{40}e^{\varphi-4(\phi_0+\phi_1)}[2e^{8\phi_1} + (\kappa + \lambda)\cosh 2\phi_2 \cosh 4\phi_3 + (\kappa - \lambda)\sinh 4\phi_3] + \frac{3p}{10}e^{-2V-\varphi+4(\phi_0-\phi_1)}, \quad (254)$$

$$\phi_0' = \frac{g}{40}e^{\varphi-4(\phi_0+\phi_1)}[2e^{8\phi_1} + (\kappa + \lambda)\cosh 2\phi_2 \cosh 4\phi_3 + (\kappa - \lambda)\sinh 4\phi_3] - \frac{3p}{10}e^{-2V-\varphi+4(\phi_0-\phi_1)}, \quad (255)$$

$$\phi_1' = -\frac{g}{8}e^{\varphi-4(\phi_0+\phi_1)}[2e^{8\phi_1} - (\kappa + \lambda)\cosh 2\phi_2 \cosh 4\phi_3 - (\kappa - \lambda)\sinh 4\phi_3] + \frac{3p}{2}e^{-2V-\varphi+4(\phi_0-\phi_1)}, \quad (256)$$

$$\phi_2' = -\frac{g}{2}e^{\phi_0-4(\phi_0+\phi_1)}(\kappa + \lambda)\sinh 2\phi_2 \text{sech} 4\phi_3, \quad (257)$$

$$\phi'_3 = -\frac{g}{4} e^{\phi_0-4(\phi_0+\phi_1)} ((\kappa+\lambda) \cosh 2\phi_2 \sinh 4\phi_3 + (\kappa-\lambda) \cosh 4\phi_3), \quad (258)$$

together with

$$\varsigma'_1 = -\frac{ge^{\varphi+4\phi_3}}{2e^{4(\phi_0+\phi_1)}} [\varsigma_1(\kappa-\lambda + (\kappa+\lambda) \cosh 2\phi_2) + \varsigma_2(\kappa+\lambda) \sinh 2\phi_2 \operatorname{sech} 4\phi_3], \quad (259)$$

$$\varsigma'_2 = -\frac{ge^{\varphi-4\phi_3}}{2e^{4(\phi_0+\phi_1)}} [\varsigma_1(\kappa+\lambda) \sinh 2\phi_2 \operatorname{sech} 4\phi_3 - \varsigma_2(\kappa-\lambda - (\kappa+\lambda) \cosh 2\phi_2)], \quad (260)$$

$$W' = -\frac{g}{4} e^{\phi_0-4(\phi_0+\phi_1)} (\kappa+\lambda) \sinh 2\phi_2 \tanh 4\phi_3. \quad (261)$$

We do not consider solutions with $k=0$ since the twist condition essentially implies $p=0$ leading to flat domain wall solutions.

For the $CSO(2,0,2) \ltimes \mathbb{R}^4$ gauge group with $\kappa=\lambda=0$, we can analytically solve the BPS equations by the same procedure as in the case of $SO(3)$ twist in $CSO(3,0,1) \ltimes \mathbb{R}^4$ gauge group. With the new radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{-3\phi_1}$, the resulting solution is given by

$$\begin{aligned} U &= -\phi_1, & V &= \frac{1}{2} \ln [6p\rho e^{4C_0}] - \phi_1 - \frac{C}{2}, & \varphi &= C + \frac{\phi_1}{5}, \\ \phi_0 &= C_0 - \frac{\phi_1}{5}, & \phi_1 &= \frac{C_0}{2} - \frac{1}{8} \ln \left[\frac{2e^C g\rho}{3} + \frac{C_1}{\rho^2} \right], \end{aligned} \quad (262)$$

with $\phi_2 = \phi_3 = W = \varsigma_1 = \varsigma_2 = 0$. Apart from some numerical factors, this solution is very similar to that of $CSO(3,0,1) \ltimes \mathbb{R}^4$ gauge group, so we only look at the behavior near the IR singularities.

As $\rho \rightarrow 0$ and $C_1 \neq 0$, the solution becomes

$$U \sim -\frac{1}{4} \ln \rho, \quad V \sim \frac{1}{4} \ln \rho, \quad \varphi \sim -\phi_0 \sim \frac{1}{20} \ln \rho, \quad \phi_1 \sim \frac{1}{4} \ln \rho. \quad (263)$$

We find the ten-dimensional metric component,

$$\hat{g}_{00} = e^{2U+2\varphi-\frac{3}{4}\phi_0-\phi_1} \sim \frac{1}{\rho^{49/80}} \rightarrow +\infty. \quad (264)$$

Therefore, the solution has an unphysical singularity.

As $\rho \rightarrow 0$ and $C_1 = 0$, we find

$$U \sim \frac{1}{8} \ln \rho, \quad V \sim \frac{5}{8} \ln \rho, \quad \varphi \sim -\phi_0 \sim -\frac{1}{40} \ln \rho, \quad \phi_1 \sim -\frac{1}{8} \ln \rho, \quad (265)$$

and

$$\hat{g}_{00} = e^{2U+2\varphi-\frac{3}{4}\phi_0-\phi_1} \sim \rho^{49/160} \rightarrow 0. \quad (266)$$

Accordingly, in this case, the IR singularity is physically acceptable, and the solution can be interpreted as a twisted compactification of five-dimensional Yang-Mills theory on K_k^4 to supersymmetric quantum mechanics in the IR.

For other gauge groups, we look for numerical solutions to the BPS equations. All solutions with an asymptotic geometry given by an $SO(4)$ symmetric flat domain wall from $SO(4) \ltimes \mathbb{R}^4$ gauge group lead to uplifted ten-dimensional solutions

with unphysical IR singularities. Therefore, we do not present these solutions here. Another class of solutions is given by those asymptotic to a locally $SO(2)$ symmetric flat domain wall of the form

$$U \sim V \sim -5\varphi \sim 5\phi_0 \sim \frac{1}{2} \ln \rho - \ln \left[-\frac{g\rho}{2} \right], \quad \phi_1 \sim \frac{1}{2} \ln \rho, \quad (267)$$

TABLE XI. Summary of satisfaction of the criterion [18] for the IR singularities of the $t \times K_k^4$ -sliced curved domain walls obtained from $SO(3)$ twist in $SO(4) \ltimes \mathbb{R}^4$, $SO(3, 1) \ltimes \mathbb{R}^4$, $SO(2, 2) \ltimes \mathbb{R}^4$, $CSO(3, 0, 1) \ltimes \mathbb{R}^4$, $CSO(2, 1, 1) \ltimes \mathbb{R}^4$, and $CSO(2, 0, 2) \ltimes \mathbb{R}^4$ gauge groups.

UV flat domain wall	Gauge group	Kahler four-cycle	Physical IR singularity
$SO(4)$	$SO(4) \ltimes \mathbb{R}^4$	CP^2, CH^2	$\times$
$SO(2)$	$SO(4) \ltimes \mathbb{R}^4$	CP^2, CH^2	$\times$
	$SO(3, 1) \ltimes \mathbb{R}^4$	CP^2	Any g
		CH^2	$g < -2.5$
	$SO(2, 2) \ltimes \mathbb{R}^4$	CP^2	Any g
		CH^2	$g < -5$
	$CSO(3, 0, 1) \ltimes \mathbb{R}^4$	CP^2, CH^2	$\times$
	$CSO(2, 1, 1) \ltimes \mathbb{R}^4$	CP^2	Any g
		CH^2	$g < -2.5$
	$CSO(2, 0, 2) \ltimes \mathbb{R}^4$ (analytic)	CP^2, CH^2	$C_1 = 0$

with $\phi_2 = \phi_3 = \varsigma_1 = \varsigma_2 = 0$ and the new radial coordinate ρ defined by $\frac{d\rho}{dr} = e^{5\phi_1 - 10\phi_0}$. It turns out that, in this case, all solutions have $\phi_2 = \varsigma_1 = \varsigma_2 = 0$ along the entire flow solutions. Moreover, all solutions in $SO(4) \ltimes \mathbb{R}^4$ and $CSO(3, 0, 1) \ltimes \mathbb{R}^4$ gauge groups are uplifted to type IIA solutions with unphysical IR singularities, so we omit numerical solutions for all of these solutions.

For $SO(3, 1) \ltimes \mathbb{R}^4$, $SO(2, 2) \ltimes \mathbb{R}^4$, and $CSO(2, 1, 1) \ltimes \mathbb{R}^4$ gauge groups, examples of numerical solutions are given in Figs. 60–65 in Appendix D 2 b. We find that all $t \times CP^2$ -sliced solutions can be uplifted to solutions of type IIA theory with physical IR singularities. On the other hand, $t \times CH^2$ -sliced solutions lead to ten-dimensional solutions with physical IR singularities only for large $|g|$. We collect all these conditions in Table XI. We also point out that the result is very similar to solutions with $SO(2)$ twist on a Riemann surface studied in [32].

V. CONCLUSIONS AND DISCUSSIONS

We have studied a large number of supersymmetric solutions interpolating between locally flat domain walls and $t \times \mathcal{M}_4$ -sliced domain walls from maximal gauged supergravity in six dimensions with $CSO(p, q, 5 - p - q)$ and $CSO(p, q, 4 - p - q) \ltimes \mathbb{R}^4$ gauged groups. The four-manifold with constant curvature $\mathcal{M}_4$ under consideration here is given by a Riemannian four-manifold, a product of two Riemann surfaces and a Kahler four-cycle. By performing different types of topological twists, various amounts of supersymmetry are preserved along the entire solutions. These solutions are expected to give holographic duals of RG flows across dimensions from the maximal SYM theories in five dimensions dual to the locally flat domain wall to supersymmetric quantum mechanics in the IR.

Upon uplifted to type IIA theory, many of these solutions possess physically acceptable IR singularities. Therefore, these solutions could be interpreted as D4-branes wrapped on $\mathcal{M}_4$ giving rise to supersymmetric quantum mechanics

via twisted compactifications of the maximal five-dimensional SYM on the world-volume of D4-branes. Due to recent interests in the holographic study of matrix quantum mechanics, we hope the results given in this paper would be useful along this line. Rather than considering supersymmetric quantum mechanics on the world-volume of D0-branes as in many previous works, we have given a new class of gravity duals to supersymmetric quantum mechanics from D4-branes wrapped on four-manifolds. This might lead to some insight into supersymmetric quantum mechanics via twisted compactifications of maximal SYM in five dimensions. With some of the solutions given analytically, we hope the present results could provide a very useful tool to further study along these and related directions.

Due to the relation between six-dimensional gauge supergravity considered here and the maximal gauged supergravity in seven dimensions via an S^1 reduction, it is natural to look for a possible relation between supersymmetric solutions with topological twists on $\mathcal{M}_4$ in both cases. In seven dimensions, a number of $AdS_3 \times \Sigma_{k_1} \times \Sigma_{k_2}$ solutions have been found in [23] from $SO(5)$ maximal gauged supergravity. These solutions describe RG flows across dimensions from $N = (2, 0)$ superconformal field theory (SCFT) in six dimensions dual to the supersymmetric AdS_7 vacuum to $N = (2, 0)$ two-dimensional SCFTs in the IR via twisted compactifications on $\Sigma_{k_1} \times \Sigma_{k_2}$. In addition, similar solutions in $CSO(p, q, 5 - p - q)$ and $CSO(p, q, 4 - p - q)$ gauge groups have been considered in [38]. Apart from the $SO(5)$ gauge group, all of these gauge groups do not lead to supersymmetric AdS_7 vacua. The resulting RG flow solutions then interpolate between $N = (2, 0)$ nonconformal field theories in six dimensions to conformal or nonconformal field theories in two dimensions.

It turns out that all of the solutions with physical IR singularities obtained in this paper correspond to seven-dimensional solutions with $AdS_3 \times \mathcal{M}_4$ fixed points.

Indeed, from the results of this paper, we have not found solutions with physical singularities only for the case of $\mathcal{M}_4 = S^2 \times H^2$ in $SO(5)$ gauge group. However, as shown in [38], the AdS_3 vacua in this case exist only in a small region of the parameter space. The corresponding six-dimensional solutions with physical singularities might exist within a very small region of the parameter space, but we have not found these solutions due to the limitation of numerical analysis. According to the result of [39], an S^1 compactification of AdS_{d+1} spaces leads to domain walls in d dimensions. We then expect that the flow solutions with physical IR singularities are those obtained from the corresponding flow solutions to AdS_3 fixed points in seven dimensions.

It would be interesting to perform a holographic study of the resulting supersymmetric quantum mechanics as in [14] and [15] using holographic renormalization to compute some correlation functions among the operators dual to various scalar fields in the six-dimensional gauged supergravity. The quantum mechanics dual to some of the solutions found here might be related to certain deformations of the Bank-Fischler-Shenker-Susskind (BFSS) matrix model which plays an important role in string/M theory, and it would be interesting to explicitly verify this. On the field theory side, it could be interesting to identify the maximal SYM theories in five dimensions dual to the flat domain walls in the presence of topological twists on $\mathcal{M}_4$. The dual quantum mechanics could be directly constructed by twisted compactifications of the maximal SYM theories in five dimensions, and it would be interesting to study both the field theory and supergravity sides of the gauge/gravity duality in this case. On the other hand, constructing the complete truncation *Ansätze* of type IIA theory on both $H^{p,q} \times \mathbb{R}^{5-p-q}$ and $H^{p,q} \times \mathbb{R}^{4-p-q} \times S^1$ is also worth considering. This could be used to uplift the solutions found here and in [32] to the complete solutions of type IIA theory in which an interpretation in terms of D4-branes wrapped on compact manifolds of different dimensions can be given.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: USEFUL FORMULAS

In this appendix, we collect relevant formulas used in the main text for convenience. All of these relations have been thoroughly described in [29], so we omit all the details here. Recall that the branching rule for an $SO(5,5)$ vector

representation under $GL(5)$ takes the form

$$\mathbf{10} \rightarrow \mathbf{5}^{+2} \oplus \mathbf{\bar{5}}^{-2}, \quad (\text{A1})$$

we can decompose the $SO(5,5)$ generators as

$$\mathbf{t}_{MN} \rightarrow (\mathbf{t}_{mn}, \mathbf{t}^m_n, \mathbf{t}^{mn}), \quad (\text{A2})$$

with $\mathbf{t}_m^n = -\mathbf{t}^n_m$.

$SO(5,5)$ generators in vector representation can be chosen as

$$(\mathbf{t}_{MN})_P^Q = 4\eta_{P[M}\delta_{N]}^Q, \quad (\text{A3})$$

with the corresponding Lie algebra of the form

$$[\mathbf{t}_{MN}, \mathbf{t}_{PQ}] = 4(\eta_{M[P}\mathbf{t}_{Q]N} - \eta_{N[P}\mathbf{t}_{Q]M}). \quad (\text{A4})$$

In terms of the shift generators $s_{mn} = \mathbf{t}_{mn}$ and the generators for the hidden symmetries $\mathbf{h}^{mn} = \mathbf{t}^{mn}$, the $SO(5,5)$ generators can be written as

$$(\mathbf{t}_{MN})_P^Q = \begin{pmatrix} \mathbf{t}^m_n & \mathbf{h}^{mn} \\ s_{mn} & -\mathbf{t}^n_m \end{pmatrix}. \quad (\text{A5})$$

The spinor and conjugate spinor representations of $SO(5,5)$ decompose as

$$\begin{aligned} \mathbf{16}_s &\rightarrow \mathbf{\bar{5}}^{+3} \oplus \mathbf{10}^{-1} \oplus \mathbf{1}^{-5} \\ \text{and } \mathbf{16}_c &\rightarrow \mathbf{5}^{-3} \oplus \mathbf{\bar{10}}^{+1} \oplus \mathbf{1}^{+5}. \end{aligned} \quad (\text{A6})$$

To concretely realize this decomposition, we introduce the following transformation matrices:

$$\mathbb{T}_{Am} = \frac{1}{2\sqrt{2}} (\Gamma_m)_{AB} \mathbf{p}_{\alpha\beta}^B \Omega^{\alpha\beta}, \quad (\text{A7})$$

$$\mathbb{T}_A^{mn} = \frac{1}{4\sqrt{2}} (\Gamma^{mn})_A{}^B \mathbf{p}_B^{\alpha\beta} \Omega_{\alpha\beta}, \quad (\text{A8})$$

$$\mathbb{T}_{A*} = \frac{1}{10} (\Gamma^m_m)_A{}^B \mathbf{p}_B^{\alpha\beta} \Omega_{\alpha\beta}. \quad (\text{A9})$$

The matrices $\mathbf{p}_A^{\alpha\beta}$ and inverse matrices $\mathbf{p}_{\alpha\beta}^A$ are given by

$$\begin{aligned} \mathbf{p}_A^{\alpha\beta} &= \delta_A^\alpha \delta_1^\beta + \delta_A^{\alpha+4} \delta_2^\beta + \delta_A^{\alpha+8} \delta_3^\beta + \delta_A^{\alpha+12} \delta_4^\beta, \\ \mathbf{p}_{\alpha\beta}^A &= \delta_\alpha^A \delta_\beta^1 + \delta_{\alpha+4}^A \delta_\beta^2 + \delta_{\alpha+8}^A \delta_\beta^3 + \delta_{\alpha+12}^A \delta_\beta^4. \end{aligned} \quad (\text{A10})$$

These matrices satisfy the relations

$$\mathbf{p}_A^{\alpha\beta} \mathbf{p}_{\alpha\beta}^B = \delta_A^B \quad \text{and} \quad \mathbf{p}_A^{\alpha\delta} \mathbf{p}_{\beta\gamma}^A = \delta_\beta^\alpha \delta_\gamma^\delta. \quad (\text{A11})$$

The inverse matrices of $\mathbb{T}_A$ are simply given by their complex conjugation $\mathbb{T}^A = (\mathbb{T}_A)^{-1} = (\mathbb{T}_A)^*$ satisfying the relations

$$\begin{aligned} \mathbb{T}^{Am}\mathbb{T}_{An} &= \delta_n^m, & \mathbb{T}_{mn}^A \mathbb{T}_A^{pq} &= \delta_m^{[p} \delta_n^{q]}, & \mathbb{T}_*^A \mathbb{T}_{A*} &= 1, \\ \mathbb{T}^{Am}\mathbb{T}_{Anp} &= 0, & \mathbb{T}^{Am}\mathbb{T}_{A*} &= 0, & \mathbb{T}_{mn}^A \mathbb{T}_{A*} &= 0, \end{aligned} \quad (\text{A12})$$

together with

$$\mathbb{T}^{Am}\mathbb{T}_{Bm} + \mathbb{T}_{mn}^A \mathbb{T}_B^{mn} + \mathbb{T}_*^A \mathbb{T}_{B*} = \delta_B^A. \quad (\text{A13})$$

With all these, we can write the following relations for $\mathbf{16}_s$ and $\mathbf{16}_c$ representations:

$$\Psi_A = \mathbb{T}_{Am} \Psi^m + \mathbb{T}_A^{mn} \Psi_{mn} + \mathbb{T}_{A*} \Psi_*, \quad (\text{A14})$$

with $\Psi_{mn} = \Psi_{[mn]}$ and

$$\Psi^A = \mathbb{T}^{Am} \Psi_m + \mathbb{T}_{mn}^A \Psi^{mn} + \mathbb{T}_*^A \Psi_*. \quad (\text{A15})$$

The $SO(5,5)$ gamma matrices $(\Gamma_M)_{AB}$ are related to $SO(5) \times SO(5)$ gamma matrices $\Gamma_{\underline{A}} = (\Gamma_a, \Gamma_{\dot{a}})$ according to the relation

$$(\Gamma_M)_{AB} = \mathbb{M}_M^{\underline{A}} (\Gamma_{\underline{A}})_{AB}, \quad (\text{A16})$$

with

$$\mathbb{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_5 & \mathbb{1}_5 \\ \mathbb{1}_5 & -\mathbb{1}_5 \end{pmatrix}. \quad (\text{A17})$$

The $SO(5,5)$ spinor indices are raised and lowered by the charge conjugation matrix c_{AB} of the form

$$c_{AB} = p_A^{\alpha\dot{\alpha}} p_B^{\beta\dot{\beta}} \Omega_{\alpha\beta} \Omega_{\dot{\alpha}\dot{\beta}}, \quad (\text{A18})$$

in which $p_A^{\alpha\dot{\alpha}}$ and $p_A^{\dot{\alpha}\alpha}$ are defined in parallel with (A10). In particular, we have

$$(\Gamma_{\underline{A}})_{AB} = (\Gamma_{\underline{A}})_A^C c_{CB}. \quad (\text{A19})$$

On the other hand, the $SO(5) \times SO(5)$ gamma matrices can also be written in terms of $SO(5)$ gamma matrices $(\gamma_a)_\alpha^\beta$ and $(\gamma_{\dot{a}})_{\dot{\alpha}}^{\dot{\beta}}$ as

$$(\Gamma_{\underline{A}})_A^B = ((\gamma_a)_A^B, (\gamma_{\dot{a}})_A^B), \quad (\text{A20})$$

with

$$(\gamma_a)_A^B = p_A^{\alpha\dot{\alpha}} (\gamma_a)_{\alpha\dot{\alpha}}^{\beta\dot{\beta}} p_{\beta\dot{\beta}}^B = p_A^{\alpha\dot{\alpha}} (\gamma_a)_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} p_{\beta\dot{\beta}}^B, \quad (\text{A21})$$

$$(\gamma_{\dot{a}})_A^B = p_A^{\alpha\dot{\alpha}} (\gamma_{\dot{a}})_{\alpha\dot{\alpha}}^{\beta\dot{\beta}} p_{\beta\dot{\beta}}^B = p_A^{\alpha\dot{\alpha}} \delta_\alpha^\beta (\gamma_{\dot{a}})_{\dot{\alpha}}^{\dot{\beta}} p_{\beta\dot{\beta}}^B. \quad (\text{A22})$$

All these relations lead to $(\Gamma_m)_{AB}$ matrices of the form

$$(\Gamma_m)_{AB} = \frac{1}{\sqrt{2}} p_A^{\alpha\dot{\alpha}} \left[(\gamma_m)_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} + \delta_\alpha^\beta (\gamma_m)_{\dot{\alpha}}^{\dot{\beta}} \right] \Omega_{\beta\gamma} \Omega_{\dot{\beta}\dot{\gamma}} p_B^{\gamma\dot{\gamma}}. \quad (\text{A23})$$

Finally, we can write the explicit form of $(\Gamma^{mn})_A^B$ and $(\Gamma^m_m)_A^B$ matrices using the following definitions:

$$(\Gamma^{mn})_A^B = \frac{1}{2} \left[(\Gamma^m)_A^C (\Gamma^n)_C^B - (\Gamma^n)_A^C (\Gamma^m)_C^B \right], \quad (\text{A24})$$

and

$$(\Gamma^m_m)_A^B = \frac{1}{2} \left[(\Gamma^m)_A^C (\Gamma_m)_C^B - (\Gamma_m)_A^C (\Gamma^m)_C^B \right], \quad (\text{A25})$$

with

$$(\Gamma_m)_A^B = \frac{1}{\sqrt{2}} p_A^{\alpha\dot{\alpha}} \left[(\gamma_m)_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} + \delta_\alpha^\beta (\gamma_m)_{\dot{\alpha}}^{\dot{\beta}} \right] p_{\beta\dot{\beta}}^B, \quad (\text{A26})$$

$$(\Gamma^m)_A^B = \frac{1}{\sqrt{2}} p_A^{\alpha\dot{\alpha}} \left[(\gamma_m)_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} - \delta_\alpha^\beta (\gamma_m)_{\dot{\alpha}}^{\dot{\beta}} \right] p_{\beta\dot{\beta}}^B. \quad (\text{A27})$$

In this paper, we use the following representation of $SO(5)$ gamma matrices:

$$\begin{aligned} \gamma_1 &= -\sigma_2 \otimes \sigma_2, & \gamma_2 &= \mathbb{1}_2 \otimes \sigma_1, & \gamma_3 &= \mathbb{1}_2 \otimes \sigma_3, \\ \gamma_4 &= \sigma_1 \otimes \sigma_2, & \gamma_5 &= \sigma_3 \otimes \sigma_2, \end{aligned} \quad (\text{A28})$$

with the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A29})$$

APPENDIX B: TYPE IIA UPLIFT

In this appendix, we give the ten-dimensional metric component $\hat{g}_{00}$ in type IIA theory relevant for determining whether a given IR singularity of six-dimensional solutions is physically acceptable upon uplifted to type IIA theory. Although, as previously mentioned, the complete truncations to six-dimensional gauged supergravity with $CSO(p, q, 5-p-q)$ and $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ gauge groups have not been worked out to date, we can at least find the explicit form of the (00)-component of the ten-dimensional metric from the partial results given in [34] and [35].

1. The embedding of $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ gauged supergravity

We begin with a more straightforward case of gauged supergravity with $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ gauge group. Using the consistent truncation to seven dimensions of type IIA theory on $H^{p,q} \times \mathbb{R}^{4-p-q}$ given in [34], we can

perform the standard dimensional-reduction on S^1 using the *Ansatz*

$$ds_7^2 = e^{2\varphi} ds^2 + e^{-8\varphi} (dz + A)^2. \quad (\text{B1})$$

The scalar φ is the dilaton in the main text while z is the seventh coordinate added to the six-dimensional metric ds^2 . In the present notation, the (00)-component of the ten-dimensional metric is given by

$$\hat{g}_{00} = e^{-3\phi_0} \Delta^{\frac{1}{3}} g_{00}^{(7)}. \quad (\text{B2})$$

Recall that ϕ_0 is the dilaton corresponding to the $SO(1, 1)$ factor in the decomposition $SL(5) \rightarrow SL(4) \times SO(1, 1)$. After a dimensional reduction on S^1 , we can write the ten-dimensional metric in terms of the six-dimensional one as

$$\hat{g}_{00} = -e^{-3\phi_0} \Delta^{\frac{1}{3}} e^{2U+2\varphi}. \quad (\text{B3})$$

We have also used the metric *Ansatz* with $g_{00} = -e^{2U}$. The warp factor Δ is defined by

$$\Delta = m^{ij} \eta_{ik} \eta_{jl} \mu^k \mu^l, \quad i, j, \dots = 1, 2, 3, 4, \quad (\text{B4})$$

in which m^{ij} is the inverse of the $SL(4)$ matrix $m_{ij} = (\tilde{\mathcal{V}} \tilde{\mathcal{V}}^T)_{ij}$. $\tilde{\mathcal{V}}$ is the $SL(4)/SO(4)$ coset representative. The coordinates μ^i on $H^{p,q} \times \mathbb{R}^{4-p-q}$ satisfy $\mu^i \mu^j \eta_{ij} = 1$ with η_{ij} being the $CSO(p, q, 4-p-q)$ invariant tensor. In all cases, the metric component $\hat{g}_{00}$ takes the same form as (B3) with different warp factor Δ due to different scalar sectors.

For $SO(2) \times SO(2)$ symmetric scalars, we have

$$\tilde{\mathcal{V}} = \text{diag}(e^{2\phi}, e^{2\phi}, e^{-2\phi}, e^{-2\phi}), \quad (\text{B5})$$

which gives

$$\Delta = e^{-4\phi} [(\mu^1)^2 + (\mu^2)^2] + \kappa e^{4\phi} [(\mu^3)^2 + (\mu^4)^2]. \quad (\text{B6})$$

In this equation, we have used the invariant tensor of $SO(4)$ and $SO(2, 2)$ given by $\eta_{ij} = \text{diag}(1, 1, \kappa, \kappa)$ for $\kappa = \pm 1$.

For the $SO(3)$ symmetric scalar, the $SL(4)/SO(4)$ coset representative is given by

$$\tilde{\mathcal{V}} = \text{diag}(e^{2\phi}, e^{2\phi}, e^{2\phi}, e^{-6\phi}), \quad (\text{B7})$$

which gives

$$\Delta = e^{-4\phi} [(\mu^1)^2 + (\mu^2)^2 + (\mu^3)^2] + \kappa e^{12\phi} (\mu^4)^2, \quad (\text{B8})$$

for $\eta_{ij} = \text{diag}(1, 1, 1, \kappa)$ and $\kappa = 0, \pm 1$.

In the case of $SO(2)$ singlet scalars, the coset representative $\tilde{\mathcal{V}}$ is much more complicated, so we restrict ourselves to the subtruncation considered in the main text. In this

subtruncation, we have only ϕ_1 , ϕ_2 , and ϕ_3 nonvanishing and

$$\tilde{\mathcal{V}} = \begin{pmatrix} e^{2\phi_1} & 0 & 0 & 0 \\ 0 & e^{2\phi_1} & 0 & 0 \\ 0 & 0 & e^{-2\phi_1+2\phi_3} \cosh \phi_2 & e^{-2\phi_1-2\phi_3} \sinh \phi_2 \\ 0 & 0 & e^{-2\phi_1+2\phi_3} \sinh \phi_2 & e^{-2\phi_1-2\phi_3} \cosh \phi_2 \end{pmatrix}. \quad (\text{B9})$$

The warp factor is then given by

$$\Delta = e^{-4\phi_1} [(\mu^1)^2 + (\mu^2)^2] + e^{4\phi_1} \{ \sinh 4\phi_3 [\lambda^2 (\mu^4)^2 - \kappa^2 (\mu^3)^2] + e^{-2\phi_2} \cosh 4\phi_3 [\lambda^2 (\mu^4)^2 + \kappa^2 (\mu^3)^2] \}, \quad (\text{B10})$$

in which we have used $\eta_{ij} = \text{diag}(1, 1, \kappa, \lambda)$ for $\kappa = 0, \pm 1$ and $\lambda = 0, \pm 1$.

2. The embedding of $CSO(p, q, 5-p-q)$ gauged supergravity

To find the embedding in type IIA theory in this case, we first recall that the 11-dimensional metric component $\hat{g}_{00}^{(11)}$ is given via a consistent truncation on $H^{p,q} \times \mathbb{R}^{5-p-q}$ by

$$\hat{g}_{00}^{(11)} = \Delta^{\frac{1}{3}} g_{00}^{(7)}. \quad (\text{B11})$$

The warp factor is defined by

$$\Delta = \mathcal{M}^{ab} \eta_{ac} \eta_{bd} \mu^c \mu^d, \quad (\text{B12})$$

with η_{ab} , $a, b = 1, 2, \dots, 5$, being the $CSO(p, q, 5-p-q)$ invariant tensor. $\mathcal{M}^{ab}$ is the inverse of the $SL(5)$ scalar matrix defined in terms of the $SL(5)/SO(5)$ coset representative as

$$\mathcal{M}_{ab} = \mathcal{V}_a^{\underline{a}} \mathcal{V}_b^{\underline{a}}. \quad (\text{B13})$$

Similar to the previous case, the coordinates μ^a satisfy $\mu^a \mu^b \eta_{ab} = 1$. Using the S^1 reduction given in (B1), we have

$$g_{00}^{(7)} = e^{2\varphi} g_{00}, \quad (\text{B14})$$

which in turn gives

$$\hat{g}_{00}^{(11)} = \Delta^{\frac{1}{3}} e^{2\varphi} g_{00}. \quad (\text{B15})$$

As in the truncation on $S^4 \times S^1$ leading to six-dimensional gauged supergravity with $SO(5)$ gauge group constructed in [36], we can perform a dimensional reduction of the 11-dimensional metric on S^1 and find the (00)-component of the ten-dimensional metric of the form

$$\hat{g}_{00} = \Omega^{\frac{1}{3}} \Delta^{\frac{1}{3}} e^{2\varphi} g_{00}, \quad (\text{B16})$$

with

$$\Omega = \Delta^{\frac{1}{3}} e^{-8\varphi} + \Delta^{-\frac{2}{3}} m_{ab} \chi^{ac} \chi^{bd} \eta_{ce} \eta_{df} \mu^e \mu^f. \quad (\text{B17})$$

It is also useful to note that the factor Ω is related to the type IIA dilaton via

$$e^\Phi = \Omega^{\frac{3}{4}}. \quad (\text{B18})$$

The axionic scalars $\chi^{ab} = -\chi^{ba}$ arise from the dimensional reduction of the $CSO(p, q, 5-p-q)$ gauge fields A_μ^{ab} in seven dimensions. These scalars transform in representation **10** of $SL(5)$ and can be identified with the shift scalars $\varsigma_1, \dots, \varsigma_{10}$ appearing in (49).

For $SO(4)$ invariant scalars, we have

$$\mathcal{V} = \text{diag}(e^{2\phi}, e^{2\phi}, e^{2\phi}, e^{2\phi}, e^{-8\phi}), \quad (\text{B19})$$

and $\eta_{ab} = \text{diag}(1, 1, 1, 1, \lambda)$ for $\lambda = 0, \pm 1$. In this case, all shift scalars vanish leading to

$$\Omega = e^{-8\varphi} \Delta, \quad (\text{B20})$$

with the warp factor Δ given by

$$\Delta = e^{-4\phi} [(\mu^1)^2 + (\mu^2)^2 + (\mu^3)^2 + (\mu^4)^2] + \lambda^2 e^{16\phi} (\mu^4)^2. \quad (\text{B21})$$

This can also be used in the case of $SU(2)_+$ invariant scalars since all $SU(2)_+$ singlets are essentially $SO(4)$ singlets.

For $SO(2) \times SO(2)$ singlet scalars, we have

$$\mathcal{V} = \text{diag}(e^{2\phi_1}, e^{2\phi_1}, e^{2\phi_2}, e^{2\phi_2}, e^{-4\phi_1-4\phi_2}), \quad (\text{B22})$$

and the nonvanishing shift scalars are identified as $\chi^{12} = \varsigma_1$ and $\chi^{34} = \varsigma_2$. These lead to

$$\Omega = e^{-8\varphi} \Delta^{\frac{1}{3}} + \Delta^{-\frac{2}{3}} \left\{ e^{4\phi_1} \varsigma_1^2 [(\mu^1)^2 + (\mu^2)^2] + e^{4\phi_2} \varsigma_2^2 [(\mu^3)^2 + (\mu^4)^2] \right\}, \quad (\text{B23})$$

with

$$\Delta = e^{-4\phi_1} [(\mu^1)^2 + (\mu^2)^2] + \kappa e^{-4\phi_2} [(\mu^3)^2 + (\mu^4)^2] + \lambda^2 e^{8\phi_1+8\phi_2} (\mu^5)^2. \quad (\text{B24})$$

We have also used the $CSO(p, q, 5-p-q)$ invariant tensor $\eta_{ab} = \text{diag}(1, 1, \kappa, \kappa, \lambda)$ with $\kappa = \pm 1$ and $\lambda = 0, \pm 1$.

Finally, in the case of $SO(3)$ singlet scalars, we have

$$\mathcal{V} = \begin{pmatrix} e^{4\phi_1} & 0 & 0 & 0 & 0 \\ 0 & e^{4\phi_1} & 0 & 0 & 0 \\ 0 & 0 & e^{4\phi_1} & 0 & 0 \\ 0 & 0 & 0 & e^{-6\phi_1+2\phi_3} \cosh \phi_2 & e^{-6\phi_1-2\phi_3} \sinh \phi_2 \\ 0 & 0 & 0 & e^{-6\phi_1+2\phi_3} \sinh \phi_2 & e^{-6\phi_1-2\phi_3} \cosh \phi_2 \end{pmatrix}. \quad (\text{B25})$$

For the solutions given in the main text, all shift scalars vanish, so we simply have the warp factor

$$\Omega = e^{-8\varphi} \Delta, \quad (\text{B26})$$

with

$$\Delta = e^{-8\phi_1} [(\mu^1)^2 + (\mu^2)^2 + (\mu^3)^2] + e^{12\phi_1} \left\{ \sinh 4\phi_3 [\lambda^2 (\mu^5)^2 - \kappa^2 (\mu^4)^2] + e^{-2\phi_2} \cosh 4\phi_3 [\lambda^2 (\mu^5)^2 + \kappa^2 (\mu^4)^2] \right\}. \quad (\text{B27})$$

In the last relation, we have used the invariant tensor $\eta_{ab} = \text{diag}(1, 1, 1, \kappa, \lambda)$ with $\kappa, \lambda = 0, \pm 1$.

APPENDIX C: NUMERICAL SOLUTIONS FROM $CSO(p, q, 5-p-q)$ GAUGED SUPERGRAVITY

In this appendix, we collect various numerical solutions from different $CSO(p, q, 5-p-q)$ gauge groups.

1. D4-branes wrapped on a product of two Riemann surfaces

a. Solutions with nonvanishing ς_1 and ς_2

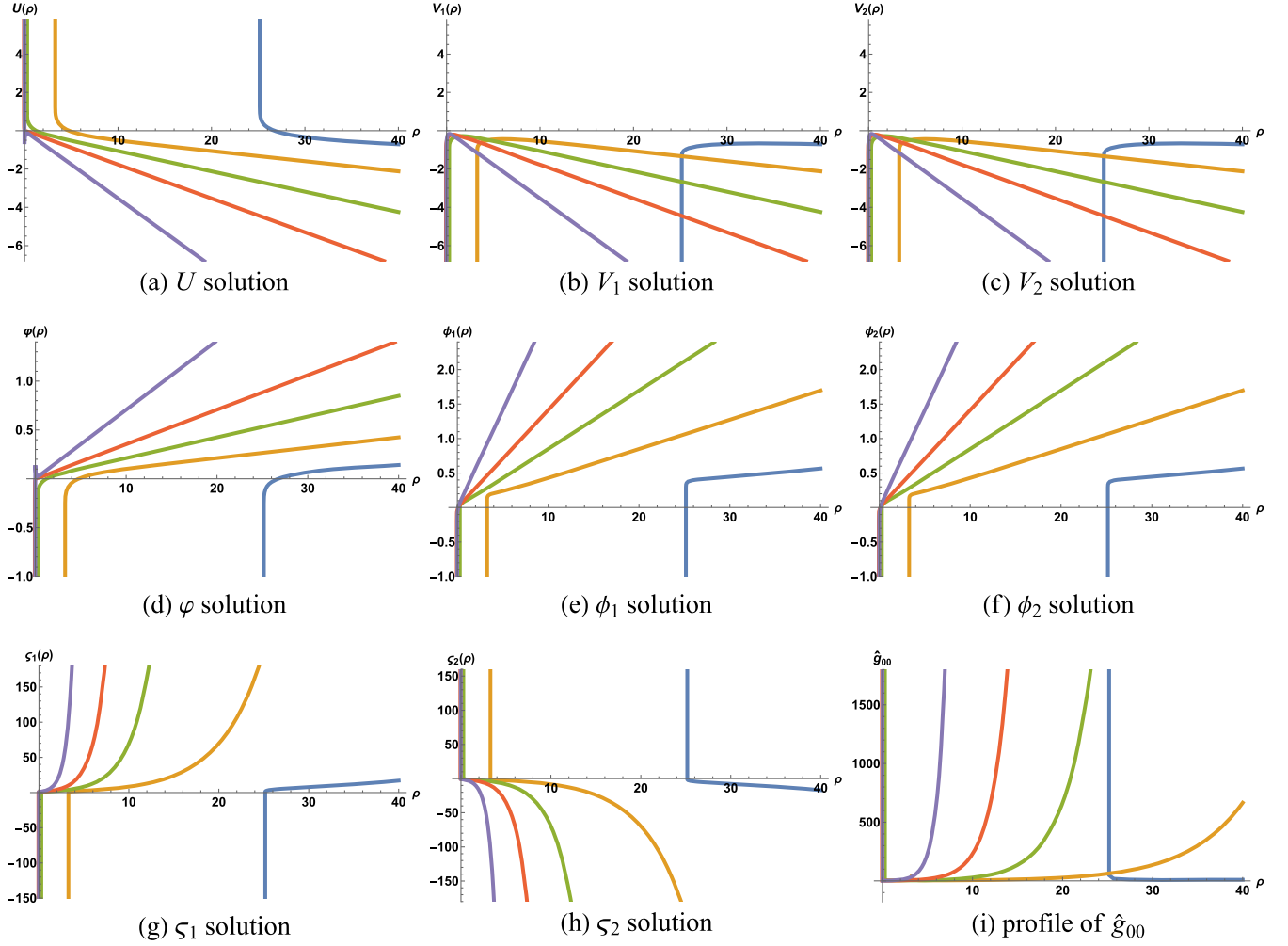


FIG. 23. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times S^2$ -sliced curved domain walls with shift scalars in $SO(4, 1)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = 0.1, 0.3, 0.6, 1, 2$.

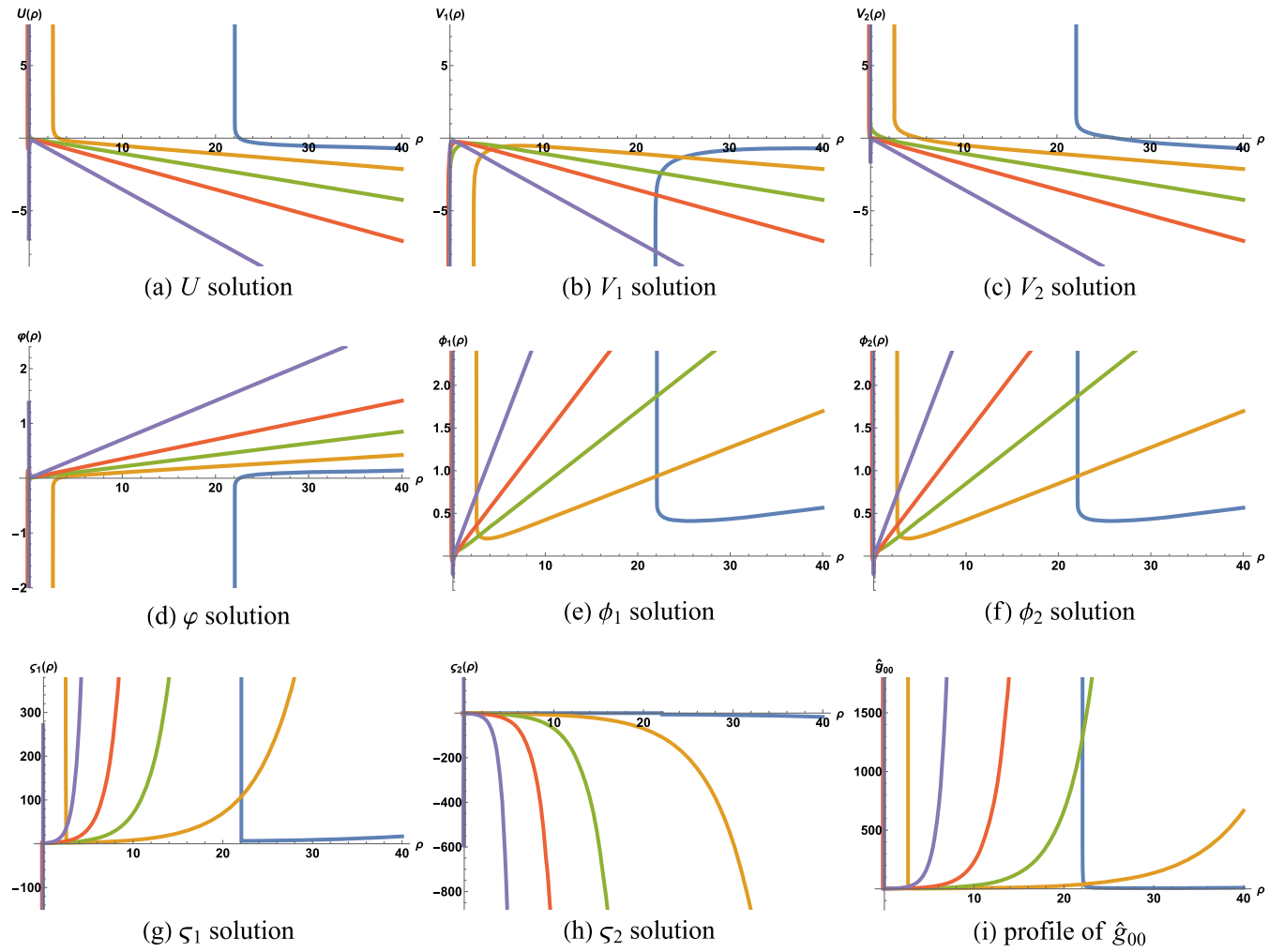


FIG. 24. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times H^2$ -sliced curved domain walls with shift scalars in $SO(4, 1)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = 0.1, 0.3, 0.6, 1, 2$.

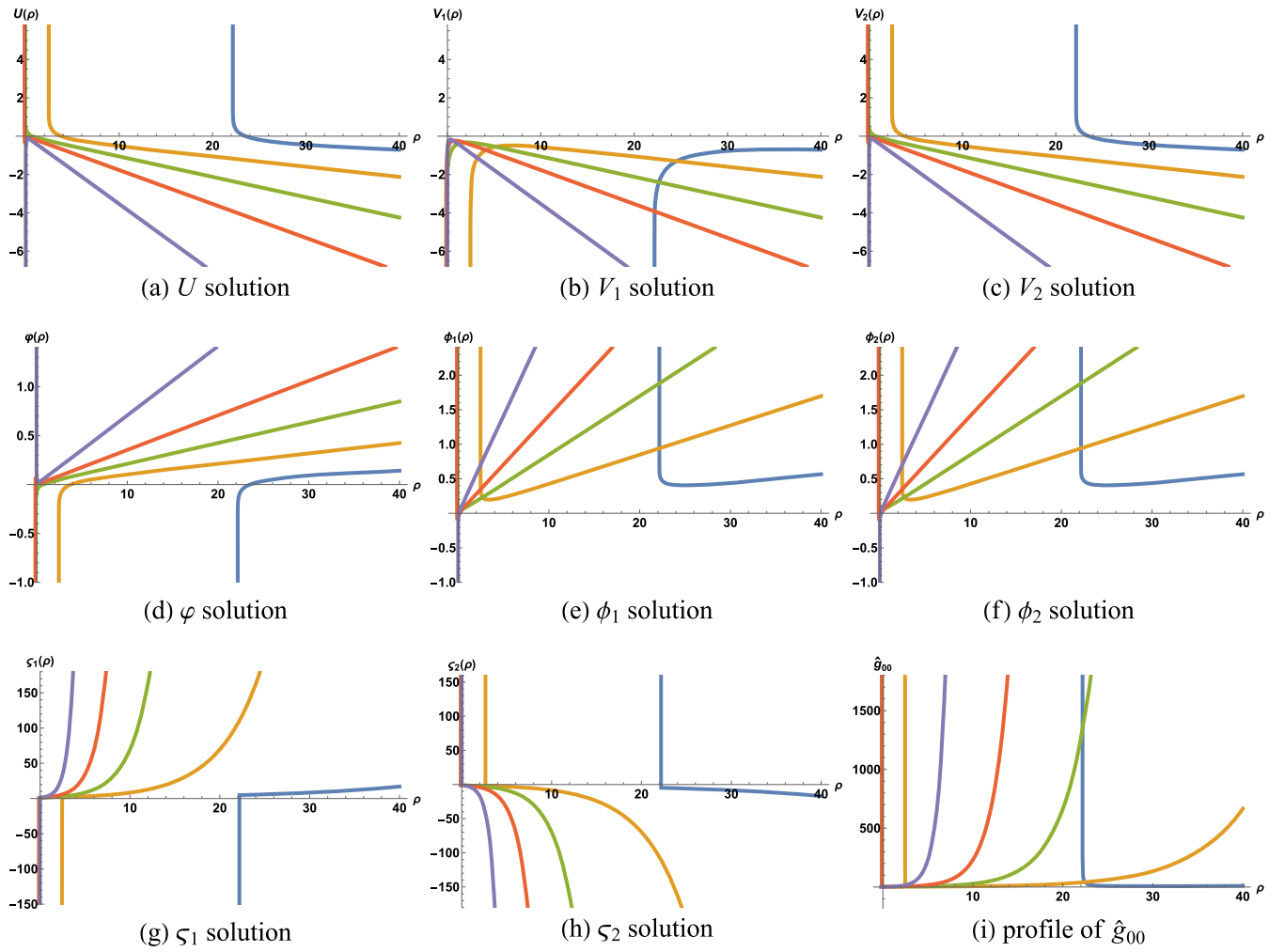


FIG. 25. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times \mathbb{R}^2$ -sliced curved domain walls with shift scalars in $SO(4, 1)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = 0.1, 0.3, 0.6, 1, 2$.

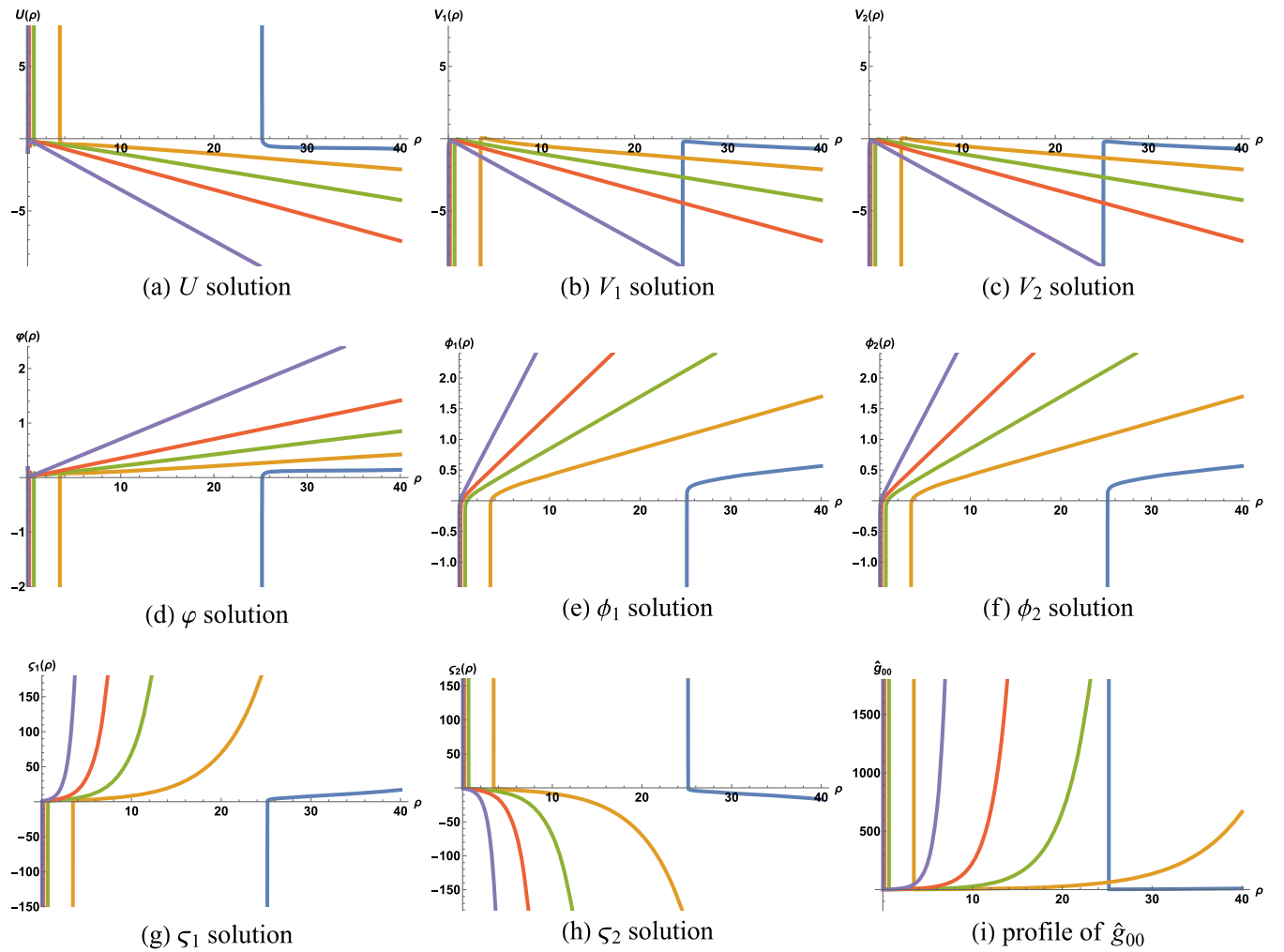


FIG. 26. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times H^2$ -sliced curved domain walls with shift scalars in $SO(4, 1)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = 0.1, 0.3, 0.6, 1, 2$.

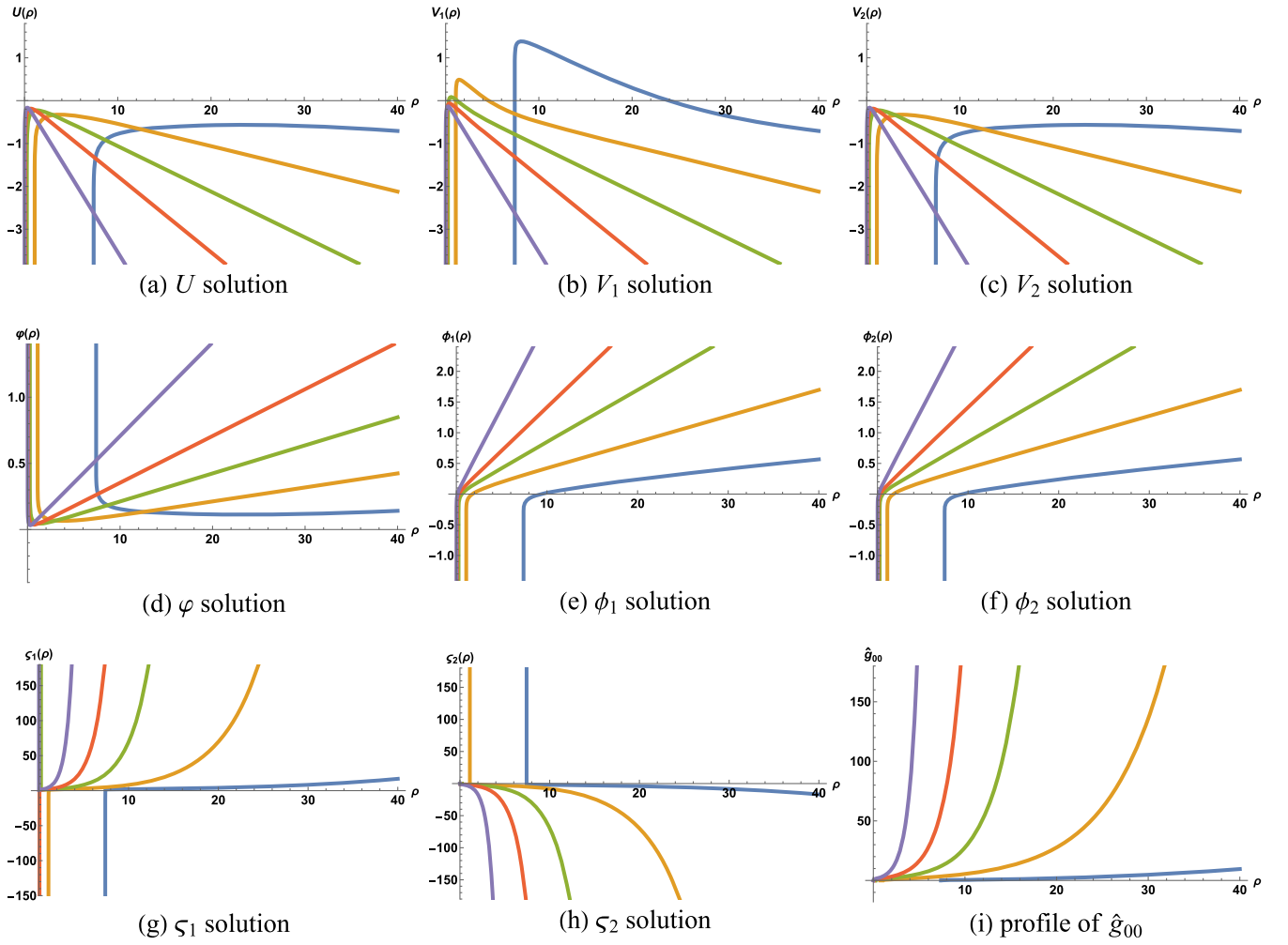


FIG. 27. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times \mathbb{R}^2$ -sliced curved domain walls with shift scalars in $SO(4, 1)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = 0.1, 0.3, 0.6, 1, 2$.

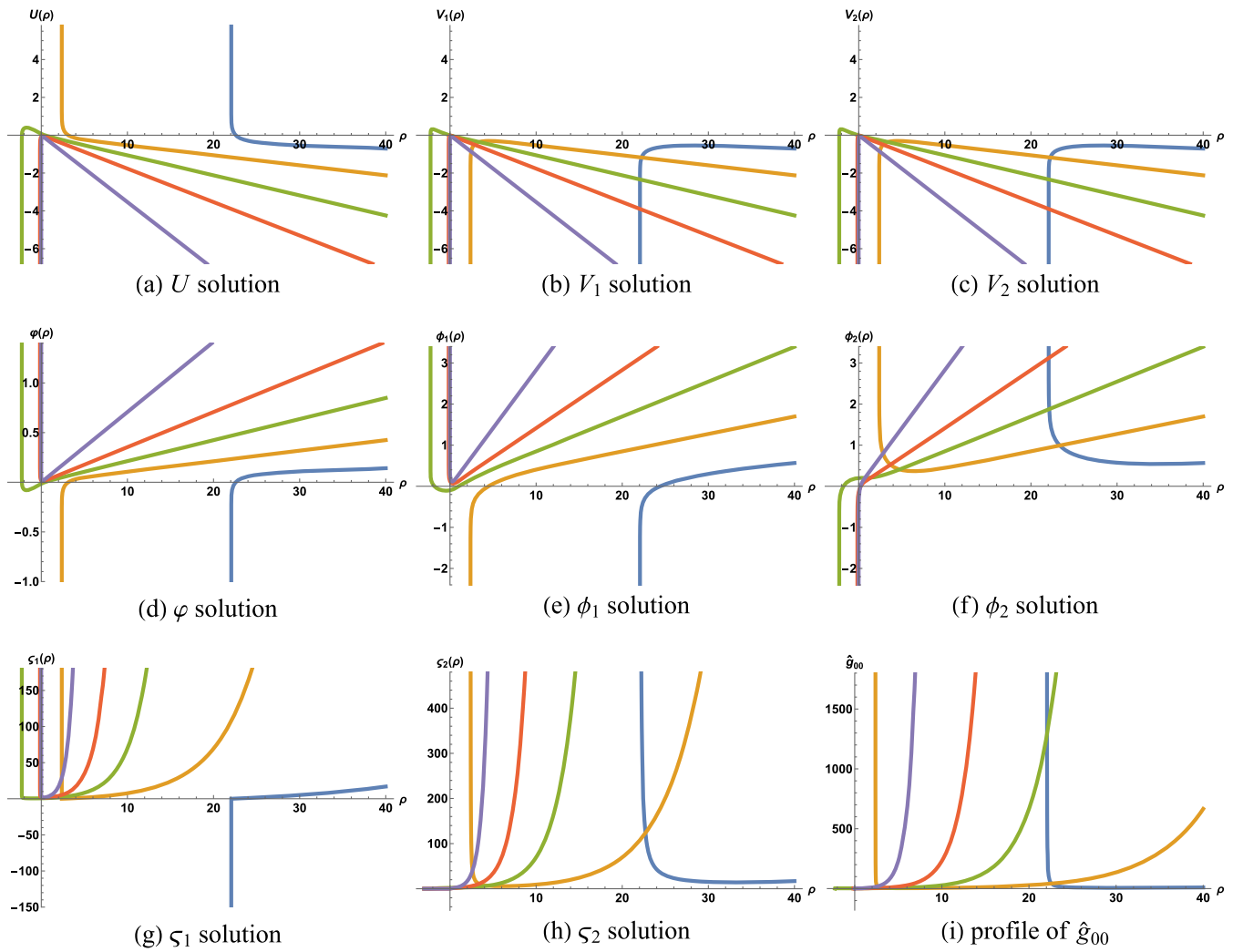


FIG. 28. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times S^2$ -sliced curved domain walls with shift scalars in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = -0.1, -0.3, -0.6, -1, -1, -2$.

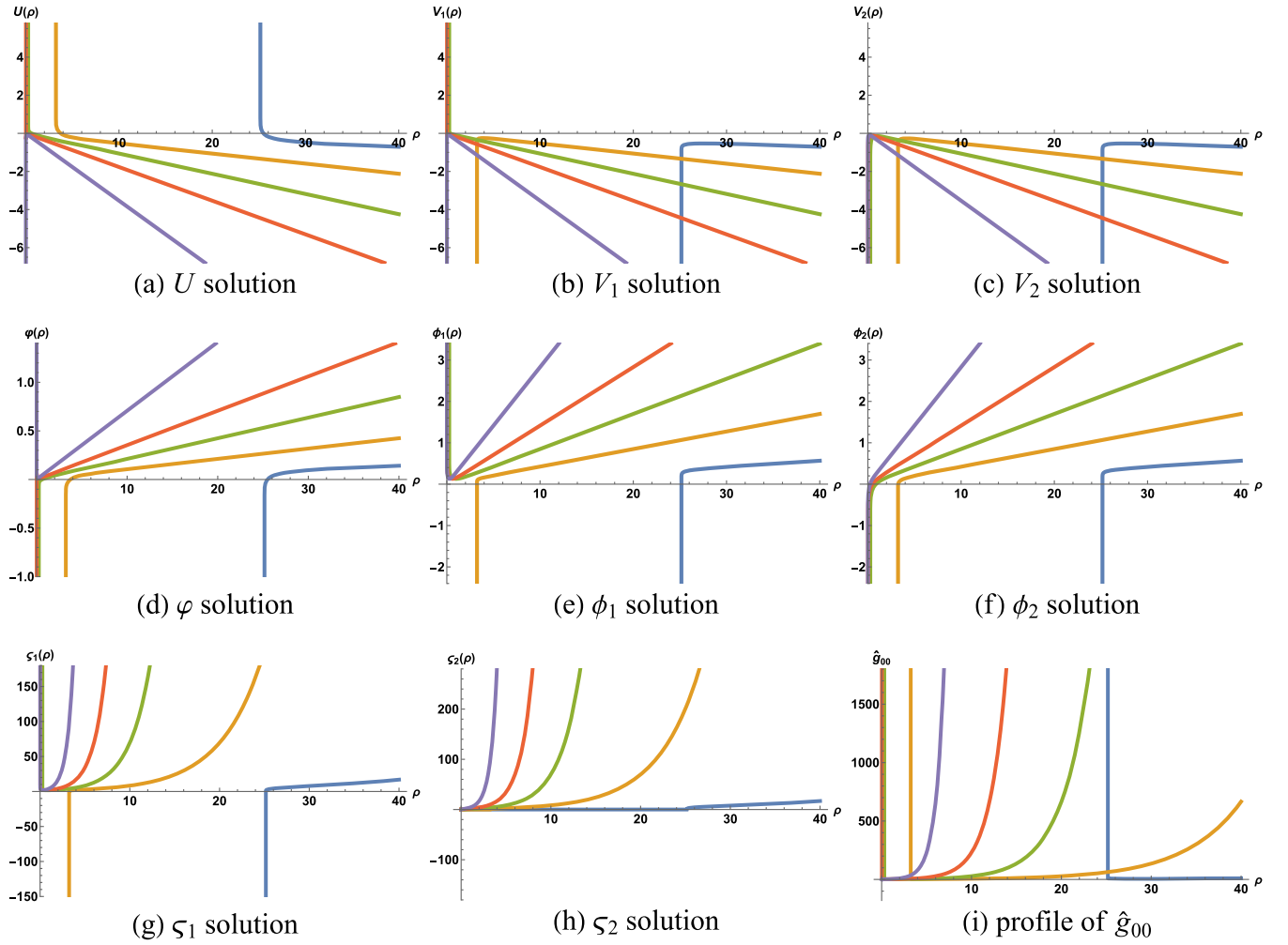


FIG. 29. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times H^2$ -sliced curved domain walls with shift scalars in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = -0.1, -0.3, -0.6, -1, -2$.

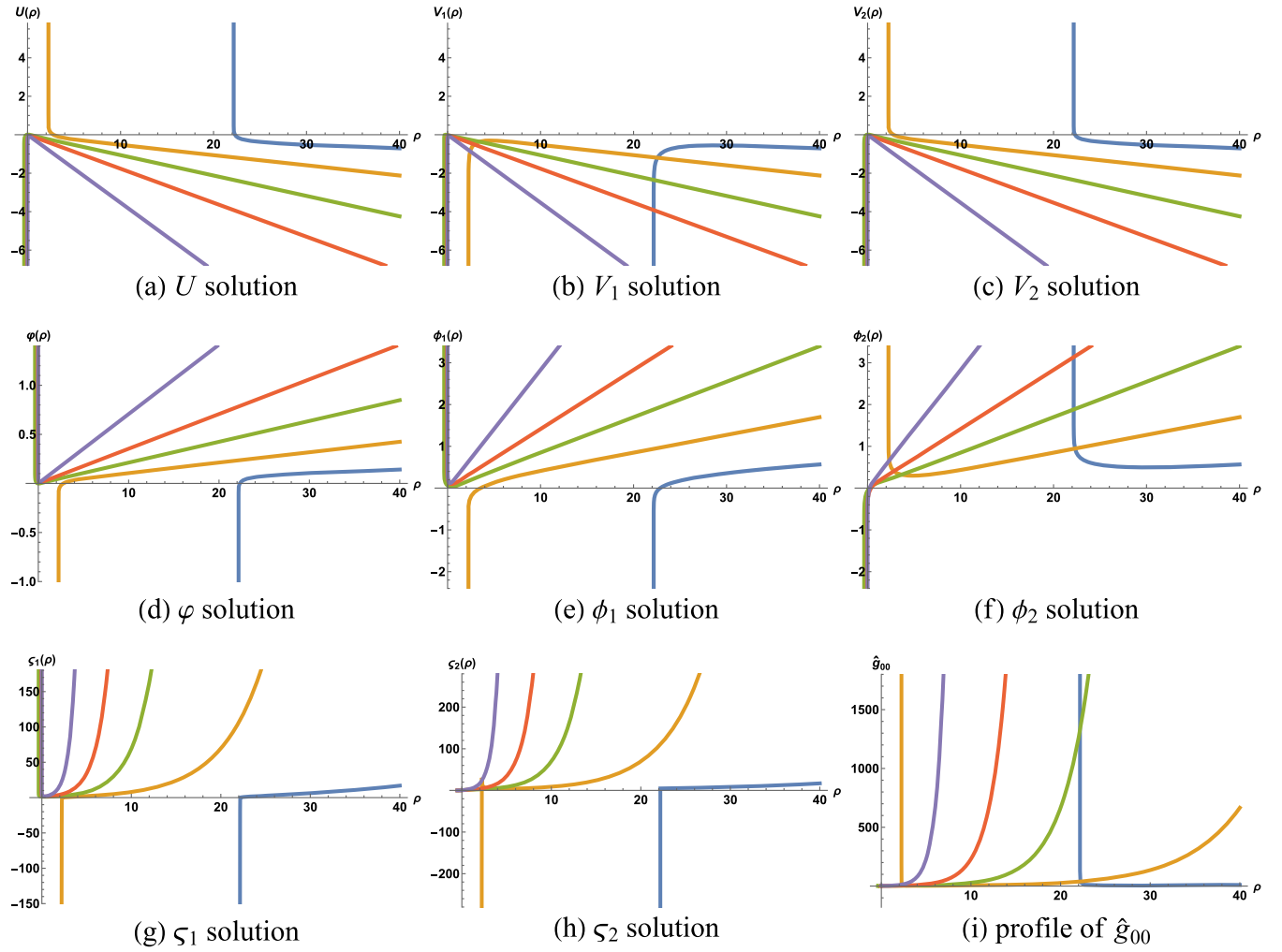


FIG. 30. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times \mathbb{R}^2$ -sliced curved domain walls with shift scalars in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = -0.1, -0.3, -0.6, -1, -2$.

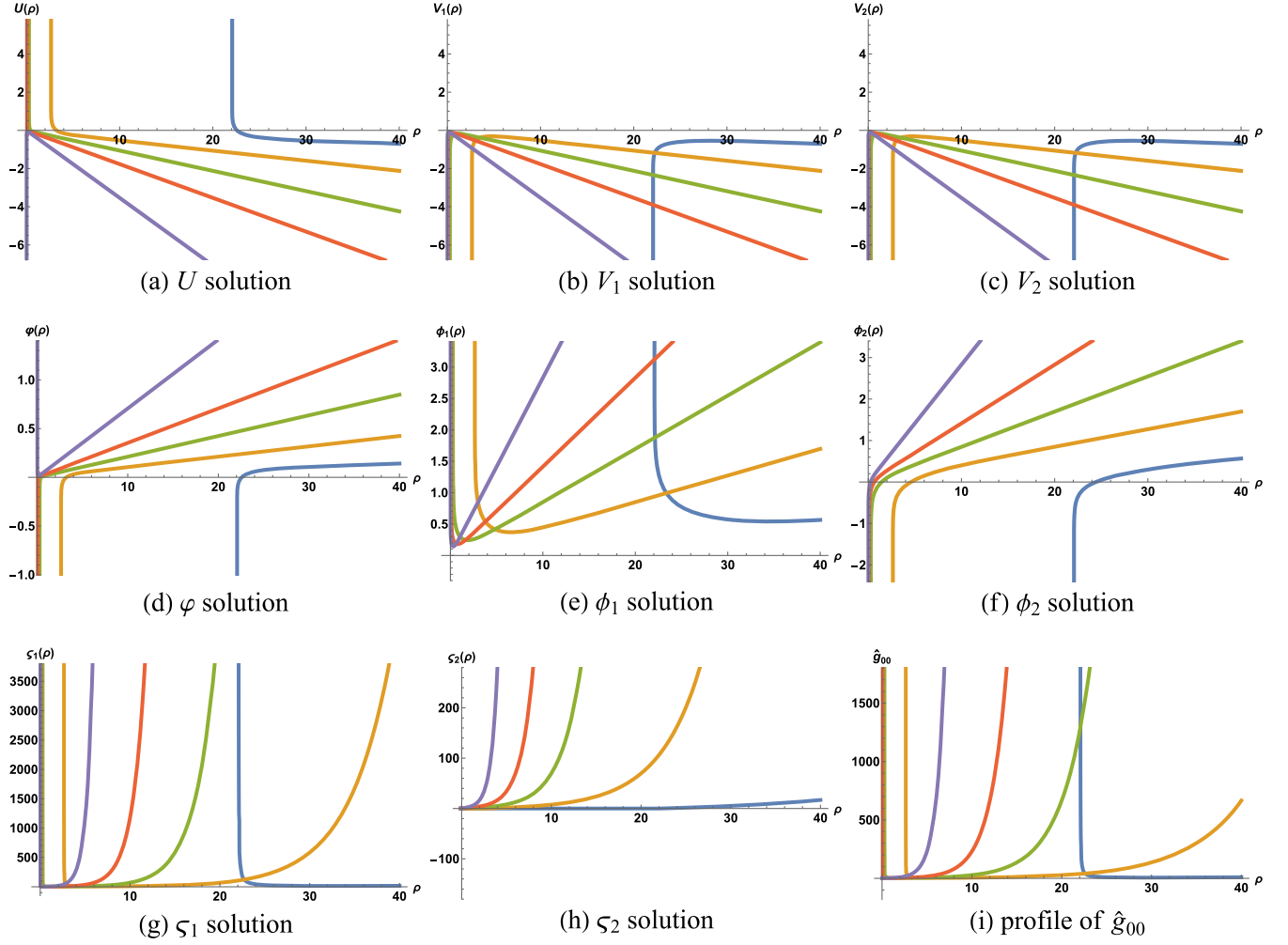


FIG. 31. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times H^2$ -sliced curved domain walls with shift scalars in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = -0.1, -0.3, -0.6, -1, -2$.

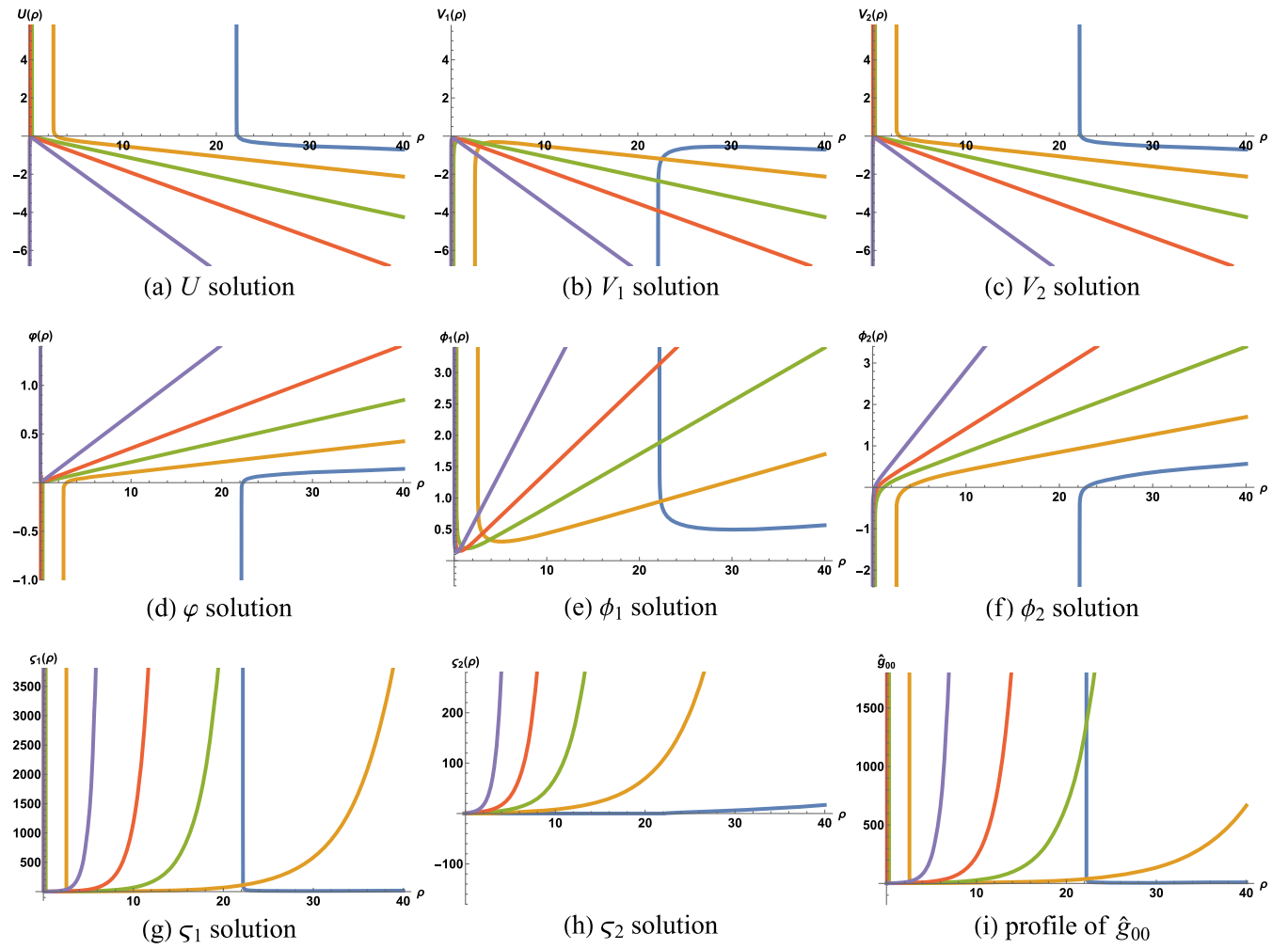


FIG. 32. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times \mathbb{R}^2$ -sliced curved domain walls with shift scalars in $SO(3,2)$ gauge group. The blue, orange, green, red, purple, and black curves refer to $g = -0.1, -0.3, -0.6, -1, -2$.

2. D4-branes wrapped on a Kahler four-cycle

a. Solutions with $SO(3)$ twist

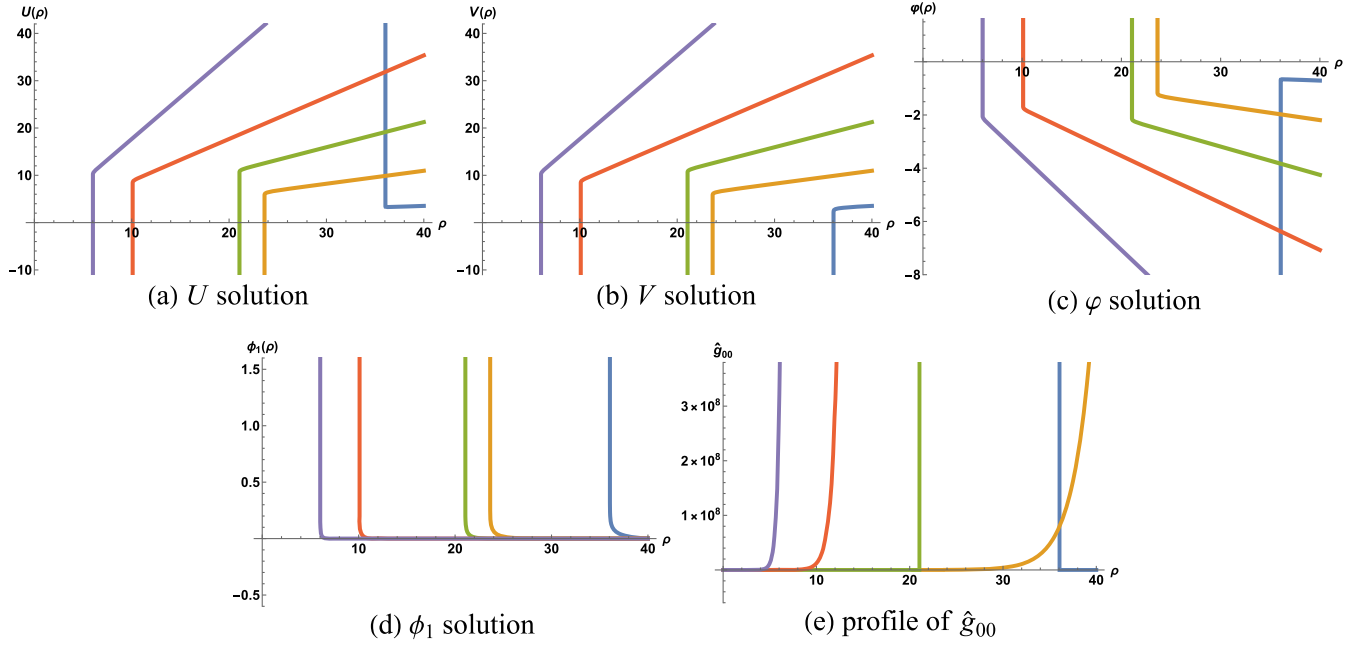


FIG. 33. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(3)$ twist in $SO(5)$ gauge group. The blue, orange, green, red, and purple curves refer to $g = 0.1, 0.3, 0.6, 1$, and 2 , respectively.

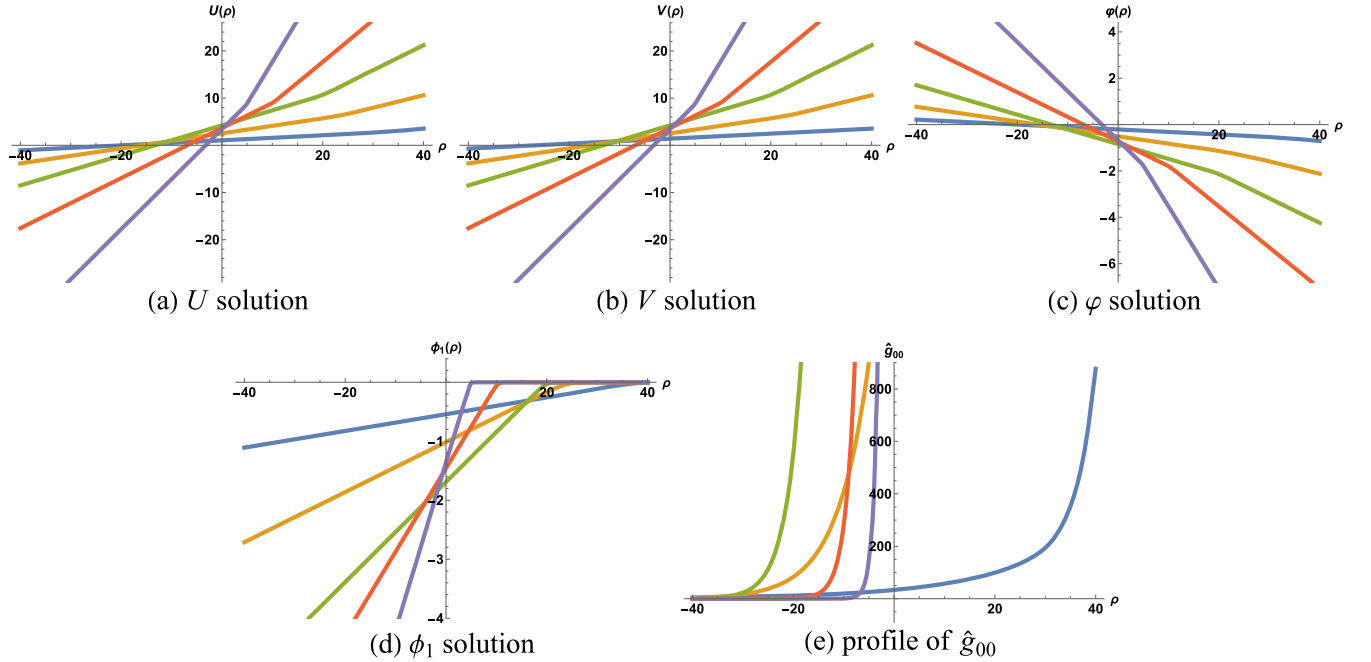


FIG. 34. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(3)$ twist in $SO(5)$ gauge group. The blue, orange, green, red, and purple curves refer to $g = 0.1, 0.3, 0.6, 1$, and 2 , respectively.

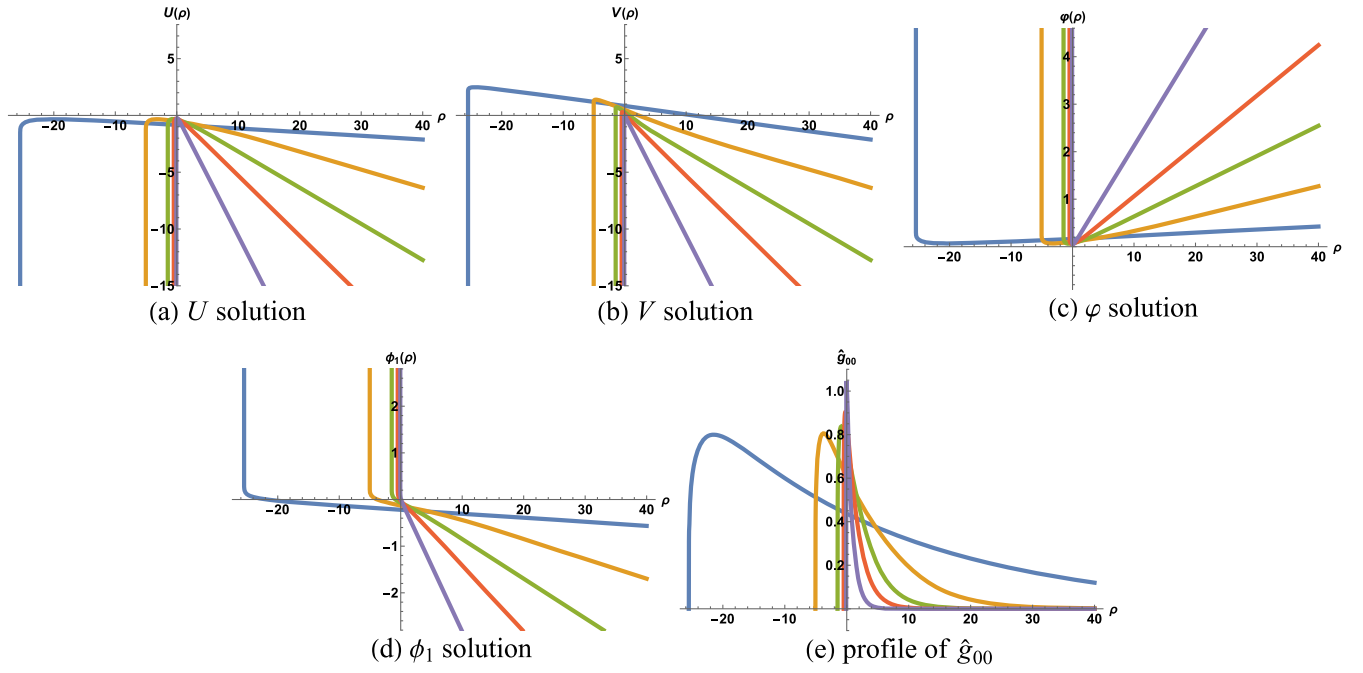


FIG. 35. Interpolating solutions between the locally $SO(3)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(3)$ twist in $SO(3,2)$ gauge group. The blue, orange, green, red, and purple curves refer to $g = -0.1, -0.3, -0.6, -1$, and -2 , respectively.

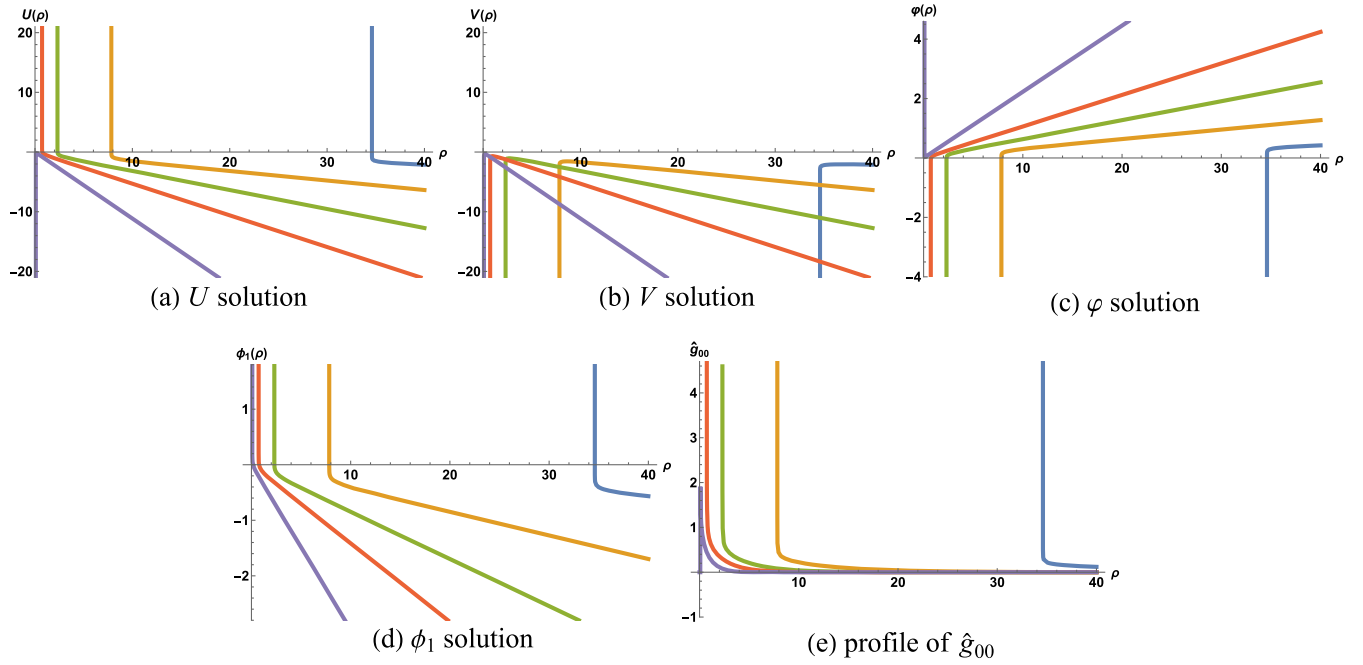


FIG. 36. Interpolating solutions between the locally $SO(3)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(3)$ twist in $SO(3,2)$ gauge group. The blue, orange, green, red, and purple curves refer to $g = -0.1, -0.3, -0.6, -1$, and -2.1 , respectively.

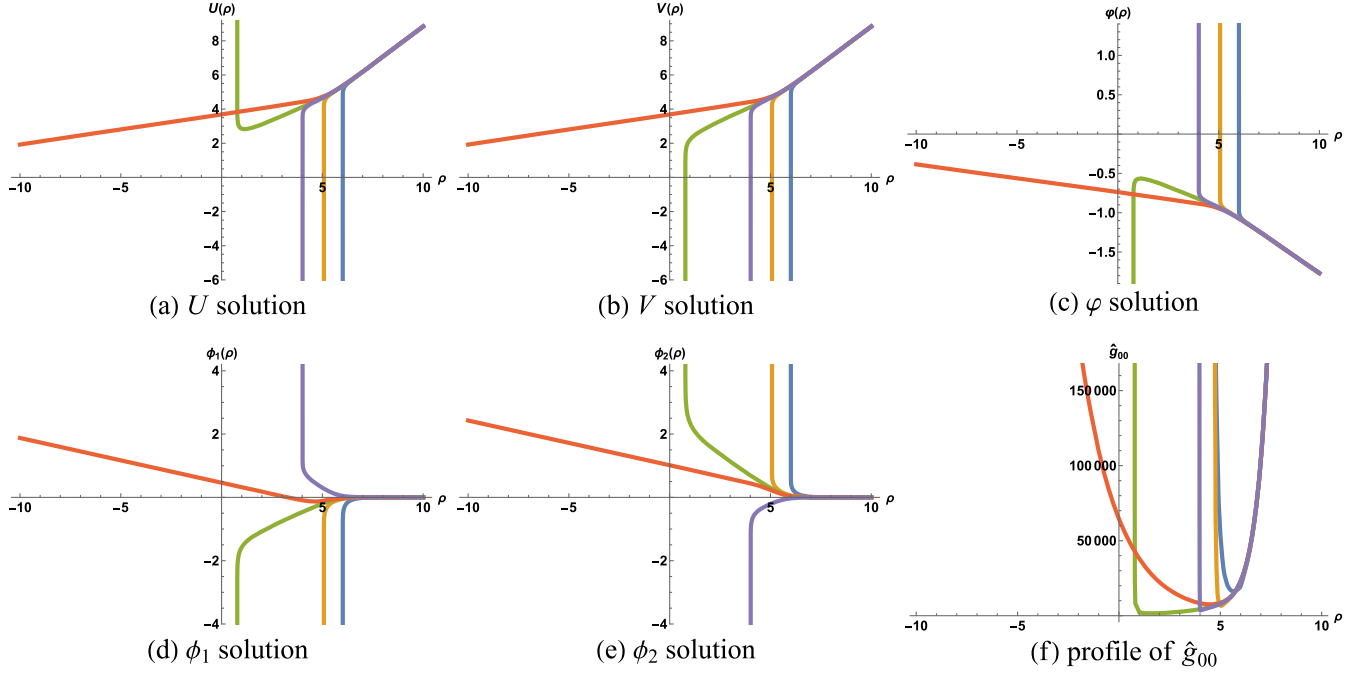
b. Solutions with $SO(2) \times SO(2)$ twist

FIG. 37. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -2.90, -0.20, 0, 0.10$, and 1.01 , respectively.

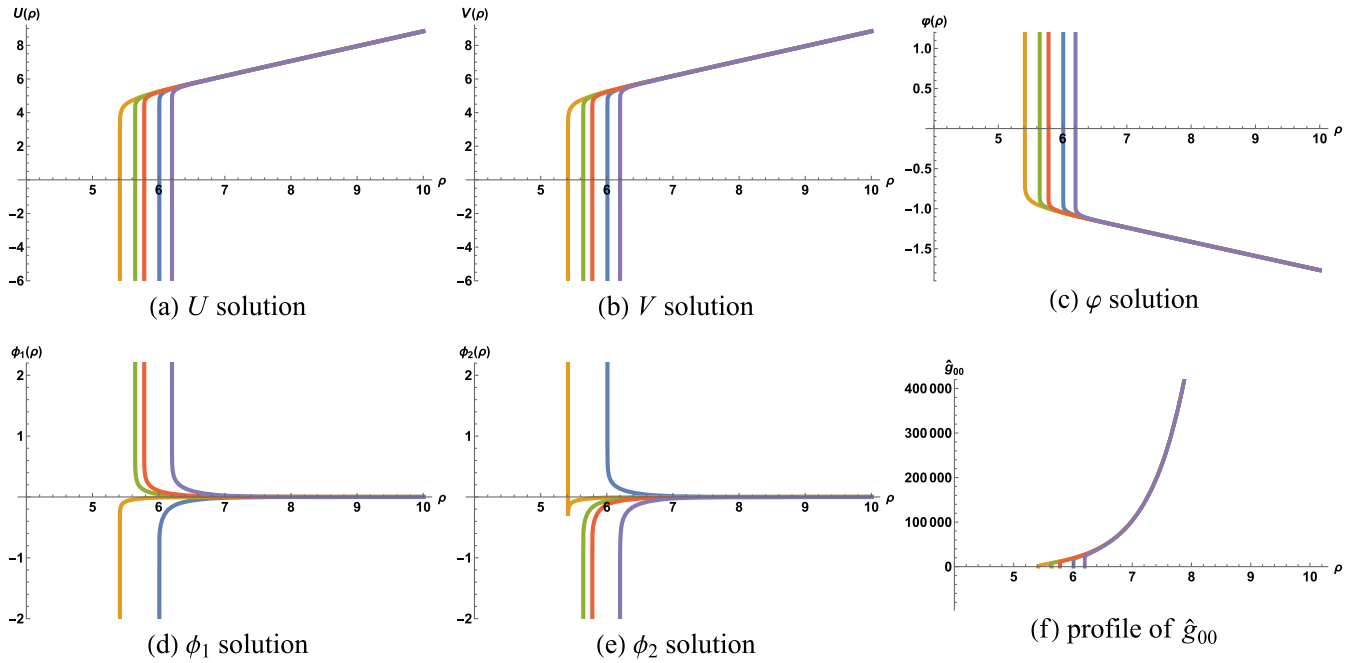


FIG. 38. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -2.9, -0.5, 0, 0.5$, and 4 , respectively.

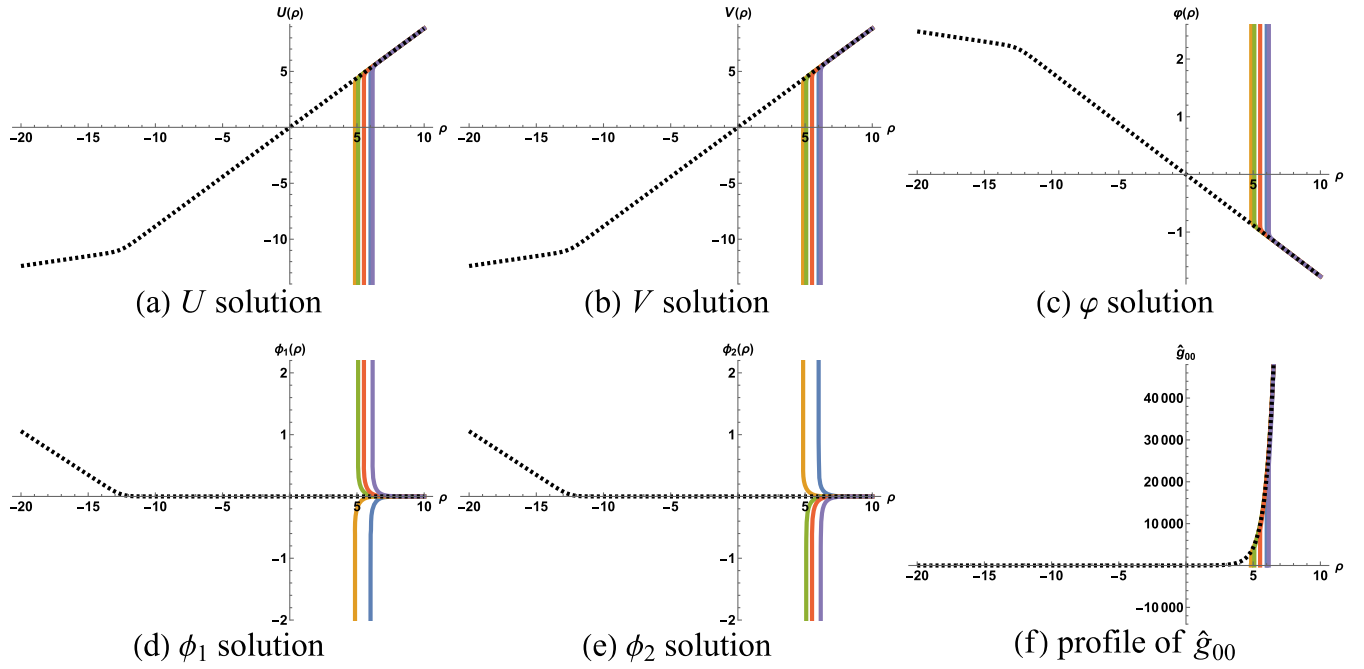


FIG. 39. Interpolating solutions between the locally $SO(5)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times \mathbb{R}^4$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(5)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -2.9, -0.1, 0.2, 0.7$, and 4.5 , respectively. The dashed curve is the $SO(2) \times SO(2)$ flat domain wall obtained from $p_1 = 0$ which implies $p_2 = k = 0$.

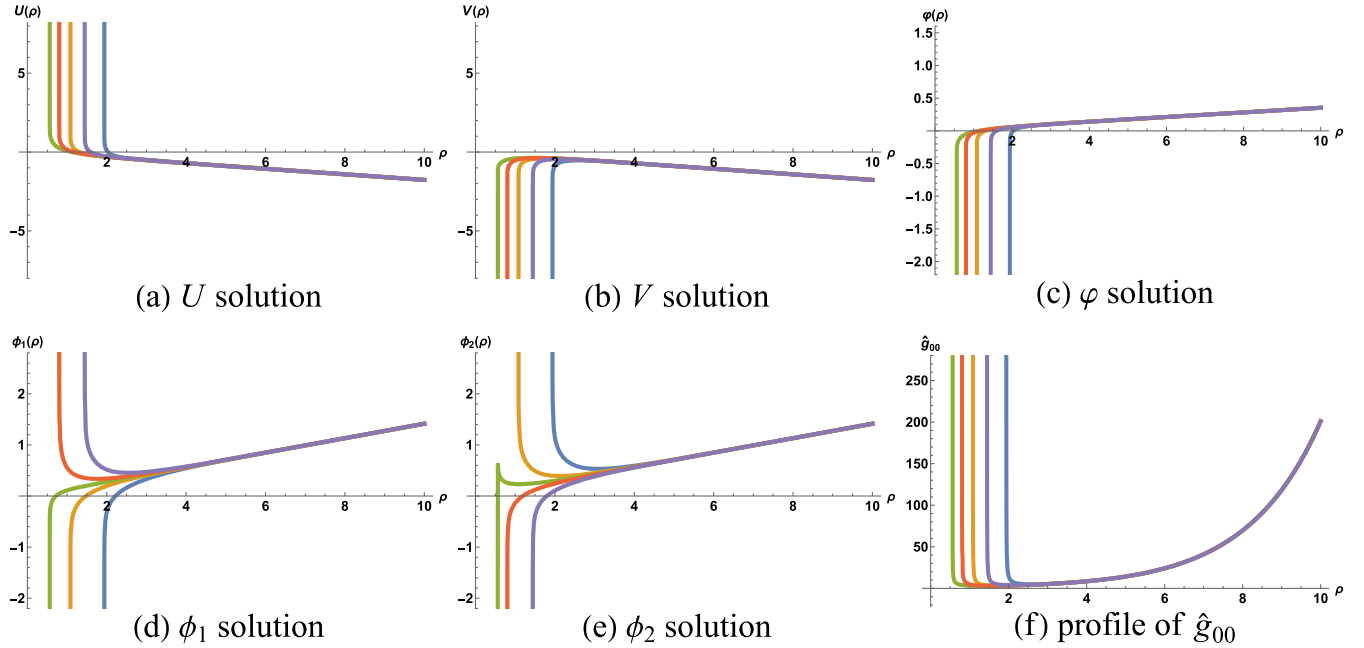


FIG. 40. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(4, 1)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -4, -0.5, 0.4, 1$, and 2.5 , respectively.

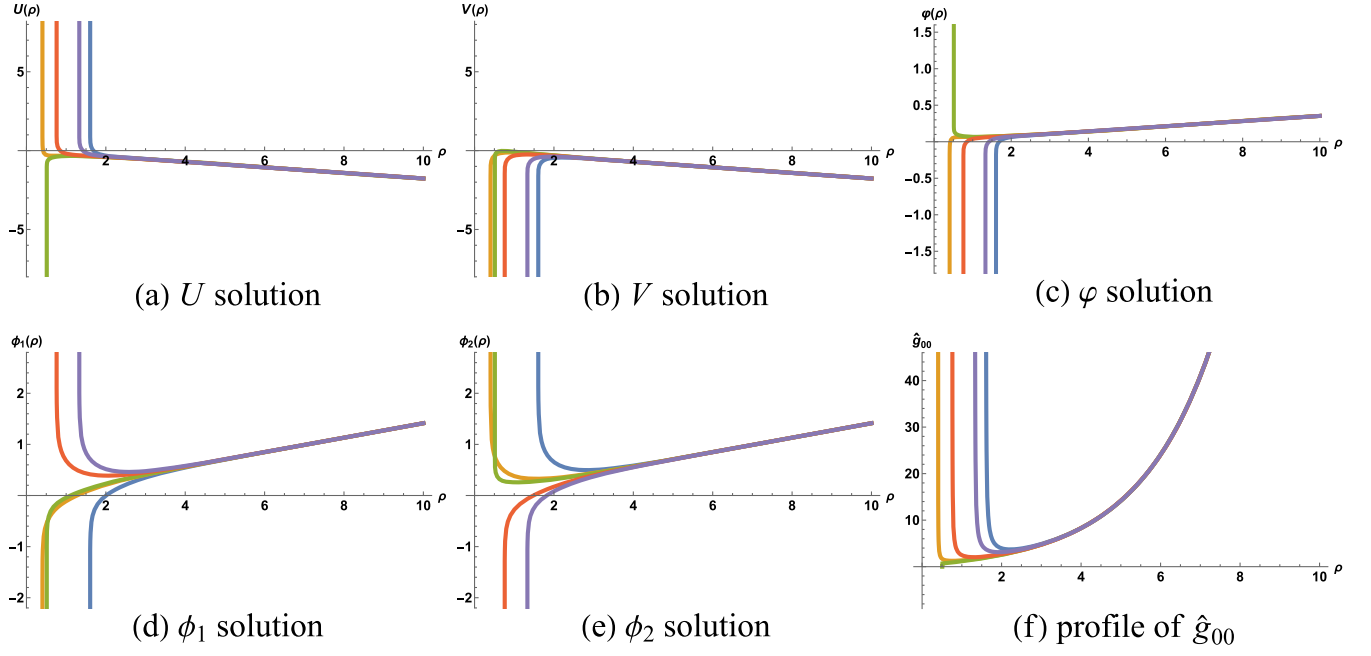


FIG. 41. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(4, 1)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -4, -1.3, -1, 0.8$, and 2 , respectively.

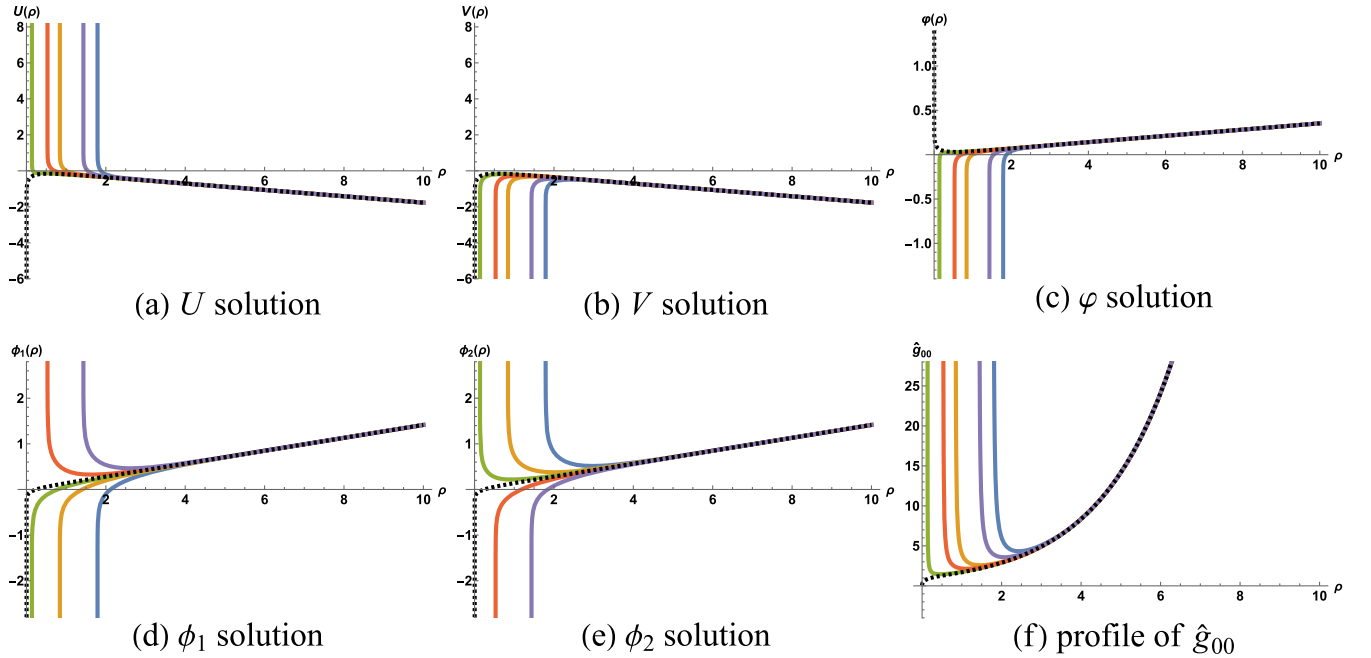


FIG. 42. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times \mathbb{R}^4$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(4, 1)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -4, -1, -0.2, 0.6$, and 2.4 , respectively. The dashed curve is the $SO(2) \times SO(2)$ flat domain wall obtained from $p_1 = 0$ which implies $p_2 = k = 0$.

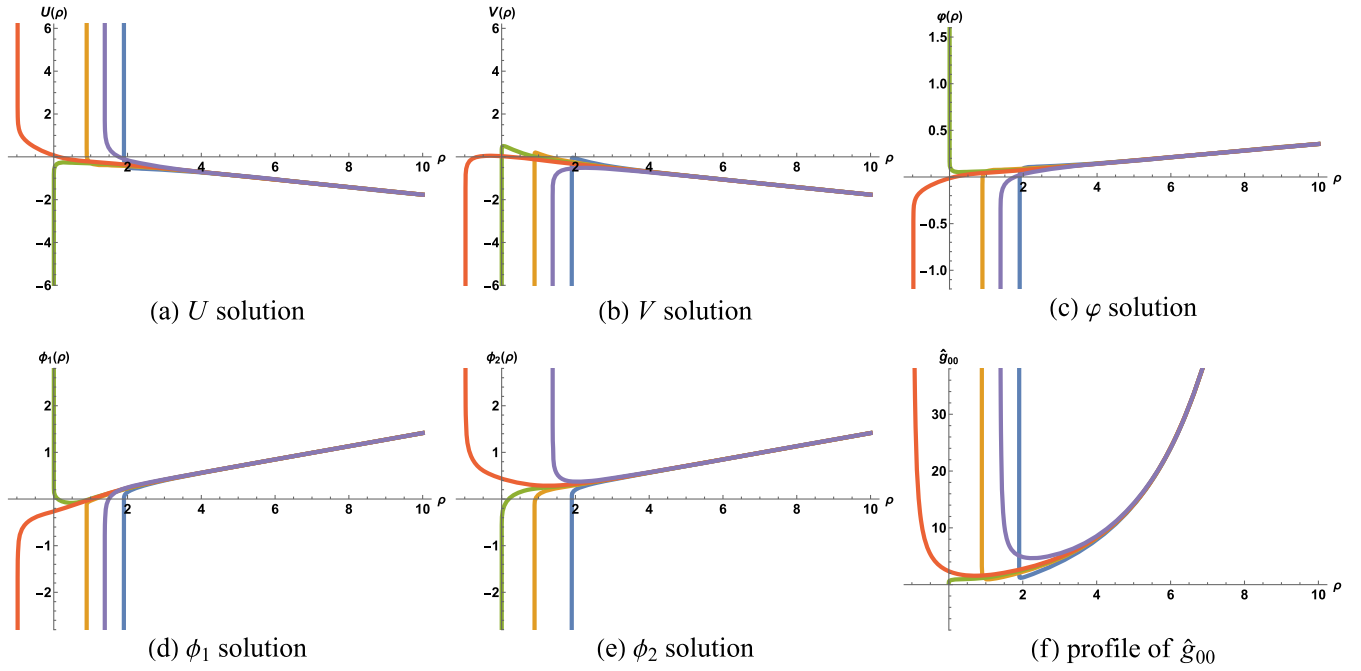


FIG. 43. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3,2)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -4, -1.5, -1, -0.7$, and 1.2 , respectively.

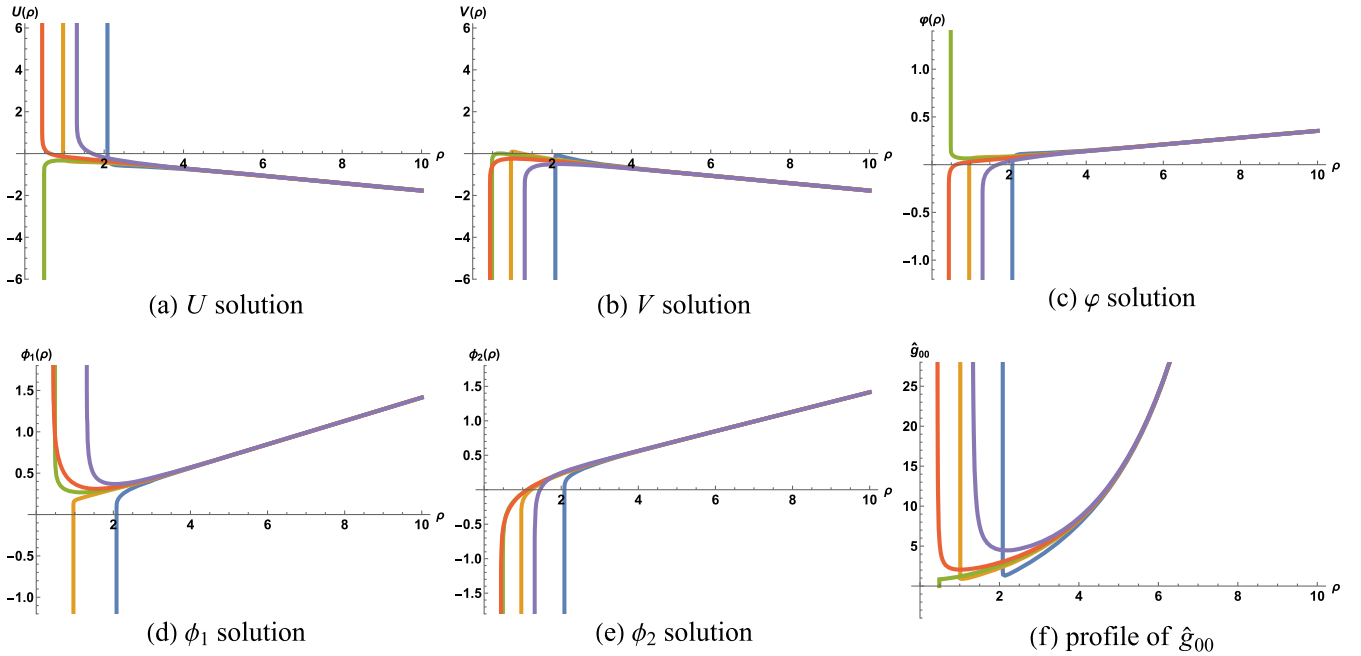


FIG. 44. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3,2)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -4, -0.5, 0, 0.5$, and 2 , respectively.

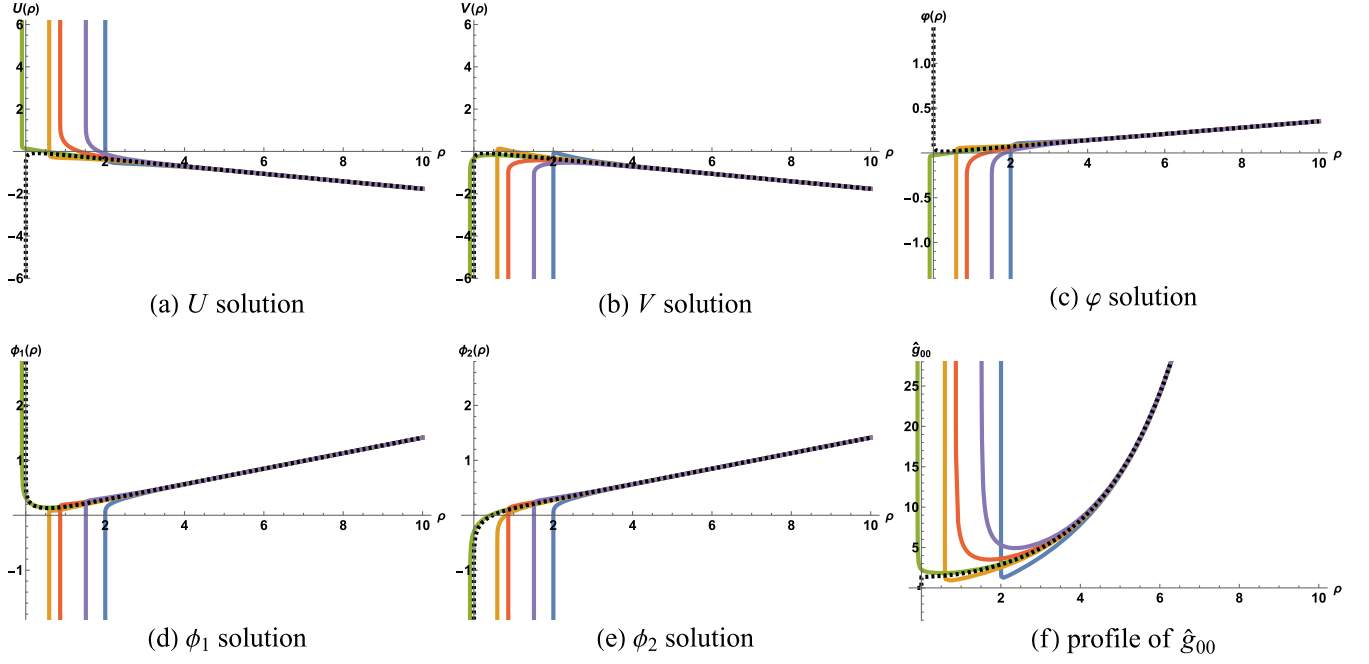


FIG. 45. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times \mathbb{R}^4$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(3,2)$ gauge group with $\varsigma_1 = \varsigma_2 = 0$. The blue, orange, green, red, and purple curves refer to $p_1 = -4, -0.5, 0.1, 0.8$, and 2 , respectively. The dashed curve is the $SO(2) \times SO(2)$ flat domain wall obtained from $p_1 = 0$ which implies $p_2 = k = 0$.

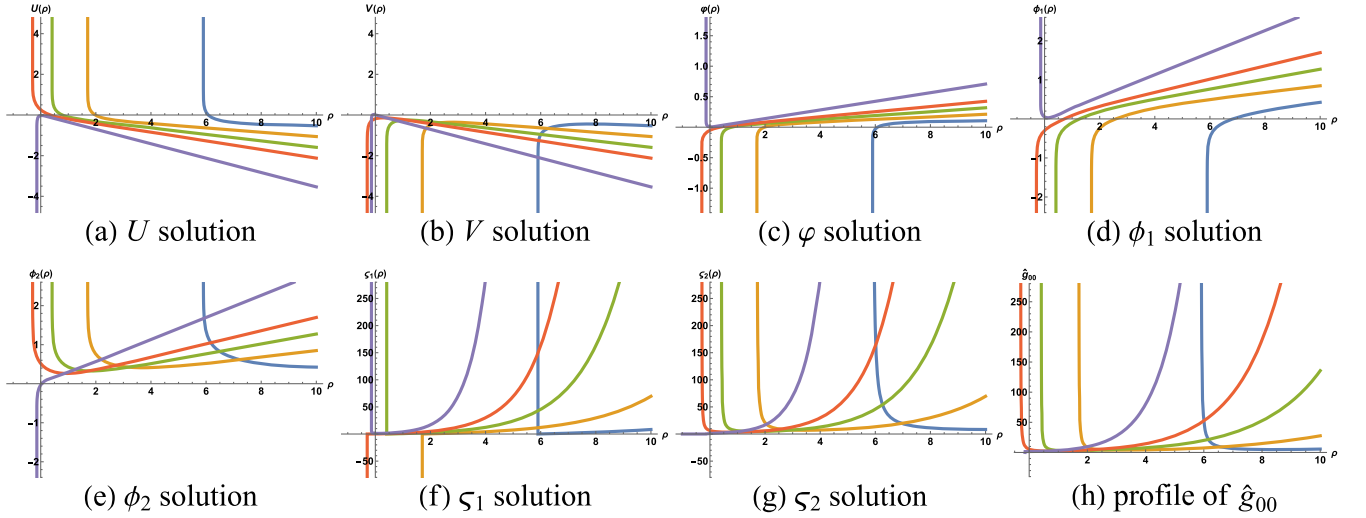


FIG. 46. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist with shift scalars in $SO(3,2)$ gauge group. The blue, orange, green, red, and purple curves refer to $p_1 = -0.3, -0.6, -0.9, -1.2$, and -2 , respectively.

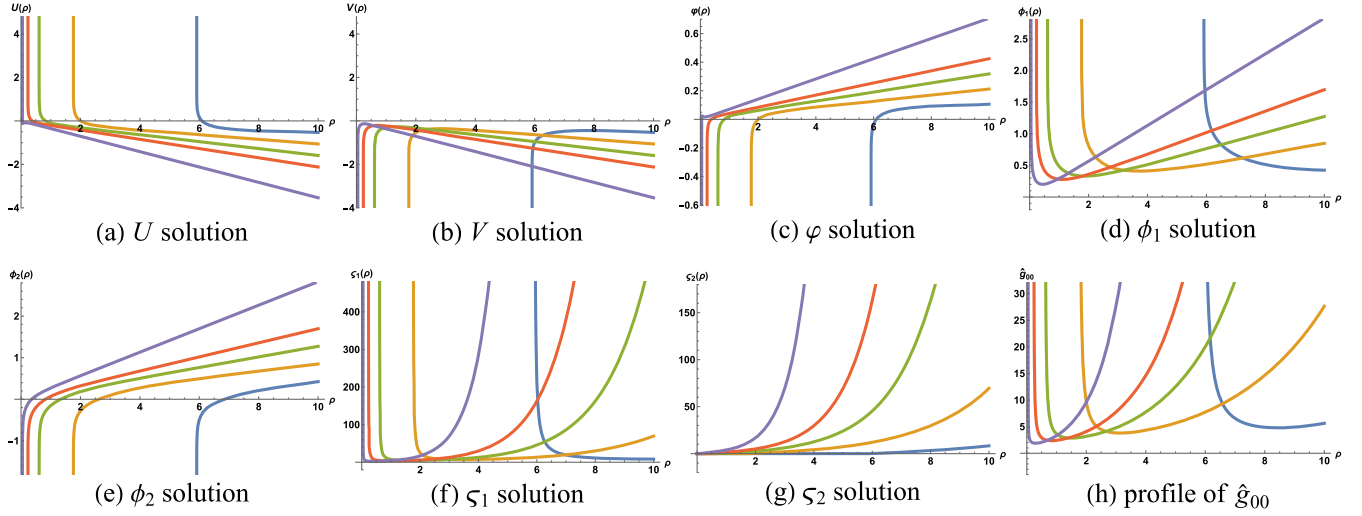


FIG. 47. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist with shift scalars in $SO(3, 2)$ gauge group. The blue, orange, green, red, and purple curves refer to $p_1 = -0.3, -0.6, -0.9, -1.2$, and -2 , respectively.

APPENDIX D: NUMERICAL SOLUTIONS FROM $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ GAUGED SUPERGRAVITY

All numerical solutions from $CSO(p, q, 4-p-q) \ltimes \mathbb{R}^4$ are given in this appendix.

1. D4-branes wrapped on a product of two Riemann surfaces

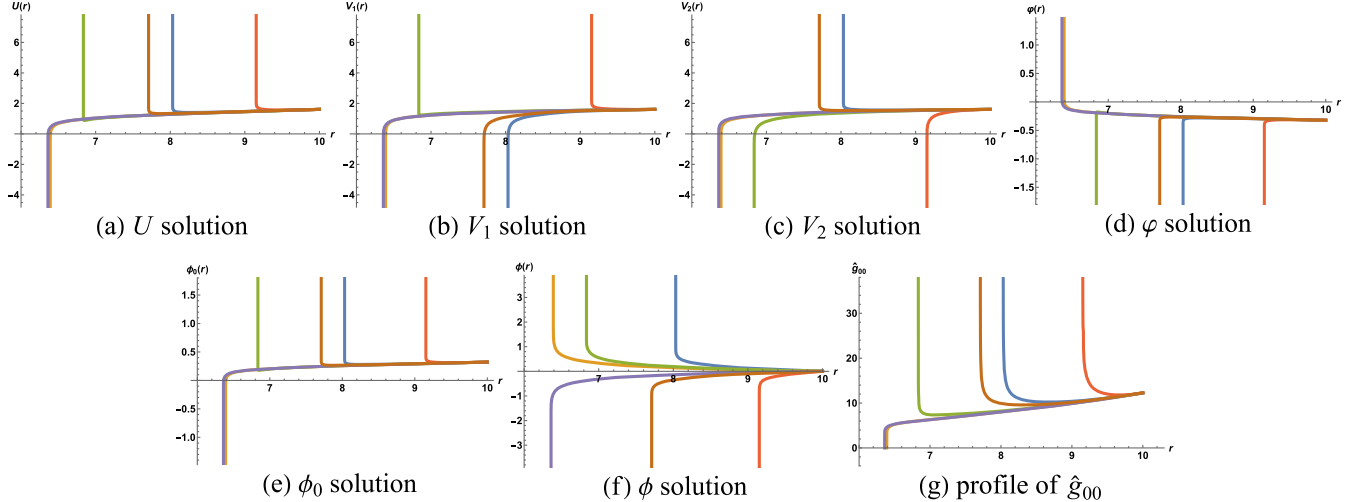


FIG. 48. Interpolating solutions between the locally $SO(4)$ flat domain wall as $r \rightarrow +\infty$ and $t \times H^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(4) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, and brown curves refer to $z_1 = -4.01, -1.01, -0.50, 0.10, 1.12$, and 3.40 , respectively.

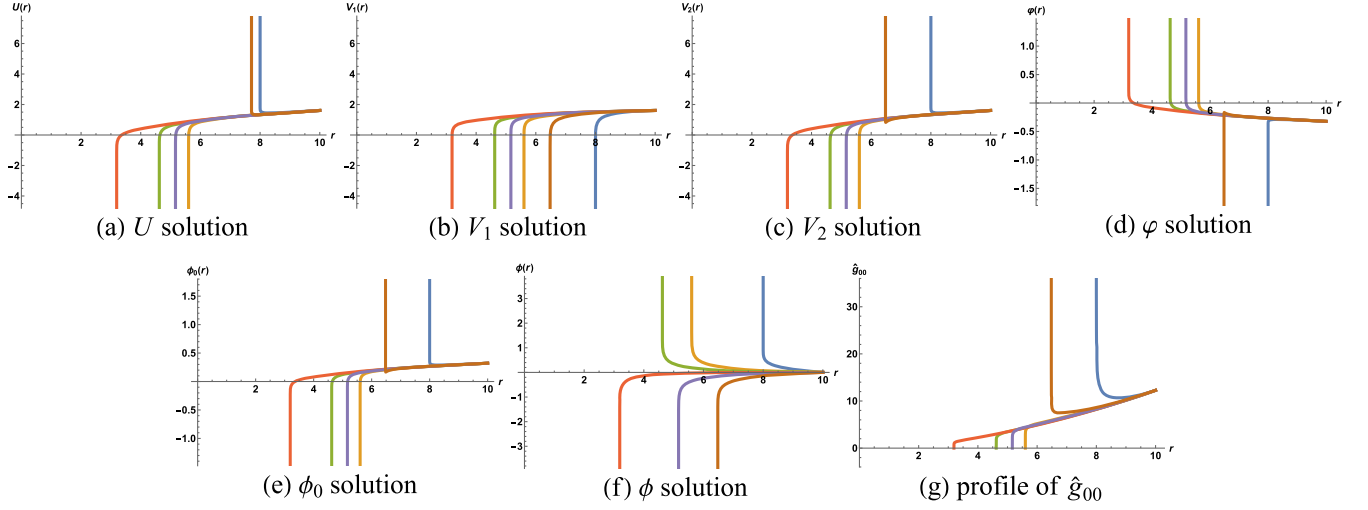


FIG. 49. Interpolating solutions between the locally $SO(4)$ flat domain wall as $r \rightarrow +\infty$ and $t \times H^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(4) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, and brown curves refer to $z_1 = -4, -1, -0.40, 0.10, 0.75$, and 1.90 , respectively.

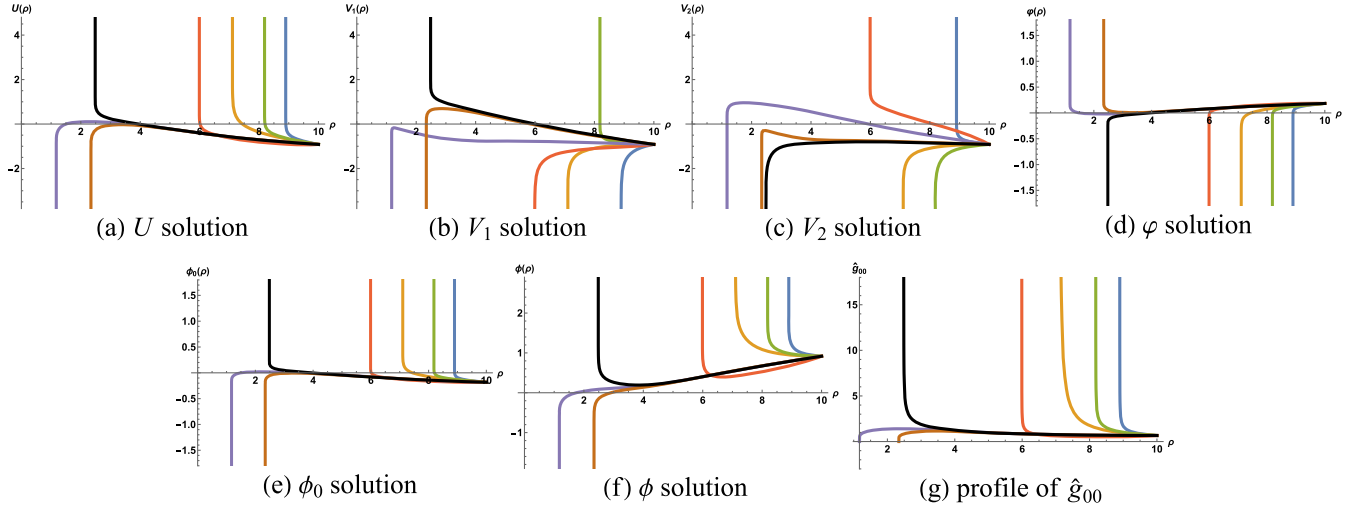


FIG. 50. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times S^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(2,2) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $z_1 = -4, -1, -0.50, 0.10, 0.48, 2$, and 2.17 , respectively.

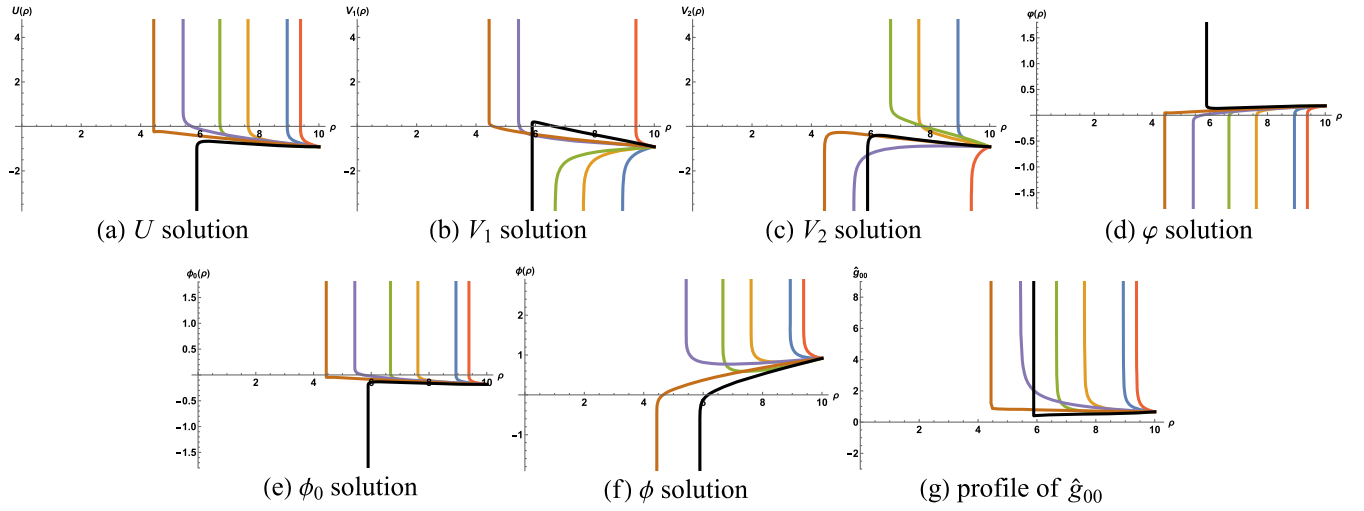


FIG. 51. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(2,2) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $z_1 = -4, -1, -0.21, 0.10, 0.50, 1.10$, and 3.90 , respectively.

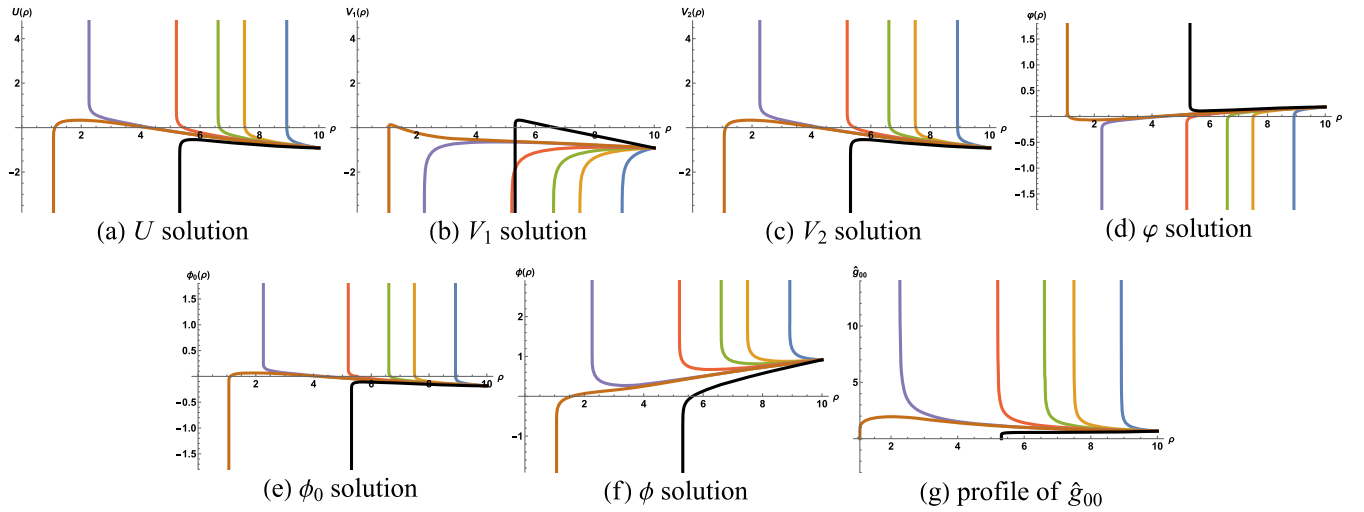


FIG. 52. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times S^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(2,2) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $z_1 = -4, -1, -0.40, 0.10, 0.50, 0.54$, and 3.70 , respectively.

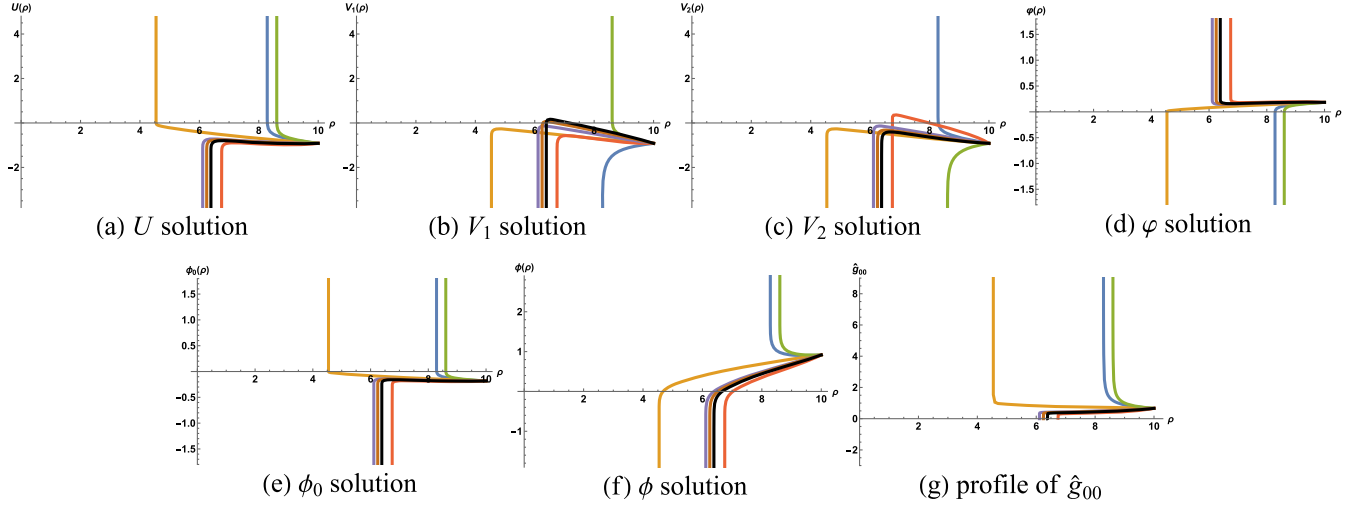


FIG. 53. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times H^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(2,2) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $z_1 = -4, -1, -0.21, 0.11, 1, 2.60$, and 4 , respectively.

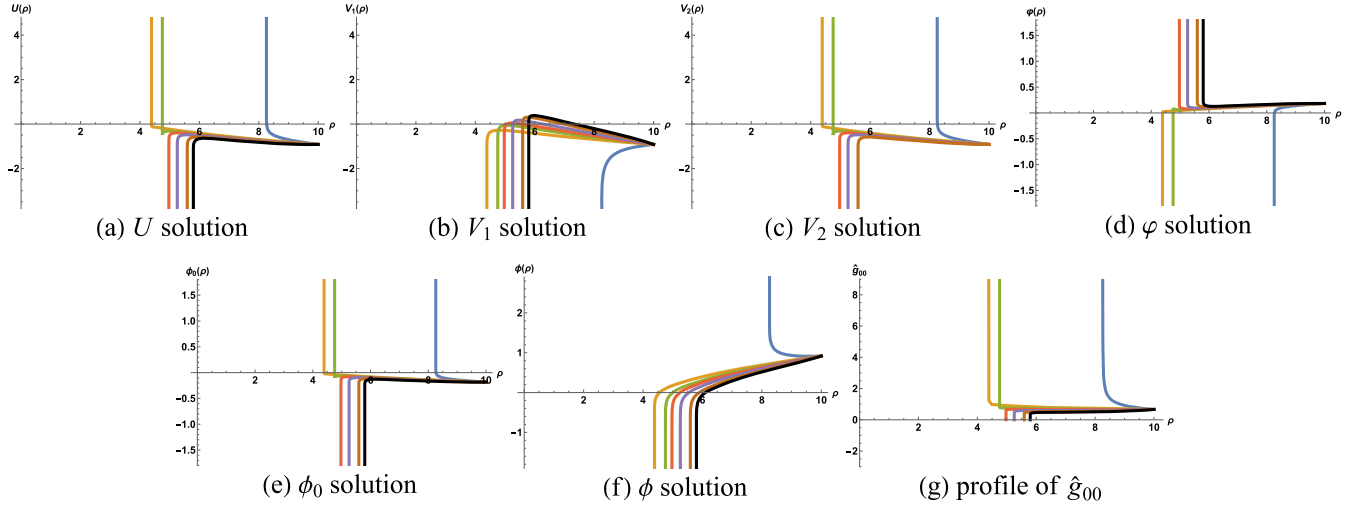


FIG. 54. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times H^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(2,2) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $z_1 = -4, -1, -0.40, 0.11, 1, 2.60$, and 4 , respectively.

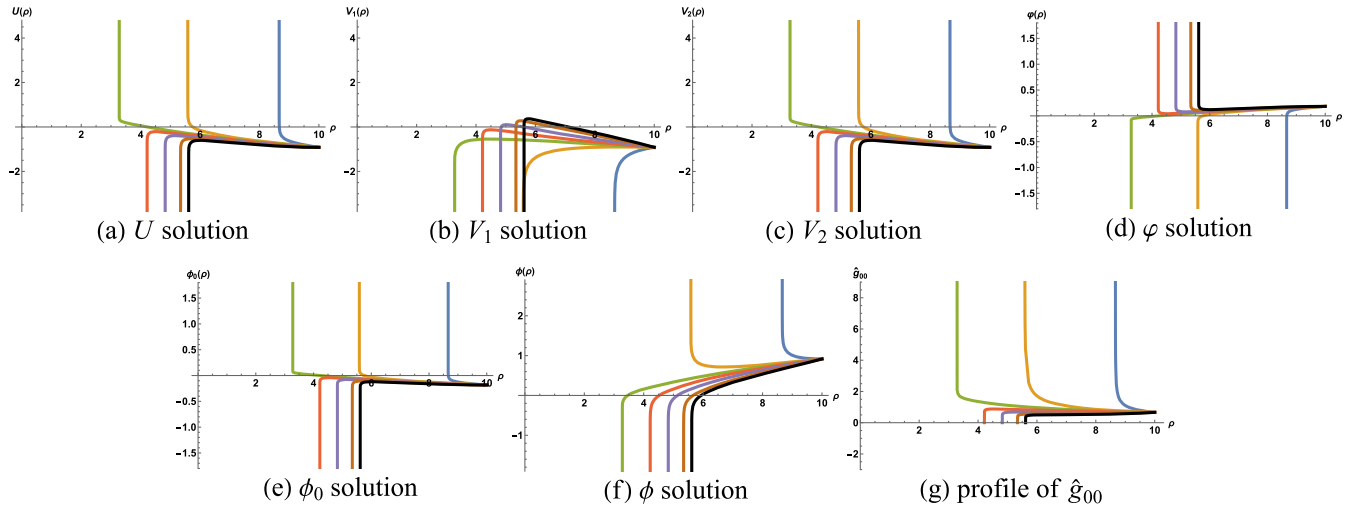


FIG. 55. Interpolating solutions between the locally $SO(2) \times SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times \mathbb{R}^2 \times \mathbb{R}^2$ -sliced curved domain walls for $SO(2) \times SO(2)$ twist in $SO(2, 2) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, brown, and black curves refer to $z_1 = -4, -1, -0.4, 0.1, 1, 2.6$, and 4 , respectively.

2. D4-branes wrapped on a Kahler four-cycle

a. Solutions with $SO(3)$ twist

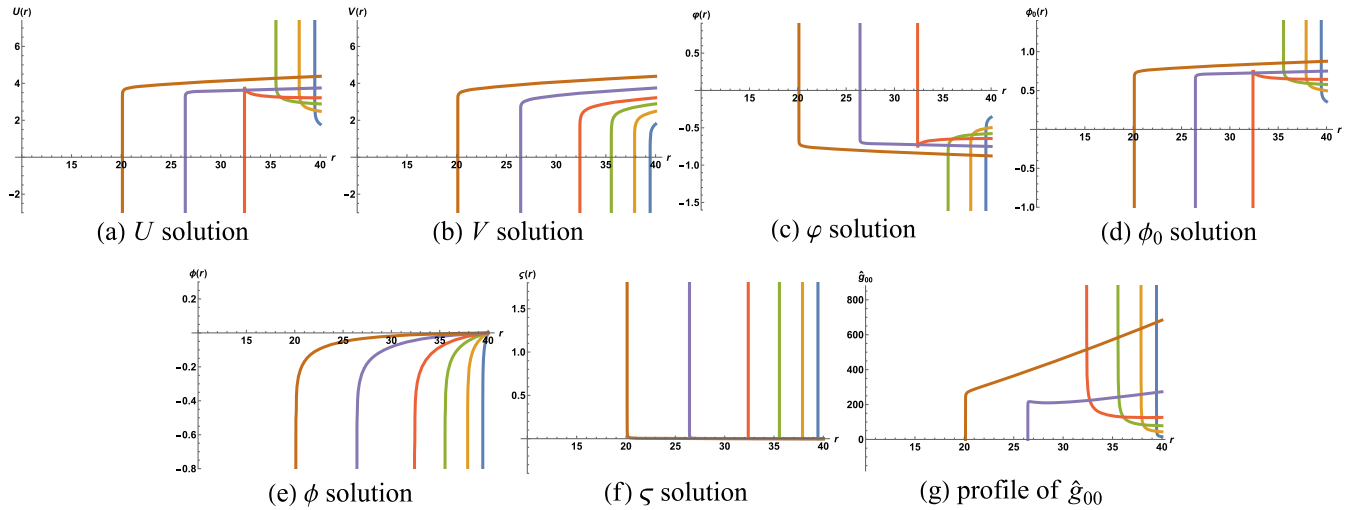


FIG. 56. Interpolating solutions between the locally $SO(4)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(3)$ twist in $SO(4) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, and brown curves refer to $g = 0.30, 0.60, 0.90, 1.25, 2.12$, and 4 , respectively.

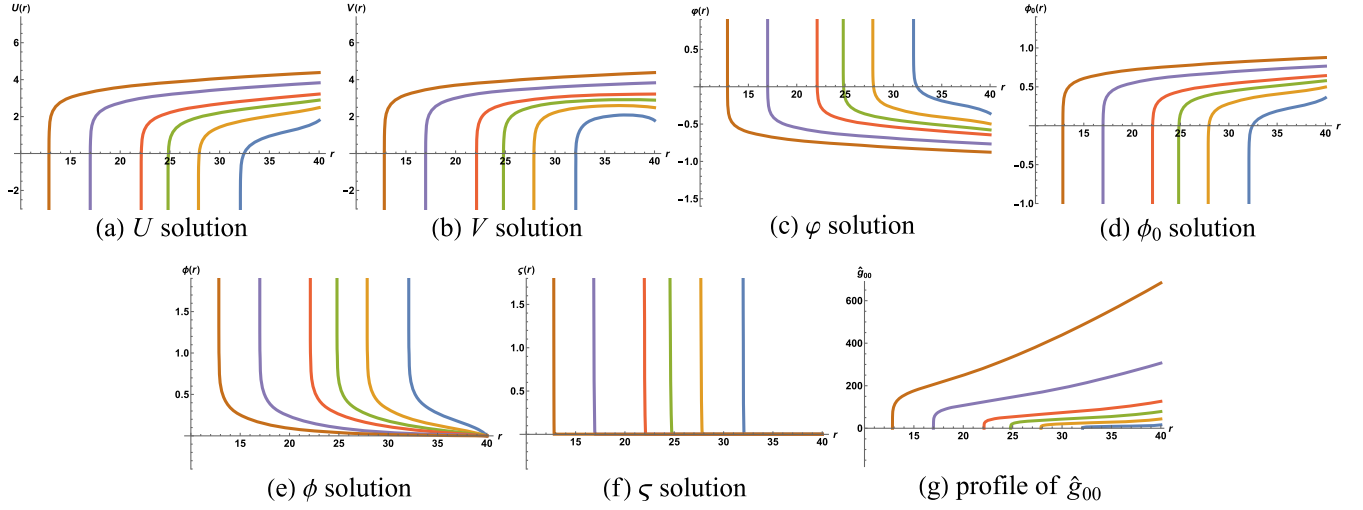


FIG. 57. Interpolating solutions between the locally $SO(4)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(3)$ twist in $SO(4) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, and brown curves refer to $g = 0.30, 0.60, 0.90, 1.25, 2.30$, and 4 , respectively.

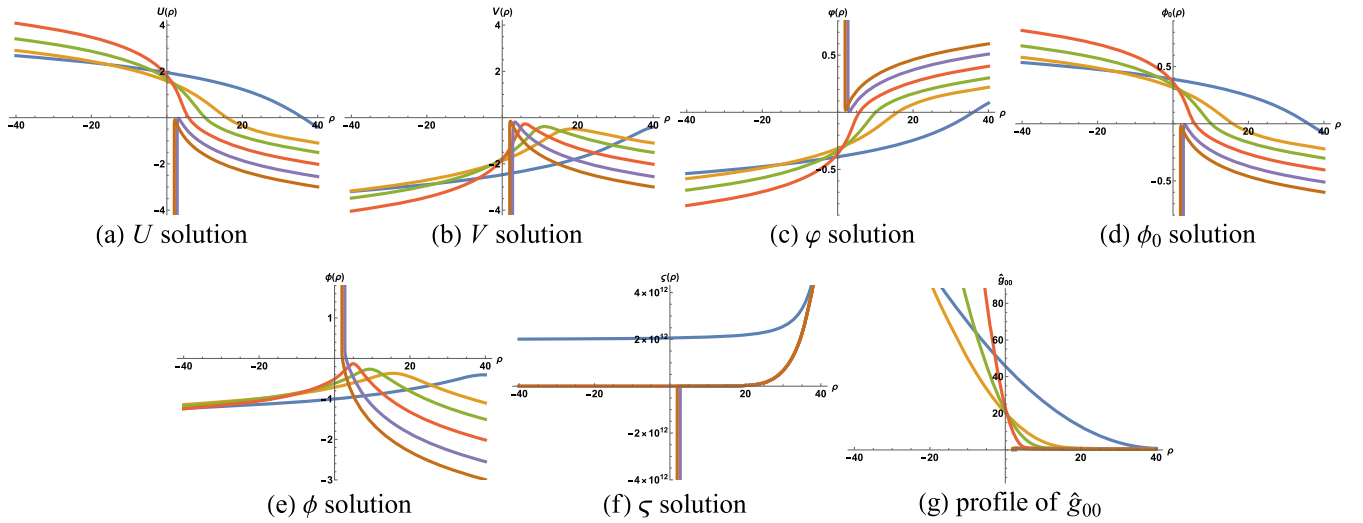


FIG. 58. Interpolating solutions between the locally $SO(3)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(3)$ twist in $SO(3, 1) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, and brown curves refer to $g = 0.3, 0.6, 0.9, 1.5, 2.56$, and 4 .

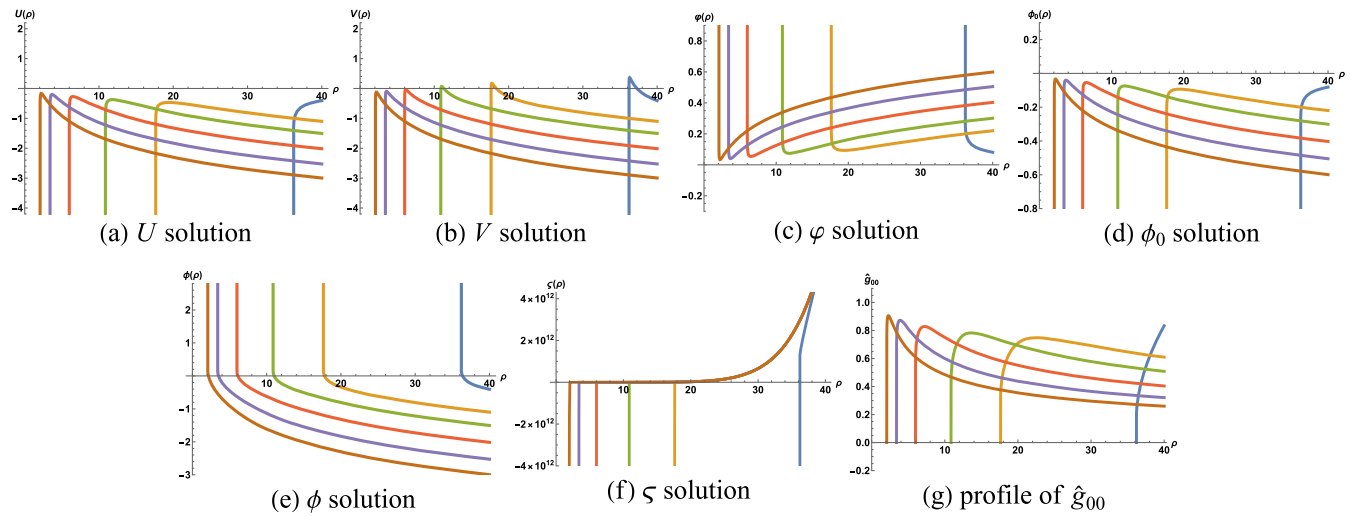


FIG. 59. Interpolating solutions between the locally $SO(3)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(3)$ twist in $SO(3, 1) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, purple, and brown curves refer to $g = 0.3, 0.6, 0.9, 1.5, 2.5$, and 4 .

b. Solutions with $SO(2)$ twist

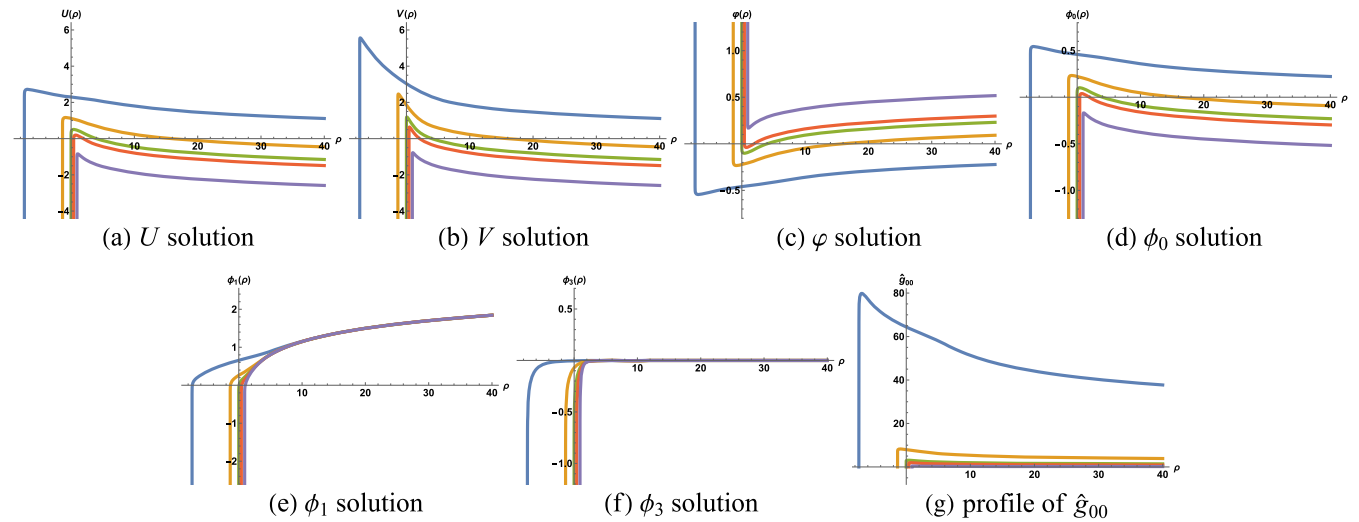


FIG. 60. Interpolating solutions between the locally $SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(2)$ twist in $SO(3, 1) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, and purple curves refer to $g = -0.1, -0.5, -1, -1.4$, and -4.2 , respectively.

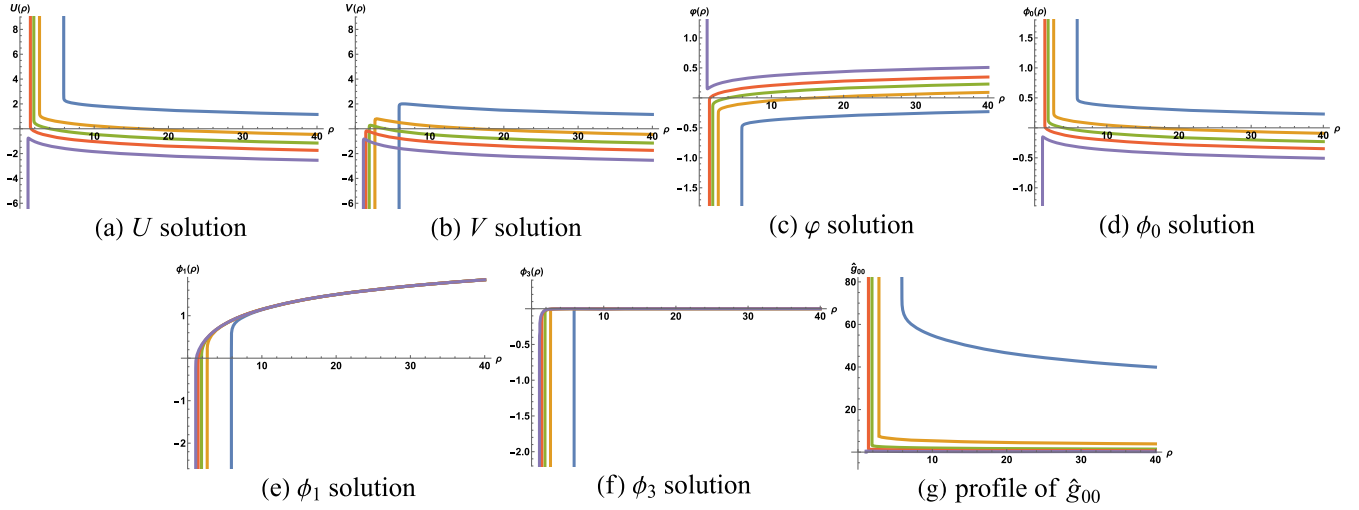


FIG. 61. Interpolating solutions between the locally $SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(2)$ twist in $SO(3, 1) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, and purple curves refer to $g = -0.1, -0.5, -1, -1.8$, and -4 , respectively.

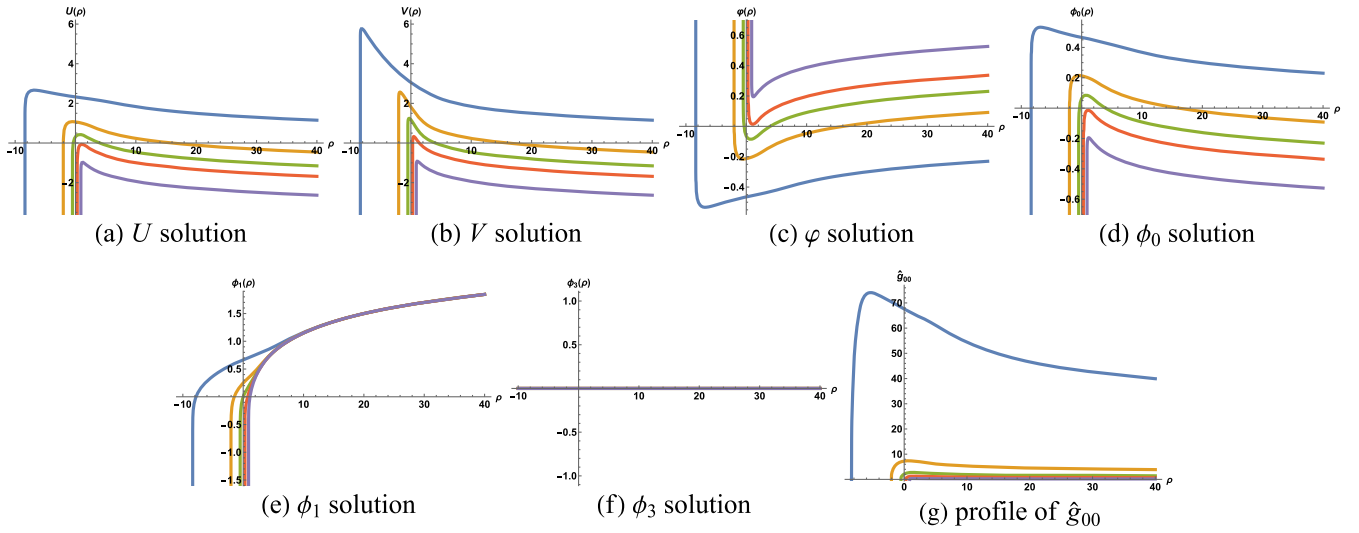


FIG. 62. Interpolating solutions between the locally $SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(2)$ twist in $SO(2, 2) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, and purple curves refer to $g = -0.1, -0.5, -1, -1.7$, and -4.4 , respectively.

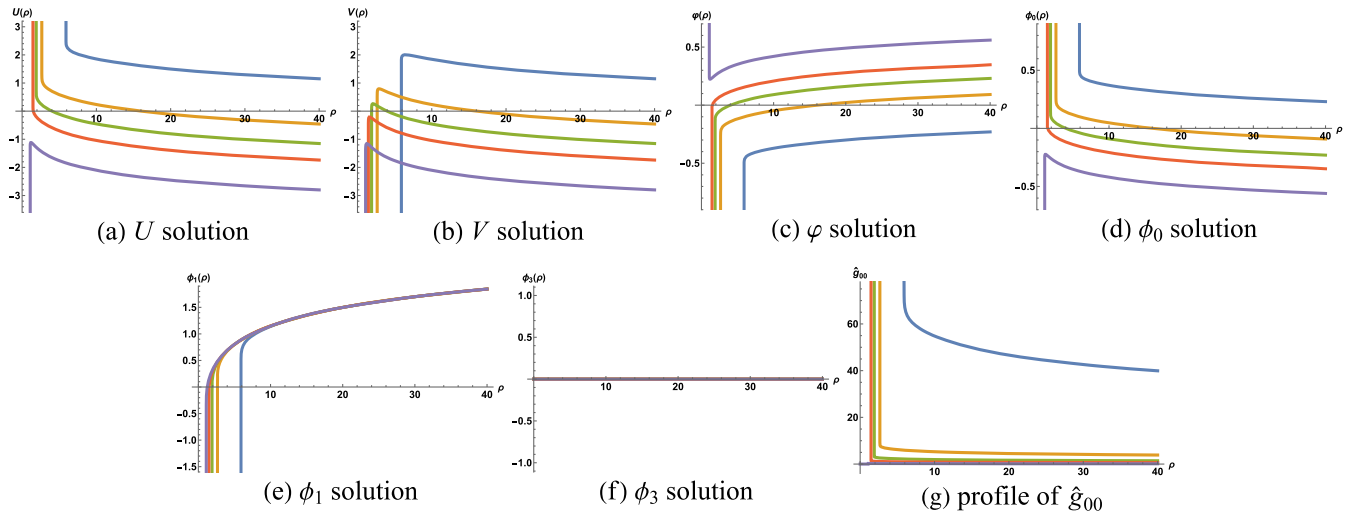


FIG. 63. Interpolating solutions between the locally $SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(2)$ twist in $SO(2, 2) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, and purple curves refer to $g = -0.1, -0.5, -1, -1.8$, and -5.2 , respectively.

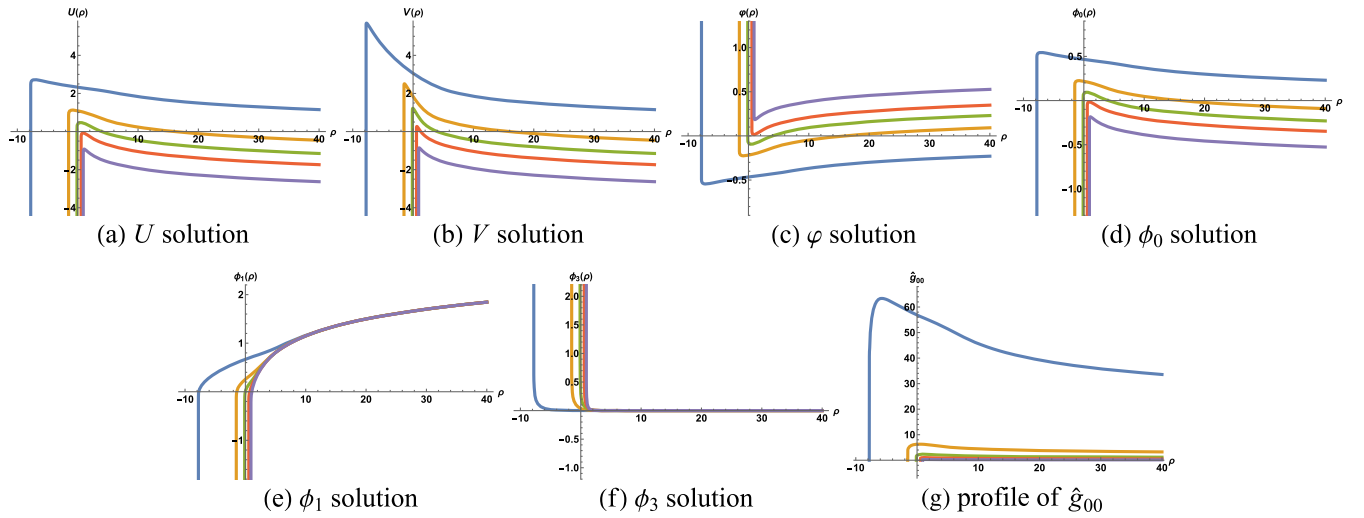


FIG. 64. Interpolating solutions between the locally $SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CP^2$ -sliced curved domain walls for $SO(2)$ twist in $CSO(2, 1, 1) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, and purple curves refer to $g = -0.1, -0.5, -1, -1.8$, and -4.4 , respectively.

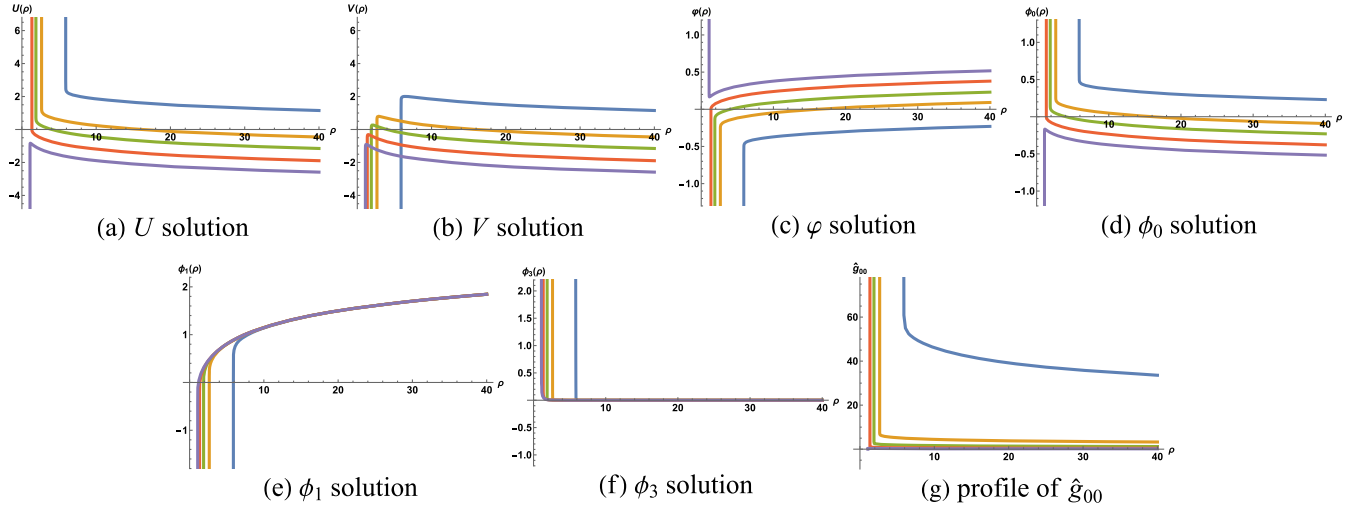


FIG. 65. Interpolating solutions between the locally $SO(2)$ flat domain wall as $\rho \rightarrow +\infty$ and $t \times CH^2$ -sliced curved domain walls for $SO(2)$ twist in $CSO(2, 1, 1) \ltimes \mathbb{R}^4$ gauge group. The blue, orange, green, red, and purple curves refer to $g = -0.1, -0.5, -1, -2.1$, and -4.2 , respectively.

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