

Heavy quark mass effects in charged-current deep-inelastic scattering at approximate NNLO in the Aivazis-Collins-Olness-Tung scheme

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The approximate SACOT- χ scheme for heavy quark production in deep-inelastic scattering was initially formulated for the neutral current structure functions F_2 and F_L . We extend this approach to the charged current case (also including F_3), and thereby complete the definitions for the most relevant inclusive structure functions. Furthermore, we implement these structure functions in the open-source code APFEL++ which provides fast numerical evaluations over a wide kinematic range; this addition to the APFEL++ code is publicly available, with details provided in the Appendix. This SACOT- χ implementation enables detailed numerical insights on the mass dependence of the structure functions and cross sections in the (x, Q^2) -plane for both neutral and charged current processes. We consider kinematic regions relevant for the experimental measurements from fixed-target ν DIS experiments (NuTeV, CCFR, and Chorus) and HERA, and also projections for the upcoming EIC. In particular, the ν DIS experiments reveal a surprisingly strong dependence on the mass effects, offering valuable insights that may help resolve long-standing challenges in accurately describing these datasets.

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I. INTRODUCTION

As we near the fifteen-year mark in the operation of the Large Hadron Collider (LHC), a wide range of measurements at the LHC have achieved very high precision. Any prediction of an observable measured at the LHC requires detailed information on the structure of the proton. Therefore, any precise theoretical prediction is crucially dependent on an accurate extraction of the proton structure from experimental data. Nowadays, the parton distribution functions (PDFs), which parametrize the structure of the proton in a simplified collinear picture, are determined using predominantly theoretical calculations at next-to-next-to-leading order (NNLO), when available. The goal is to determine the structure of the proton with an accuracy of 1% or better.

The cornerstone of any determination of PDFs are the double-differential cross-section data for inclusive deep-inelastic scattering (DIS). The most precise data on proton DIS come from the combination of measurements from H1 and ZEUS experiments at the HERA collider [1–4]. To best utilize the precision of the data, one uses the most accurate theoretical predictions for the corresponding F_2 , F_3 and F_L DIS structure functions to interpret the data in terms of PDFs.

Current state-of-the-art descriptions of the DIS structure functions are given using general-mass variable-flavor-number schemes (GM-VFNS) [5–19] at NNLO. Variable-flavor-number (VFN) schemes dynamically modify the number of active quarks depending on the kinematic region to ensure optimal precision across different energy scales. While the zero-mass variable-flavor-number scheme (ZM-VFNS) [20] neglects the quark-mass effects in the matrix elements, the GM-VFNS approach extends and improves on the ZM-VFNS by incorporating relevant quark-mass effects both in kinematics and in the underlying matrix elements. However, at NNLO, fully accounting for quark-mass effects becomes highly complex, leading to results that are cumbersome to implement efficiently. Several procedures have

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been designed that leverage the simpler ZM matrix elements, but apply the quark-mass effects in the kinematics [21–23]. These sometimes called “intermediate mass”-schemes yield results closer to the fully massive calculation [24] than the ZM matrix elements, but retain their numerical simplicity.

The present analysis aims for an efficient implementation of the DIS structure functions up to NNLO. For that we first utilize (and extend) the “intermediate mass”-scheme that was developed in Ref. [23] in the context of the ACOT-scheme [5–7] for the NNLO contribution. Secondly, we effectively factorize out the PDF dependence of the calculation and store the PDF-independent result in interpolation tables as implemented in the open-source code APFEL++ [25–27]. In this framework, the precomputed tables, which are computationally the most expensive ingredient, are easily combined with the PDFs to yield the final physical prediction; using the tables, rather than a full recalculation, can speed up the calculation by orders of magnitude.

The simplified NNLO calculation of the structure functions in the ACOT-scheme, here called *aSACOT- χ* (for approximate, Simplified ACOT- χ), was developed for the neutral current structure functions F_2 and F_L . In this paper we extend the approach to cover the neutral current F_3 , and also the complete set of charged current structure functions. Thereby, the *aSACOT- χ* scheme is presented for the most relevant inclusive DIS structure functions.¹ The calculation requires the *SACOT- χ* scheme up to NLO. Thus, the additions to APFEL++, accompanying this paper, include *SACOT- χ* (NLO) and *aSACOT- χ* (NNLO) for all nine structure functions; $F_{2,L,3}$ for neutral-current, and charged-current neutrino and antineutrino DIS.

Implementing all nine structure functions not only enables us to assess the impact of the *SACOT- χ* scheme, but also allows to apply the scheme to cross-section predictions. With an emphasis on charged current predictions, we investigate the impact on current experimental measurements including HERA II [3] and neutrino DIS [29–32], as well as projected measurements at the upcoming Electron Ion Collider (EIC) [33,34]. As the kinematic coverage of the neutrino DIS data sets is in the region where the heavy-quark masses impact the predictions most, we place special emphasis on analyzing these predictions.

The outline of our article is as follows. In Sec. II, we present the main elements of the *aSACOT- χ* scheme, and the treatment of the heavy-quark masses. In Sec. III, we outline the efficient numerical implementation, and demonstrate the impact for a variety of experimental measurements including fixed-target ν DIS (CCFR, NuTeV, CHORUS), HERA,

and the upcoming EIC. In Sec. IV, we recap the key results of this investigation.

We provide a set of appendices that further detail the implementation and application of the *aSACOT- χ* scheme. In Appendix A, we present the details of the numerical implementation; this is an essential step to efficiently include NNLO corrections. In Appendix B, we display the Feynman diagrams, including the new channels that are present at NNLO. In Appendix C, we explore the details of the n -scaling and use this to estimate the theoretical uncertainty. In Appendix D, we provide the key components required to use the *aSACOT- χ* implementation within the APFEL++ framework; the code can be downloaded from the GitHub repository [26].

II. THEORETICAL METHODS

This section introduces the main concepts of *aSACOT- χ* for neutral current interactions, and we also extend this to charged current interactions. For a detailed introduction to the scheme, we refer the reader to the original formulation in Ref. [23].

The *aSACOT- χ* scheme can be summarized as

$$aSACOT-\chi \equiv SACOT-\chi[\mathcal{O}(\alpha_s^{0+1})] + \text{ZM-VFNS}[\mathcal{O}(\alpha_s^2)]|_{\chi(n)}. \quad (1)$$

That is, we apply the exact scheme at LO and NLO in α_s , and at NNLO we use the ZM-VFNS Wilson coefficients, but restrict the (lower) integration bound in the convolution to the generalized $\chi(n)$ -scaling variable (see below). As a reminder, in the *SACOT- χ* scheme, the dynamic mass of an incoming heavy quark—or of a heavy quark appearing in an internal line with an on-shell cut—is set to zero. This procedure is not an approximation but due to an internal freedom in the formulation of the ACOT scheme, as shown in Ref. [35]. The only approximation made here is to set the heavy quark mass to zero in the NNLO terms (e.g., for incoming gluons) except for the integration boundaries (implemented via the χ -variable).

In Ref. [23] it has been shown that this treatment captures the dominant mass effects.² In technical terms, we separate the phase-space mass from the dynamic mass. The phase-space mass is the kinematic mass that constrains the effective phase space, and we retain these contributions. The dynamic mass is the mass appearing in the hard-scattering cross section $\hat{\sigma}(m)$; this is neglected for those higher-order terms where the massless Wilson coefficients are implemented.

¹Note that there are also the structure functions F_4 and F_5 . Their contribution is suppressed by the mass of the lepton and therefore less studied. A calculation of F_4 and F_5 in the ACOT scheme at NLO is currently underway [28].

²To be more precise, it has been shown that, up to $\mathcal{O}(\alpha_s)$, the phase-space mass is the dominant contribution (cf. Figs. 6 and 7 of Ref. [23]). We are working under the assumption that this observation holds for higher orders as well.

Thus, the convolution between a generic Wilson coefficient C^λ and a PDF $f(x, Q^2)$ is given by

$$[C^\lambda \otimes f](x, Q^2) = \int_{\chi(n)}^1 \frac{dz}{z} C^\lambda(z, Q^2) f\left(\frac{x}{z}, Q^2\right). \quad (2)$$

Following the original formulation, we choose the generalized $\chi(n)$ -scaling variable to be

$$\chi(n) = x \left(1 + \frac{(nm_H)^2}{Q^2} \right), \quad (3)$$

where x is the Bjorken- x (the lower integration limit in the massless treatment) and m_H is the mass of the heavy quark. For the following discussion, $n = \{0, 1, 2, 3\}$ is a scaling factor, which can be interpreted as having $n = 0, 1, 2, 3$ heavy quarks produced, and can be used to measure the impact of heavy quark contributions on the physical predictions. The special case $n = 0$ replicates the massless case. In our numerical implementation n can be chosen freely. See Appendix C for details on the n -scaling.

The key ingredient of the $aSACOT\text{-}\chi$ scheme is the decomposition of the structure functions into individual flavor contributions both for the initial and final states. This allows one to apply the appropriate $\chi(n)$ -rescaling factor with m_H set to the mass of the participating heavy quark. This has to be done case by case for the individual structure functions, which we describe in the following sections starting with the neutral current example.

Note that for this paper we assume the charm, bottom and top quarks are massive and the remaining quarks are massless. The numerical implementation however is general, such that the massive/massless quarks can be chosen freely.

A. Definition of $aSACOT\text{-}\chi$ in neutral current DIS

This section reviews the core definitions of the neutral current $aSACOT\text{-}\chi$ in a manner updated to coincide with the present numerical implementation, and to prepare the relevant equations for extension to charged current interactions. The last subsection adapts these concepts to define F_3 in the $aSACOT\text{-}\chi$ scheme.

For neutral current interactions at NNLO, a structure function F_λ with $\lambda \in \{2, L\}$ is constructed as

$$F_\lambda = \sum_{k=0}^6 \sum_{l=1}^6 F_{\lambda,k}^l, \quad (4)$$

where the index k represents initial-state partons with $k = 0$ denoting the gluon and $k = 1, 2, 3, \dots$ denoting $d, u, s, \dots$ quarks and antiquarks. The index l runs over the final-state (anti)quarks and does not include the gluon as they do not impose phase space constraints.

In the following, we discuss the partonic decomposition and the resulting $\chi(n)$ -prescription starting from the massless expressions. Generally a structure function in the ZM-VFNS can be calculated from

$$x^{-1} F_\lambda = C_{ns}^\lambda \otimes q_{ns} + \langle a^2 \rangle (C_s^\lambda \otimes q_s + C_g^\lambda \otimes g), \quad (5)$$

where the *nonsinglet* (ns) coefficient C_{ns}^λ couples to the ns -combination

$$q_{ns} = \sum_{i=1}^{n_f} (a_i^2 - \langle a^2 \rangle) q_i^+ = \sum_{i=1}^{n_f} (a_i^2 - \langle a^2 \rangle) (q_i + \bar{q}_i), \quad (6)$$

the *singlet* (s) coefficient C_s^λ to the s -combination

$$q_s = \sum_{i=1}^{n_f} q_i^+ \quad (7)$$

and the gluon coefficient C_g^λ to the gluon distribution. a_i are the full electroweak couplings³ and the n_f -average over the squares $\langle a^2 \rangle$ is defined by

$$\langle a^2 \rangle \equiv \langle a^2 \rangle^{(n_f)} = \frac{1}{n_f} \sum_{i=1}^{n_f} a_i^2. \quad (8)$$

Isolating the contribution to F_λ from an incoming quark of flavor k we find

$$x^{-1} F_{\lambda,k} = [a_k^2 C_{ns}^\lambda + \langle a^2 \rangle C_{ps}^\lambda] \otimes q_k^+ \quad (9)$$

with the *purely singlet* (ps) coefficient C_{ps}^λ given by

$$C_{ps}^\lambda = C_s^\lambda - C_{ns}^\lambda. \quad (10)$$

The gluon-initiated contribution is instead given by

$$x^{-1} F_{\lambda,0} = \langle a^2 \rangle C_g^\lambda \otimes g. \quad (11)$$

As a next step, we disentangle the final-state flavors. As all contributions from the individual final-state flavors are the same in the massless scheme, explicit factors of n_f appear inside the Wilson coefficients, where the contributions have been summed up. At NNLO, we find the relations

$$C_{ns}^\lambda = \tilde{C}_{ns,A}^\lambda + n_f \tilde{C}_{ns,B}^\lambda \quad (12a)$$

$$C_{ps}^\lambda = n_f \tilde{C}_{ps}^\lambda \quad (12b)$$

$$C_g^\lambda = n_f \tilde{C}_g^\lambda, \quad (12c)$$

³The full electroweak couplings include the γ -couplings, the γ/Z interference and the Z -couplings, see, e.g., Eqs. (B9)–(B11) of Ref. [23].

where the ns -coefficient splits into a contribution not proportional to n_f (index A) and a contribution proportional to n_f (index B). The ps - and the gluon coefficient are entirely proportional to n_f .

Finally, we can perform the final-state flavor decomposition. Starting with the initial parton being a quark of flavor k , we find

$$x^{-1}F_{\lambda,k}^l = \{a_k^2[\tilde{C}_{ns,A}^\lambda\delta_{kl} + \tilde{C}_{ns,B}^\lambda] + a_l^2\tilde{C}_{ps}^\lambda\} \otimes_{kl} q_k^+, \quad (13)$$

where the Kronecker δ is necessary to avoid double counting. For the case of the initial parton being a gluon, we find

$$x^{-1}F_{\lambda,0}^j = a_j^2\tilde{C}_g^\lambda \otimes_j g. \quad (14)$$

Note that we have indexed the Mellin convolutions, $\otimes_{kl}$ and $\otimes_j$, with the participating flavor indices. This notation makes it explicit that the convolution (*cf.* Eq. (2)) has to be performed with the lower integration bound set to

$$\chi_{kl}(n) = x\left(1 + \frac{[n \max(m_k, m_l)]^2}{Q^2}\right) \quad (15)$$

in the quark-initiated case and to

$$\chi_j(n) = x\left(1 + \frac{[nm_j]^2}{Q^2}\right) \quad (16)$$

in the gluon-initiated case. Summing over all initial- and final-state partons as prescribed in Eq. (4) yields the total structure function in the $aSACOT$ - χ scheme.

1. Extension of $aSACOT$ - χ to F_3

The original formulation of the $aSACOT$ - χ scheme was restricted to F_2 and F_L only. However, the parton decomposition as described in the steps above can be repeated for F_3 . Only a few differences have to be taken into account:

- (i) instead of the total distributions, the convolution has to be performed with the difference distributions: $q_k^+ \rightarrow q_k^- = (q_k - \bar{q}_k)$
- (ii) the full electroweak charges have to be replaced by the parity-violating couplings: $a_k^2 \rightarrow \tilde{a}_k^2$
- (iii) the ns -coefficients C_{ns}^λ have to be replaced with the appropriate coefficients for F_3 : $C_{ns}^\lambda \rightarrow C_{ns}^3$

Applying these changes together with the fact that both gluon and ps -coefficients vanish in the massless scheme, one finds

$$F_{3,k}^l = \tilde{a}_k^2[\tilde{C}_{ns,A}^3\delta_{kl} + \tilde{C}_{ns,B}^3] \otimes_{kl} q_k^- \quad (17a)$$

$$F_{3,0}^l = 0. \quad (17b)$$

B. Extension of $aSACOT$ - χ to charged current

In this section, we extend the $aSACOT$ - χ scheme to charged current interactions. The general principle of dissecting the massless Wilson coefficients into the individual flavor contributions remains the same. However, as we have a flavor change at the electroweak vertex, the flavor structure becomes more involved and we have to keep track of an additional flavor j

$$F_\lambda = \sum_{k=0}^6 \sum_{l=1}^6 \sum_{j=1}^6 F_{\lambda,k}^{j,l}. \quad (18)$$

As an example, consider one of the diagrams of Fig. 7(b) for a $\nu_\mu \rightarrow \mu^-$ interaction (i.e., with a W^+ boson): The incoming flavor k has to be of down-type ($k \in \{d, s, b\}$) and the outgoing flavor j of up-type ($j \in \{u, c, t\}$). However, the radiated flavor l can be of any type. Therefore, we have to consider cases where, e.g., $k = b$, $j = t$ and $l = c$ and keep track of three masses simultaneously. The $\chi(n)$ -scaling variable is generalized to

$$\chi_{klj}(n) = x\left(1 + \frac{[n \max(m_k, m_l, m_j)]^2}{Q^2}\right) \quad (19)$$

for convolutions with three flavor indices denoted by $\otimes_{klj}$.

As in the neutral current case, we start the flavor decomposition from the formulation in the massless scheme: the structure functions for an incoming neutrino $F_\lambda(\nu)$ and an incoming antineutrino $F_\lambda(\bar{\nu})$ are conveniently computed using the linear combinations

$$\begin{aligned} F_\lambda^+ &= \frac{1}{2}[F_\lambda(\nu) + F_\lambda(\bar{\nu})] \\ &= \sum_U \sum_D^{u,c,t \ d,s,b} |V_{UD}|^2 (C_{+,ns}^\lambda + C_{ps}^\lambda) \otimes (D^+ + U^+) \\ &\quad + \left[\sum_U \sum_D^{u,c,t \ d,s,b} |V_{UD}|^2 \right] 4C_g^\lambda \otimes g \end{aligned} \quad (20)$$

$$\begin{aligned} F_\lambda^- &= \frac{1}{2}[F_\lambda(\nu) - F_\lambda(\bar{\nu})] \\ &= \sum_U \sum_D^{u,c,t \ d,s,b} |V_{UD}|^2 C_{-,ns}^\lambda \otimes (D^- - U^-), \end{aligned} \quad (21)$$

where the sums run over up-/down-type quarks. Note that the ps - and gluon coefficients do not contribute to the “−”-combination. To disentangle the final-state flavors, we notice that the massless Wilson coefficients have the same n_f -dependence as in the neutral current case given in Eq. (12). In the following, we investigate the flavor structure of the ns -, ps - and gluon coefficients separately.

ns -coefficients: The relevant Feynman diagrams are given in Fig. 7. Following the flavor notation of these

diagrams, we note that k and j are connected: if k is of down-type, j has to be up-type. The third flavor l is independent analogously to the neutral current relations. Thus, we arrive at the decomposition

$$F_{\lambda,k}^{+,ns,j,l} = \begin{cases} |V_{kj}|^2 [\tilde{C}_{+,ns,A}^{\lambda} \delta_{kl} + \tilde{C}_{+,ns,B}^{\lambda}] \otimes_{kjl} q_k^+ & \text{if } k \in U, j \in D \\ |V_{jk}|^2 [\tilde{C}_{+,ns,A}^{\lambda} \delta_{kl} + \tilde{C}_{+,ns,B}^{\lambda}] \otimes_{kjl} q_k^+ & \text{if } k \in D, j \in U \\ 0 & \text{else,} \end{cases} \quad (22a)$$

$$F_{\lambda,k}^{-,ns,j,l} = \begin{cases} -|V_{kj}|^2 [\tilde{C}_{-,ns,A}^{\lambda} \delta_{kl} + \tilde{C}_{-,ns,B}^{\lambda}] \otimes_{kjl} q_k^- & \text{if } k \in U, j \in D \\ |V_{jk}|^2 [\tilde{C}_{-,ns,A}^{\lambda} \delta_{kl} + \tilde{C}_{-,ns,B}^{\lambda}] \otimes_{kjl} q_k^- & \text{if } k \in D, j \in U \\ 0 & \text{else.} \end{cases} \quad (22b)$$

Note the extra minus sign in $F_{\lambda,k}^{-,ns,j,l}$ when k is of up-type.

ps-coefficient: From the graphs in Fig. 8 one can see that in these contributions the flavor indices j and l are connected. The incoming flavor k is independent. Thus we find

$$F_{\lambda,k}^{+,ps,j,l} = \begin{cases} |V_{jl}|^2 \tilde{C}_{ps}^{\lambda} \otimes_{kjl} q_k^+ & \text{if } j \in U, l \in D \\ |V_{lj}|^2 \tilde{C}_{ps}^{\lambda} \otimes_{kjl} q_k^+ & \text{if } j \in D, l \in U \\ 0 & \text{else.} \end{cases} \quad (23)$$

Gluon coefficient: In the gluon-initiated contributions, shown in Fig. 9, j and l are connected yielding

$$F_{\lambda,0}^{+,j,l} = \begin{cases} 4|V_{jl}|^2 \tilde{C}_g^{\lambda} \otimes_{jl} g & \text{if } j \in U, l \in D \\ 4|V_{lj}|^2 \tilde{C}_g^{\lambda} \otimes_{jl} g & \text{if } j \in D, l \in U \\ 0 & \text{else.} \end{cases} \quad (24)$$

Final formula: The final structure functions are constructed by summing over the three classes of coefficients, which yields

$$F_{\lambda}^{+} = \sum_{k=0}^6 \sum_{j=1}^6 \sum_{l=1}^6 [F_{\lambda,k}^{+,ns,j,l} + F_{\lambda,k}^{+,ps,j,l} + F_{\lambda,k}^{+,g,j,l}] \quad (25a)$$

$$F_{\lambda}^{-} = \sum_{k=0}^6 \sum_{j=1}^6 \sum_{l=1}^6 F_{\lambda,k}^{-,ns,j,l}. \quad (25b)$$

1. *aSACOT- χ* for charged current F_3

The same derivation can be made for F_3 with the replacement $\bar{q}_k \rightarrow -\bar{q}_k$. Both the *ps*-coefficient and the gluon-initiated coefficient are zero, such that the massless result reads

$$F_3^{+} = \sum_U^{u,c,t} \sum_D^{d,s,b} |V_{UD}|^2 C_{+,ns}^3 \otimes (D^{-} + U^{-}) \quad (26a)$$

$$F_3^{-} = \sum_U^{u,c,t} \sum_D^{d,s,b} |V_{UD}|^2 C_{-,ns}^3 \otimes (D^{+} - U^{+}) \quad (26b)$$

However, since the flavor structure is analogous to $F_{2,L}$, we can recycle Eqs. (22a), (22b) by making the replacement $q_k^+ \leftrightarrow q_k^-$ and write immediately

$$F_3^{+} = \sum_{k=0}^6 \sum_{j=1}^6 \sum_{l=1}^6 F_{3,k}^{+,ns,j,l} \quad (27a)$$

$$F_3^{-} = \sum_{k=0}^6 \sum_{j=1}^6 \sum_{l=1}^6 F_{3,k}^{-,ns,j,l}. \quad (27b)$$

III. RESULTS

In this section, we present the numerical implementation of the *aSACOT- χ* scheme in APFEL++ [25–27] and show the impact on physical predictions using the scheme. First, we focus on the effect of the χ -scaling variable on the structure functions for neutral and charged currents at NNLO. We then examine charged-current cross sections for kinematics that have been measured at HERA, and also consider pseudodata sets for the EIC. The final subsection highlights the NNLO effects for neutrino DIS data sets from NuTeV, CCFR, and Chorus, which are measured at lower Q^2 where the impact of heavy quarks can be significant.

A. Numerical implementation

The numerical implementation was performed in the open-source code APFEL++ [25–27]. The code base allows for a numerically efficient evaluation of the structure functions by means of precomputed interpolation tables. The computational advantage lies in interpolating the PDFs with

TABLE I. The available VFN-schemes for the individual structure functions in APFEL++ [25–27] at NLO and NNLO. The public release of the code included the ZM and FONLL-C structure functions as indicated by the check marks. The remaining check marks for the full ACOT, SACOT- χ and aSACOT- χ schemes are newly added.

VFNS	Neutral current			Charged current		
	F_2	F_L	F_3	F_2	F_L	F_3
<i>NLO</i>						
full ACOT	✓					
SACOT- χ	✓	✓	✓	✓	✓	✓
<i>NNLO</i>						
ZM	✓	✓	✓	✓	✓	✓
FONLL-C	✓	✓				
aSACOT- χ	✓	✓	✓	✓	✓	✓

$$f(x) = \sum_{\alpha=0}^{N_x} w_{\alpha}(x) f(x_{\alpha}), \quad (28)$$

on a fixed x -grid $\{x_{\alpha}\}$ and storing the time-expensive convolution of the Wilson coefficients with the interpolating functions $w_{\alpha}(x)$. At evaluation time, only the interpolation step has to be performed, which reduces to a simple matrix-vector multiplication—an extremely fast operation on modern CPUs. This setup allows one to recompute the structure functions with different PDF sets very quickly. The details of the implementation have been deferred from the main text and can be found in Appendix A. An overview of current and newly implemented schemes is given in Table I.

The APFEL++ framework currently provides structure functions in the *ZM-VFNS* for both neutral- and charged-current interactions up to NNLO. Additionally, the massive neutral-current F_2 and F_L structure functions in the NNLO scheme *FONLL-C* [13] are also available in the code.

We implemented the full ACOT scheme for neutral-current F_2 and the SACOT- χ scheme for the complete set $\{F_2, F_L, F_3\}$ for neutral and charged current at NLO. The fully massive coefficients have been taken from Refs. [5–7]. The implementation has been compared to the *nCTEQ* code-base (which was in turn benchmarked in, e.g., Ref. [36]) and agrees to better than 1‰ with an evaluation speed that is $\mathcal{O}(100)$ faster. The remaining difference can be attributed to the interpolation effects, and can thus be made arbitrarily small.

At NNLO we implemented the $\{F_2, F_L, F_3\}$ structure functions for both neutral- and charged-current interactions in the *aSACOT- χ* scheme. It is to be noted that for the numerical implementation we do not use the exact formulas for the NNLO massless coefficients presented in Refs. [37–39], but an x -space parametrization, which was provided in Refs. [40,41]. These are accurate enough for

numerical applications, but significantly faster to evaluate. The neutral current structure functions have been found to agree to better than 1‰ with an evaluation speed that is of $\mathcal{O}(500)$ faster compared to the benchmark code used in the original formulation in Ref. [23].

B. Mass effects on structure functions at NNLO

The strength of the impact of the dynamic mass at NNLO is controlled by the n -scaling variable $\chi(n)$. To compare this effect for the structure functions in an isolated setup, we form a ratio of the *aSACOT- χ* scheme and a combination made up of the standard SACOT- χ at NLO with the massless scheme at NNLO

$$\frac{F_{\lambda}^{\text{LO+NLO}}(\text{SACOT}_{\chi}) + F_{\lambda}^{\text{NNLO}}(a\text{SACOT}_{\chi}(n))}{F_{\lambda}^{\text{LO+NLO}}(\text{SACOT}_{\chi}) + F_{\lambda}^{\text{NNLO}}(\text{ZM})} \quad (29)$$

for all structure functions. Thereby, the ratio is not sensitive to the impact of using NNLO over NLO predictions, but to the mass effects introduced in the *aSACOT- χ* scheme as this is the only difference between numerator and denominator.

Structure functions depend solely on the (negative) virtuality of the vector boson Q^2 and the partonic longitudinal momentum fraction x . Therefore, we consider the ratio as a heat map on a double-logarithmic grid of (400×400) -nodes corresponding to the ranges $Q^2 \in [1.3^2, 250^2]$ GeV² and $x \in [10^{-5}, 1]$. Note that experimental determinations are additionally constrained by the center-of-mass collision energy $\sqrt{s}$ (or the energy of the incoming lepton E_l). This restricts the measurable region to a wedge in the bottom-right corner of the (Q^2, x) -plane, as the kinematics fulfill the relation

$$Q^2 = xy(s - M^2) \quad \text{with} \quad 0 \leq y \leq 1, \quad (30)$$

where M is the mass of the struck hadron. Therefore, any (Q^2, x) -pair that lies above this linear relation cannot be measured. In the following, we display the ratio for the complete set of structure functions $\{F_2, F_3, F_L\}$ for $n = \{1, 2, 3\}$ in a (3×3) -matrix. Neutral current interactions are considered in Fig. 1 and charged current in Fig. 2 for a W^+ and in Fig. 3 for a W^- exchange. The heat map is colored in red if the ratio is above unity, white if the ratio is equal to unity, and blue if the ratio is below unity. Dotted (dash-dotted) lines indicate contours of 2.5% (7.5%) difference. We use the CT18 [42] NNLO PDF set to evaluate the structure functions. We use the heavy quark masses as specified in the PDF determination.

Going from top to bottom in Fig. 1 (neutral current), we can confirm the expectation that the strength of the mass effects increases with increasing n . More precisely, when the ratio is enhanced/suppressed in the first row, then the enhancement/suppression increases in the second and third row. Further, we find that these effects are strongest for low

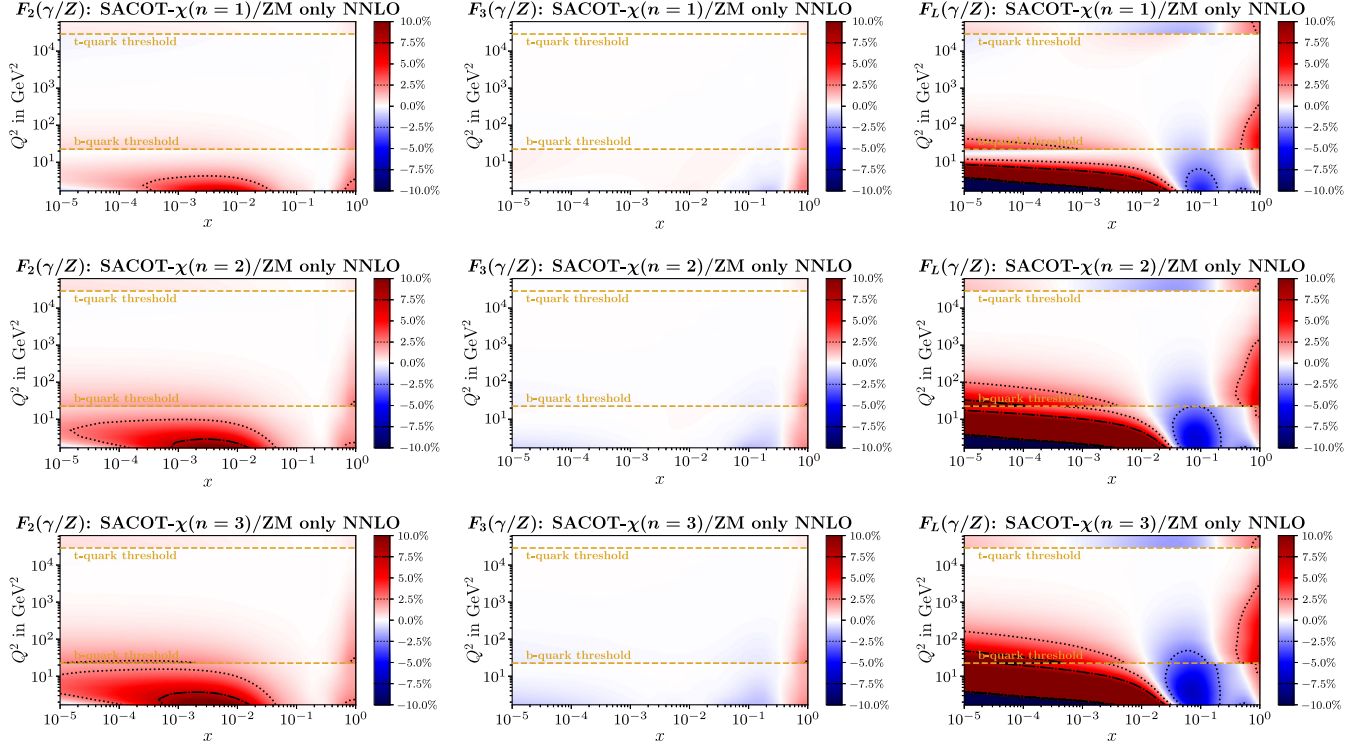


FIG. 1. The ratio of neutral current structure functions F_2 , F_3 , and F_L (from left to right) to the ZM NNLO-coefficient for $n = \{1, 2, 3\}$ (top to bottom). The ratio is defined in Eq. (29). The results are obtained with the CT18 [42] NNLO proton PDFs. We also indicate the mass thresholds for the bottom- and top-quark, as the ZM-coefficients are discontinuous at these values.

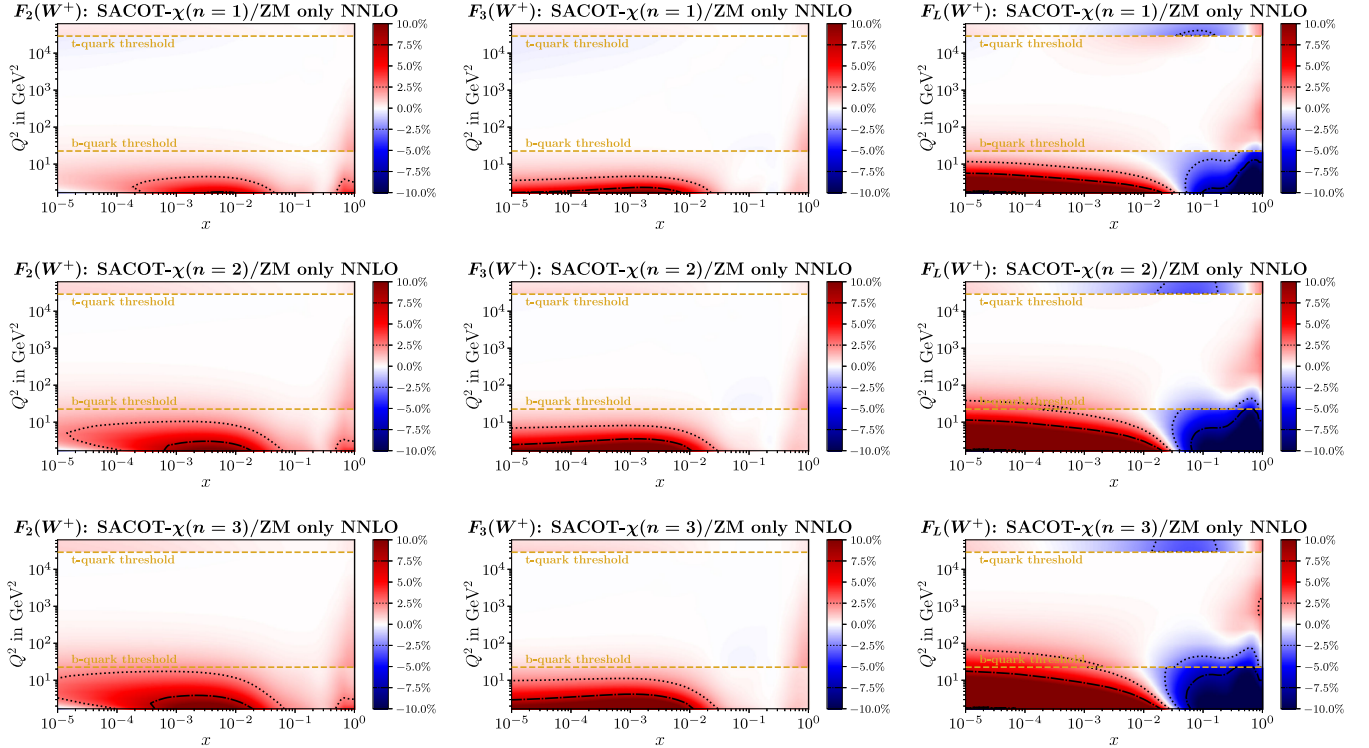


FIG. 2. The same as Fig. 1 but for charged current with a W^+ exchange.

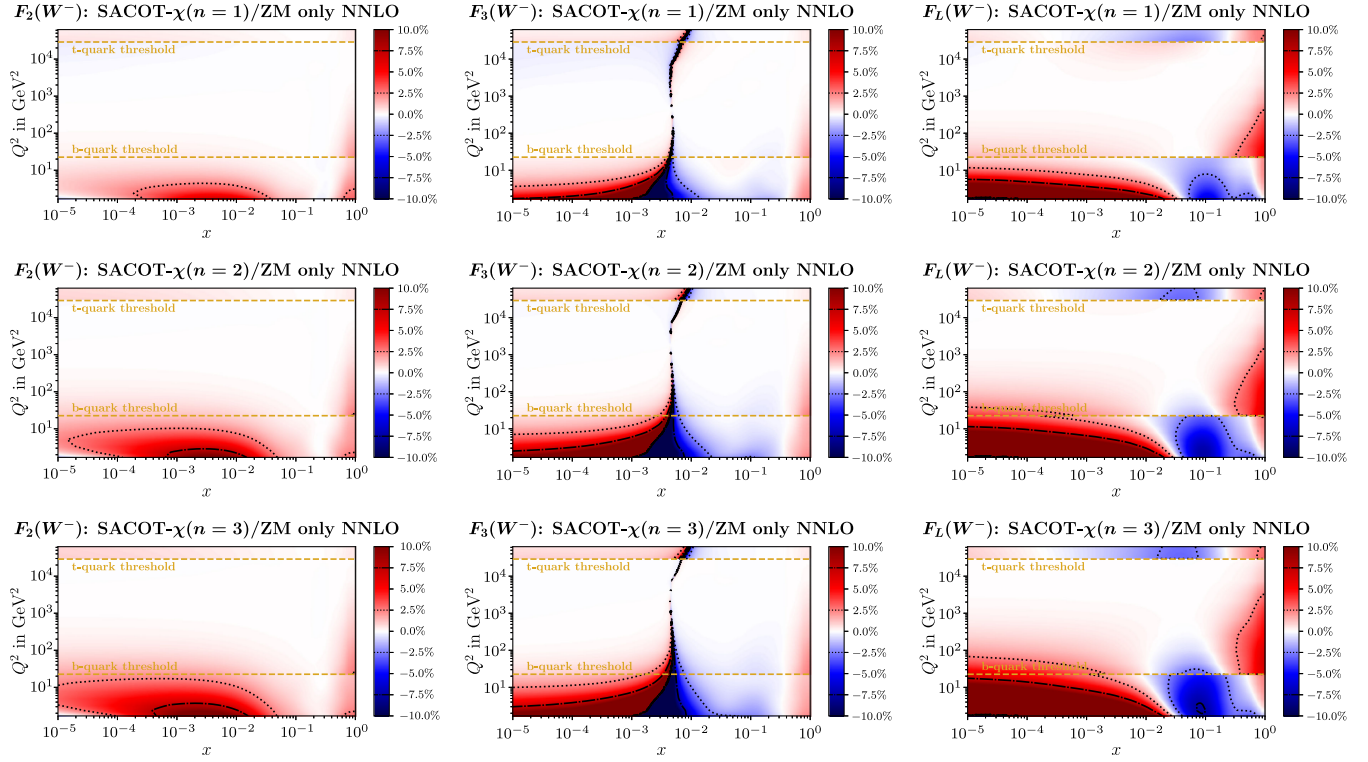


FIG. 3. The same as Fig. 1 but for charged current with a W^- exchange. The ratio for $F_3(W^-)$ exhibits a distinct feature at $x \sim 0.005$. This is a numerical artifact, as the structure function turns negative below this x -value and the ratio is not well-defined in this point (see Appendix C 3).

Q^2 values, as expected since the mass is divided by Q in the scaling variable. Interestingly, we observe almost no difference between the massless NNLO corrections and $aSACOT\text{-}\chi$ for the structure function F_3 . The strongest impact on the ratio is found for F_L , which vanishes in the $SACOT\text{-}\chi$ scheme at LO and is therefore effectively one order lower in the perturbative expansion.⁴ Thus the effects are enhanced relative to F_2 and F_3 .

Moving to the charged current structure functions in Figs. 2, 3, we note that mass effects are of similar size compared to neutral current structure functions, except for F_3 . For $F_3(W^+)$, the extent in the (Q^2, x) -plane and the relative size of mass effects are similar compared to F_2 . In the case of a W^- exchange, we observe a numerical artifact at $x \sim 0.005$. This is a result of the ratio being ill-defined in these kinematics since $F_3(W^-)$ crosses zero at this value of x (see Appendix C 3).

C. Physical cross sections

Having implemented all charged current structure functions, the effect of the n -scaling can be investigated for the full cross section. In the following, we compute theoretical

predictions for measurements taken at HERA and the upcoming EIC.

We first study the mass effects only in the NNLO correction. We define a ratio similar to the structure-function ratio of the previous subsection

$$\frac{\sigma^{\text{LO+NLO}}(SACOT_\chi) + \sigma^{\text{NNLO}}(aSACOT_\chi(n))}{\sigma^{\text{LO+NLO}}(SACOT_\chi) + \sigma^{\text{NNLO}}(ZM)}. \quad (31)$$

In Fig. 4 we show the ratio in the (Q^2, x) -plane as a heat map. In order to make a connection with a specific experiment, we consider the measurements at HERA [3], with $\sqrt{s} = 318$ GeV.

Unphysical regions are indicated by gray patches. The predictions were again obtained from the CT18 NNLO PDF set, which has an initial scale of $Q_{\min} = 1.3$ GeV. The data set consists of measurements of an incoming e^- , shown in the left column with the location of the measurements as blue circles, and an incoming e^+ , shown in the right column as green circles. We find that the data points are taken at such a high Q^2 that the mass effect is negligible. In fact, performing a χ^2 -evaluation of the data sets with the ZM scheme at all orders compared to using $aSACOT\text{-}\chi$ (i.e., with mass effects at every order) results in a difference of 0.2 χ^2 -points for the 82 charged-current data

⁴Note that the LO contribution to F_L does not vanish in the ACOT-scheme or if target mass corrections are included [43].

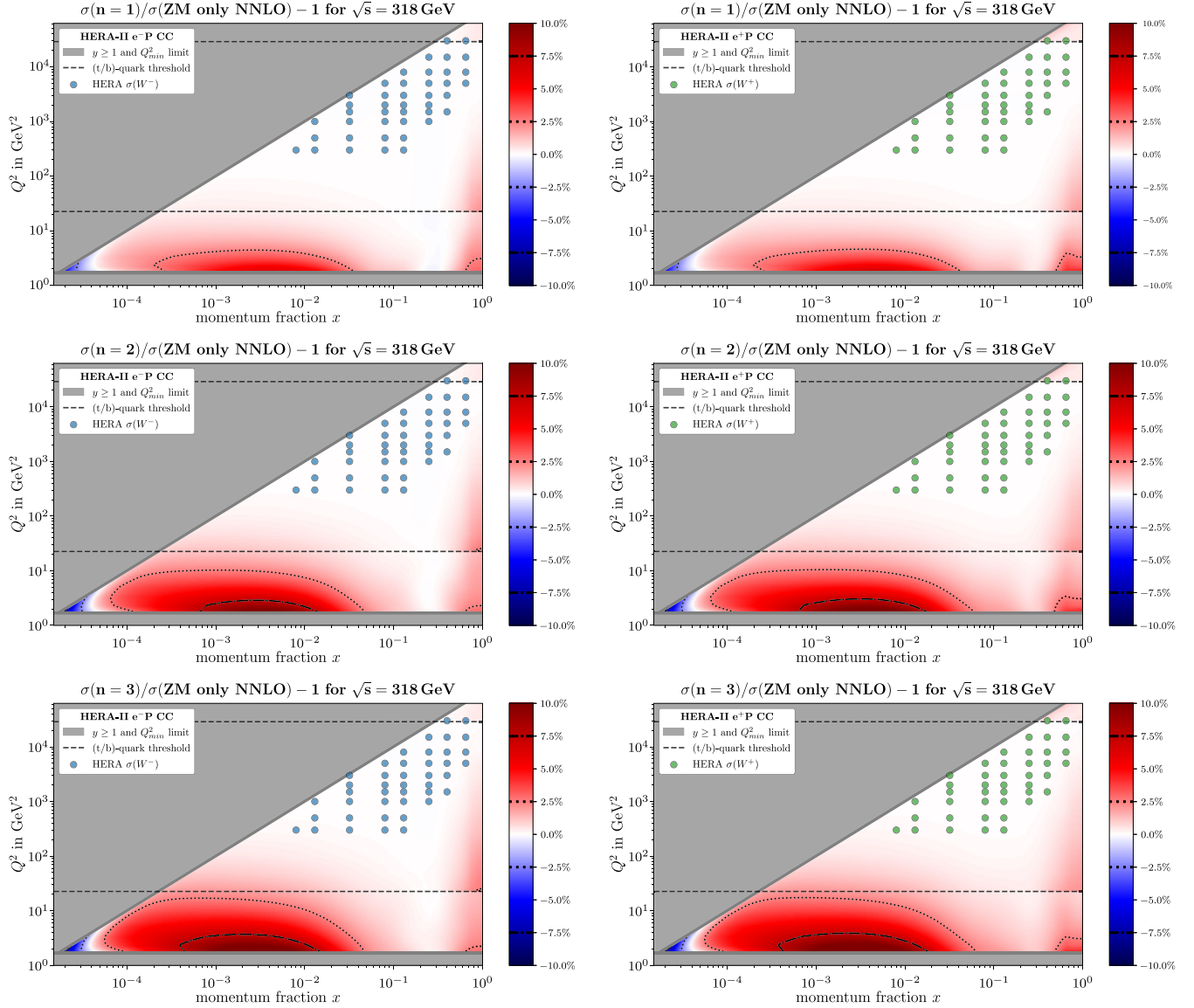


FIG. 4. The reduced CC DIS cross-section measurements from HERA for an incoming e^- (left, blue circles) and e^+ (right, green circles). The CMS energy is set to $s = 318^2 \text{ GeV}^2$. From top to bottom we set the scaling variable $n = \{1, 2, 3\}$.

points. This indicates that mass effects are not resolved by the measurements.

Now we investigate the impact of including mass effects in the whole tower of contributions. For that we use the example of EIC instead. We modify the heat map to display the ratio

$$\frac{\sigma^{\text{LO+NLO}}(\text{SACOT}_\chi) + \sigma^{\text{NNLO}}(a\text{SACOT}_\chi(n))}{\sigma^{\text{LO+NLO+NNLO}}(\text{ZM})}, \quad (32)$$

i.e., we compare the $a\text{SACOT}_\chi$ against the massless predictions at all orders. For the EIC predictions, we use the pseudodata sets presented in Refs. [33,34]. The results for EIC kinematics are shown in Fig. 5. The layout is the

same as in the former HERA plots. Comparing the two we find that mass effects are more pronounced but, in spite of the lower Q^2 compared to the HERA data, the impact is less than 2.5%. In the case of an incoming e^- , data might be sensitive to mass effects in the lowest Q^2 bins and high- x region, but only for an experimental accuracy of $\sim 1\%$.

D. Neutrino DIS cross sections

As discussed in the previous section, mass effects are dominant in the low- Q^2 regime. Neutrino DIS measurements have been performed in this region to a precision that is comparable to the relative size of mass effects. To show this, we consider measurements from the NuTeV [29] and CCFR [30,31] Collaborations taken on an iron target.

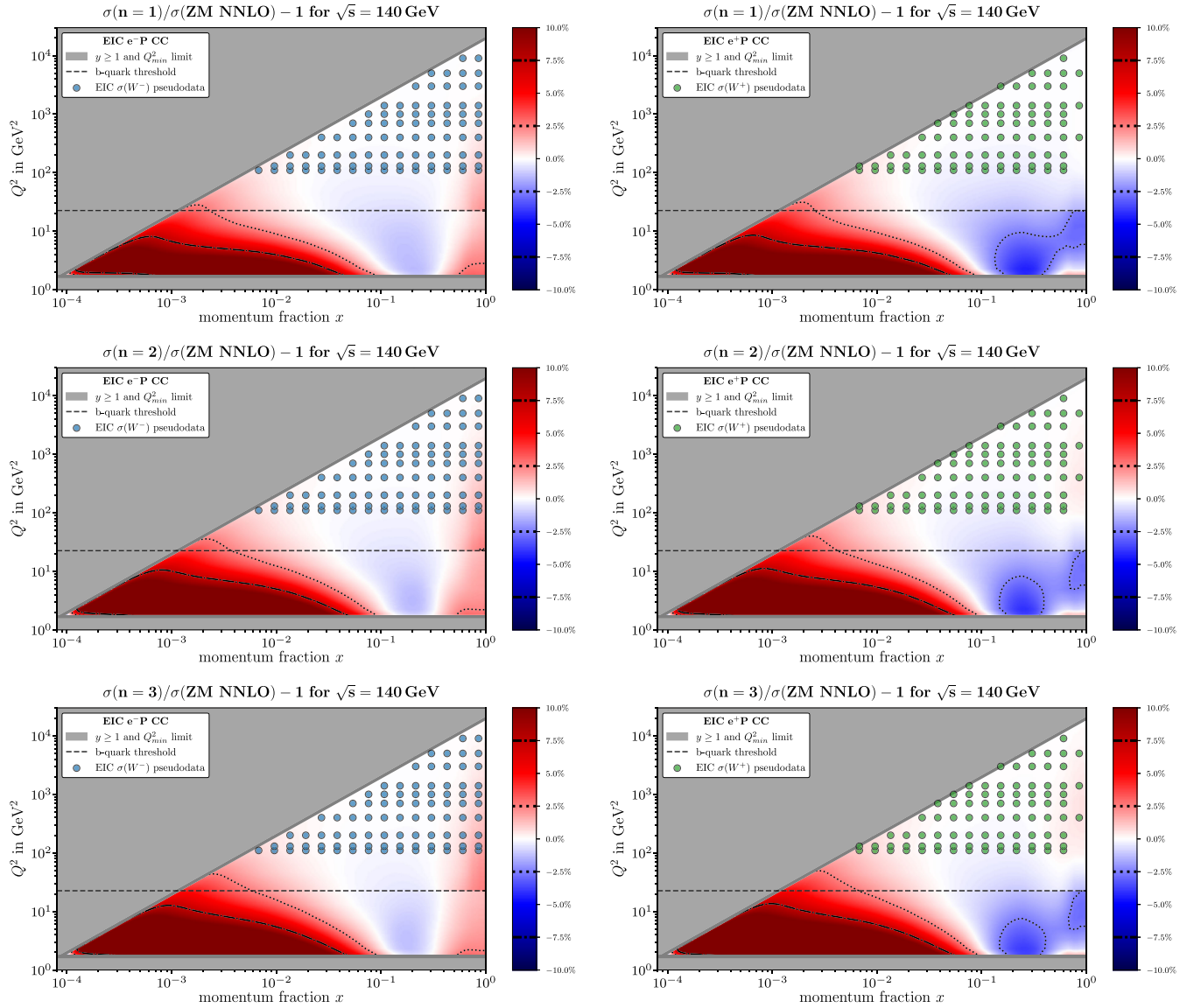


FIG. 5. We display pseudodata for charged current interactions at the EIC [33,34]. The left column assumes an incoming e^- (location of pseudodata as blue circles) and the right column an e^+ (green circles) with the CMS energy set to $s = 140^2 \text{ GeV}^2$. The heat maps show Eq. (32) of $a\text{SACOT-}\chi$ against the ZM scheme (at all orders up to NNLO). From top to bottom we set the scaling variable to $n = \{1, 2, 3\}$.

Additionally, we include measurements from the Chorus Collaboration [32] taken on a lead target. In Table II, we evaluate the χ^2 -function to quantify the quality of the description of different data sets by theory calculations in different schemes. We additionally split the data sets into incoming ν (left) and $\bar{\nu}$ (right). For the χ^2 evaluation we use the nCTEQ15HQ NLO PDFs [44] for the appropriate nuclei and cut data points measured below the initial scale of the PDF set, i.e., $Q_{\min} = 1.3 \text{ GeV}$. As before, we use the heavy quark masses as specified in the PDF determination. The available correlated uncertainties for NuTeV and Chorus have been consistently accounted for. We do not

apply target-mass corrections [43] or higher-twist effects of any kind.

The table lists the number of data points after cuts and the χ^2 -value per number of data points (upper value) alongside with the difference with respect to the SACOT- χ NLO scheme (lower value) for the schemes:

- (1) ZM at LO and NLO
- (2) SACOT- χ NLO
- (3) ZM at LO, NLO and NNLO
- (4) SACOT- χ NLO + ZM at NNLO (called “ZM only NNLO”) equivalent to $a\text{SACOT-}\chi$ with $n = 0$
- (5–7) $a\text{SACOT-}\chi$ with $n \in \{1, 2, 3\}$

TABLE II. χ^2 -goodness of fit criterion per number of data points for the NuTeV [29], CCFR [30,31] and Chorus [32] neutrino and antineutrino datasets for various mass schemes. The upper entry in each row gives the χ^2 per number of points and the lower entry, given in slanted font, the difference to the SACOT- χ NLO scheme per number of data points. The values are obtained with the Iron (NuTeV and CCFR) and Lead (Chorus) nCTEQ15HQ [44] sets. We note that the absolute χ^2 values depend on the PDF set used, but the qualitative trend especially the difference with respect to the SACOT- χ NLO scheme does not. Also note that “SACOT- χ NLO + ZM NNLO” is equivalent to “aSACOT- $\chi(n = 0)$.”

	NuTeV		CCFR		Chorus	
	ν	$\bar{\nu}$	ν	$\bar{\nu}$	ν	$\bar{\nu}$
Data Points	1371	1146	1282	1273	534	534
<i>NLO</i>						
ZM NLO	3.68	1.56	1.59	1.04	1.74	1.33
	<i>0.74</i>	<i>0.20</i>	<i>-0.17</i>	<i>-0.09</i>	<i>-0.16</i>	<i>-0.05</i>
SACOT- χ NLO	2.95	1.36	1.75	1.13	1.90	1.38
	...	...	...	...	...	...
<i>NNLO</i>						
ZM NNLO	3.10	1.51	2.54	1.24	1.89	1.35
	<i>0.15</i>	<i>0.15</i>	<i>0.79</i>	<i>0.12</i>	<i>-0.01</i>	<i>-0.03</i>
SACOT- χ NLO + ZM NNLO	2.81	1.65	2.77	1.38	2.10	1.48
	<i>-0.13</i>	<i>0.29</i>	<i>1.02</i>	<i>0.26</i>	<i>0.20</i>	<i>0.09</i>
aSACOT- $\chi(n = 1)$	2.60	1.50	2.52	1.32	1.92	1.40
	<i>-0.35</i>	<i>0.14</i>	<i>0.77</i>	<i>0.19</i>	<i>0.01</i>	<i>0.02</i>
aSACOT- $\chi(n = 2)$	2.50	1.41	2.37	1.27	1.84	1.37
	<i>-0.44</i>	<i>0.05</i>	<i>0.62</i>	<i>0.14</i>	<i>-0.06</i>	<i>-0.01</i>
aSACOT- $\chi(n = 3)$	2.48	1.38	2.32	1.25	1.84	1.37
	<i>-0.46</i>	<i>0.02</i>	<i>0.56</i>	<i>0.12</i>	<i>-0.06</i>	<i>-0.01</i>

We choose to highlight the difference with respect to the SACOT- χ NLO scheme, since this is the current state-of-the-art choice to predict neutrino DIS cross-section measurements in nuclear PDF extractions, see, e.g., Refs. [45,46].

We also note that PDF extractions depend on the heavy-quark mass scheme used in the fit because the numerical optimization cannot distinguish between PDF information and heavy-quark mass effects. Therefore, the minimizer tends to “overfit” the PDFs to the employed mass scheme (see Ref. [21]) and in order to make conclusive statements about which mass scheme describes the data best a new global fit with each of the mass schemes would be necessary. This is beyond the scope of this paper. The SACOT- χ NLO scheme was used in the nCTEQ15HQ extraction for the DIS predictions.

We repeated the same procedure with different PDF sets and found that, although the absolute χ^2 values differ from one set to another, the qualitative trend, especially the difference with respect to the SACOT- χ NLO, remains the same.

For the NuTeV and CCFR measurements, we observe a significant impact on the χ^2 -value when moving from a NLO scheme to a NNLO scheme. Indeed, χ^2 -values per number of data points may differ by an amount that ranges between 12% and 102%. Furthermore, we notice that the difference between using massless and massive schemes for the NNLO contribution [SACOT- χ NLO + ZM NNLO vs

aSACOT- $\chi(n = 1)$] is important as well, since χ^2 -value shifts by up to ~ 341 units are observed. The impact on the Chorus data set is less pronounced, which can be explained by the fact that the data lies at higher Q^2 values and is less precise. We conclude that the experimental neutrino data is sensitive to the quark-mass effects. In order to fully understand the effect of heavy quarks at NNLO, in the following we investigate the NuTeV data in more detail.

In Fig. 6, the aSACOT- $\chi(n = 1)$ scheme is compared to SACOT- χ NLO, ZM (all orders) and ZM only in the NNLO contribution from top to bottom. These ratios are obtained by setting the incoming lepton energy E_l to the median value of the NuTeV dataset: $E_l^2 = 170 \text{ GeV}^2$. The blue (incoming neutrino) and green (incoming antineutrino) circles display the position of the individual data points, including those that do not belong to the median lepton energy. The gray patches indicate regions that are excluded, either due to the initial scale of the PDF set ($Q_{\min} = 1.3 \text{ GeV}$) or due to kinematic constraints ($y \geq 1$). The dotted and dash-dotted lines indicate the 2.5% and 7.5% contours of the ratio. The ratio is evaluated on the same (400×400) -grid as the structure functions in Sec. III B.

Focusing on the upper row first, the heat map clearly indicates why the difference in χ^2 is so significant: (1) almost the full range of (Q^2, x) -pairs, that capture the scheme differences (i.e., ratio $\neq 1$, colored patches) is

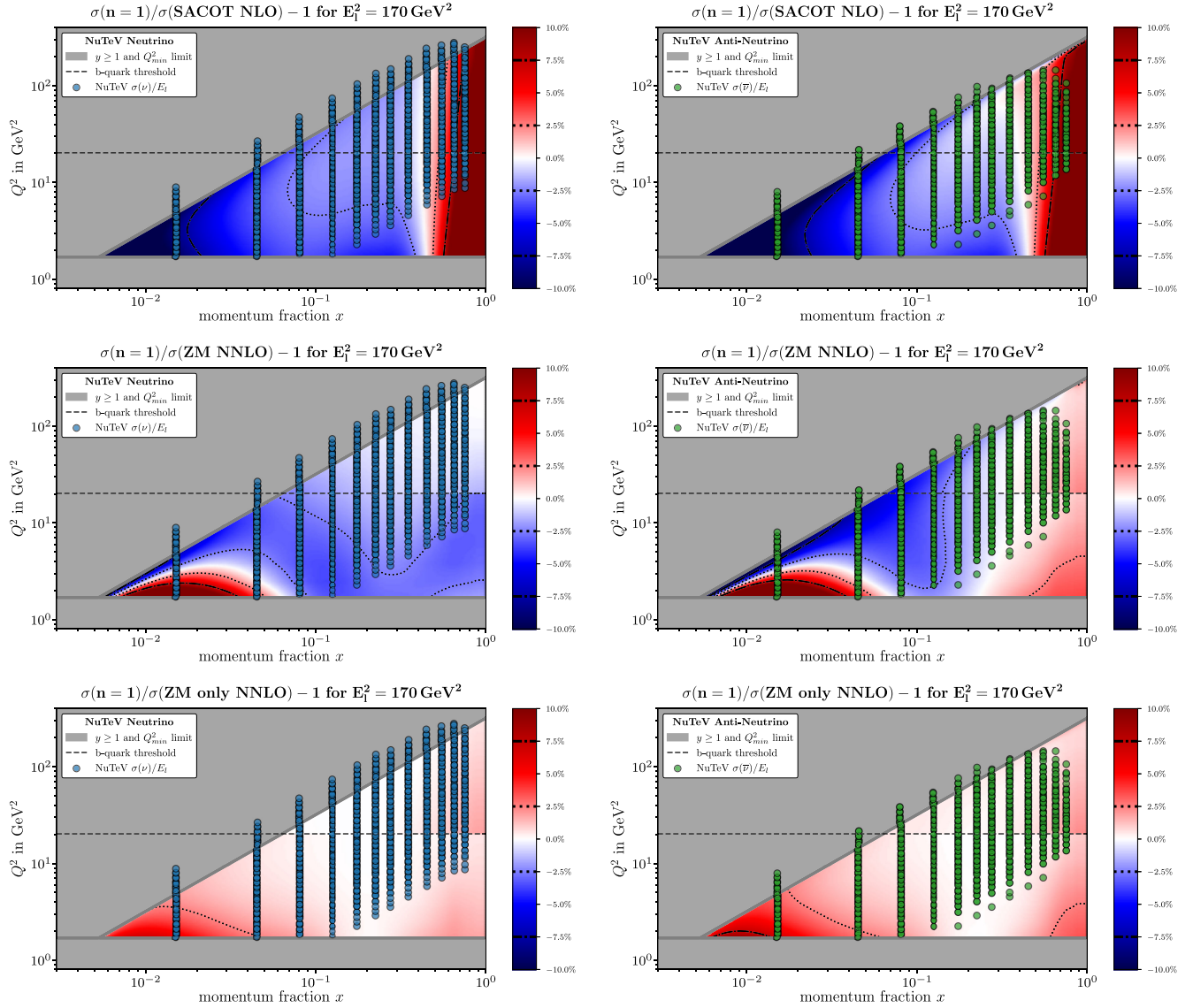


FIG. 6. The ratio of $aSACOT\text{-}\chi(n=1)$ to $SACOT\text{-}\chi$ NLO in the first row, ZM NNLO in the second row and $SACOT\text{-}\chi$ NLO + ZM NNLO in the last row with NuTeV inspired kinematics in the (x, Q^2) -plane as a heat map. The left column displays the ratio for an incoming ν and the right column an incoming $\bar{\nu}$ alongside with the corresponding measurements. We assume a lepton energy of $E_l^2 = 170 \text{ GeV}^2$, which is the median of the dataset. The blue/green circles are the positions of every data available (not only those with $E_l^2 = 170 \text{ GeV}^2$). The gray patches indicate the kinematic limits for the prediction, with $Q_{\min}^2$ originating from the parametrization scale of the nCTEQ15HQ Iron-PDF set used here.

covered by the data points; (2) further, the experimental precision is comparable to the size of the effect.

The second row indicates the total mass effects for all orders and, although the ratio is less than 5% from unity for most of the kinematic region, mass effects are still significant for almost every data point.

Finally, the last row shows the mass effects arising from the NNLO corrections. The absolute size decreases even more compared to the first and second row, but can reach more than 2.5% and affects a large amount of data points.

Overall, we conclude that mass effects are essential for describing the neutrino DIS measurements, and a more

detailed investigation is required. This is in particular important in view of the fact that all current nuclear PDFs [45,47,48] use NLO calculations for predictions for the ν -DIS data, and they have problems with describing these measurements [45,49–51].

IV. CONCLUSIONS

We extended the definition of the $aSACOT\text{-}\chi$ scheme for DIS structure functions to charged-current interactions at NNLO and performed the corresponding calculations. The basic principles of the neutral current definition can be

transferred to the charged current case by accounting for the more involved flavor decomposition. Furthermore, we defined F_3 which was missing in the neutral current definition of the scheme. With this extension, the three most important structure functions are now available for all interactions and the corresponding cross section can be calculated for the first time in this scheme.

The numerical implementation has been performed in the open-source framework of APFEL++, which allows for an efficient evaluation of structure functions by means of interpolation techniques. The implementation will be made publicly available with an update of the code accompanying this paper. Additionally, as APFEL++ is integrated into the xFitter framework [52,53], these results can also be studied with the xFitter package.

The results of the $aSACOT\chi$ calculation have been compared in detail to the available zero-mass calculations and in the context of physical cross-section predictions for HERA, the upcoming EIC and neutrino-DIS measurements by the NuTeV, CCFR and Chorus collaborations. We concluded that the charged-current measurements at HERA and the EIC are at momentum transfers too large for the heavy-quark mass effects to be resolved. However, in the case of neutrino-DIS the process energy is lower and mass effects play a significant role in the description of the data, which can have important phenomenological effects, e.g., on nuclear PDFs. A more detailed investigation is required, since the measurements have been proven to be sensitive to other kinematic effects like target mass corrections [43] or general higher twist effects. A systematic nuclear PDF fit at NNLO precision with the these effects included (and then individually turned off) could also shed new light on the issue.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [26].

APPENDIX A: BASIC PRINCIPLES OF THE NUMERICAL IMPLEMENTATION

This appendix gives a brief introduction to the numerical methods used in the implementation of structure functions in APFEL++. A more detailed description of the technology can be found in Refs. [25,27] or alternatively [54,55].

The computation of structure functions reduces to Mellin convolutions between a Wilson coefficient C^λ and a parton distribution f . The integral to solve is structured as

$$[C^\lambda \otimes f](x, Q^2) = \int_x^1 \frac{dz}{z} C^\lambda(z, Q^2) f\left(\frac{x}{z}, Q^2\right). \quad (\text{A1})$$

Since the Wilson coefficient consists of nontrivial functions and distributions, these integrals are not straightforward to compute and the numerical evaluation is time intensive. In practical applications, parton distributions change constantly (e.g., in a global PDF fit) and thus repeated evaluations of these integrals are required. Conversely, Wilson coefficients do not change. The technology employed in APFEL++ is a “PDF independent” formulation of the integrals, where the integration over the Wilson coefficients is precomputed and the relevant information is stored in look-up tables.

The definition of these look-up tables is first given at fixed Q^2 , and in the following generalized to variable Q^2 .

1. Interpolation for fixed Q^2

In this subsection, we consider Eq. (A1) for fixed values of Q^2 , which we temporarily drop from our notation. In order to remove the parton distributions from the integration, we interpolate $f(x)$ on a predefined x -grid

$$g_x = \{x_0, \dots, x_{N_x}\} \quad (\text{A2})$$

with interpolating functions $w_\alpha(x)$ as given in Eq. (28). The interpolating functions are uniquely defined by the grid g_x through Lagrange interpolation of arbitrary degree. By inserting the interpolated PDF into Eq. (A1), we find

$$\begin{aligned}
[C^\lambda \otimes f](x) &= \sum_{\alpha}^{N_x} \int_x^1 \frac{dz}{z} C^\lambda(z) w_\alpha\left(\frac{x}{z}\right) f(x_\alpha) \\
&= \sum_{\alpha}^{N_x} f(x_\alpha) \int_x^1 \frac{dz}{z} C^\lambda(z) w_\alpha\left(\frac{x}{z}\right), \quad (\text{A3})
\end{aligned}$$

which removes the PDF dependence from the integrand.

Finally, to obtain a prediction for the Mellin convolution at every x -value without the need of recalculating, we interpolate the convolution on the same grid g_x

$$[C^\lambda \otimes f](x) = \sum_{\beta}^{N_x} w_\beta(x) [C^\lambda \otimes f](x_\beta). \quad (\text{A4})$$

Inserting in Eq. (A3) yields the final interpolation table $\Gamma_{\alpha\beta}$

$$[C^\lambda \otimes f](x, Q^2) = \sum_{\beta}^{N_x} \sum_{\alpha}^{N_x} w_\beta(x) f(x_\alpha, Q^2) \Gamma_{\alpha\beta}(Q^2) \quad (\text{A5})$$

with

$$\Gamma_{\alpha\beta}(Q^2) = \int_{x_\beta}^1 \frac{dz}{z} C^\lambda(z, Q^2) w_\alpha\left(\frac{x_\beta}{z}\right), \quad (\text{A6})$$

where, for completeness, we have reintroduced the Q^2 dependence.

2. Interpolation for any Q^2

In order to predict the structure functions at any Q^2 without needing to recalculate $\Gamma_{\alpha\beta}(Q^2)$, we interpolate our results from above on a *separate* Q^2 -grid g_Q

$$g_Q = \{Q_0^2, \dots, Q_{N_Q}^2\}, \quad (\text{A7})$$

with associated interpolation functions given by $\tilde{w}(Q^2)$. Thus to calculate Eq. (2) at any Q^2 we use

$$[C^\lambda \otimes f](x, Q^2) = \sum_{\gamma}^{N_Q} \tilde{w}_\gamma(Q^2) [C^\lambda \otimes f](x, Q_\gamma^2). \quad (\text{A8})$$

And with the shorthand notation

$$\begin{aligned}
\Gamma_{\alpha\beta\gamma} &= \Gamma_{\alpha\beta}(Q_\gamma^2) \\
&= \int_{x_\beta}^1 \frac{dz}{z} C^\lambda(z, Q_\gamma^2) w_\alpha\left(\frac{x_\beta}{z}\right) \quad (\text{A9})
\end{aligned}$$

we find the final formula

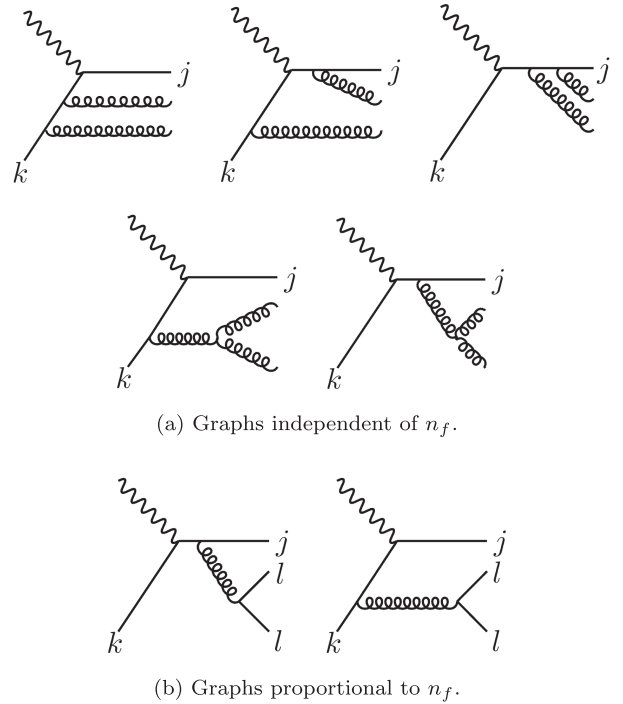


FIG. 7. $\mathcal{O}(\alpha_s^2)$ contributions to C_{ns}^λ . The coefficient can be written as $C_{ns}^\lambda = C_{ns,A}^\lambda + n_f C_{ns,B}^\lambda$, where the graphs contributing to $C_{ns,A}^\lambda$ are given in (a) and the graphs contributing to $C_{ns,B}^\lambda$ are given in (b). (a) Graphs independent of n_f . (b) Graphs proportional to n_f .

$$[C^\lambda \otimes f](x, Q^2) = \sum_{\gamma}^{N_Q} \sum_{\beta}^{N_x} \sum_{\alpha}^{N_x} \tilde{w}_\gamma(Q^2) w_\beta(x) f(x_\alpha, Q_\gamma^2) \Gamma_{\alpha\beta\gamma}. \quad (\text{A10})$$

The entries of $\Gamma_{\alpha\beta\gamma}$ are numbers that can be computed once and for all and stored. Thus, to arrive at a final evaluation of the integral it is sufficient to evaluate the simple vector-matrix operations of Eq. (A10).

APPENDIX B: FEYNMAN DIAGRAMS FOR CC DIS AT NNLO

This section displays the relevant Feynman diagrams for charged-current DIS at NNLO with the flavor labeling corresponding to Sec. II B. Note that, at this order in α_s , graphs with up to three different flavors, namely in Figs. 7(b) and 8, are possible for the first time.

APPENDIX C: APPROXIMATE HIGHER-ORDER MASS CONTRIBUTIONS

As demonstrated in Ref. [23], the n -rescaling of Eq. (3) yields an approximate mechanism to adjust for the reduced phase space available when heavy quarks are produced in

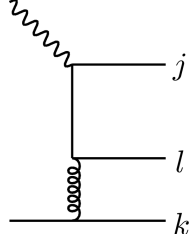


FIG. 8. $\mathcal{O}(\alpha_s^2)$ contribution to C_{ps}^λ . The graph is proportional to n_f and therefore C_{ps}^λ is as well.

the final state.⁵ The n -scaling not only provides us with an approximation of the proper phase-space suppression, but, by examining the variation of the structure functions with respect to n , we can also estimate both the uncertainty of this approximation and the impact of the heavy-quark masses. In the following we discuss this aspect in detail.

1. Interpretation of n

We start by identifying the correspondence between $n = \{0, 1, 2, 3\}$ and the underlying heavy-quark processes. For example, when $n = 0$, there is no rescaling and this corresponds to the case of a massless quark (ZM).⁶ Similarly, $n = 1$ introduces the typical factor observed in the production of a single heavy quark such as the $q \rightarrow WQ$ charged-current process [6]. For a gluon producing a pair of heavy quarks, such as $\gamma g \rightarrow Q\bar{Q}$, the $n = 2$ factor yields the appropriate phase-space suppression. Finally, at NNLO we have a new combination as illustrated in Fig. 8 where we can have 3 heavy quarks in the final state corresponding to $n = 3$.

Note that n is not required to be an integer and could be an “effective” number of heavy quarks instead. In this paper, we restrict the discussion to whole numbers as it reduces the possibilities but still covers the range of physically motivated choices. Our numerical implementation allows the user to freely choose the value of n .

2. Quality of approximation

We begin by examining the structure function relative ratio plots of Figs. 10–12. This will allow us to both (i) gauge the impact of the mass effects on the $\{F_2, F_3, F_L\}$

⁵In particular, see Table I of Ref. [23] for an analysis of the NLO phase space factors.

⁶Note that in the ZM scheme the gluon coefficient receives contributions from the heavy-quark even below its mass threshold. However, for technical reasons in many implementations (also in APFEL++) a step function of the form $\tilde{C}_g^\lambda \rightarrow \theta(Q^2 - m_H^2)\tilde{C}_g^\lambda$ is introduced and the heavy quark contribution appears only above the threshold. Our $n = 0$ implementation includes the step function in the gluon coefficient to align with the ZM implementation. However, for $n \neq 0$ the heavy-quark also contributes below the threshold.

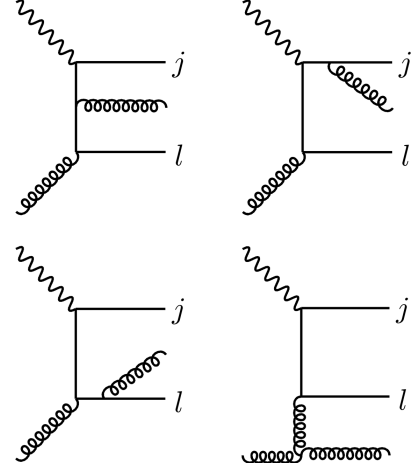


FIG. 9. $\mathcal{O}(\alpha_s^2)$ contributions to C_g^λ . All graphs are proportional to n_f and therefore C_g^λ is as well.

structure functions, and (ii) estimate the uncertainty due to missing higher-order mass contributions.

The (ZM only NNLO)/($n = 1$) case: We start with the (ZM only NNLO)/($n = 1$) comparisons displayed in the top row of the figures. Recall that the dotted (dash-dotted) lines indicate 2.5% (7.5%) contours. We know that the ZM massless result (effectively $n = 0$) is not a reasonable estimate since the quark mass effects are essential to consider, especially in the low Q^2 region. Therefore, these plots show us the minimum effect of the mass terms we are missing when we use the ZM-VFN scheme.

F_2 : For example, the F_2 structure functions display a significant change ($\gtrsim 2.5\%$) for low Q^2 , especially in the intermediate x -range ($\sim 10^{-3} - 10^{-2}$). There is also a lesser change ($\lesssim 2.5\%$) for high x -values which is especially evident above the b -quark threshold. As F_2 is often the dominant contribution to the physical cross section, we cannot neglect the mass terms in these regions.

F_3 : For the parity-violating neutral current structure function F_3 , the mass effects are essentially zero. This is because the only parity-violating contributions come from the Z-boson, which is suppressed relative to the photon due to its large mass M_Z .

Conversely, for charged current processes ($W^\pm$), we do observe significant mass effects at low Q^2 on a relative scale; however, on an absolute scale, these contributions will be suppressed by a factor (Q^2/M_W^2) due to the large M_W mass in a manner similar to the neutral current Z-contribution above.

Note in Fig. 11 the F_3 structure function passes through zero at an intermediate x -value ($\sim 10^{-2}$), and yields the observed artifact in the plot. In contrast to F_2 which is positive definite, there is no constraint on the sign of F_3 , and we discuss this further in Appendix C 3.

F_L : The impact of the mass terms is much more pronounced in the ratios for F_L exceeding $\gtrsim 7.5\%$ below

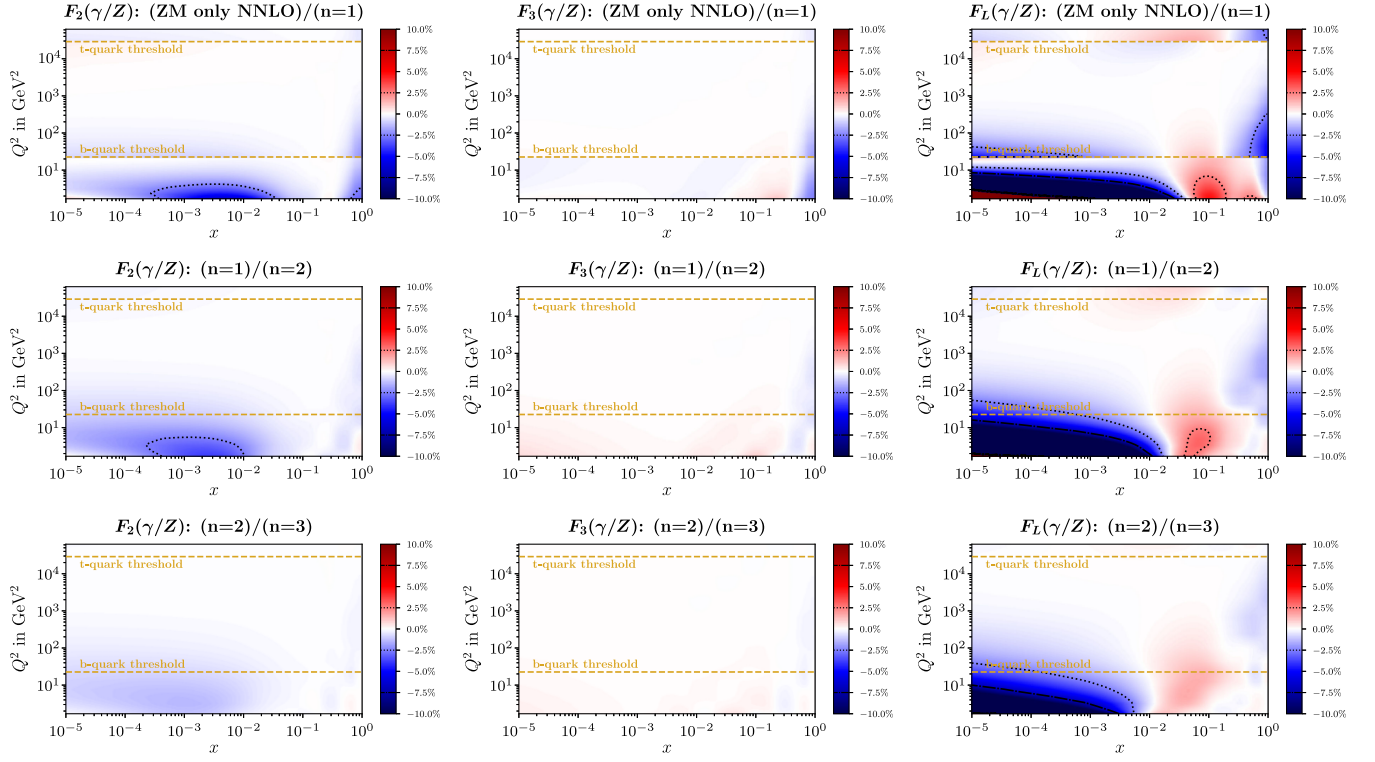


FIG. 10. The relative ratio of NNLO neutral current structure functions F_2 , F_3 , and F_L from left to right. From top to bottom, we consider the ratios for ZM/ $n = 1$, $n = 1/n = 2$, and $n = 2/n = 3$ as the NNLO contributions, respectively. The LO and NLO contributions are always given in the $SACOT_\chi$ scheme.

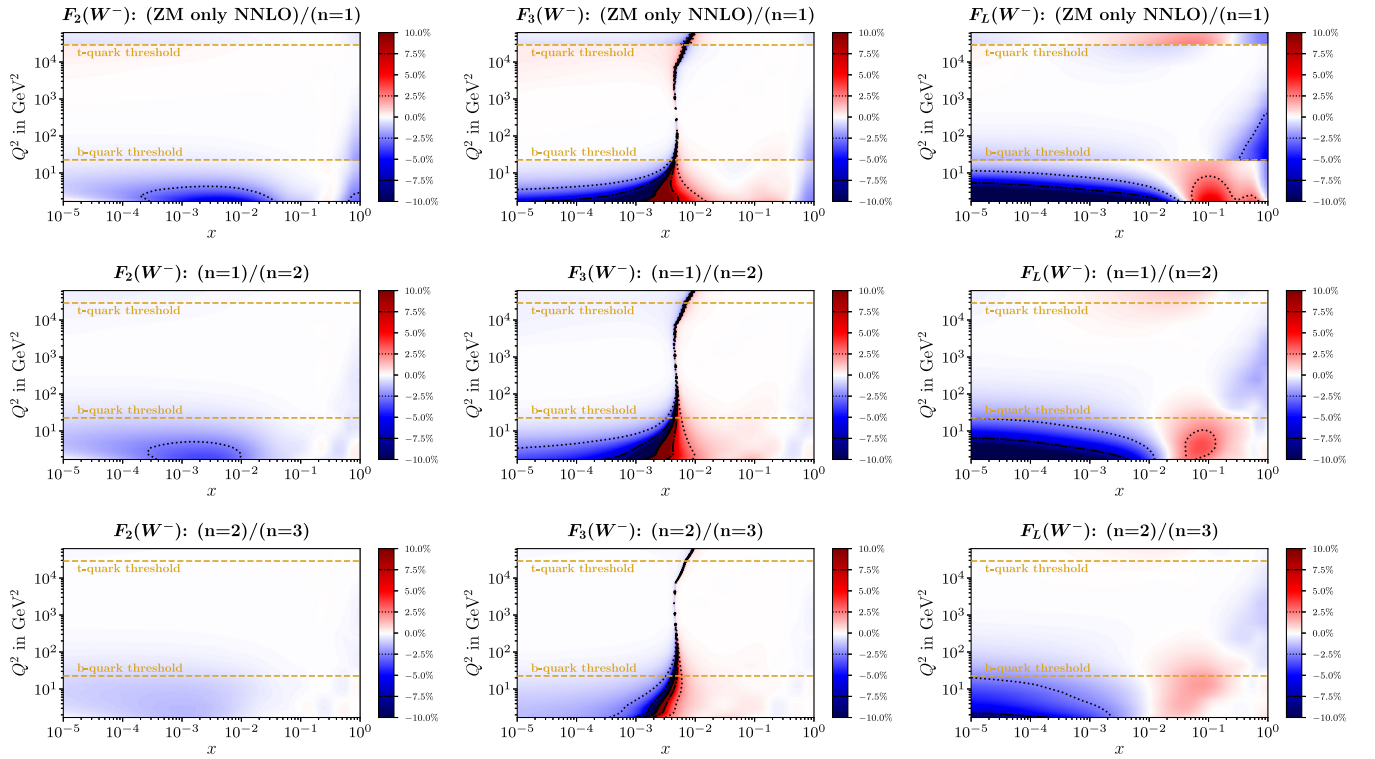


FIG. 11. The same as Fig. 10 but for charged current with a W^- exchange.

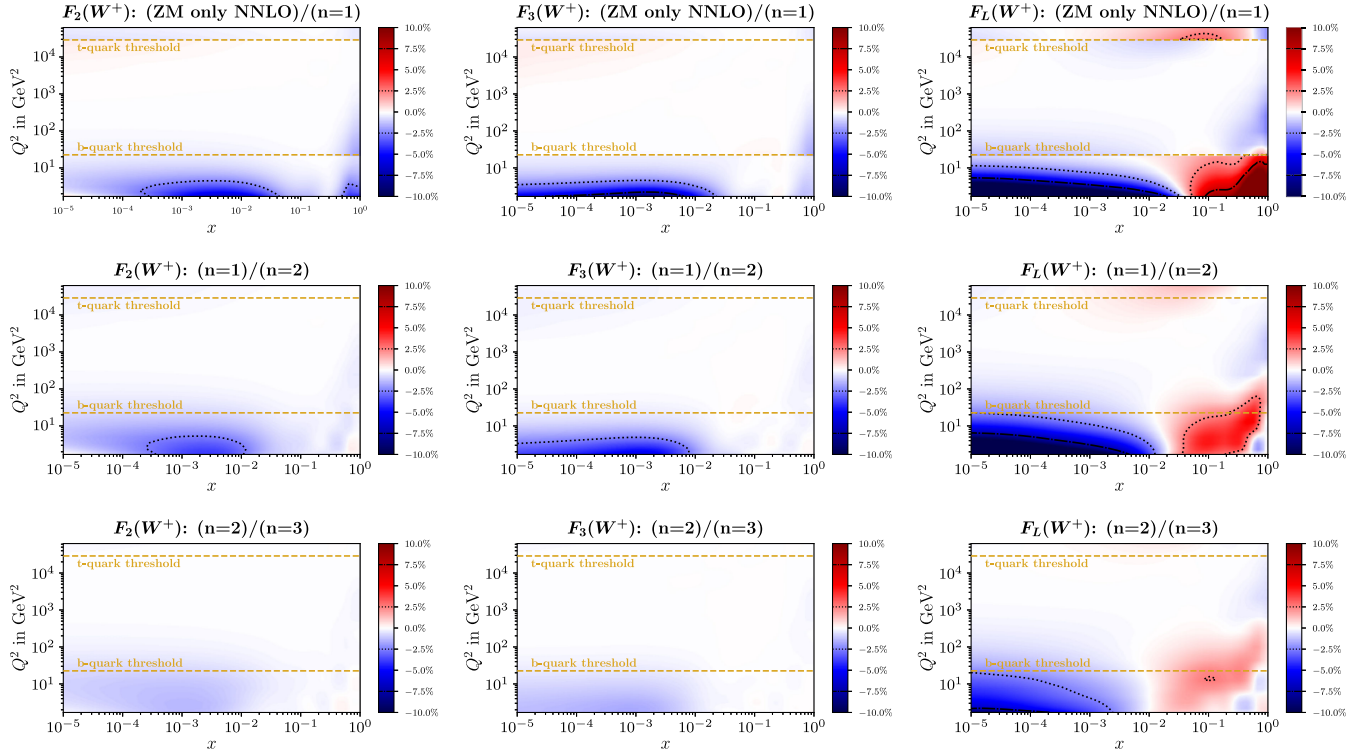


FIG. 12. The same as Fig. 10 but for charged current with a W^+ exchange.

the bottom threshold, and still exceeding $\gtrsim 2.5\%$ above. Since the LO contributions to F_L are suppressed relative to the F_2 terms,⁷ our calculation for this quantity is effectively one-order lower than for F_2 ; hence the effect of the mass correction is enlarged. Furthermore, we note that F_L is close to zero in the $x < 10^{-2}$ regime below the bottom threshold, thus small differences between the schemes are artificially enlarged. We observe large corrections ($\gtrsim 7.5\%$) below the b -quark threshold, and these persist above this threshold at both large and small x -values.

The $(n=2)/(n=3)$ case: Turning now to examine the $n=3$ case, the suppression factor corresponds to an incoming heavy quark Q , producing a $Q'\bar{Q}'$ pair via gluon splitting; thus, there are 3 heavy quarks in the final state. Given that the PDF for a heavy quark is significantly suppressed compared to the gluon and lighter quarks, we expect contributions from this channel to be minimal; therefore, $(n=3)$ represents an extreme case which is most likely beyond reasonable expectations for the mass uncertainty.

F_2 : For the F_2 structure function, we examine the bottom row of Figs. 10, 11, 12 and observe the relative difference between the $n=2$ and $n=3$ result is minimal ($\lesssim 2.5\%$). As implied above, this suggests that the heavy-quark contributions are already strongly suppressed in the

$n=2$ instance so that the additional suppression for $n=3$ is minimal.

F_3 : For the neutral current F_3 structure function, the difference between the $n=2$ and $n=3$ result also is minimal due to the (Q^2/M_Z^2) suppression of the parity-violating Z contribution. Conversely, for the charged-current F_3 structure function we do observe relative differences in the region of low Q^2 ; however, the absolute contributions will again be suppressed by the (Q^2/M_W^2) factor due to the $W^\pm$ boson mass.

F_L : The F_L also displays significant differences due to the fact that this is essentially a lower-order result as compared to F_2 . Again, this behavior is similar to the $ZM/(n=1)$ instance.

The $(n=1)/(n=2)$ case: The above observations of $n=\{0,3\}$ suggest that our best estimate for the mass uncertainty of our approximation can be obtained by examining the range between $n=\{1,2\}$. Therefore, we examine the middle rows of Figs. 10 to 12, to focus on the $(n=1)/(n=2)$ ratio as an estimate of the neglected higher-order mass contributions.

F_2 : For the F_2 structure function, as before we see a significant change ($\gtrsim 2.5\%$) for low Q^2 , especially in the intermediate x -range. There is also a lesser change above the b -quark threshold; across all vector bosons the difference is less than 1%. As F_2 is often the dominant contribution to the physical cross section, the ratios

⁷Contributions to F_L at LO are suppressed due to helicity conservation, and these vanish in the massless limit [5].

displayed in these plots represent our best estimate of the uncertainty due to our *aSACOT- χ* scheme.

F_3 : The neutral current F_3 relative ratio of Fig. 10 displays minimal dependence on the treatment of the heavy quark mass. As before, the charged current F_3 structure functions (Figs. 11, 12) do show relative differences in the region of low Q^2 ; however, the absolute contributions will again be suppressed by the (Q^2/M_W^2) factor due to the $W^\pm$ boson mass.

F_L : As before, the impact of the mass terms is even more pronounced in the ratios for F_L exceeding $\gtrsim 7.5\%$ below the bottom threshold, and still exceeding $\gtrsim 2.5\%$ above the bottom threshold. This reflects the fact that the absolute size of the F_L structure function are small compared to F_2 ; hence, the relative uncertainty is large.

This is also evident when comparing the experimental uncertainty of F_2 as compared to F_L which is more challenging to measure [56]. While current theoretical uncertainties align with the precision of existing experimental data, F_L stands out as an observable that would significantly benefit from a more refined treatment of heavy-quark masses.

Recap: In summary, these figures display the kinematic regions where the structure functions are sensitive to the mass dependence. As observed in Sec. III, the impact on the charged current HERA and EIC data (which is typically at larger Q^2 values) is rather minimal. But the effect on the ν DIS data can be significant, as shown in Fig. 6 for NuTeV.

Additionally, by examining the variation between the $(n=1)$ and $(n=2)$ results, we can also gauge the theoretical uncertainty of our approximations. For the inclusive F_2 structure function, these are typically small; for $Q^2 > 4 \text{ GeV}^2$ (a typical cut in global fit analyses), the variation is $\lesssim 2.5\%$.

However, for F_L , the impact of the mass contributions, and the associated uncertainties, are significantly larger; hence, this quantity can benefit from updated theoretical calculations as the experimental accuracy is improved.

The *aSACOT- χ* scheme and the *APFEL++* framework are designed with the flexibility to seamlessly incorporate new calculations as they become available. Additionally, the precomputed grid technique enables fast and efficient numerical evaluation, ensuring both adaptability and computational speed in future analyses.

3. The parity-violating F_3

We observed that the F_3 structure function crosses zero at intermediate x -values, and this appears as an artifact in

Fig. 11. We briefly examine the physics behind this phenomenon.

For charged current processes at leading order, we have

$$F_3^{W^+} = 2 \sum_D^{d,s,b} \sum_U^{u,c,t} |V_{DU}|^2 (f_D(x) - f_{\bar{U}}(x)) \quad (\text{C1})$$

$$F_3^{W^-} = 2 \sum_D^{d,s,b} \sum_U^{u,c,t} |V_{DU}|^2 (f_U(x) - f_{\bar{D}}(x)). \quad (\text{C2})$$

Focusing on the dominant first- and second-generation $\{u, d, s, c\}$ quarks, in the region of small- x (that is, below the valence region), we have effectively

$$F_3^{W^+} \simeq 2(d_{\text{sea}} - u_{\text{sea}} + s - c) \simeq -F_3^{W^-} \quad (\text{C3})$$

That is, the dominant contributions to $F_3^{W^+}$ and $F_3^{W^-}$ will have opposite signs; thus, there is no constraint that F_3 be positive. This is in contrast to F_2 , which can be written as a sum of (positive definite) longitudinal and transverse cross sections.

Additionally, we can relate F_3 to the left- and right-helicity structure functions (which are positive) via the relation [5,6,57,58]

$$F_3 \propto (-F_+ + F_-) \quad (\text{C4})$$

where $F_\pm$ represents the right/left-helicity structure-function. Thus, when we change from W^+/W^- , we change from a dominantly right/left coupling, and the sign of F_3 will flip. This is precisely the source of the zero we observe in F_3 for the charged W^- exchange process, and explicitly demonstrates the parity-violating nature of F_3 .

Finally, note that at the level of the cross section the sign in front of F_3 changes depending on the incoming lepton: “+” for a lepton and “−” for an antilepton. Thus for ν -DIS (W^+ exchange) F_3 contributes with a positive prefactor and for $\bar{\nu}$ -DIS (W^- exchange) with a negative prefactor.

APPENDIX D: LIBRARY DOCUMENTATION

The code can be downloaded from the git repository under the URL [26].

The full documentation can be found under this link as well. Currently, the implementation is located in the “ACOT” branch; to switch to the branch we use the commands

```
1 cd /location/of/apfelxx
   git checkout ACOT
```

The code is installed with the following procedure

```
cd build
2 cmake -DCMAKE_INSTALL_PREFIX=/your/installation/path ..
make && make install
```

Here, `/your/installation/path` is the path where the user would like to install APFEL++. Note that the two full stops at the end are important. The installation can be tested with the command

```
1 make test
```

Before tabulating a structure function, a few initializations have to be performed, namely

```
1 // set perturbative order
const int pto = 2;
3 // define a x-grid through a two subgrids that have 100/50 interpolation nodes,
// x_min=1e-5/1e-1 and are of degree 3/3
5 const apfel::Grid g{{apfel::SubGrid{100,1e-5,3},apfel::SubGrid{50,1e-1,3}}};
// define a vector of mass thresholds -> needed to not interpolate across a discontinuity
7 const std::vector<double> Thresholds = {0,0,0,mc,mb,mt};
// define a function that returns the effective electro-weak charges as a function of Q
9 const auto fEW = [=] (double const& Q) -> std::vector<double> {
return apfel::ElectroWeakCharges(Q,false);}; // use predefined function within APFEL++
```

For the prediction of structure functions, a set of PDFs is required. In the following we assume that we have a function called `my_PDF(x, Q)` that returns a `std::map<int, double>`, which maps the flavor indices in the LHAPDF-format⁸ [59] to the values of the respective distribution at $\{x, Q\}$. The set of PDF needs to be rotated into the QCD evolution basis, which can be done with a helper function part of the library:

```
// rotate the PDFs into the QCD evolution basis
2 const auto PDFrotated = [&] (double const& x, double const& Q) -> std::map<int, double>{
return apfel::PhysToQCDEv(my_PDF(x,Q));};
```

As the PDF set is usually aligned with a specific implementation of the strong coupling, the library is built such that a user-defined function can be used. For our example, we define the function `strongCoupling` with the call structure `(double const& mu) -> double`. A built-in implementation of the strong coupling is given through the class `AlphaQCD`.

In the following, we describe how to obtain fast evaluations of the neutral current F_2 structure function in the *aSACOT- χ* scheme. In APFEL++ the structure functions are built in two steps to decouple the expensive integration over the operators from the convolution with the PDFs.

First, the operators that make up the structure function are precalculated on the x - and Q -grids. For NC F_2 in the *aSACOT- χ* scheme this is done using the function `InitF2NCsimACOT_NNLO`. Other structure functions or other schemes can be constructed using different functions with a similar naming scheme that can be found in the full documentation. We call

```
// Construct the operator grids for the F2 structure function
2 const auto F2objects = apfel::InitF2NCsimACOT_NNLO(
g, Thresholds, IntEps=1e-5, nQ=100, Qmin=2, Qmax=225, intdeg=3, n=1);
```

The variables $\{nQ, Qmin, Qmax, intdeg\}$ define the Q -grid, `IntEps` is the integration accuracy and the variable `n` is the scaling variable from Eq. (3). This step usually takes a few seconds as we are calculating the expensive integrals over the operators, but has to be done only once.

⁸I.e., we have the integers $\{-6, -5, \dots, -2, -1, 1, 2, \dots, 5, 6, 21\}$ that map to the flavors $\{\bar{t}, \bar{b}, \dots, \bar{u}, \bar{d}, d, u, \dots, b, t, g\}$.

Second, we build the F_2 structure function from the operators tables, a process that is very fast. This is done by combining `F2objects` with the rotated PDFs, the strong coupling, the EW-charges and selecting the perturbative order.

```
// Build the structure function
2 const std::map<int, apfel::Observable<>> F2 = apfel::BuildStructureFunctions(
    F2objects, PDFrotated, pto, strongCoupling, fEW);
```

This map allows one to access the total F_2 through

```
1 // Evaluate the total F2 at x=0.1 and Q=10
    F2.at(0).Evaluate(x=0.1, Q=10);
```

Note that we have accessed the map at the key “0.” In general, the possible keys are $\{0, 1, 2, 3, 4\}$ and result in the total, light, charm, bottom, or top contribution to the structure function, respectively. For the *aSACOT- χ* scheme we implemented only the total structure functions (key “0”).

Now, every time the function `Evaluate(x, Q)` is called, the operator tables are convolved with the PDFs. A final speed-up can be achieved by interpolating the Q -dependence of the final result as well:

```
// Interpolate the Q-dependence of the total F2
2 const apfel::TabulateObject<apfel::Distribution> F2total {
    [&] (double const& Q) -> apfel::Distribution{return F2.at(0).Evaluate(Q);},
4    nQ=50, Qmin=2, Qmax=225, intdeg=3, Thresholds};
```

where we substitute the evaluation of F_2 as an anonymous function alongside with the definition of the Q -grid and the mass thresholds to the interpolation routine. F_2 can now be evaluated as

```
// Evaluate the Q-interpolated total F2 at x=0.1 and Q=10
2 F2total.Evaluate(x=0.1, Q=10);
```

-
- [1] H. Abramowicz *et al.* (H1 and ZEUS Collaborations), Combination and QCD analysis of charm production cross section measurements in deep-inelastic ep scattering at HERA, *Eur. Phys. J. C* **73**, 2311 (2013).
 - [2] H. Abramowicz *et al.* (ZEUS Collaboration), Measurement of beauty and charm production in deep inelastic scattering at HERA and measurement of the beauty-quark mass, *J. High Energy Phys.* **09** (2014) 127.
 - [3] H. Abramowicz *et al.* (H1 and ZEUS Collaborations), Combination of measurements of inclusive deep inelastic $e^\pm p$ scattering cross sections and QCD analysis of HERA data, *Eur. Phys. J. C* **75**, 580 (2015).
 - [4] H. Abramowicz *et al.* (H1 and ZEUS Collaborations), Combination and QCD analysis of charm and beauty production cross-section measurements in deep inelastic ep scattering at HERA, *Eur. Phys. J. C* **78**, 473 (2018).
 - [5] M. A. G. Aivazis, F. I. Olness, and W.-K. Tung, Leptoproduction of heavy quarks. 1. General formalism and kinematics of charged current and neutral current production processes, *Phys. Rev. D* **50**, 3085 (1994).
 - [6] M. A. G. Aivazis, J. C. Collins, F. I. Olness, and W.-K. Tung, Leptoproduction of heavy quarks. 2. A Unified QCD formulation of charged and neutral current processes from fixed target to collider energies, *Phys. Rev. D* **50**, 3102 (1994).
 - [7] S. Kretzer and I. Schienbein, Heavy quark initiated contributions to deep inelastic structure functions, *Phys. Rev. D* **58**, 094035 (1998).
 - [8] W.-K. Tung, S. Kretzer, and C. Schmidt, Open heavy flavor production in QCD: Conceptual framework and implementation issues, *J. Phys. G* **28**, 983 (2002).
 - [9] B. A. Kniehl, G. Kramer, I. Schienbein, and H. Spiesberger, Inclusive B-meson production at the LHC in the GM-VFN scheme, *Phys. Rev. D* **84**, 094026 (2011).
 - [10] R. S. Thorne and R. G. Roberts, An ordered analysis of heavy flavor production in deep inelastic scattering, *Phys. Rev. D* **57**, 6871 (1998).

- [11] R. S. Thorne, A variable-flavor number scheme for NNLO, *Phys. Rev. D* **73**, 054019 (2006).
- [12] M. Cacciari, M. Greco, and P. Nason, The p_T spectrum in heavy-flavour hadroproduction., *J. High Energy Phys.* **05** (1998) 007.
- [13] S. Forte, E. Laenen, P. Nason, and J. Rojo, Heavy quarks in deep-inelastic scattering, *Nucl. Phys.* **B834**, 116 (2010).
- [14] S. Höche, J. Krause, and F. Siegert, Multijet merging in a variable flavor number scheme, *Phys. Rev. D* **100**, 014011 (2019).
- [15] R. Gauld, A massive variable flavour number scheme for the Drell-Yan process, *SciPost Phys.* **12**, 024 (2022).
- [16] M. Guzzi, P. M. Nadolsky, H.-L. Lai, and C. P. Yuan, General-mass treatment for deep inelastic scattering at two-loop accuracy, *Phys. Rev. D* **86**, 053005 (2012).
- [17] J. Gao, T. J. Hobbs, P. M. Nadolsky, C. Sun, and C. P. Yuan, General heavy-flavor mass scheme for charged-current DIS at NNLO and beyond, *Phys. Rev. D* **105**, L011503 (2022).
- [18] M. Guzzi, P. Nadolsky, L. Reina, D. Wackeroth, and K. Xie, General mass variable flavor number scheme for Z boson production in association with a heavy quark at hadron colliders, *Phys. Rev. D* **110**, 114030 (2024).
- [19] A. Barontini, A. Candido, F. Hekhorn, G. Magni, and R. Stegeman, An FONLL prescription with coexisting flavor number PDFs, *J. High Energy Phys.* **10** (2024) 004.
- [20] J. C. Collins and W.-K. Tung, Calculating heavy quark distributions, *Nucl. Phys.* **B278**, 934 (1986).
- [21] R. S. Thorne and W. K. Tung, PQCD formulations with heavy quark masses and global analysis, in *HERA and the LHC: 4th Workshop on the Implications of HERA for LHC Physics* (2008), pp. 332–351, [arXiv:0809.0714](https://arxiv.org/abs/0809.0714).
- [22] P. M. Nadolsky and W.-K. Tung, Improved formulation of global QCD analysis with Zero-mass matrix elements, *Phys. Rev. D* **79**, 113014 (2009).
- [23] T. Stavreva, F. I. Olness, I. Schienbein, T. Jezo, A. Kusina, K. Kovarik, and J. Y. Yu, Heavy quark production in the ACOT Scheme at NNLO and N3LO, *Phys. Rev. D* **85**, 114014 (2012).
- [24] K. Xie, Massive elementary particles in the Standard Model and its supersymmetric triplet Higgs extension, Ph.D. thesis, Southern Methodist University, 2019.
- [25] V. Bertone, APFEL++: A new PDF evolution library in C++, *Proc. Sci. DIS2017* (2018) 201 [[arXiv:1708.00911](https://arxiv.org/abs/1708.00911)].
- [26] V. Bertone, APFELXX, <https://github.com/vbertone/apfelxx> (2017).
- [27] V. Bertone, S. Carrazza, and J. Rojo (APFEL Collaboration), APFEL: A PDF evolution library with QED corrections, *Comput. Phys. Commun.* **185**, 1647 (2014).
- [28] E. Spezzano *et al.*, Heavy quark contributions to the DIS structure functions F_4 and F_5 in the ACOT scheme at NLO (to be published), see <https://indico.cern.ch/event/1436959/contributions/6352265/>.
- [29] M. Tzanov *et al.* (NuTeV Collaboration), Precise measurement of neutrino and anti-neutrino differential cross sections, *Phys. Rev. D* **74**, 012008 (2006).
- [30] U.-K. Yang *et al.* (CCFR/NuTeV Collaboration), Measurements of F_2 and $xF_3^\nu - xF_3^{\bar{\nu}}$ from CCFR $\nu_\mu - \text{Fe}$ and $\bar{\nu}_\mu - \text{Fe}$ data in a physics model independent way, *Phys. Rev. Lett.* **86**, 2742 (2001).
- [31] U.-K. Yang, A measurement of differential cross sections in charged current neutrino interactions on iron and a global structure functions analysis, Ph.D. thesis, Rochester University, 2001.
- [32] A. Kayis-Topaksu *et al.* (CHORUS Collaboration), Leading order analysis of neutrino induced dimuon events in the CHORUS experiment, *Nucl. Phys.* **B798**, 1 (2008).
- [33] R. Abdul Khalek *et al.*, Science requirements and detector concepts for the electron-ion collider: EIC Yellow report, *Nucl. Phys.* **A1026**, 122447 (2022).
- [34] R. A. Khalek, J. J. Ethier, E. R. Nocera, and J. Rojo, Self-consistent determination of proton and nuclear PDFs at the electron ion collider, *Phys. Rev. D* **103**, 096005 (2021).
- [35] M. Krämer, F. I. Olness, and D. E. Soper, Treatment of heavy quarks in deeply inelastic scattering, *Phys. Rev. D* **62**, 096007 (2000).
- [36] T. Binoth *et al.* (SM, NLO Multileg Working Group), The SM and NLO multileg working group: Summary report, in *Proceedings of the 6th Les Houches Workshop on Physics at TeV Colliders* (2010), pp. 21–189, [arXiv:1003.1241](https://arxiv.org/abs/1003.1241).
- [37] W. L. van Neerven and E. B. Zijlstra, Order $\alpha - s^2$ contributions to the deep inelastic Wilson coefficient, *Phys. Lett. B* **272**, 127 (1991).
- [38] E. B. Zijlstra and W. L. van Neerven, Contribution of the second order gluonic Wilson coefficient to the deep inelastic structure function, *Phys. Lett. B* **273**, 476 (1991).
- [39] E. B. Zijlstra and W. L. van Neerven, Order $\alpha - s^2$ QCD corrections to the deep inelastic proton structure functions F2 and F(L), *Nucl. Phys.* **B383**, 525 (1992).
- [40] W. L. van Neerven and A. Vogt, NNLO evolution of deep inelastic structure functions: The nonsinglet case, *Nucl. Phys.* **B568**, 263 (2000).
- [41] W. L. van Neerven and A. Vogt, NNLO evolution of deep inelastic structure functions: The singlet case, *Nucl. Phys.* **B588**, 345 (2000).
- [42] T.-J. Hou *et al.* (CTEQ-TEA Collaboration), New CTEQ global analysis of quantum chromodynamics with high-precision data from the LHC, *Phys. Rev. D* **103**, 014013 (2021).
- [43] R. Ruiz *et al.*, Target mass corrections in lepton–nucleus DIS: Theory and applications to nuclear PDFs, *Prog. Part. Nucl. Phys.* **136**, 104096 (2024).
- [44] P. Duwentäster, T. Ježo, M. Klasen, K. Kovařík, A. Kusina, K. F. Muzakka, F. I. Olness, R. Ruiz, I. Schienbein, and J. Y. Yu, Impact of heavy quark and quarkonium data on nuclear gluon PDFs, *Phys. Rev. D* **105**, 114043 (2022).
- [45] K. F. Muzakka *et al.*, Compatibility of neutrino DIS data and its impact on nuclear parton distribution functions, *Phys. Rev. D* **106**, 074004 (2022).
- [46] M. Klasen and H. Paukkunen, Nuclear PDFs after the first decade of LHC data, *Annu. Rev. Nucl. Part. Sci.* **74**, 49 (2024).
- [47] R. Abdul Khalek, R. Gauld, T. Giani, E. R. Nocera, T. R. Rabemananjara, and J. Rojo, nNNPDF3.0: Evidence for a modified partonic structure in heavy nuclei, *Eur. Phys. J. C* **82**, 507 (2022).
- [48] K. J. Eskola, P. Paakkinen, H. Paukkunen, and C. A. Salgado, EPPS21: A global QCD analysis of nuclear PDFs, *Eur. Phys. J. C* **82**, 413 (2022).

- [49] I. Schienbein, J. Y. Yu, K. Kovarik, C. Keppel, J. G. Morfin, F. Olness, and J. F. Owens, PDF nuclear corrections for charged and neutral current processes, *Phys. Rev. D* **80**, 094004 (2009).
- [50] H. Paukkunen and C. A. Salgado, Compatibility of neutrino DIS data and global analyses of parton distribution functions, *J. High Energy Phys.* **07** (2010) 032.
- [51] K. Kovarik, I. Schienbein, F. I. Olness, J. Y. Yu, C. Keppel, J. G. Morfin, J. F. Owens, and T. Stavreva, Nuclear corrections in neutrino-nucleus DIS and their compatibility with global NPDF analyses, *Phys. Rev. Lett.* **106**, 122301 (2011).
- [52] S. Alekhin *et al.*, HERAFitter, *Eur. Phys. J. C* **75**, 304 (2015).
- [53] H. Abdolmaleki *et al.* (xFitter Collaboration), xFitter: An Open Source QCD Analysis Framework. A resource and reference document for the Snowmass study, [arXiv:2206.12465](https://arxiv.org/abs/2206.12465).
- [54] S. Carrazza, E. R. Nocera, C. Schwan, and M. Zaro, PineAPPL: combining EW and QCD corrections for fast evaluation of LHC processes, *J. High Energy Phys.* **12** (2020) 108.
- [55] C. Schwan, T. R. Rabemananjara, A. Candido, F. Hekhorn, T. Sharma, S. Carrazza, A. Barontini, J. Wissmann, and J. M. Cruz-Martinez, Nnpdf/pineappl: v1.0.0., [10.5281/zenodo.15635174](https://doi.org/10.5281/zenodo.15635174) (2025).
- [56] V. Andreev *et al.* (H1 Collaboration), Measurement of inclusive ep cross sections at high Q^2 at $\sqrt{s} = 225$ and 252 GeV and of the longitudinal proton structure function F_L at HERA, *Eur. Phys. J. C* **74**, 2814 (2014).
- [57] F. I. Olness and W.-K. Tung, Factorization of helicity amplitudes and angular correlations for electroweak processes, *Phys. Rev. D* **35**, 833 (1987).
- [58] M. A. G. Aivazis, F. I. Olness, and W.-K. Tung, QCD formulation of charm production in deep inelastic scattering and the sea quark—gluon dichotomy, *Phys. Rev. Lett.* **65**, 2339 (1990).
- [59] A. Buckley, J. Ferrando, S. Lloyd, K. Nordström, B. Page, M. Rüfenacht, M. Schönherr, and G. Watt, LHAPDF6: Parton density access in the LHC precision era, *Eur. Phys. J. C* **75**, 132 (2015).