

New horizons in effective field theory?

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We consider the most general parity symmetric effective scalar tensor theory in four dimensions containing terms up to fourth derivative order in the Lagrangian. It has been shown [H. S. Reall, *Phys. Rev. D* **103**, 084027 (2021)] that this theory has three polarizations generically governed by different (nested) propagation cones, neither of which, in general, coincides with the light cone as defined by the metric. Consequently, the notion of a black hole horizon must be defined relative to the widest propagation cone and not with respect to the metric. We provide two theorems stating that, nevertheless, the horizon of a *stationary* black hole is null with respect to the metric and that, in fact, all three propagation cones touch on the horizon. The conditions in these theorems allow for rotating black holes. Our theorems thereby suggest that the notion of a Killing horizon, central in most discussions of black hole thermodynamics, retains its fundamental status and that certain thermodynamic paradoxes associated with multiple propagation cones are evaded.

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I. INTRODUCTION

In relativistic physics, the propagation of disturbances as described by, e.g., the scalar wave equation $g^{ab}\nabla_a\nabla_b\Phi = 0$ is well known to be causal. That means that the retarded respectively advanced solutions of the corresponding equation with a delta-function source located at a point $x \in \mathcal{M}$ in a causally well-behaved spacetime¹ $(\mathcal{M}, g_{ab})$ vanish outside the causal future $J^+(x)$ respectively past $J^-(x)$ of this point. The causal future respectively past consists of all events that can be connected to x by a causal (i.e., timelike or null) past respectively future directed curve. Since a curve is timelike or null if its tangent vector u^a satisfies² $g_{ab}u^au^b \leq 0$, the notion of causality for the scalar field propagation is tied to the light cones of the metric, g_{ab} . An equivalent formulation is that any solution to $g^{ab}\nabla_a\nabla_b\Phi = 0$ is uniquely determined by its initial data on some subset $\mathcal{U} \subset \Sigma$ of a Cauchy-surface Σ within the domain of dependence, $D(\mathcal{U})$, of $\mathcal{U}$. Again, $D(\mathcal{U})$ is determined by the metric as the set of all points such that any inextendible future or past directed causal curve must

intersect $\mathcal{U}$. For modern accounts of these well-known facts, see, e.g., [2].

Up to issues of gauge, i.e., after passing to suitable gauge equivalence classes or after suitably fixing the gauge, the same concept of causality applies to solutions A_a of the Maxwell equations or solutions h_{ab} of the linearized vacuum Einstein equations. In fact, stating causality in terms of the domain of dependence, the same holds true even at the nonlinear level for the coupled Einstein–Maxwell-scalar wave equation system, see, e.g., [2]. Given that information is thought of as being transported by waves, it is thereby justified to define the notion of a black hole region in such a theory as $\mathcal{M} \setminus J^-(\mathcal{I}^+)$, where $\mathcal{I}^+$ is future null infinity. The future horizon thus is

$$\mathcal{H}^+ = \partial[\mathcal{M} \setminus J^-(\mathcal{I}^+)]. \quad (1)$$

Since the notion of a causal past, J^- , depends on the metric g_{ab} , so obviously does the notion of a black hole.

In alternative theories of gravity, there may be several different metrics, or even if there is only a single metric, various “polarizations” of the fields of the theory may propagate according to a different “effective” metric due to the structure of the highest derivative terms in the equations of motion. As a simple minded toy model of such situations, one may consider two scalar fields Φ_A , Φ_B obeying

$$\begin{aligned} g_A^{ab}\nabla_a\nabla_b\Phi_A - V_{,\Phi_A}(\Phi_A, \Phi_B) &= 0, \\ g_B^{ab}\nabla_a\nabla_b\Phi_B - V_{,\Phi_B}(\Phi_A, \Phi_B) &= 0, \end{aligned} \quad (2)$$

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¹For example, the spacetime should be globally hyperbolic, see, e.g., [1].

²Our signature convention is $(-+++)$.

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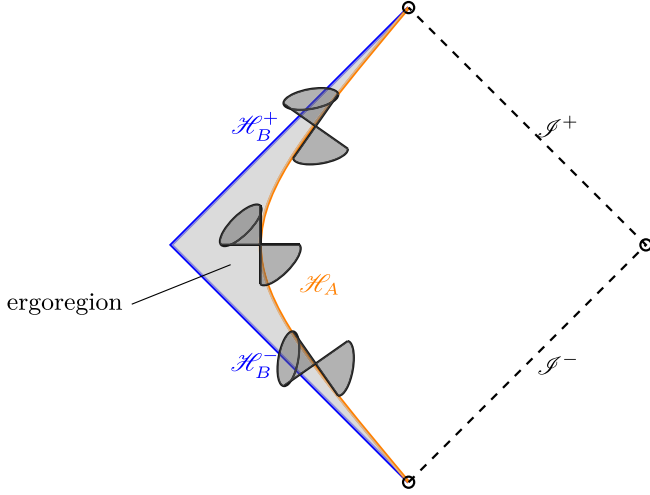


FIG. 1. A hypothetical spacetime containing two nested horizons $\mathcal{H}_A$ and $\mathcal{H}_B$ defined with respect to g_{Aab} and g_{Bab} , respectively. Since χ^a is a “boostlike” Killing field, it does not vanish on $\mathcal{H}_A$, meaning that this set must be *either* a future *or* a past horizon with respect to g_{Aab} . In particular, this metric cannot have a “time reflection” symmetry, indicating a fundamental time asymmetry of the setup.

where g_{Aab} , g_{Bab} are different metrics, and V is a potential coupling the two fields. In such a theory, the notion of a causal past and future should obviously be defined with respect to the curves whose tangent is causal with respect to at least one—but possibly not the other—metric, i.e., the “widest” light cone at every point.

For the sake of illustration, envisage a hypothetical black hole spacetime as in Fig. 1 with nested horizons $\mathcal{H}_A$ and $\mathcal{H}_B$ defined relative to the A and B metrics. We assume that there is a vector field χ^a which Lie derives *both* metrics and that both metrics are asymptotically flat, with χ^a timelike near null infinity (simultaneously for both metrics). We assume further that the B metric has the widest light cones.

Even though in our hypothetical spacetime the inner horizon, $\mathcal{H}_B$, is thus of course the true black hole horizon, the nested structure of $\mathcal{H}_A$ and $\mathcal{H}_B$ may give rise to interesting effects and even paradoxes, as has been noted by several authors [3–5], see also [6] for further discussion and references.

A first observation, which by itself does not raise a paradox, is that if the (outer) horizon $\mathcal{H}_A$ is a nondegenerate Killing horizon with respect to χ^a , i.e., $\mathcal{H}_A$ is not rotating relative to infinity and χ^a is tangent and normal to $\mathcal{H}_A$ with respect to the A metric, then the horizon Killing field χ^a must be spacelike with respect to the A metric somewhere between the A and B horizons.³ Thus, the region between the A and B horizons includes a kind of ergoregion for A

³This immediately follows, e.g., from the equation $g_A^{ac}\nabla_c(\chi_b\chi^b) = -2\kappa_A\chi^a$ and the fact that $\kappa_A > 0$, see, e.g., [1] [sec. 12.5].

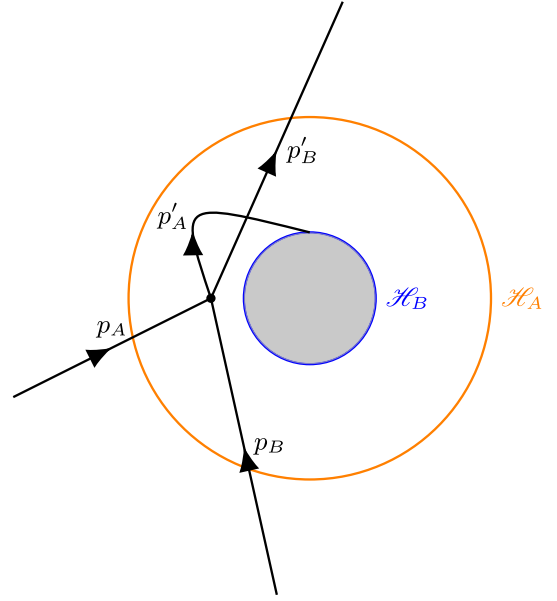


FIG. 2. The collision between two particles with 4-momenta p_{Aa} and p_{Ba} .

particles, which enables an analog of the Penrose process for extracting energy from black holes, see, e.g., [1] for an explanation of the standard version of this process in the Kerr black hole ergoregion.

More concretely, in a mechanical version of this process, consider an A and a B particle with momenta p_{Aa} and p_{Ba} , moving along lightlike geodesics in the A respectively B metrics. They collide inside the region between the A and B horizons at a point where χ^a is spacelike with respect to the A metric. After the collision, the particles have lightlike momenta p'_{Aa} and p'_{Ba} relative to the respective metrics. One can arrange that $E'_A = -\chi^a p'_{Aa} < 0$ and that the outgoing B particle can reexit the A horizon, while the outgoing A particle necessarily falls into the black hole, see Fig. 2. Obviously, by energy conservation,⁴ $E_A + E_B = E'_A + E'_B$ and $E_A > 0$ since χ^a is timelike near infinity, so $E'_B > E_B$. For a closely related setting see, e.g., [4].

It is not inconceivable that there could be a field-theoretic analog of this setup choosing high-frequency Gaussian beam solutions Φ_A and Φ_B whose initial support is close to the trajectories cotangent to p_{Aa} and p_{Ba} . Indeed, before and after the collision, the potential terms in the coupled wave equations (2) can be neglected, and the solutions are approximately equal to free Gaussian beams, as described, e.g., in [7]. Matching these solutions across the collision might be possible.

While interesting and potentially observable via an analog of the superradiance effect [8], energy extraction from a black hole by itself does not raise any physical

⁴This follows from the fact that χ^a Lie derives both the A and B metrics.

paradox. However, as argued in [3–5], by exploiting the hypothetical situation with nested A and B horizons further, one may enable perpetual motion creating tensions with the generally accepted laws of thermodynamics.

Furthermore, while it is not totally clear how well motivated one should consider theories with several metrics, different *effective metrics* frequently arise in effective field theory (EFT) type generalizations of Einstein-matter theories with higher derivative terms in the action and/or theories with unconventional fields such as Einstein-aether theories, see, e.g., [9]. Although there is only a single dynamical metric g_{ab} in such theories, different “polarizations” of the coupled field theoretic content are governed by different effective metrics.⁵

While the notion of propagation cones in EFTs is thus, in general, different from the standard notion of light cone with respect to g_{ab} , it may of course happen that for *stationary*, regular, asymptotically flat black hole solutions all those notions precisely coincide *at the event horizon*. Evidence that this may be the case has been provided by [11–14]. These authors study various scalar-tensor theories in the Horndeski class [11,12] or higher dimensions [13] with second order equations of motion or various EFTs for just a metric [14]. They demonstrate at various levels of generality that a Killing horizon relative to g_{ab} is automatically a characteristic surface of a suitably defined propagation cone of the theory. Since the characteristic directions determine the propagation of discontinuities [15], hence the boundary of the domain of dependence of the equation, it follows that a Killing horizon can be the causal edge just as in minimally coupled Einstein-scalar theories.

While these results provide an interesting link between Killing and propagation cone horizons, the fundamental question is whether, vice versa, any propagation cone horizon is also a Killing horizon. If true, this would eliminate any of the concerns [3–5] with perpetual motion. It would also reinforce the usual laws of black hole mechanics in alternative or effective gravity theories with higher derivatives and/or additional unconventional fields, since these laws are usually formulated with regard to Killing horizons [16–19].

Positive evidence for this scenario has been given by [11] for spherically symmetric static black hole solutions in the Horndeski class. In the present paper, we investigate *rotating* black holes in the class of scalar-tensor EFTs in dimension $d = 4$ described by the Lagrange density,

$$\begin{aligned} \mathcal{L} = & [-V(\Phi) + R + X]\epsilon \\ & + \frac{1}{2}\ell^2\alpha(\Phi)X^2\epsilon + \frac{1}{16}\ell^2\beta(\Phi)L_{\text{GB}}, \end{aligned} \quad (3)$$

⁵As has been shown by [10], the effective propagation cones may not even be defined by a quadratic equation arising from an effective metric but by higher order, e.g., quartic, polynomials arising from the principal symbol of the highest order operator in the Euler–Lagrange equations. We refer to Secs. II B and II C for a short review of some of his results.

where $X = -\frac{1}{2}g^{ab}\nabla_a\Phi\nabla_b\Phi$ is the kinetic term of a scalar field Φ , where L_{GB} is the Gauss–Bonnet density, and where α, β, V are, in principle, free real functions. The above Lagrangian is a generalization of the usual Einstein-scalar field theory—the first line—containing additional terms with four derivatives in the second line. The overall size of the four-derivative couplings is dictated by the constant ℓ which has the dimension of a length. It has been shown [20] that this theory in the Horndeski class is, up to field redefinitions, the most general scalar-tensor EFT with up to four derivative terms in the action under the assumption of parity symmetry; thus, it represents a quite general class of models.

The Euler–Lagrange equations of (3) contain terms with at most two derivatives acting on each occurrence of (Φ, g_{ab}) [10]. The nature of the propagation cones of these equations has been analyzed in detail by [10] (we recall some of these results in Sec. II B). In particular, it has been shown [10] that the theory has, at least at the linearized level, three distinct “polarizations” with distinct propagation cones. The covectors in these cones, which are co-tangent to the characteristics, are defined as solutions to either a quadratic or quartic equation, with the inner sheet of the quartic cone determining the “fastest” polarization, see fig. 5. Thus, the characteristics of the fastest polarization determine the physical notion of black hole horizon, $\mathcal{H}^+$.

In the present paper, we provide two theorems showing that this horizon must in fact be a Killing horizon. Both theorems assume that the black hole solution (Φ, g_{ab}) is a real analytic, stationary (possibly rotating), and given to us as a real analytic family in the EFT parameter ℓ . Furthermore, it is assumed that the $\ell = 0$ solution of the usual Einstein-scalar theory is a stationary black hole spacetime with nondegenerate horizon. Theorem A also assumes that the generators of the characteristic flow on $\mathcal{H}^+$ —the propagation cone analog of null geodesics—are collinear with a Killing vector field, χ^a , at all points of $\mathcal{H}^+$, and that β' does not vanish. On the other hand, Theorem B assumes that the metric has a “ t - ϕ ” reflection symmetry analogous to Kerr and that $V \geq 0$. The assumptions are discussed in detail in Sec. III (Theorem A) and Sec. IV (Theorem B).

Both theorems conclude that $\mathcal{H}^+$ is actually a Killing horizon with respect to g_{ab} with constant surface gravity for sufficiently small ℓ . At this horizon, the three-sheet structure of the various propagation cones degenerates, as shown in Fig. 3.

Due to the technical assumptions underlying our theorems, these are not totally definite results, but we believe that they provide evidence of a censorship mechanism which forces stationary propagation cone horizons to be actual Killing horizons, thereby saving the laws of black hole thermodynamics from the paradoxes raised in [3–5].

This paper is organized as follows. In Sec. II, we review our EFT and its various propagation cones. In Secs. III and

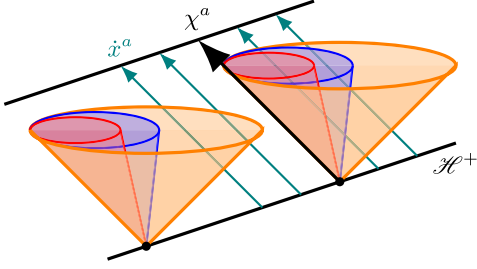


FIG. 3. The red, orange, and blue cones indicate different propagation cones of the theory (3) in the tangent space (quartic and quadratic cones). At the horizon $\mathcal{H}^+$, these touch at the horizon Killing vector field χ^a pointing along $\mathcal{H}^+$. At these points, χ^a is null with respect to g_{ab} and satisfies $\chi^a \nabla_a \chi^b = \kappa \chi^b$ with constant $\kappa > 0$. χ^a is also colinear with the tangent $\dot{x}^a$ of the null geodesics ruling $\mathcal{H}^+$. This illustration is exaggerated because all cones should be close to each other in a weakly coupled (small $|\ell|$) theory.

IV, we present our theorems. Some lengthy technical arguments and formulas needed for the proofs are deferred to various Appendixes. Our notations and conventions are the same as in [1].

II. SETUP

A. Equations and principal symbols of EFT

We consider configurations $(\mathcal{M}, \Phi, g_{ab})$ where $(\mathcal{M}, g_{ab})$ is a 4-dimensional spacetime and Φ a real-valued scalar field, which satisfy second order equations of motion (EoMs) produced by the Lagrangian (3) of our EFT. Written out more fully, this Lagrangian is

$$\begin{aligned} L = (X - V + R)\epsilon \\ + \left(\frac{1}{2} \ell^2 \alpha X^2 + \frac{1}{16} \ell^2 \beta \delta_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} R_{a_1 a_2}{}^{b_1 b_2} R_{a_3 a_4}{}^{b_3 b_4} \right) \epsilon. \end{aligned} \quad (4)$$

$V(\Phi), \alpha(\Phi), \beta(\Phi)$ are real analytic functions of Φ . $X = -\frac{1}{2} g^{ab} \nabla_a \Phi \nabla_b \Phi$ is shorthand for the kinetic term of Φ . $\delta_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} / 4!$ is the projector onto the totally antisymmetric part of a rank four tensor. The constant ℓ governs the strength of the higher derivative terms in the second line and has the dimension of a length.

The EoMs obtained from varying (3) with respect to g_{ab} and Φ are given by $E_g^{ab} = E_\Phi = 0$, where [10]

$$\begin{aligned} E_g^{ab} = G^{ab} - \left(\frac{1}{2} + \alpha X \right) \nabla^a \Phi \nabla^b \Phi \\ - \frac{1}{2} g^{ab} \left(V - X + \frac{1}{2} \ell^2 \alpha X \right) \\ - \frac{1}{4} \ell^2 \epsilon^{ac_1 c_2 c_3} \epsilon^{bd_1 d_2 d_3} R_{c_1 c_2 d_1 d_2} (\nabla^2 \beta)_{c_3 d_3}, \end{aligned} \quad (5)$$

(G^{ab} is the Einstein tensor, and indices are raised with g^{ab} throughout this paper), and

$$\begin{aligned} E_\Phi = -\nabla_a \nabla^a \Phi (1 + \ell^2 \alpha X) - \ell^2 \nabla^a \Phi \nabla_a (\alpha X) + V' \\ - \frac{1}{2} \ell^2 \alpha' X^2 - \frac{1}{16} \ell^2 \beta' \delta_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} R_{a_1 a_2}{}^{b_1 b_2} R_{a_3 a_4}{}^{b_3 b_4}. \end{aligned} \quad (6)$$

The coefficient β only appears through its derivatives in the EoMs, consistent with the fact that the Gauss–Bonnet density by itself is a topological term. From a phenomenological perspective, the most natural choices might be $\beta = \beta_0 \Phi$ as it is minimalistic and leads to a shift symmetric theory ($\Phi \rightarrow \Phi + c$) for $V \equiv 0$, or $\beta = e^{b_0 \Phi}$.

The principal symbol captures the structure of the highest derivative terms of the EoM of the EFT. It is made up of the following tensors [10]:

$$\begin{aligned} P_{gg}^{abcdef} &= \frac{\partial E_g^{ab}}{\partial (\partial_e \partial_f g_{cd})}, & P_{g\Phi}^{abcd} &= \frac{\partial E_g^{ab}}{\partial (\partial_c \partial_d \Phi)}, \\ P_{\Phi\Phi}^{ab} &= \frac{\partial E_\Phi}{\partial (\partial_a \partial_b \Phi)}, & P_{\Phi g}^{abcd} &= \frac{\partial E_\Phi}{\partial (\partial_c \partial_d g_{ab})}. \end{aligned} \quad (7)$$

In these equations, ∂_a denotes an arbitrary background derivative operator such as, e.g., the flat coordinate derivative associated with some coordinate system.

As shown by [10], the above expressions are expressible in terms of certain tensors C^{abcdef} , C^{abcd} , P^{ab} and the so called *effective metric* C_{ab} via

$$P_{gg}^{abcdef} = C^{a(c|e|b|d|f)}, \quad (8a)$$

$$C^{abcd} = P_{\Phi g}^{abcd}, \quad (8b)$$

$$P^{ab} = P_{\Phi\Phi}^{ab}, \quad (8c)$$

$$C^{a_1 a_2 a_3 b_1 b_2 b_3} = -\frac{1}{2} \epsilon^{a_1 a_2 a_3 a} \epsilon^{b_1 b_2 b_3 b} C_{ab}, \quad (8d)$$

where ϵ_{abcd} is the volume form of $(\mathcal{M}, g_{ab})$.

Reference [10], Sec. 4.1] has shown that the effective metric and the tensors C^{abcd} and P^{ab} of our EFT (4) take the following form:

$$C_{ab} = g_{ab} - \ell^2 \nabla_a \nabla_b \beta, \quad (9a)$$

$$C^{abcd} = -\frac{1}{4} \ell^2 \beta' \epsilon^{abe_1 e_2} \epsilon^{cdf_1 f_2} R_{e_1 e_2 f_1 f_2}, \quad (9b)$$

$$P^{ab} = -(1 + \ell^2 \alpha X) g^{ab} + \ell^2 \alpha \nabla^a \Phi \nabla^b \Phi. \quad (9c)$$

B. Propagation cones of EFT

It has been shown by [21] that the EoMs of the EFT admit a well-posed initial value formulation in a certain gauge, assuming that the theory is suitably “weakly

coupled”. By this, one means that given a codimension one surface $\Sigma \subset \mathcal{M}$ that is “noncharacteristic” in a suitable sense described more fully below, and given suitable initial data comprised of (Φ, g_{ab}) and its first derivatives off the surface Σ subject to the constraints of the theory (3), then one locally (in some open neighborhood of Σ) has one and only one solution to the EoMs.

In the case of a quasilinear system of partial differential equations (PDEs) for some set of fields Ψ^I , i.e., the highest derivative terms in the EoMs can be written as $P_J^{lab}(\Psi, \partial\Psi)\partial_a\partial_b\Psi^J + \dots = 0$, one calls a covector ξ_a at a point p *noncharacteristic* if the “principal symbol”, $P_J^{lab}(\Psi, \partial\Psi)\xi_a\xi_b|_p$, is an invertible matrix. This means that for a coordinate x^0 locally defining Σ by $x^0 = \text{const.}$ such that $\xi_a|_p = \nabla_a x^0|_p$, we can solve the equation for $\partial_0^2\Psi^J$ and iteratively determine all derivatives of Ψ^J at p off of Σ in terms of up to one derivative with respect to x^0 and the derivatives tangent to Σ , i.e., with respect to some coordinates $x^j, j = 1, 2, 3$ parameterizing Σ . Thus, if the initial data on Σ are smooth, then so will be the solution (if it exists) near p .

On the other hand, if a covector ξ_a at a point p is characteristic, which means that $P_J^{lab}(\Psi, \partial\Psi)\xi_a\xi_b|_p$ is not invertible, then we cannot solve the equation for $\partial_0^2\Psi^J$. As a consequence, the solution may develop discontinuities at characteristic surfaces, i.e., ones such that their (locally) defining function x^0 is such that $\nabla_a x^0 = \xi_a$ is characteristic at each point of the surface. In particular, it follows from such a consideration that the edge of the causal shadow of a point, or the boundary of the domain of dependence, must be characteristic at each point in the case of a quasilinear PDE, see, e.g., [15] for details. Even if the fields Ψ^I contain a Lorentz metric g_{ab} , the characteristic surfaces need not have anything to do with the light cones of this metric; see Fig. 4.

The EoMs of our EFT (4), viewed as a system for $\Psi^I = (\Phi, g_{ab})$, are not quasilinear, but one may still define a principal symbol by linearizing the EoMs, and one may still define the notion of a characteristic covector. In fact, as in [10][Sec. 3.2] we say that a covector ξ_a is *characteristic* if there exists a *nontrivial* (i.e., nongauge) $T = ([t_{ab}], t)$ such that $P(\xi)T = 0$, where

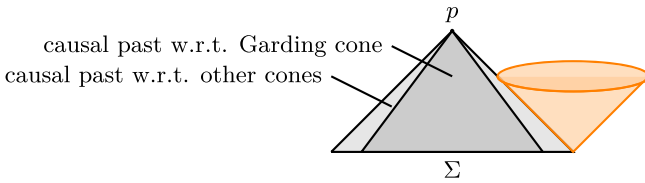


FIG. 4. The figure shows the causal pasts of a point p with respect to various cones terminating on some Cauchy surface Σ . Obviously, the domains of the dependence will be different.

$$P(\xi)T = \begin{pmatrix} P_{gg}^{abcdef}\xi_e\xi_f & P_{\Phi g}^{abef}\xi_e\xi_f \\ P_{g\Phi}^{cdef}\xi_e\xi_f & P_{\Phi\Phi}^{ef}\xi_e\xi_f \end{pmatrix} \begin{pmatrix} t_{cd} \\ t \end{pmatrix}, \quad (10)$$

and $P(\xi)$ is called the *principal symbol* of $E_g^{ab} = E_\Phi = 0$. The equivalence class is defined by $t_{ab} \sim t'_{ab}$ if there exists a covector X_a such that $t'_{ab} = t_{ab} + \xi_{(a}X_{b)}$, noting that $\xi_{(a}X_{b)}$ always is in the kernel of P . Such elements correspond to infinitesimal gauge transformations. A hypersurface $\Sigma \subset \mathcal{M}$ is said to be a *characteristic hypersurface* if its conormal is characteristic.

Just as for quasilinear equations, the edge of the domain of dependence of a suitably gauge-fixed [21] version of the linearized EoMs is characteristic in the above sense. It seems plausible to us that this will remain true even in the full nonlinear EFT (4), at least in the *weakly coupled* case. (We call the solution of our EFT weakly coupled if the contributions of the 4-derivative terms to $P(\xi)$ are small compared to the 2-derivative contributions, see [10], Sec. 4.1.) We leave this as an open problem and use the above notion of a characteristic covector to define the propagation cone of the theory, which, at any rate, will be reasonable for the solutions of the linearized theory, and which should be a reasonable approximation for small ℓ .

Reference [10], Sec. 4.3] showed that there is an equivalent condition for ξ_a to be characteristic in the following way: Let ξ_a be a nonzero real covector. Then, ξ_a is characteristic if and only if [10], Sec. 3.2]

$$p(\xi) = C^{-1}(\xi)Q(\xi) = 0, \quad (11)$$

where $C^{-1}(\xi) = (C^{-1})^{ab}\xi_a\xi_b$ and

$$Q(\xi) = \frac{1}{4!} Q^{abcd}\xi_a\xi_b\xi_c\xi_d, \quad (12a)$$

$$Q^{abcd} = \det(Cg^{-1})(C^{-1})^{(ab}P^{cd)} + (2C_{a_1a_2}C_{b_1b_2} - C_{a_1b_1}C_{a_2b_2})C^{a_1(ab|a_2}C^{b_1|cd)b_2}. \quad (12b)$$

Note that in the case of a 2-derivative theory [standard Einstein-scalar having $\ell = 0$ in (4)] we have $(C^{-1})^{ab} = -P^{ab} = g^{ab}$ and $C^{abcd} = 0$ [10], Sec. 4.4], so that the *characteristic cone* at a point $x \in \mathcal{M}$, $\{\xi_a \in T_x^*\mathcal{M} : p(\xi)|_x = 0\}$, is simply the null cone of the physical metric $g^{ab}|_x$ in $T_x^*\mathcal{M}$. Therefore, a covector ξ_a in a 2-derivative theory is characteristic if and only if ξ_a is null with respect to g^{ab} . In the following, we refer to $\{\xi_a \in T_x^*\mathcal{M} : C^{-1}(\xi)|_x = 0\}$ as the *quadratic cone* and to $\{\xi_a \in T_x^*\mathcal{M} : Q(\xi)|_x = 0\}$ as the *quartic cone*. As illustrated in Ref. [5], both cones are subsets of the characteristic cone, and the quartic cone consists of two sheets between which the quadratic cone lies [10], Sec. 4.4].

The characteristic cone provides the following notion of causality in the cotangent space:

Definition II.1. [10][Sec. 4.4]: Let $x \in \mathcal{M}$. Assume that the theory is weakly coupled (such that $g^{ab}|_x \approx (C^{-1})^{ab}|_x$) and fix a future pointing⁶ causal covector u_a in $T_x^*\mathcal{M}$ with respect to $(C^{-1})^{ab}|_x$. We call the *connected component* of $\{\xi_a \in T_x^*\mathcal{M} : p(\xi)|_x \neq 0\}$ that contains $\mp u_a$ the *Garding cone* $\Gamma_x^\pm$ at the point $x \in \mathcal{M}$.

The Garding double cone $\Gamma_x^+ \cup \Gamma_x^-$ is the region inside the innermost sheet of the quartic cone; i.e., in a 2-derivative theory, it is simply the set of covectors that are timelike with respect to the physical metric at a point [10][Sec. 4.4]. Starting from the Garding double cone, one can define a notion of causality in the tangent space:

Definition II.2. [10][Sec. 4.6, 4.7]: For $x \in \mathcal{M}$ define the *causal cone* $\mathcal{C}_x^\pm$ in $T_x\mathcal{M}$ as the dual of $\Gamma_x^\pm$, i.e.,

$$\mathcal{C}_x^\pm = \{X^a \in T_x\mathcal{M} : \xi_a X^a|_x \leq 0 \quad \forall \xi_a \in \Gamma_x^\pm\}. \quad (13)$$

A vector $X^a \in T_x\mathcal{M}$ is said to be

- (i) future- (past-) directed causal at x if and only if $X^a \neq 0$ and $X^a \in \mathcal{C}_x^+ (\mathcal{C}_x^-)$,
- (ii) future- (past-) directed timelike at x if and only if $X^a \in \overset{\circ}{\mathcal{C}}_x^+ (\overset{\circ}{\mathcal{C}}_x^-)$.

Here, a circle above a set means its interior.

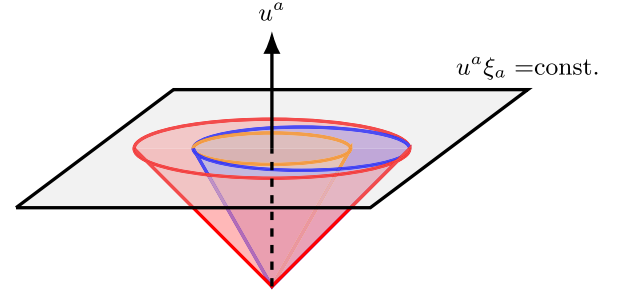


FIG. 5. Generic behavior of the characteristic cone in the *cotangent* space for a nonvanishing Weyl-like tensor (15) [10]. The red and orange cones illustrate the two sheets of the quartic cone. The blue cone is the quadratic part of the characteristic cone, which smoothly touches the quartic cones along two principal null directions of the Weyl-like tensor. The orange, innermost cone corresponds to the fastest degree of freedom, because its dual cone in the *tangent* space would be the outermost one. See also Sec. II B below for details.

A vector field X^a is called future- (past-) directed causal (timelike) if $X^a|_x$ is future- (past-) directed causal (timelike) for all $x \in \mathcal{M}$. A smooth curve γ on $\mathcal{M}$ is said to be future- (past-) directed causal (timelike) if its tangent vector field is future- (past-) directed causal (timelike). Furthermore, let S be some subset of $\mathcal{M}$. The sets

$$J^\pm(S) = \{x \in \mathcal{M} : \text{there exists a future- (past-) directed causal curve from } S \text{ to } x\}, \quad (14a)$$

$$I^\pm(S) = \{x \in \mathcal{M} : \text{there exists a future- (past-) directed timelike curve from } S \text{ to } x\}. \quad (14b)$$

are called *causal future (past)* and *chronological future (past)*.

In order to interpret the various cones, it is helpful to be aware that an inclusion relation between cones in the cotangent space is reversed when going to the dual cones, i.e., the tangent space.

C. Slowness surface of EFT

Taking any cross section of constant $\xi_a u^a$ of the characteristic cone at a point, one obtains the so-called *slowness surfaces* [10], consisting of three ellipsoids where the innermost one corresponds to the “fastest degree of freedom” (note the similarity with the null cone in 2-derivative theories) [10][Sec. 4.4]. On this surface there are special points at which two or more sheets of the slowness surface touch each other as shown in Fig. 5. Such

points on the slowness surface then correspond to a special direction of the characteristic cone by scaling freedom. They may be classified in the following way:

Definition II.3. [10][Sec. 4.4]: Suppose that ξ_{0a} belongs to the characteristic cone. Then ξ_{0a} is called

- (i) a singular direction if $\partial p / \partial \xi_a|_{\xi=\xi_0} = 0$ or, equivalently, ξ_{0a} belongs to the quadratic and the quartic cone,
- (ii) a double direction if ξ_{0a} is a singular direction of the characteristic cone and $\partial^2 p / \partial \xi_a \partial \xi_b|_{\xi=\xi_0} \neq 0$,
- (iii) a triple direction if ξ_{0a} is a singular direction of the quartic cone and $\partial^2 p / \partial \xi_a \partial \xi_b|_{\xi=\xi_0} = 0$.

Generically, at a triple point all three sheets of the slowness surface coincide, so the propagation speed is equal for all three polarizations [10][Sec. 4.4]. At a double point, only two sheets of the slowness surface—one of the quadratic- and one of a quartic cone—touch [10][Sec. 4.4], see Fig. 5.

Within the expression for $Q(\xi)$, one can split off a part that is determined by a “Weyl-like” tensor W^{abcd} given by

⁶We assume a time orientable setting in which a continuous choice of u_a can be made globally.

$$W^{abcd} = C^{abcd} - (C^{-1})^{a[c} D^{d]b} + (C^{-1})^{b[c} D^{d]a} + \frac{1}{3} D(C^{-1})^{a[c} (C^{-1})^{d]b}, \quad (15)$$

where D^{ab} is the “Ricci-like tensor”, and D is the “Ricci-like scalar” formed from C^{abcd} using the effective metric C_{ab} [10][Sec. 3.2]. Reference [10][Sec. 4.4] showed that a singular direction ξ_{0a} must be a principal null direction of this Weyl-like tensor.

D. Bicharacteristic curves of EFT

We have seen that the innermost sheet $\partial(\Gamma^+ \cup \Gamma^-)$ of the quartic cone plays the role of the fastest propagation direction. Thus, in analogy to the null cone in 2-derivative theories, we define the event horizon $\mathcal{H}^\pm = \partial(\mathcal{M} \setminus J^\mp(\mathcal{I}^\pm))$ just as in (1) but where the notions $J^\pm$ of the causal future (past) are now defined as above instead of by the physical metric. Thus, a horizon is a characteristic hypersurface with normal $\xi_a \in \partial\Gamma^\pm$.

A notion appearing in the assumptions of Theorem A below which we have not reviewed yet is that of a bicharacteristic curve.

Definition II.4. [10][Sec. 4.5]: A *bicharacteristic curve* of Q (12a) is a pair $(x(s), \xi_a(s))$ satisfying Hamilton’s equations

$$\dot{x}^a = \frac{\partial Q}{\partial \xi_a}, \quad \dot{\xi}_a = -\frac{\partial Q}{\partial x^a} \quad (16)$$

with the initial condition that $(x(0), \xi_a(0))$ belongs to the quartic cone. Here, a dot means derivative with respect to s , and $x(s)$ is called the *projection of a bicharacteristic curve*.

If we start a bundle of bicharacteristic curves on a codimension one surface on a noncharacteristic 3-surface Σ , then these curves locally generate a characteristic surface. In particular, the future/past horizons $\mathcal{H}^\pm$ locally are ruled by bicharacteristic curves.

Bicharacteristic curves of Q , see (12a), in 4-derivative theories are in close analogy to null geodesics of g_{ab} in 2-derivative theories which would be obtained from the Hamiltonian $H = g^{ab} \xi_a \xi_b$ instead of Q . For example, the future horizon (usually generated by null geodesics in standard GR) is generated by bicharacteristic curves of the innermost quartic sheet $\partial\Gamma^\pm$ in 4-derivative theories.

A difference between the usual case of null geodesics is that there is no canonical parametrization of the bicharacteristics associated with Q analogous to affine parametrization, in general. Furthermore, it may—and actually, in our situation below, will—happen that the Hamiltonian flow is trivial in the sense that $\dot{x}^a = 0$ on the characteristic surface.

E. Killing horizons in EFT

A *Killing horizon* in a spacetime $(\mathcal{M}, g_{ab})$ with a Killing vector field χ^a is a codimension one null surface $\mathcal{N}$ such that χ^a is both tangent and normal (meaning $\chi_a v^a = 0$ for all $v^a \in T\mathcal{N}$) to $\mathcal{N}$. Such a Killing vector field χ^a automatically satisfies

$$\chi^a \nabla_a \chi^b = \kappa \chi^b \text{ on } \mathcal{N}, \quad (17)$$

where κ is called the *surface gravity*. Equation (17) immediately gives $\chi^a \nabla_a \kappa = 0$ on $\mathcal{N}$. Furthermore, it is known [1] that if the Einstein tensor G_{ab} of g_{ab} satisfies the dominant energy condition, then κ is actually constant on all of $\mathcal{N}$. This property is called the zeroth law black hole thermodynamics in the case of black holes with Killing horizons.

In a theory with scalars, it is natural to expect that the zeroth law includes a statement that Φ is constant on the horizon. As [10][p.31] showed, the zeroth law holds in this sense within the setting of our EFT (4) for any weakly coupled solution (Φ, g_{ab}) to the EoMs that are Lie derived by χ^a ,

$$\mathcal{L}_\chi g_{ab} = \mathcal{L}_\chi \Phi = 0, \quad (18)$$

for any $\mathcal{N}$ which is a Killing horizon relative to χ^a . As a corollary⁷ of the rigidity theorem for general EFTs by [22], this property holds, in fact, much more generally for the horizon $\mathcal{H}$ defined with respect to g_{ab} of any stationary, asymptotically flat, real analytic, nondegenerate ($\kappa > 0$) black hole with compact horizon cross sections, of *any* EFT constructed from a diffeomorphism covariant Lagrangian, to arbitrary orders in the EFT scale ℓ . See also [23] for model-independent proofs of the zeroth law based on a t - q reflection symmetry.

If we have two Killing horizons ${}^{L,R}\mathcal{N}$ with the same χ^a and surface gravity $\kappa > 0$, intersecting in a (compact) 2-surface $\mathcal{B}$, one speaks of a nondegenerate (compactly generated) *bifurcate Killing horizon* [24], see Fig. 6. (A) deSitter, Minkowski, the extended subextremal Kerr, and Schwarzschild spacetimes including their (A)dS counterparts all host such structures.

Proposition II.5. Let $\mathcal{N}$ be a nondegenerate ($\kappa > 0$) bifurcate Killing horizon with Killing vector field χ^a which Lie derives the solution (Φ, g_{ab}) of the EFT (4). Then $\mathcal{N}$ is characteristic with respect to the quartic and quadratic cone, $Q^{abcd} \xi_a \xi_b \xi_c \xi_d = 0 = (C^{-1})^{ab} \xi_a \xi_b$ for all conormals ξ_a to $\mathcal{N}$. Furthermore,

⁷Reference [22] shows more generally that the horizon actually is a Killing horizon with respect to some Killing vector field χ^a that is not necessarily assumed from the beginning.

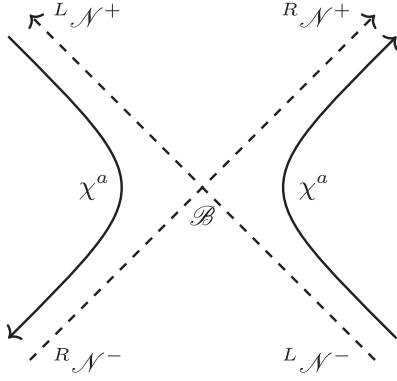


FIG. 6. Bifurcate Killing horizon.

$$\left. \frac{\partial Q}{\partial \xi_a} \right|_{\mathcal{N}} = 0, \quad (19)$$

for all characteristic covectors ξ_a at $\mathcal{N}$, all of which are singular triple directions in the sense of Definition II.3. *Proof.* The proof is quite similar in spirit to arguments by [11–13]. By the definition of Killing horizon, χ_a is conormal to $\mathcal{N}$. Thus, we must show $Q^{abcd}\chi_a\chi_b\chi_c\chi_d = 0 = (C^{-1})^{ab}\chi_a\chi_b$ on $\mathcal{N}$. Since Q^{abcd}, C_{ab} are locally and covariantly constructed out of (Φ, g_{ab}) , we have

$$\mathcal{L}_\chi Q^{abcd} = \mathcal{L}_\chi (C^{-1})^{ab} = 0. \quad (20)$$

Since $\mathcal{L}_\chi \chi_a = 0$, we thereby have $\mathcal{L}_\chi [Q^{abcd}\chi_a\chi_b\chi_c\chi_d] = 0$, $\mathcal{L}_\chi [(C^{-1})^{ab}\chi_a\chi_b] = 0$. Thus, the quantities in square brackets are constant along the orbits of χ^a . However, $Q^{abcd}\chi_a\chi_b\chi_c\chi_d = 0 = (C^{-1})^{ab}\chi_a\chi_b$ on $\mathcal{B}$, since χ^a vanishes there and since $Q^{abcd}, (C^{-1})^{ab}$ are smooth, in particular, finite, at $\mathcal{B}$.

To show (19), we restrict attention to ${}^R\mathcal{N}^+$, say, and introduce a second null vector field N^a uniquely defined by [18][Sec. II. B]

$$\nabla_a \chi_b = 2\kappa N_{[a}\chi_{b]}, \quad (21)$$

at points of ${}^R\mathcal{N}^+$. By uniqueness, we have $\mathcal{L}_\chi N^a = 0$. We complete χ^a, N^a to a null tetrad by introducing two additional vectors $s_i^a, i = 1, 2$ such that $\mathcal{L}_\chi s_i^a = 0$, $s_i^a s_{ja} = \delta_{ij}$, $s_i^a N_a = s_i^a \chi_a = 0$. By [18][Sec. II. B], the limits of $s_{ia}, N_a \chi_b$ toward $\mathcal{B}$ exist, and $Q^{abcd}N_a \chi_b \chi_c \chi_d$, as well as $Q^{abcd}s_{ia}\chi_b \chi_c \chi_d$, are Lie derived by χ^a . Since the limit of χ^a at $\mathcal{B}$ vanishes, these quantities vanish on ${}^R\mathcal{N}^+$, and by a similar argument for the other parts of $\mathcal{N}$, on all of $\mathcal{N}$. Since we already know that $Q^{abcd}\chi_a\chi_b\chi_c\chi_d = 0$ on $\mathcal{N}$, the proof of (19) is complete.

In view of $p(\xi) = C^{-1}(\xi)Q(\xi)$, the previous results imply that χ_a is a triple singular direction.

Note that the proof used neither the detailed structures of Q_{abcd}, C_{ab} nor the fact that (Φ, g_{ab}) is a solution to the

EoMs. In this paper, we are interested in the *converse* of Proposition II.5; i.e., we want to show that a horizon defined with respect to the propagation cones of our EFT is actually a Killing horizon. This will require the detailed structure of the propagation cone in our EFT and the EoMs in Secs. III, IV, and Appendix D.

F. Generalized Gaussian null coordinates

Whether or not g_{ab} satisfies any EoMs or other special equations, one can set up a convenient coordinate system in the neighborhood of any (part of a) Killing horizon referred to as *Gaussian null coordinates* (GNCs). We now review this construction; see, e.g., [25] for details. We assume that the Killing vector field χ^a is complete on $\mathcal{N}$, which we should think of as being either ${}^{R,L}\mathcal{N}^+$ in this section. Thus, its flow can be used to set up a diffeomorphism between $\mathcal{N}$ and $N \times \mathbb{R}$ such that the flow of χ^a is along the $\mathbb{R}$ Cartesian factor. We refer to N as a *cut* of $\mathcal{N}$; the orbits of χ^a must be transversal to this N .

First, we define local coordinates $x^A, A = 1, 2$ covering some subset of N . Let $\{F_v\}$ be the 1-parameter flow of isometries associated to χ^a and define $N(v) := F_v[N]$, i.e., on $\mathcal{N}$ we have $\chi^a = (\partial_v)^a$. Finally, at each point of $\mathcal{N}$, define a null vector field l^a satisfying $l^a \chi_a = 1$ and $l^a m_a = 0$ for all $m^a \in TN(v)$. By construction, $l^a = (\partial_u)^a$. Let u be the affine parameter of null geodesics off of each $N(v)$ with initial velocity l^a .

Then, in (v, u, x^A) coordinates, the metric takes the form [25]

$$g = -f dv^2 + 2dvdu + 2k_A dx^A dv + h_{AB} dx^A dx^B, \quad (22)$$

and u is a defining coordinate for $\mathcal{N}$.

We define the following tensors:

$$h_{ab} = h_{AB}(dx^A)_a(dx^B)_b, \quad k_a = k_A(dx^A)_a, \quad (23)$$

and q^{ab}, p^a_b are defined by

$$p^a_b = (\partial_A)^a(dx^A)_b, \quad q^{ab} = (h^{-1})^{AB}(\partial_A)^a(\partial_B)^b, \quad (24)$$

These tensors are independent of the chosen local coordinates x^A on N and thereby globally defined near $\mathcal{N}$. They satisfy

$$p^a_b k_a = k_b, \quad p^a_b p^c_d h_{ac} = h_{bd}, \quad p^a_b p^b_c = p^a_c, \quad (25a)$$

$$p^a_b = q^{ac} h_{cb}, \quad p^a_b l^b = p^a_b \chi^b = 0, \quad (25b)$$

$$\mathcal{L}_\chi k_a = \mathcal{L}_\chi f = \mathcal{L}_\chi l^a = \mathcal{L}_\chi q^{ab} = \mathcal{L}_\chi p^a_b = 0, \quad (25c)$$

near $\mathcal{N}$, as well as

$$k_a|_{\mathcal{N}} = f|_{\mathcal{N}} = 0, \quad p^a_b|_{\mathcal{N}} = g_{bc} q^{ca}|_{\mathcal{N}}. \quad (26)$$

The surface gravity κ is given by

$$\kappa = \frac{1}{2} \mathcal{L}_\ell f|_{\mathcal{N}}. \quad (27)$$

Conversely, if a metric is defined by a set of coordinates (v, u, x^A) as in (22) with metric components that are independent of v , then $u = 0$ defines a Killing horizon $\mathcal{N}$ with the Killing vector field $\chi^a = (\partial_v)^a$, provided $f, k_a = 0$ on $\mathcal{N}$.

The above construction can be repeated in an analogous manner also in the case that χ^a is tangent to $\mathcal{N}$ but not necessarily normal, as would have to be the case if $\mathcal{N}$ was not a null surface. This construction, which we need to apply to a characteristic surface $\mathcal{N}$ of the Garding cone below—not *a priori* known to be null—again yields coordinates (u, v, x^A) such that g_{ab} satisfies (22) and tensors $f, k_a, h_{ab}, p^a_b, q^{ab}$. If the pull back of g_{ab} to N is a Riemannian metric, as will always be the case for sufficiently small $|\ell|$ below, then so is h_{ab} , as for a Killing horizon. However, contrary to the case of a Killing horizon, f, k_a will *not*, in general, vanish on $\mathcal{N}$. We refer to these (u, v, x^A) as *generalized Gaussian null coordinates* (gGNC) in the following.

Below, we consider real analytic 1-parameter families of spacetimes $(\mathcal{M}(\ell), g_{ab}(\ell, x))$, surfaces $\mathcal{N}(\ell)$ tangent to a Killing field $\chi^a(\ell)$, cuts $N(\ell)$, as above, etc. Then, for each ℓ , we can introduce gGNC based on $N(\ell)$, which cover an open neighborhood of each $\mathcal{N}(\ell)$. We may then introduce an identification (diffeomorphism) between these neighborhoods by declaring points for different ℓ to be equal if their gGNCs coincide. After such an identification (“gauge choice”), we are dealing with only *one* spacetime, in which the locations of $\mathcal{N}$ and N , or χ^a , no longer vary with ℓ .

It should be noted, however, that if $\mathcal{N}$ was, e.g., a bifurcate Killing horizon (Fig. 6) for the $\ell = 0$ member of the family of metrics, then the metrics $g_{ab}(\ell, x)$ will *not* be smooth at $\mathcal{B}$ if $\mathcal{N}$ is not a Killing horizon (e.g., if it is not null) for $\ell \neq 0$.

Below, it is convenient to use a “purely angular” derivative operator D_a associated with gGNCs acting on purely angular tensors $T_{ab\dots c}$, i.e., ones which are projected by p^a_b . Its definition is

$$D_b T_{a_1\dots a_r} = p^c_b p^{d_1}_{a_1} \cdots p^{d_r}_{a_r} \nabla_c T_{d_1\dots d_r}. \quad (28)$$

Then, we have

$$0 = [\mathcal{L}_\chi, D_a] = D_a h_{bc} = D_a q^{bc} = D_a p^b_c = 0. \quad (29)$$

The components of the Riemann tensor R_{abcd} , contracted into p^a_b, χ^a , or l^a in arbitrary ways, may thereby be expressed in terms of contractions of k_a, f , their D_a

covariant derivatives, and $p^a_b, q^{ab}, \chi^a, l^a$. These expressions are provided in the Appendix A for later use.

III. THEOREM A

A. Assumptions

For Theorem A, we assume the following:

- (i) $(\mathcal{M}(\ell), \Phi(\ell, x), g_{ab}(\ell, x))$ is a 1-parameter family of solutions to the EFT (4) that is jointly real analytic in (x, ℓ) under some 1-parameter family of diffeomorphisms identifying each $\mathcal{M}(\ell)$ with some fixed $\mathcal{M}$.
- (ii) For $\ell = 0$, the solution has an ordinary, nondegenerate (i.e., $\kappa > 0$) Killing horizon, $\mathcal{H}^+$, with Killing vector field K^a .
- (iii) There is a family $\mathcal{H}^+(\ell)$ for $|\ell| < \ell_0$ ruled by (projections of) bicharacteristic curves of Q i.e., $\mathcal{H}^+(\ell)$ is characteristic (cotangent to the Garding cone of $(\Phi(\ell), g_{ab}(\ell))$).
- (iv) $\mathcal{H}^+(\ell)$ is analytically diffeomorphic to $H \times \mathbb{R}$ via a family of analytic diffeomorphisms depending analytically on ℓ . H is diffeomorphic to $\mathbb{S}^2$.
- (v) There are commuting vector fields $t^a(\ell, x), \phi^a(\ell, x)$, jointly analytic in (x, ℓ) , which Lie derive (18) the solution $(\Phi(\ell, x), g_{ab}(\ell, x))$ for all ℓ . $\phi^a(\ell)$ generates an isometric action of $U(1)$ and is tangent to the H Cartesian factor in (iv). (H, h_{ab}) has no Killing vector fields other than ϕ^a . $t^a(\ell)$ is complete on $\mathcal{H}^+(\ell)$ and may be chosen to be tangent to the $\mathbb{R}$ Cartesian factor in (iv).
There is a linear combination $\chi^a(\ell) = t^a(\ell) + \Omega(\ell)\phi^a(\ell)$ such that $\dot{\chi}^a = c\chi^a$; i.e., it is tangent to the bicharacteristics of Q (see Definition II.4), and either $c = 0$ or $c \neq 0$ on all of $\mathcal{H}^+(\ell)$.
- (vi) β' has no zeros.

See Sec. II for notions such as quartic cone, Garding cone, bicharacteristics, etc. referring to the propagation of signals in our EFT.

Before we state Theorem A, we comment on these assumptions. Analyticity in (i) is just a technical assumption which is needed since our proof works order by order in ℓ . If we were to assume that the $\ell = 0$ solution describes a stationary, analytic, asymptotically Minkowskian, nondegenerate black hole, then the Killing horizon structure would be a consequence of [25]. Here, we prefer not to make such a global assumption.

Assumption (iii) corresponds to the fact that, for an asymptotically Minkowskian or AdS spacetime, the (future) horizon $\mathcal{H}^+$ would be the boundary of $\mathcal{M} \setminus J^-(\mathcal{I}^+)$, where J^- is defined by the Garding cone (14a). In this setting, (iii) would simply be a consequence of this fact. Again, because our constructions related to Theorem A are entirely local, we prefer not to require the existence of $\mathcal{I}^+$, and to state our assumptions as local conditions on $\mathcal{H}^+$.

Assumption (iv) is required to set up our gGNC system (A1) in the proof and is of a purely technical nature. As already described in Sec. II F, we set up the gGNC based on $H(\ell)$ —the image of H under the identification in (iii)—for each ℓ based on some choice of Killing vector field $K^a(\ell)$ [compatible with $\phi^a(\ell)$] with complete orbits for each ℓ . Afterwards, we make an identification of the spacetimes $\mathcal{M}(\ell)$ based on these gGNCs. In this “gauge”, the spacetimes, $K^a, \phi^a, \mathcal{H}^+$, etc., will then be *independent* of ℓ . Analyticity of the identification is required for the same reason as in (i).

Assumption (v) is the essential and nontrivial one, in the following sense. We know [22] that for any stationary, analytic, asymptotically flat, nondegenerate black hole with the horizon defined relative to g_{ab} , and in any EFT, there exist K^a, ϕ^a, ω such that $K^a = t^a + \omega \phi^a$ is tangent to the null generators of this horizon (different *a priori* from $\mathcal{H}^+$), i.e., $\dot{x}^a \propto K^a$ for the ordinary characteristic curves (null geodesics) with tangent $\dot{x}^a$. We are effectively claiming that a similar statement remains true with regard to the characteristics defined by the propagation cone (quartic cone) in our EFT, for which K^a need not be equal to χ^a and Ω need not be equal to ω *a priori*.

B. Statement

Theorem A is the following statement:

Theorem III.7. Under the assumptions (i)–(vi), $\mathcal{H}^+$ is an ordinary Killing horizon with Killing vector field χ^a with respect to the metric g_{ab} for sufficiently small $|\ell| < \ell_0$.

Remarks. (1) We have already mentioned in Sec. II that the surface gravity κ of a Killing horizon in our EFT must be constant, as must be Φ . By general results about Killing horizons, χ^a is furthermore a repeated principal null direction of the Weyl tensor⁸ of g_{ab} , see, e.g., [1][Sec. 12.5].

(2) In the following, all statements are valid for sufficiently small $|\ell| < \ell_0$, where ℓ_0 is some positive number, and we no longer repeat this.

We can now apply the results of [10][Sec. 4.7] or Proposition II.5 to conclude:

Corollary III.9. χ_a is a triple direction (see Sec. II) of the characteristic cone on $\mathcal{H}^+$, and

$$\left. \frac{\partial^2 Q}{\partial \xi_a \partial \xi_b} \right|_{\xi=\chi} \propto \chi^a \chi^b. \quad (30)$$

In particular, the two sheets of the quartic cone and that of the quadratic cone all touch at points of $\mathcal{H}^+$ in the cotangent vector χ_a . See Fig. 3 for an illustration.

⁸By the results of [10][Sec. 4.4], it is at the same time a principal null direction of the Weyl-like tensor (15).

C. Outline of proof

For the proof, we construct gGNCs around $\mathcal{H}^+$ just as in Sec. II E. In the present case, $\mathcal{H}^+$ is not known *a priori* to be null with respect to g_{ab} , but the construction can nevertheless be carried just the same. To show that $\mathcal{H}^+$ is a Killing horizon, we must show that $f = k_a = 0$ on $\mathcal{H}^+$.

We expand all tensor fields or constants as power series in ℓ , as, e.g., in

$$k_a = \sum_{j=0}^{\infty} \ell^j k_{(j)a}, \quad (31a)$$

$$f = \sum_{j=0}^{\infty} \ell^j f_{(j)}, \quad (31b)$$

$$\Omega = \sum_{j=0}^{\infty} \ell^j \Omega_{(j)}, \quad (31c)$$

etc. These expansions are absolutely convergent by our analyticity assumptions. In our proof, we set up our gGNC system as in (22) at first not with respect to $\chi^a = t^a + \Omega \phi^a$ but initially instead for

$$K^a = t^a + \Omega_{(0)} \phi^a. \quad (32)$$

In the course of our inductive proof provided in detail in the Appendix D, we update the definition of K^a at each subsequent order in ℓ taking into account corrections to $\Omega_{(0)}$. By assumption, K^a Lie derives (Φ, g_{ab}) . Furthermore, by assumption, $\mathcal{H}^+$ is a Killing horizon for the zeroth order metric $g_{(0)ab}$ with respect to K^a , and as a consequence, we get that

$$f_{(0)}|_{\mathcal{H}^+} = k_{(0)a}|_{\mathcal{H}^+} = 0. \quad (33)$$

The idea of the proof is to verify the analogous statement for $f_{(j)}, k_{(j)a}$ for increasing $j = 1, 2, \dots$. For this, we must use the EoMs of our EFT, as well as our conditions for the characteristic surface $\mathcal{H}^+$ in our EFT.

These conditions are best analyzed by decomposing the metric, and its various expansion orders in ℓ , as in (22), with corresponding tensors $p^a_b, q^{ab}, l^a, h_{ab}, k^a = q^{ab}k_b$ and functions (coordinates) u, v , all of which are Lie derived by K^a and by ϕ^a , see Sec. II E and Appendix A. This analysis is rather involved, and the full technical details are therefore moved to Appendix D.

A series of lemmas first leads to the conclusion that $k^a_{(1)}|_{\mathcal{H}^+} = \omega_{(1)} \phi^a|_{\mathcal{H}^+}$, where $\omega_{(1)}$ is constant on $\mathcal{H}^+$, i.e. $k^a_{(1)}$ is an infinitesimal rotation on $\mathcal{H}^+$. It is also shown that

$$f_{(1)}|_{\mathcal{H}^+} = f_{(2)}|_{\mathcal{H}^+} = 0, \quad (34)$$

and that $(f + k^2)_{(3)}|_{\mathcal{H}^+} = 0$. Based on these insights, we redefine

$$K^a \rightarrow K^a - \ell \omega_{(1)} \phi^a, \quad (35)$$

which still Lie derives (Φ, g_{ab}) . Having made this redefinition, we set up our gGNC system as in (22) with respect to this new K^a . The corresponding new k_a, f are seen to satisfy all previous properties, and additionally,

$$f_{(3)}|_{\mathcal{H}^+} = k_{(1)a}|_{\mathcal{H}^+} = 0. \quad (36)$$

The remainder of the proof repeats the arguments for increasing orders in ℓ , assuming inductively that for some $n \geq 0$,

$$f_{(j)}|_{\mathcal{H}^+} = 0 \quad \text{for all } j \leq n + 3, \quad (37a)$$

$$k_{(j)a}|_{\mathcal{H}^+} = 0 \quad \text{for all } j \leq n + 1. \quad (37b)$$

By similar arguments as in the base case, these are seen to imply that $(f + k^2)_{(n+4)}|_{\mathcal{H}^+} = 0$ and that $k_{(n+2)}^a|_{\mathcal{H}^+} = \omega_{(n+2)} \phi^a|_{\mathcal{H}^+}$, where $\omega_{(n+2)}$ is constant on $\mathcal{H}^+$; i.e., it is proportional to a rotation.

Similarly, as in the base case, we redefine

$$K^a \rightarrow K^a - \ell^{n+2} \omega_{(n+2)} \phi^a, \quad (38)$$

which still Lie derives (Φ, g_{ab}) . Having made this redefinition, we set up our gGNC system as in (22) with respect to this new K^a . The corresponding new k_a, f are seen to satisfy all the previous properties, and additionally,

$$f_{(n+4)}|_{\mathcal{H}^+} = k_{(n+2)a}|_{\mathcal{H}^+} = 0, \quad (39)$$

thereby closing the induction loop.

After our inductive readjustments, the final Killing field K^a and gGNCs are by construction such that $K_a|_{\mathcal{H}^+} = (du)_a|_{\mathcal{H}^+}$ at points of $\mathcal{H}^+$; i.e., it is characteristic, and $\mathcal{H}^+$ is a Killing horizon with respect to K^a . Strictly speaking, we still need to show that the series defining K^a converges. This follows from analyticity in a fairly straightforward manner, but we do not go through the somewhat tedious details here.

IV. THEOREM B

A. Assumptions

For the formulation of Theorem B, we introduce the *domain of outer communication* $\mathcal{D} = I^+(\mathcal{I}^-) \cap I^-(\mathcal{I}^+)$ with respect to the propagation cones of our EFT (Sec. II B). We then define the future and past horizons as $\mathcal{H}^\pm = \partial\mathcal{D} \cap I^\pm(\mathcal{I}^\mp)$, as usual, but with respect to the propagation cones of our EFT.

We assume the following:

- (i) $(\mathcal{M}(\ell), \Phi(\ell, x), g_{ab}(\ell, x))$ is a 1-parameter family of solutions to the EFT (4) that is jointly real analytic in (x, ℓ) under some 1-parameter family of diffeomorphisms identifying each $\mathcal{M}(\ell)$ with some fixed $\mathcal{M}$.
- (ii) Each $\mathcal{M}(\ell)$ is asymptotically Minkowskian, and there is a vector field $t^a(\ell)$ that Lie derives $(\Phi(\ell), g_{ab}(\ell))$, that is jointly analytic in (x, ℓ) , that has complete orbits, and that is timelike in the asymptotic region. We require that $\mathcal{M}(\ell)$ includes an open neighborhood of $\mathcal{H}^+(\ell)$.
- (iii) For $\ell = 0$, $\mathcal{H}^+$ forms a nondegenerate ($\kappa > 0$) Killing horizon. The pull back of $g_{ab}(\ell)$ to $\mathcal{H}^+(\ell)$ has an isometry ι fixing some cut $H(\ell)$, such that⁹ $\iota_* t^a(\ell) = -t^a(\ell)$, $\iota^2 = id$. The complex structure of the cut $H(\ell)$ satisfies

$$\iota^* J^a_b = -J^a_b. \quad (40)$$

- (iv) For the $\ell = 0$ solution, we have

$$V(\Phi) \geq 0 \quad (41)$$

on $\mathcal{H}^+$.

- (v) The restriction of t^a tangent to $\mathcal{B}$ in the $\ell = 0$ solution is not vanishing identically; i.e., the black hole is rotating.

As we explain in more detail in the Appendix E, (iii') states in a covariant way that ι is a kind of “ t - φ ” reflection isometry on $\mathcal{H}^+$, without actually introducing¹⁰ coordinates t or φ . Note that we only require the symmetry ι locally, i.e., for the pull back to $\mathcal{H}^+$. This geometric feature is present in Kerr, but the reflection ι on $\mathcal{H}^+$ is different from the global t - φ reflection isometry of Kerr which exchanges $\mathcal{H}^\pm$, assuming such a global symmetry would enable a “purely kinematical” proof similar to that of Proposition II. 5.

To picture the hypothetical case that $\mathcal{H}^+$ might not be a Killing horizon, we may imagine that $\mathcal{H}^+$ as defined by the Garding cone is a timelike surface with respect to g_{ab} tangent to a boostlike Killing field K^a similar to Fig. 1. The future pointing Garding cones must then tilt inwards relative to those of g_{ab} everywhere along $\mathcal{H}^+$, so that $\mathcal{H}^- = \emptyset$. We note that there is an evident asymmetry between past and future, which is already somewhat at odds with thermodynamic intuition.

Apart from the global assumptions such as the reflection isometry and asymptotic flatness, or topology of $\mathcal{H}^+$, the main differences between these assumptions for Theorem B and those formulated for Theorem A in Sec. III are as follows: we do not require, *a priori*, any rotational vector field that Lie derives (Φ, g_{ab}) , and we do not require that

⁹ ι could depend discontinuously on ℓ .

¹⁰Note that we do not know at this stage that there is a Killing field “ $\phi^a = (\partial_\varphi)^a$ ”.

the bicharacteristics are tangent to a vector field that Lie derives (Φ, g_{ab}) . On the other hand, we need the stability requirement $V \geq 0$.

If we were to assume instead of (iv') that the $\ell = 0$ black hole solution is *nonrotating*, then the solution would be static [26]. As shown in the proofs of the uniqueness of static black hole solutions for a wide class of Einstein-scalar field theories, see, e.g., [27–29], the $\ell = 0$ solution $(\mathcal{M}, \Phi, g_{ab})$ would in fact have to be *spherically symmetric*. An expansion argument in ℓ would then show that this remains true for $\ell \neq 0$ [22][Sec. 4]. We do not treat explicitly this comparatively easier case, but see, e.g., [11] for corresponding arguments, leading to the same conclusion as our Theorem B in the spherically symmetric case.

B. Statement

Theorem B is the following statement:

Theorem IV.10. Under the assumptions (i')–(v'), $\mathcal{H}^+$ is an compactly generated Killing horizon with respect to the metric g_{ab} .

Remark. (1) Again, we also have Corollary III. 2, and the proof also shows that the symmetry group of (Φ, g_{ab}) is (at least) $\mathbb{R} \times U(1)$.

(2) If we make the assumptions fully symmetric in past and future, then we analogously get a *bifurcate* Killing horizon.

(3) By (iv'), the scalar field stress energy tensor of the $\ell = 0$ theory satisfies the dominant energy condition on $\mathcal{H}^+$. By Hawking's topology theorem [30], therefore $\mathcal{B} \cong \mathbb{S}^2$.

C. Proof

We argue in a broadly similar way as in the proof of Theorem A. First, we get a second continuous symmetry of (Φ, g_{ab}) . For $\ell = 0$, t^a is tangent to $\mathcal{B} = \partial\mathcal{H}^+$; i.e., its restriction to $\mathcal{B}$ must be $\Omega\phi^a$, where ϕ^a has 2π -periodic orbits. By the arguments of [22], this ϕ^a extends in view of (i')–(v'), to a rotational vector field $\phi^a(\ell)$ Lie deriving $(\Phi(\ell), g_{ab}(\ell))$ on the entire spacetime for $|\ell| < \ell_0$ and a sufficiently small but positive ℓ_0 .¹¹

Next, we introduce gGNCs (A1) around $\mathcal{H}^+(\ell)$, based on the cut $H(\ell)$ as in (iii'), initially taking $K^a = t^a + \Omega_{(0)}\phi^a$. (As in the proof of theorem A, this K^a is subsequently updated.) Again, we identify points of $\mathcal{M}(\ell)$ for different values of ℓ if these gGNCs coincide. Thereby, we get the corresponding tensor fields $l^a, K^a, \phi^a, p^a_b, k_a, q^{ab}, h_{ab}$, see (24) in Sec. IIF and

¹¹Reference [22] considered the horizons defined by the light cones of the metric g_{ab} , not the Garding cones. Of course, we are free also in our situation to consider the metric light cones to make the argument, since the Killing field extends to as a single-valued object (due to $\pi_1(\mathcal{D}) = 0$ [31]) to all of $\mathcal{D}$ by analyticity.

Appendix A for details. After our identification, $K^a, p^a_b, t^a, \phi^a, H, \mathcal{H}^+$ are independent of ℓ .

As in the proof of Theorem A, we next make power series expansions of all quantities as in, e.g., (31a), with the aim of showing $f = k_a = 0$ on $\mathcal{H}^+$ —this implies that K^a is tangent and normal to $\mathcal{H}^+$; i.e., it is a Killing horizon.

Since for $\ell = 0$, we have an ordinary Einstein-scalar field theory, the results by [25] show that $\mathcal{H}^+$ is a Killing horizon with respect to K^a with constant surface gravity $\kappa = \frac{1}{2}\mathcal{L}_l f_{(0)}$ and constant $\Phi_{(0)}$. Thus, we have $D_a\Phi^{(0)} = f_{(0)} = k_{(0)a} = K_{(0)ab} = 0$ on $\mathcal{H}^+$.

Analogously to the proof of Theorem A in the Appendix D, we can also evaluate the condition $Q^{abcd}(du)_a(du)_b(du)_c(du)_d = 0$, expressing that $\mathcal{H}^+$ is characteristic, at subsequent orders in ℓ . At order ℓ^2 , (D31) first shows that $(f + k^2)_{(1)} = 0$ on $\mathcal{H}^+$. Since we already know that $f_{(0)} = k_{(0)a} = 0$ on $\mathcal{H}^+$, we get $f_{(1)} = 0$ on $\mathcal{H}^+$.

Consider a change $v \rightarrow v + \lambda(x^A)$; i.e., we change the cut H of $\mathcal{H}^+$ by an angle-dependent amount. The gGNC associated with the new cut is such that

$$\mathcal{L}_l k_{(0)a} \rightarrow \mathcal{L}_l k_{(0)a} - 2\kappa D_a \lambda. \quad (42)$$

There is a real analytic solution λ to

$$D^a D_a \lambda = \frac{1}{2\kappa} D^a \mathcal{L}_l k_{(0)a} \text{ on } H, \quad (43)$$

by standard results about elliptic partial differential equations, see, e.g., [32]. Here and in the rest of the proof, D_a temporarily denotes the angular covariant derivative with respect to $h_{(0)ab}$. In the new real analytic gGNC, we have

$$D^a \mathcal{L}_l k_{(0)a} = 0 \Rightarrow \mathcal{L}_l k_{(0)a} = \epsilon_{ab} D^b F_{(1)}, \quad (44)$$

for some $F_{(1)}$ for which $\mathcal{L}_\phi F_{(1)} = \mathcal{L}_K F_{(1)} = 0$ on $\mathcal{H}^+$. This implies that

$$(D_a \mathcal{L}_l k_{(0)b}) \phi^a \phi^b = 0, \quad (45)$$

on $\mathcal{H}^+$. We also know that $k^a_{(1)} = \omega_{(1)} \phi^a$ for some function $\omega_{(1)}$ since we have ι in (iii'), see Appendix E.

Our aim is to show that $\omega_{(1)}$ is constant on $\mathcal{H}^+$. For this, we first transvect the EoM (5) of our EFT for the metric into $K^a p^b_c$. Evaluating the result at order ℓ yields

$$R_{(1)ab} K^a p^b_c = \frac{1}{2} k_{(1)c} V(\Phi_{(0)}). \quad (46)$$

For the Ricci-tensor component on the left side, we employ the expressions in the Appendix A at order ℓ , which yields

$$\begin{aligned}
0 = D^b D_{[a} k_{(1)b]} - \left[2\kappa \bar{K}_{(0)ab} + \frac{1}{2} h_{(0)ab} V(\Phi_{(0)}) + \frac{1}{2} \mathcal{L}_l k_{(0)a} \mathcal{L}_l k_{(0)b} \right] k_{(1)}^b \\
+ \mathcal{L}_l k_{(0)[a} D_{b]} k_{(1)}^b - \frac{1}{2} D_a \mathcal{L}_l f_{(1)} + \left[D_b \mathcal{L}_l k_{(0)a} - \frac{1}{2} D_a \mathcal{L}_l k_{(0)b} \right] k_{(1)}^b, \quad (47)
\end{aligned}$$

on $\mathcal{H}^+$. Here, we use the shorthand $k^a = q^{ab} k_b$, disregarding temporarily our usual convention that indices are raised with g^{ab} .

Next, we transvect (47) with ϕ^a using (45) and $k_{(1)}^a = \omega_{(1)} \phi^a$. In terms of the uniformizing angular coordinates (ϑ, φ) on $H \cong \mathbb{S}^2$ [see Eq. (D44)], setting $\xi = \cos \vartheta$ and $y = k_{(1)a} \phi^a$, we obtain the ordinary differential equation,

$$\left[\frac{d}{d\xi} \left(\Psi^{-1} \frac{d}{d\xi} \right) + W \right] y = 0, \quad (48)$$

where

$$W = \frac{4\kappa \langle \phi \otimes \phi, \bar{K}_{(0)} \rangle + V(\Phi_{(0)}) \|\phi\|^2 + \langle \phi, \mathcal{L}_l k_{(0)} \rangle^2}{\|\phi\|^2}. \quad (49)$$

Here, we use the compact notations $\langle \cdot, \cdot \rangle, \|\cdot\|$ for angular tensor contractions introduced in (D16) and (D17) (applied to $q_{(0)}^{ab}$).

The function $W = W(\xi)$ is regular on $[-1, 1]$ up to and including the boundary, as is $\Psi^{-1} = \Psi(\xi)^{-1}$ [see (D44)]. $y = y(\xi)$ is regular and vanishes at the boundary. There can be at most one such function up to rescaling, and this function is, in fact, given by $y = \|\phi\|^2$. To see this, note that if we were to change $K^a \rightarrow K^a - \ell \omega_{(1)} \phi^a$ for a *constant* $\omega_{(1)}$ and then used this new K^a to construct or gGNCs, then in these coordinates $k_{(1)}^a$ would be replaced by $k_{(1)}^a - \omega_{(1)} \phi^a$, see Lemma D.10. Thus, $\omega_{(1)} \phi^a$ with constant $\omega_{(1)}$ solves (47). This must in fact be the only solution. since any other solution $y = k_{(1)a} \phi^a$ to (48) would be nonvanishing at least at one boundary point $\xi = \pm 1$, corresponding to a $k_{(1)}^a$ that is irregular at the north/south pole of the sphere H .

As in the proof of Theorem A in the Appendix D, we next update our initial guess of K^a to $K^a \rightarrow K^a - \ell \omega_{(1)} \phi^a$, where $\omega_{(1)}$ is the constant such that $k_{(1)}^a = \omega_{(1)} \phi^a$. In the gGNCs based on the new K^a , we then have $k_{(0)}^a = k_{(1)}^a = f_{(0)} = f_{(1)} = 0$ on $\mathcal{H}^+$.

The next step is to evaluate the condition (D31) at order ℓ^4 , giving that $(f + k^2)_{(2)} = 0$ on $\mathcal{H}^+$. Thus, we additionally get $f_{(2)} = 0$ on $\mathcal{H}^+$. Then furthermore $(f + k^2)_{(3)} = 0$ by (D37a).

This set of arguments is now repeated in an inductive manner similar to Appendix D 3 b for Theorem A. We

assume that, on $\mathcal{H}^+$, $f_{(j)} = 0$ for all $j \leq n+1$, $k_{(j)a} = 0$ for all $j \leq n$, and $(f + k^2)_{(n+2)} = 0$. Again, we transvect the EoM (5) of our EFT for the metric into $K^a p^b{}_c$ and now evaluate the result at $O(\ell^{n+1})$. This yields (46) with $k_{(1)a} \rightarrow k_{(n+1)a}$. The reason for this is the following: the terms with explicit powers of ℓ in (5) all vanish at this order by the induction hypothesis, as a consequence of the fact¹² that a contraction $p^a{}_b K^c T_{ac}$ of a local covariant tensor T_{ab} built from (Φ, g_{ab}) vanishes on a Killing horizon with respect to K^a .

Proceeding next along the same lines as before gives $k_{(n+1)}^a = \omega_{(n+1)} \phi^a$ for a constant $\omega_{(n+1)}$, which we use to update $K^a \rightarrow K^a - \ell^{n+1} \omega_{(n+1)} \phi^a$. In the new gGNC based on this new K^a , we additionally have $k_{(n+1)a} = 0$ on $\mathcal{H}^+$. When combined with $(f + k^2)_{(n+2)} = 0$ and the induction hypothesis, we get $f_{(n+2)} = 0$ on $\mathcal{H}^+$. Finally, (D37a) which holds at this stage with the replacements $k_{(1)a} \rightarrow k_{(n+1)a}$ and $(f + k^2)_{(3)} \rightarrow (f + k^2)_{(n+3)}$ then tells us that $(f + k^2)_{(n+3)} = 0$, thus closing the induction loop.

V. OUTLOOK

While our complementary Theorems A and B cover a fairly general class of EFTs and solutions, it would be interesting to remove some of their technical assumptions. Concerning theories, one should try to include, at least at the derivative order in the Lagrangian considered here, multiple scalar fields and Abelian vector fields while staying within the class of theories with second order EoMs.

Concerning the assumptions in Theorem B, the reflection symmetry condition at $\mathcal{H}^+$ makes intuitive sense for equilibrium solutions in theories as our EFTs that do not introduce an explicit time direction such as Einstein-aether theories¹³ [9]. The other assumption, $V \geq 0$, is also well motivated as a stability condition and, e.g., required in other standard theorems in general relativity, such as the positive mass theorem.

Concerning Theorem A, the most restrictive assumption is that the Killing orbits line up with the bicharacteristics of the propagation cone. To remove this, one would evidently

¹²The proof of this well known result is of the same nature as the proof of Proposition II.5.

¹³In fact, [9] have found stationary black hole solutions in such theories which do not have this reflection symmetry and which do not have Killing horizons.

have to bring in the global structure of spacetime, which is not considered in Theorem A as all its assumptions are of a local nature.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this paper. Requests for further information or data should be sent to the authors.

APPENDIX A: CURVATURE COMPONENTS IN GGNCS

Below we provide the expressions for the curvature components of the following metric given by

$$g = 2dvdu - f dv^2 + 2k_A dx^A dv + h_{AB} dx^A dx^B. \quad (A1)$$

(For equivalent expressions in a different notation, see [19].) The nonzero components of the inverse metric are

$$g^{uu} = (f + k^2), \quad g^{uv} = 1, \quad g^{uA} = -k^A, \quad g^{AB} = h^{AB}, \quad (A2)$$

where h^{AB} is the inverse of h_{AB} , $k^A \equiv h^{AB} k_B$ and $k^2 \equiv k_A k^A$. We do not assume in this section that $k_A = f = 0$ for $u = 0$, as would be the case for the usually defined GNCs off of the null surface $\mathcal{N}$ defined by $u = 0$.

We introduce $l^a = (\partial_u)^a$, $K^a = (\partial_v)^a$, p^a_b , h_{ab} , D_a as in (24), (28), (23), and the following variables:

$$K_{ab} \equiv \frac{1}{2} \mathcal{L}_K h_{ab}, \quad \bar{K}_{ab} \equiv \frac{1}{2} \mathcal{L}_l h_{ab}, \quad (A3)$$

and

$$K = q^{ab} K_{ab}, \quad \bar{K} = q^{ab} \bar{K}_{ab}. \quad (A4)$$

In this Appendix, K^a is not assumed to be a Killing vector field, though in the main part of the paper, it is, resulting in simplified expressions obtained by dropping $\mathcal{L}_K k_a$, $\mathcal{L}_K f$, K_{ab} , K everywhere.

The nonzero Christoffel symbols are given by

$$K^a l^b \nabla_a (du)_b = -\frac{1}{2} \mathcal{L}_l f - \frac{1}{2} q^{ab} k_b \mathcal{L}_l k_a, \quad (A5a)$$

$$K^a l^b (\partial_A)^c \nabla_a (dx^A)_b = \frac{1}{2} q^{ca} \mathcal{L}_l k_a, \quad (A5b)$$

$$l^a p^b_c \nabla_a (du)_b = \frac{1}{2} \mathcal{L}_l k_c - q^{ab} k_b \bar{K}_{ca}, \quad (A5c)$$

$$l^a p^b_c (\partial_A)^d \nabla_a (dx^A)_b = \bar{K}^d_c, \quad (A5d)$$

$$K^a K^b \nabla_a (dv)_b = \frac{1}{2} \mathcal{L}_l f, \quad (A5e)$$

$$K^a K^b \nabla_a (du)_b = -\frac{1}{2} \mathcal{L}_K f + \frac{1}{2} (f + k^2) \mathcal{L}_l f - q^{ab} k_b \left(\mathcal{L}_K k_a - \frac{1}{2} D_{af} \right), \quad (A5f)$$

$$K^a K^b (\partial_A)^c \nabla_a (dx^A)_b = \frac{1}{2} q^{ca} k_a \mathcal{L}_l f + q^{ca} \left(\mathcal{L}_K k_a + \frac{1}{2} D_{af} \right), \quad (A5g)$$

$$K^a p^b_c \nabla_a (dv)_b = -\frac{1}{2} \mathcal{L}_l k_c, \quad (A5h)$$

$$K^a p^b_c \nabla_a (du)_b = -\frac{1}{2} (f + k^2) \mathcal{L}_l k_c - \frac{1}{2} D_{cf} - q^{ab} k_b D_{[c} k_{a]} - q^{ab} k_b K_{ca}, \quad (A5i)$$

$$K^a p^b_c (\partial_A)^d \nabla_a (dx^A)_b = \frac{1}{2} q^{da} k_a \mathcal{L}_l k_c - q^{da} D_{[a} k_{c]} + q^{da} K_{ac}, \quad (A5j)$$

$$p^a_b p^c_d \nabla_a (dv)_c = -\bar{K}_{bd}, \quad (A5k)$$

$$p^a_b p^c_d \nabla_a (du)_c = -(f + k^2) \bar{K}_{bd} + D_{(b} k_{d)} - K_{bd}, \quad (A5l)$$

$$p^a_b p^c_d (\partial_A)^e \nabla_a (dx^A)_c = p^a_b p^c_d (\partial_A)^e D_a (dx^A)_c + q^{ea} k_a \bar{K}_{bd}. \quad (A5m)$$

The components of the Riemann tensor are as follows:

$$l^a K^b l^c K^d R_{abcd} = \frac{1}{4} q^{ab} \mathcal{L}_l k_a \mathcal{L}_l k_b + \frac{1}{2} \mathcal{L}_l \mathcal{L}_l f, \quad (A6a)$$

$$l^a K^b l^c p^d_e R_{abcd} = \frac{1}{2} q^{ab} \bar{K}_{eb} \mathcal{L}_l k_a - \frac{1}{2} \mathcal{L}_l \mathcal{L}_l k_e, \quad (A6b)$$

$$l^a K^b K^c p^d_e R_{abcd} = -\frac{1}{2} \mathcal{L}_K \mathcal{L}_l k_e - \frac{1}{2} q^{ab} K_{ea} \mathcal{L}_l k_b + q^{ab} \bar{K}_{ea} \mathcal{L}_K k_b - \frac{1}{4} q^{ab} k_a \mathcal{L}_l k_e \mathcal{L}_l k_b - \frac{1}{2} q^{ab} \mathcal{L}_l k_b D_{[e} k_{a]} - \frac{1}{2} D_e \mathcal{L}_l f + \frac{1}{2} q^{ab} \bar{K}_{eb} D_{af} - \frac{1}{2} q^{ab} k_b \bar{K}_{ea} \mathcal{L}_l f, \quad (A6c)$$

$$l^a p^b{}_c l^d p^e{}_f R_{abde} = q^{ab} \bar{K}_{cb} \bar{K}_{fa} - \mathcal{L}_l \bar{K}_{cf}, \quad (\text{A6d})$$

$$\begin{aligned} l^a p^b{}_c K^d p^e{}_f R_{abde} = & -\mathcal{L}_K \bar{K}_{cf} + q^{ab} K_{cb} \bar{K}_{fa} + q^{ab} k_b \bar{K}_{f[a} \mathcal{L}_l k_{c]} + q^{ab} \bar{K}_{fb} D_{[c} k_{a]} + \frac{1}{2} D_f \mathcal{L}_l k_c - \frac{1}{4} \mathcal{L}_l k_c \mathcal{L}_l k_f \\ & - \frac{1}{2} \bar{K}_{cf} \mathcal{L}_l f, \end{aligned} \quad (\text{A6e})$$

$$\begin{aligned} K^a p^b{}_c K^d p^e{}_f R_{abde} = & -\mathcal{L}_K K_{cf} + q^{ab} K_{cb} K_{fa} - 2q^{ab} k_b \bar{K}_{c[f} \mathcal{L}_l k_{a]} - 2q^{ab} k_b \bar{K}_{f[c} \mathcal{L}_l k_{a]} + \frac{1}{4} (f + k^2) \mathcal{L}_l k_c \mathcal{L}_l k_f \\ & + \frac{1}{2} \mathcal{L}_l k_{(c} D_{f)} f - q^{ab} k_b \mathcal{L}_l k_{(c} D_{f)} k_a - q^{ab} K_{a(c} D_{b} k_{f)} + q^{ab} K_{a(c} D_{f)} k_b + D_{(f} \mathcal{L}_K k_{c)} \\ & + \frac{1}{2} D_f D_c f - \frac{1}{2} \bar{K}_{cf} q^{ab} k_b D_a f + \frac{1}{2} q^{ab} k_b \mathcal{L}_l k_{(c} D_a k_{f)} + q^{ab} D_{[a} k_{c]} D_{[b} k_{f]} + \frac{1}{2} K_{cf} \mathcal{L}_l f \\ & + \frac{1}{2} (f + k^2) \bar{K}_{cf} \mathcal{L}_l f + \frac{1}{2} D_{(c} k_{f)} \mathcal{L}_l f - \frac{1}{2} \bar{K}_{cf} \mathcal{L}_K f, \end{aligned} \quad (\text{A6f})$$

$$\begin{aligned} p^a{}_b p^c{}_d K^e p^f{}_g R_{acef} = & -2D_{[b} K_{d]g} + \mathcal{L}_l k_{[b} K_{d]g} - 2q^{ac} K_{a[b} \bar{K}_{d]g} k_c - \mathcal{L}_l k_{[b} \bar{K}_{d]g} (f + k^2) + \bar{K}_{g[b} D_{d]} f \\ & + \frac{1}{2} \mathcal{L}_l k_{[b} D_{d]} k_g - \frac{1}{2} D_g k_{[b} \mathcal{L}_l k_{d]} - D_g D_{[b} k_{d]} + q^{ac} \bar{K}_{g[b} k_c D_{d]} k_a - q^{ac} k_a \bar{K}_{g[b} D_c k_{d]}, \end{aligned} \quad (\text{A6g})$$

$$p^a{}_b p^c{}_d l^e p^f{}_g R_{acef} = 2D_{[d} \bar{K}_{b]g} - 2q^{ac} \bar{K}_{a[b} \bar{K}_{d]g} k_c + \mathcal{L}_l k_{[b} \bar{K}_{d]g}, \quad (\text{A6h})$$

$$\begin{aligned} p^a{}_b p^c{}_d p^e{}_f p^g{}_h R_{aceg} = & R[h]_{bdfh} + 2K_{d[f} \bar{K}_{h]b} + 2K_{b[h} \bar{K}_{f]d} + 2(f + k^2) \bar{K}_{b[h} \bar{K}_{f]d} - \bar{K}_{h[d} D_{b]} k_f - \bar{K}_{f[d} D_{b]} k_h \\ & + \bar{K}_{d[h} D_{f]} k_b - \bar{K}_{b[h} D_{f]} k_d. \end{aligned} \quad (\text{A6i})$$

The components of the Ricci tensor are given by the following expressions:

$$\begin{aligned} K^a K^b R_{ab} = & -q^{ab} \mathcal{L}_K K_{ab} + q^{ac} q^{bd} K_{ab} K_{cd} - \bar{K} q^{ab} k_b \mathcal{L}_l k_a + 2q^{ac} q^{bd} \bar{K}_{cd} k_a \mathcal{L}_l k_b + q^{ac} q^{bd} \bar{K}_{ab} k_c D_d f - \frac{1}{2} \bar{K} q^{ab} k_b D_a f \\ & - q^{ab} q^{cd} k_b \mathcal{L}_l k_c D_a k_d + q^{ab} D_b \mathcal{L}_K k_a - q^{ab} k_b D_a \mathcal{L}_l f + q^{ab} q^{cd} k_b \mathcal{L}_l k_c D_d k_a + \frac{1}{2} q^{ab} D_b D_a f - \frac{1}{2} \bar{K} \mathcal{L}_K f \\ & - q^{ac} q^{bd} D_{[a} k_{b]} D_d k_c + \frac{1}{2} K \mathcal{L}_l f + \frac{1}{2} (f + k^2) \bar{K} \mathcal{L}_l f - q^{ab} k_b \mathcal{L}_K \mathcal{L}_l k_a - \bar{K}_{ab} k^a k^b \mathcal{L}_l f + \frac{1}{2} q^{ab} D_a k_b \mathcal{L}_l f \\ & + \frac{1}{2} (f + k^2) \mathcal{L}_l \mathcal{L}_l f + \frac{1}{2} (f + k^2) q^{ab} \mathcal{L}_l k_a \mathcal{L}_l k_b - \frac{1}{2} q^{ab} q^{cd} k_a k_c \mathcal{L}_l k_b \mathcal{L}_l k_d + \frac{1}{2} q^{ab} \mathcal{L}_l k_a D_b f, \end{aligned} \quad (\text{A7a})$$

$$\begin{aligned} l^a K^b R_{ab} = & -q^{ab} \mathcal{L}_K \bar{K}_{ab} + q^{ac} q^{bd} K_{cd} \bar{K}_{ab} - \frac{1}{2} \bar{K} q^{ab} k_b \mathcal{L}_l k_a + q^{ac} q^{bd} \bar{K}_{cd} k_a \mathcal{L}_l k_b - \frac{1}{2} \mathcal{L}_l \mathcal{L}_l f - \frac{1}{2} q^{ab} \mathcal{L}_l k_a \mathcal{L}_l k_b \\ & + \frac{1}{2} q^{ab} D_b \mathcal{L}_l k_a - \frac{1}{2} \bar{K} \mathcal{L}_l f - \frac{1}{2} q^{ab} k_b \mathcal{L}_l \mathcal{L}_l k_a, \end{aligned} \quad (\text{A7b})$$

$$l^a l^b R_{ab} = q^{ac} q^{bd} \bar{K}_{ab} \bar{K}_{cd} - q^{ab} \mathcal{L}_l \bar{K}_{ab}, \quad (\text{A7c})$$

$$l^a p^b{}_c R_{ab} = q^{ab} D_b \bar{K}_{ca} - D_c \bar{K} + \frac{1}{2} \mathcal{L}_l \mathcal{L}_l k_c + \frac{1}{2} \bar{K} \mathcal{L}_l k_c - q^{ab} \bar{K}_{cb} \mathcal{L}_l k_a + q^{ab} k_b (2q^{de} \bar{K}_{ce} \bar{K}_{ad} - \bar{K}_{ca} \bar{K} - \mathcal{L}_l \bar{K}_{ca}), \quad (\text{A7d})$$

$$\begin{aligned} K^a p^b{}_c R_{ab} = & q^{ab} D_a K_{cb} - D_c K - \frac{1}{2} \mathcal{L}_K \mathcal{L}_l k_c - \frac{1}{2} K \mathcal{L}_l k_c + q^{ab} \bar{K}_{cb} \mathcal{L}_K k_a + \mathcal{L}_l k_{[c} \bar{K}_{a]b} k_d k_e q^{ad} q^{be} - \frac{1}{2} q^{ab} k_b \mathcal{L}_l k_c \mathcal{L}_l k_a \\ & + (f + k^2) q^{ab} \bar{K}_{cb} \mathcal{L}_l k_a - \bar{K} q^{ab} k_b D_{[c} k_{a]} + q^{ab} \mathcal{L}_l k_{[c} D_{a]} k_b + 2q^{ad} q^{be} \bar{K}_{b[c} k_d D_e k_{a]} - 2q^{ad} q^{be} \bar{K}_{b[c} k_d D_{a]} k_e \\ & + 2q^{ad} q^{be} K_{ce} \bar{K}_{ab} k_d - q^{ab} K_{ca} \bar{K}_{kb} - \frac{1}{2} D_c \mathcal{L}_l f + q^{ab} D_b D_{[c} k_{a]} + q^{ab} \bar{K}_{ca} D_b f - \frac{1}{2} \bar{K} D_c f \\ & + q^{ab} k_b D_a \mathcal{L}_l k_c - \frac{1}{2} q^{ab} k_b D_c \mathcal{L}_l k_a - q^{ab} \bar{K}_{ca} k_b \mathcal{L}_l f - \frac{1}{2} (f + k^2) (\mathcal{L}_l \mathcal{L}_l k_c + \bar{K} \mathcal{L}_l k_c) - q^{ab} k_b \mathcal{L}_K \bar{K}_{ca}, \end{aligned} \quad (\text{A7e})$$

$$\begin{aligned}
p^a{}_b p^c{}_d R_{ac} = & R[h]_{bd} - 2\mathcal{L}_K \bar{K}_{bd} - K \bar{K}_{bd} + 4q^{ac} K_{(b|c} \bar{K}_{|d)a} - K_{bd} \bar{K} + (f + k^2)(2q^{ac} \bar{K}_{bc} \bar{K}_{da} - \bar{K}_{bd} \bar{K} - \mathcal{L}_l \bar{K}_{bd}) \\
& - 4q^{ad} q^{ce} \bar{K}_{b[a} \bar{K}_{d]c} k_d k_e + 2q^{ac} \mathcal{L}_l k_{(b} \bar{K}_{d)a} k_c - 2q^{ac} \bar{K}_{bd} k_c \mathcal{L}_l k_a - \bar{K}_{bd} \mathcal{L}_l f - 2q^{ac} k_c D_{[b} \bar{K}_{a]d} \\
& - 2q^{ac} k_c D_{[d} \bar{K}_{a]b} + \bar{K}_{d(b} k_{b)} + \frac{1}{2} D_{(d} \mathcal{L}_l k_{b)} - \frac{1}{2} \mathcal{L}_l k_b \mathcal{L}_l k_d + \bar{K}_{bd} q^{ac} D_a k_c - 2q^{ac} \bar{K}_{a(b} D_c k_{|d)}. \quad (A7f)
\end{aligned}$$

APPENDIX B: BIG-O NOTATION

In the proofs of Theorem A and Theorem B, we have to analyze whether certain tensor fields vanish on a Killing horizon, $\mathcal{H}^+$, with respect to a Killing vector field K^a . These tensors will be built from q^{ab}, h_{ab}, f, k_a , their Lie along K^a, l^a , and their angular covariant derivatives D_a , where these quantities refer to gGNCs, see Sec. II E. On a Killing horizon $\mathcal{H}^+$, we have $f, k_a = 0$, so also their D_a and $\mathcal{L}_K$ derivatives vanish as these point along $\mathcal{H}^+$. On the other hand, the Lie derivative $\mathcal{L}_l$ need not vanish. In the proofs of Theorem A and Theorem B, we do not *a priori* know that $f, k_a = 0$, and that their D_a and $\mathcal{L}_K$ derivatives vanish at $\mathcal{H}^+$, because we precisely want to prove that $\mathcal{H}^+$ actually is a Killing horizon.

In order to retain control of the order (in the gGNC u defining $\mathcal{H}$ through $u = 0$) at which certain expressions vanish in various perturbative arguments, we must keep track of the powers of k_a, f and their D_a and $\mathcal{L}_K$ derivatives.

For this, we now introduce a convenient notation: if the power of f is r and that of k_a is s , we write that such an expression is $O(f^r k^s)$. More precisely, a tensor $T^{a\dots b}{}_{c\dots d}$ built from $q^{ab}, h_{ab}, f, k_a, K^a, l^a$, their Lie derivatives along K^a, l^a , and their angular covariant derivatives D_a , is said to be $T = O(f^r k^s)$, if it is a linear combination of simultaneous eigenvectors of eigenvalues ($\geq r, \geq s$) of the operators

$$N_f = \sum_{p,q} \mathcal{L}_K^p D_{(a_1\dots a_q)}^q f \frac{\partial}{\partial (\mathcal{L}_K^p D_{(a_1\dots a_q)}^q f)}, \quad (B1)$$

$$N_k = \sum_{p,q} \mathcal{L}_K^p D_{(a_1\dots a_q)}^q k_c \frac{\partial}{\partial (\mathcal{L}_K^p D_{(a_1\dots a_q)}^q k_c)}. \quad (B2)$$

We also write, e.g., $T = O(kf, k^2)$ if it is a sum of terms of the order $O(k^2)$ and $O(kf)$. As examples,

$$q^{cb} \mathcal{L}_K D_c f D_a k_b = O(fk), \quad (B3)$$

but

$$q^{cb} \mathcal{L}_K D_c f D_a \mathcal{L}_l k_b = O(f), \quad (B4)$$

since terms hit by $\mathcal{L}_l$ are not counted. At a Killing horizon, we have $T = O(f^r k^s) \Rightarrow T = O(u^{r+s})$ where $O(u^n)$ is a function such that $\lim_{u \rightarrow 0} u^{-n+\delta} O(u^n) = 0$ for any $\delta > 0$.

APPENDIX C: CONTRACTIONS OF Q_{abcd}, C_{ab}

For the analysis of the quartic propagation cone in our EFT (3), we need the following two contractions (C3), (C4), where we define

$$\epsilon^{ab} = \epsilon^{AB} (\partial_A)^a (\partial_B)^b, \quad (C1)$$

so that

$$\epsilon^{abcd} = 6K^{[a} l^b \epsilon^{cd]}, \quad (C2)$$

in terms of the 4-dimensional volume element,

$$\begin{aligned}
& (2C_{a_1 a_2} C_{b_1 b_2} - C_{a_1 b_1} C_{a_2 b_2}) (du)_{e_1} (du)_{e_2} (du)_{f_1} (du)_{f_2} \epsilon^{a_1 e_1 g_1 g_2} \epsilon^{e_2 a_2 h_1 h_2} R_{g_1 g_2 h_1 h_2} \epsilon^{b_1 f_1 i_1 i_2} \epsilon^{f_2 b_2 j_1 j_2} R_{i_1 i_2 j_1 j_2} \\
& = (K^a K^b C_{ab})^2 (\epsilon^{cd} \epsilon^{ef} R_{cdef})^2 + 2K^a K^b K^c C_{da} C_{bc} \epsilon^{de} \epsilon^{fg} K^h R_{hefg} \epsilon^{ij} \epsilon^{kl} R_{ijkl} + 2K^a K^b K^c C_{da} C_{bc} \epsilon^{ef} \epsilon^{dg} K^h R_{efhg} \epsilon^{ij} \epsilon^{kl} R_{ijkl} \\
& + 2K^a K^b K^c C_{da} C_{bc} \epsilon^{ef} \epsilon^{gh} R_{efgh} \epsilon^{di} \epsilon^{jk} K^l R_{lijl} + 2K^a K^b K^c C_{da} C_{bc} \epsilon^{ef} \epsilon^{gh} R_{efgh} \epsilon^{ij} \epsilon^{dk} K^l R_{ijlk} \\
& + 4K^a K^b (2C_{cd} C_{ab} - C_{ca} C_{db}) \epsilon^{ce} \epsilon^{df} K^g K^h R_{gehf} \epsilon^{ij} \epsilon^{kl} R_{ijkl} + 4K^a K^b (2C_{ca} C_{db} - C_{cd} C_{ab}) \epsilon^{ce} \epsilon^{fg} K^h R_{hefg} \epsilon^{di} \epsilon^{jk} K^l R_{lijl} \\
& + 4K^a K^b C_{ca} C_{db} \epsilon^{ce} \epsilon^{fg} K^h R_{hefg} \epsilon^{ij} \epsilon^{dk} K^l R_{ijlk} + 4K^a K^b (2C_{ca} C_{db} - C_{ab} C_{cd}) \epsilon^{ef} \epsilon^{cg} K^h R_{efhg} \epsilon^{ij} \epsilon^{dk} K^l R_{ijlk} \\
& + 4K^a K^b C_{ca} C_{db} \epsilon^{ef} \epsilon^{cg} K^h R_{efhg} \epsilon^{di} \epsilon^{jk} K^l R_{lijl} + 4K^a K^b (2C_{ab} C_{cd} - C_{ca} C_{bd}) \epsilon^{ef} \epsilon^{gh} R_{efgh} \epsilon^{ci} \epsilon^{dj} K^k K^l R_{kilj} \\
& + 8K^a (2C_{ba} C_{cd} - C_{ca} C_{bd}) \epsilon^{ef} \epsilon^{bg} K^h R_{efhg} \epsilon^{ci} \epsilon^{dj} K^k K^l R_{kilj} + 8K^a (2C_{ba} C_{cd} - C_{bc} C_{da}) \epsilon^{be} \epsilon^{fg} K^h R_{hefg} \epsilon^{ci} \epsilon^{dj} K^k K^l R_{kilj} \\
& + 8K^a (2C_{bc} C_{da} - C_{ba} C_{cd}) \epsilon^{be} \epsilon^{cf} K^g K^h R_{gehf} \epsilon^{ij} \epsilon^{dk} K^l R_{ijlk} + 8K^a (2C_{bc} C_{da} - C_{bd} C_{ac}) \epsilon^{be} \epsilon^{cf} K^g K^h R_{gehf} \epsilon^{di} \epsilon^{jk} K^l R_{lijl} \\
& + 16(2C_{ab} C_{cd} - C_{ac} C_{bd}) \epsilon^{ae} \epsilon^{bf} K^g K^h R_{gehf} \epsilon^{ci} \epsilon^{dj} K^k K^l R_{kilj}, \quad (C3)
\end{aligned}$$

and

$$\begin{aligned}
& (2C_{a_1a_2}C_{b_1b_2} - C_{a_1b_1}C_{a_2b_2})(du)_{e_1}(du)_{e_2}(du)_{f_1}p^c_{f_2}\epsilon^{a_1e_1g_1g_2}\epsilon^{e_2a_2h_1h_2}R_{g_1g_2h_1h_2}\epsilon^{b_1f_1i_1i_2}\epsilon^{f_2b_2j_1j_2}R_{i_1i_2j_1j_2} \\
& = -2(K^aK^bC_{ab})^2\epsilon^{de}\epsilon^{fg}R_{defg}\epsilon^{hi}\epsilon^{cj}l^kR_{hikj} \\
& \quad + 4K^aK^bK^dC_{ea}C_{bd}\epsilon^{ef}\epsilon^{gh}K^iR_{ifgh}\epsilon^{jk}\epsilon^{cl}l^mR_{jkml} + 4K^aK^bK^dC_{ea}C_{bd}\epsilon^{fg}\epsilon^{eh}K^iR_{fghi}\epsilon^{jk}\epsilon^{cl}l^mR_{jkml} \\
& \quad + 4K^aK^bK^dC_{ea}C_{bd}\epsilon^{fg}\epsilon^{hi}R_{fghi}\epsilon^{ej}\epsilon^{ck}K^l l^mR_{ljmk} + 4K^aK^bK^dC_{ea}C_{bd}\epsilon^{fg}\epsilon^{hi}R_{fghi}\epsilon^{jk}\epsilon^{ce}K^l l^mR_{jklm} \\
& \quad - 8K^aK^b(2C_{de}C_{ab} - C_{da}C_{eb})\epsilon^{df}\epsilon^{eg}K^hK^iR_{hfig}\epsilon^{jk}\epsilon^{cl}l^mR_{jkml} \\
& \quad - 8K^aK^b(2C_{da}C_{eb} - C_{de}C_{ab})\epsilon^{df}\epsilon^{gh}K^iR_{ifgh}\epsilon^{ej}\epsilon^{ck}K^l l^mR_{ljmk} \\
& \quad + 8K^aK^bC_{da}C_{eb}\epsilon^{df}\epsilon^{gh}K^iR_{ifgh}\epsilon^{jk}\epsilon^{ce}K^l l^mR_{jklm} + 8K^aK^b(2C_{da}C_{eb} - C_{ab}C_{db})\epsilon^{fg}\epsilon^{dh}K^iR_{fghi}\epsilon^{jk}\epsilon^{ce}K^l l^mR_{jklm} \\
& \quad - 8K^aK^bC_{da}C_{eb}\epsilon^{fg}\epsilon^{dh}K^iR_{fghi}\epsilon^{ej}\epsilon^{ck}K^l l^mR_{ljmk} + 8K^aK^b(2C_{ab}C_{de} - C_{ad}C_{be})\epsilon^{fg}\epsilon^{hi}R_{fghi}\epsilon^{dj}\epsilon^{ce}K^kK^l l^mR_{kjlm} \\
& \quad + 16K^a(2C_{bd}C_{ea} - C_{be}C_{da})\epsilon^{bf}\epsilon^{dg}K^hK^iR_{hfig}\epsilon^{ej}\epsilon^{ck}K^l l^mR_{ljmk} \\
& \quad - 16K^a(2C_{bd}C_{ea} - C_{be}C_{da})\epsilon^{bf}\epsilon^{dg}K^hK^iR_{jfig}\epsilon^{jk}\epsilon^{ce}K^l l^mR_{jklm} \\
& \quad - 16K^a(2C_{ba}C_{de} - C_{bd}C_{ae})\epsilon^{bf}\epsilon^{gh}K^iR_{ifgh}\epsilon^{dj}\epsilon^{ce}K^kK^l l^mR_{kjlm} \\
& \quad + 16K^a(2C_{ba}C_{de} - C_{bd}C_{ae})\epsilon^{fg}\epsilon^{bh}K^iR_{fghi}\epsilon^{dj}\epsilon^{ce}K^kK^l l^mR_{kjlm} \\
& \quad + 32(C_{ab}C_{de} - C_{ad}C_{be})\epsilon^{af}\epsilon^{bg}K^hK^iR_{hfig}\epsilon^{dj}\epsilon^{ce}K^kK^l l^mR_{kjlm} \\
& \quad - 2K^aK^bK^d l^e C_{ab}C_{de}\epsilon^{fg}\epsilon^{hi}R_{fghi}\epsilon^{jk}\epsilon^{cl}K^mR_{jkml} + 4K^aK^b l^d(2C_{ea}C_{bd} - C_{eb}C_{ad})\epsilon^{fg}\epsilon^{eh}K^iR_{fghi}\epsilon^{jk}\epsilon^{cl}K^mR_{jkml} \\
& \quad - 4K^aK^b l^d C_{ea}C_{bd}\epsilon^{ef}\epsilon^{gh}K^iR_{ifgh}\epsilon^{jk}\epsilon^{cl}K^mR_{jkml} - 4K^aK^b l^d(2C_{ab}C_{ed} - C_{ae}C_{bd})\epsilon^{fg}\epsilon^{hi}R_{fghi}\epsilon^{ej}\epsilon^{ck}K^l K^mR_{ljmk} \\
& \quad + 8K^a l^b(2C_{de}C_{ab} - C_{da}C_{eb})\epsilon^{df}\epsilon^{eg}K^hK^iR_{hfig}\epsilon^{jk}\epsilon^{cl}K^mR_{jkml} \\
& \quad - 8K^a l^b(2C_{da}C_{eb} - C_{de}C_{ab})\epsilon^{df}\epsilon^{gh}K^iR_{ifgh}\epsilon^{ej}\epsilon^{ck}K^l K^mR_{ljmk} \\
& \quad + 8K^a l^b(2C_{da}C_{eb} - C_{de}C_{ab})\epsilon^{fg}\epsilon^{dh}K^iR_{fghi}\epsilon^{ej}\epsilon^{ck}K^l K^mR_{ljmk} \\
& \quad + 16l^a(C_{bd}C_{ea} - C_{be}C_{da})\epsilon^{bf}\epsilon^{dg}K^hK^iR_{hfig}\epsilon^{ej}\epsilon^{ck}K^l K^mR_{ljmk} \\
& = O(k^2, kf, f^2). \tag{C4}
\end{aligned}$$

We also need to compute certain contractions with $(C^{-1})^{ab}$. For this, we make use of the well-known formula for the inverse in terms of the adjugate of C^{-1} , i.e.,

$$\det(Cg^{-1})(C^{-1})^{ab} = \text{adj}(C)^{ab} = \frac{1}{3!}\epsilon^{aa_1a_2a_3}\epsilon^{bb_1b_2b_3}C_{a_1b_1}C_{a_2b_2}C_{a_3b_3}. \tag{C5}$$

Contractions that we need are

$$\begin{aligned}
& \det(Cg^{-1})(du)_a(du)_b(C^{-1})^{ab} \\
& = \frac{1}{3!}\epsilon^{ab}\epsilon^{cd}K^eK^f l^g(3C_{ef}C_{ac}C_{bd} - C_{ae}C_{fc}C_{bd} + C_{ae}C_{bc}C_{fd} - C_{ec}C_{af}C_{bd} + C_{ec}C_{ad}C_{bf} - C_{ac}C_{ed}C_{bf} - C_{ac}C_{be}C_{fd}), \tag{C6}
\end{aligned}$$

and

$$\begin{aligned}
& \det(Cg^{-1})p^a_b(du)_c(C^{-1})^{bc} \\
& = \frac{1}{3!}\epsilon^{ab}\epsilon^{cd}K^eK^f l^g(C_{ge}C_{fc}C_{bd} - C_{ef}C_{gc}C_{bd} + C_{ef}C_{bc}C_{gd} - C_{be}C_{fc}C_{gd} + C_{be}C_{gc}C_{fd} - C_{ge}C_{bc}C_{fd} - C_{gc}C_{ef}C_{bd} \\
& \quad + C_{gc}C_{ed}C_{bf} + C_{ec}C_{ge}C_{bd} - C_{ec}C_{gd}C_{bf} - C_{ec}C_{bf}C_{gd} + C_{ec}C_{bd}C_{fg} + C_{bc}C_{ef}C_{gd} - C_{bc}C_{ed}C_{gf} - C_{bc}C_{ge}C_{fd} \\
& \quad + C_{bc}C_{gd}C_{ef} + C_{gc}C_{be}C_{fd} - C_{gc}C_{bd}C_{ef}). \tag{C7}
\end{aligned}$$

Both expressions can be simplified by applying the big- O notation introduced earlier and by using the identity $\epsilon^a_c\epsilon^{cb} = q^{ab}$.

APPENDIX D: PROOF OF THEOREM A

By assumptions (iv), we may construct a gGNC system (u, v, x^A) , see (A1), near $\mathcal{H}^+$. We must show that $f, k_a = k_A(dx^A)_a$ vanish on $\mathcal{H}^+$. The proof follows the inductive scheme outlined in Sec. III C. We take our gGNC system to be such that $K^a = (\partial_v)^a$, where $K^a = t^a + \omega\phi^a$, with ω a constant. Initially, $\omega = \Omega_{(0)}$, which is subsequently modified. At any stage, K^a, ϕ^a Lie derive (Φ, g_{ab}) by (v).

1. Consequences of $\dot{x}^a = c\chi^a$ on $\mathcal{H}^+$

First, we investigate the consequences of (iii) and (v).

Lemma D.1. On $\mathcal{H}^+$ let $c = c(\ell, x)$ be as in (v). Then,

$$c = (du)_a(du)_b(du)_c(dv)_d Q^{abcd} \quad (D1)$$

and $\mathcal{L}_\phi c|_{\mathcal{H}^+} = 0 = \mathcal{L}_K c|_{\mathcal{H}^+}$ and either $c = 0$ or $c > 0$ on all of $\mathcal{H}^+$. On $\mathcal{H}^+$, we have

$$(du)_a(du)_b(du)_c(du)_d Q^{abcd} = 0, \quad (D2a)$$

$$(du)_a(du)_b(du)_c p^e{}_d Q^{abcd} = c(\Omega - \omega)\phi^e. \quad (D2b)$$

Proof. By construction, $(du)_a$ is normal to $\mathcal{H}^+$. As $\mathcal{H}^+$ is assumed to be a characteristic hypersurface by (iii), $(du)_a$ must be a characteristic covector, i.e., $Q(du) = 0$. Furthermore, by Hamilton's equations, we have

$$\begin{aligned} \dot{x}^a &= \left. \frac{\partial Q}{\partial \xi_a} \right|_{\xi=du} = (du)_b(du)_c(du)_d Q^{bcd a} \\ &= c\chi^a, \end{aligned} \quad (D3)$$

using (v) in the last step. Also note that $t^a = (\partial_v)^a - \omega\phi^a$, and therefore,

$$\chi^a = (\partial_v)^a + (\Omega - \omega)\phi^a. \quad (D4)$$

By dotting (D3) into $(du)_a, (dv)_a, p^b{}_a$ in turn, we obtain all statements in the lemma. Note that these tensors, as well as Q^{abcd} , are Lie derived by ϕ^a , so that we get, e.g.,

$$\begin{aligned} 0 &= \mathcal{L}_\phi[(du)_a(du)_b(du)_c(dv)_d Q^{abcd}] \\ &= \mathcal{L}_\phi c. \end{aligned} \quad (D5)$$

Thus, $\mathcal{L}_\phi c = 0$ on $\mathcal{H}^+$. Since K^a Lie derives the solution, the same argument works to show that $\mathcal{L}_K c|_{\mathcal{H}^+} = 0$. ■

Note that by (v), either $c(\Omega - \omega) = 0$ or $c(\Omega - \omega) \neq 0$ everywhere on $\mathcal{H}^+$, so it has the same general property as c . In the following, we rename $c(\Omega - \omega) \rightarrow c$.

2. Consequences of $Q = 0$ on $\mathcal{H}^+$

The aim is now to evaluate the relations (D2a) and (D2b) in such a way that we can eventually deduce $c, f, k_a = 0$ on $\mathcal{H}^+$ by using the big- O notation introduced above.

Lemma D.2. Define $\beta_{ab} = \nabla_a \nabla_b \beta$. The contractions

$$K^a K^b \beta_{ab}, \quad (D6a)$$

$$p^a{}_b K^c \beta_{ac}, \quad (D6b)$$

$$K^a p^b{}_c K^d p^e{}_f R_{abde}, \quad (D6c)$$

$$K^a p^b{}_c p^d{}_e p^f{}_g R_{abdf}, \quad (D6d)$$

$$K^a l^b K^c p^d{}_e R_{abcd} \quad (D6e)$$

are all of order $O(k, f)$ on $\mathcal{H}^+$.

Proof. First note that since K^a is Killing on $\mathcal{H}^+$, $\mathcal{L}_K \beta(\Phi) = \beta'(\Phi) \mathcal{L}_K \Phi = 0$, and $K_{ab} = \frac{1}{2} \mathcal{L}_K h_{ab} = 0$, meaning that the curvature expressions in the Appendix C simplify significantly. Thus, the statements for (D6a)–(A6d) follow immediately from the following formulas:

$$K^a K^b \beta_{ab} = -\frac{1}{2}(f + k^2) \mathcal{L}_l f \mathcal{L}_l \beta - \frac{1}{2} q^{ab} k_b D_a f \mathcal{L}_l \beta - \frac{1}{2} q^{ab} k_b \mathcal{L}_l f D_a \beta - \frac{1}{2} q^{ab} D_b f D_a \beta, \quad (D7)$$

$$p^a{}_b K^c \beta_{ac} = -\left[-\frac{1}{2}(f + k^2) \mathcal{L}_l k_b - \frac{1}{2} D_b f - q^{ac} k_a D_{[b} k_{c]} \right] \mathcal{L}_l \beta - \left[\frac{1}{2} k_a \mathcal{L}_l k_b - D_{[a} k_{b]} \right] q^{ac} D_c \beta, \quad (D8)$$

$$\begin{aligned} K^a p^b{}_c K^d p^e{}_f R_{abde} &= -2q^{ab} k_a \bar{K}_{c[f} \mathcal{L}_l k_{b]} - 2q^{ab} k_a \bar{K}_{f[c} \mathcal{L}_l k_{b]} + \frac{1}{4}(f + k^2) \mathcal{L}_l l_c \mathcal{L}_l k_f + \frac{1}{2} \mathcal{L}_l k_{(c} D_{f)} f \\ &\quad - q^{ab} k_a \mathcal{L}_l k_{(c} D_{f)} k_b + \frac{1}{2} D_c D_f f - \frac{1}{2} \bar{K}_{cf} q^{ab} k_a D_b f + \frac{1}{2} q^{ab} k_a \mathcal{L}_l k_{(c} D_{b]} k_{f]} \\ &\quad + q^{ab} D_{[a} k_{c]} D_{b]} k_{f]} + \frac{1}{2}(f + k^2) \bar{K}_{cf} \mathcal{L}_l f - \frac{1}{2} D_{(c} k_{f)} \mathcal{L}_l f, \end{aligned} \quad (D9)$$

$$K^a p^b{}_c p^d{}_e p^f{}_g R_{abdf} = -\mathcal{L}_l k_{[e} \bar{K}_{g]c} (f + k^2) + \bar{K}_{c[e} D_{g]} f + \frac{1}{2} \mathcal{L}_l k_{[e} D_{g]} k_c - \frac{1}{2} D_c k_{[e} \mathcal{L}_l k_{g]} - D_c D_{[e} k_{g]} + \bar{K}_{c[e} q^{ab} k_a D_{|g]} k_b - k_a \bar{K}_{c[e} q^{ab} D_b k_{|g]}. \quad (D10)$$

This is not as clear for the last contraction (A6e). We continue by considering the EoMs [by (i)],

$$0 = K^a q^{bc} G_{ac} - \left(\frac{1}{2} + \alpha X \right) \mathcal{L}_K \Phi q^{bc} \nabla_c \Phi - \frac{1}{2} K^a q^{bc} g_{ac} \left(X - V + \frac{1}{2} \ell^2 \alpha X \right) - \frac{1}{4} \ell^2 K^a \epsilon_a{}^{e_1 e_2 e_3} q^{bc} \epsilon_c{}^{f_1 f_2 f_3} R_{e_1 e_2 f_1 f_2} \beta_{e_3 f_3}. \quad (D11)$$

The Einstein tensor satisfies (recall that $p^a{}_b K^b = 0$)

$$K^a q^{bc} G_{ac} = l^a K^c K^d q^{bf} R_{acdf} g^{gh} (du)_g (du)_h + l^a K^c K^d q^{bf} R_{acdf} g^{gh} (du)_g (dv)_h + K^a K^c l^d q^{bf} R_{acdf} g^{gh} (dv)_g (du)_h + l^a K^c p^d{}_e q^{bf} R_{acdg} g^{eh} (du)_h + p^a{}_c K^d l^e q^{bf} R_{ade f} g^{ch} (du)_h + K^a q^{bc} R_{daec} q^{de}. \quad (D12)$$

The third term vanishes due to the first Bianchi identity, and the second term is the important one, since all other terms are $O(k, f)$. Thus, defining $\epsilon^{ab} = \epsilon^{AB} (\partial_A)^a (\partial_B)^b$, we have using the EoMs for g_{ab} ,

$$\begin{aligned} K^a l^c K^d q^{be} R_{acde} &= -K^a q^{bc} G_{ac} + O(k, f) \\ &= -\frac{1}{4} \ell^2 [-4\epsilon^{ac} \epsilon^{bd} K^e l^f K^g p^h{}_a R_{efgh} \beta_{cd} + \epsilon^{ac} \epsilon^{bd} K^e p^f{}_a K^g p^h{}_d R_{efgh} l^i \beta_{ic} \\ &\quad + (du)_a (dv)_c p^b{}_d p^e{}_f (\epsilon^{aa_1 a_2 f} \epsilon^{db_1 b_2 c} + \epsilon^{aa_1 a_2 c} \epsilon^{db_1 b_2 f}) R_{a_1 a_2 b_1 b_2} K^g \beta_{eg} \\ &\quad - 2\epsilon^{ac} \epsilon^{bd} K^e p^f{}_d p^g{}_a p^h{}_c R_{efgh} K^i K^j \beta_{ij} + (du)_a (du)_c (dv)_d p^b{}_e \epsilon^{aa_1 a_2 c} \epsilon^{eb_1 b_2 d} R_{a_1 a_2 b_1 b_2} K^f K^g \beta_{fg}] \\ &\quad + O(k, f). \end{aligned} \quad (D13)$$

The prefactors arise from the possible permutations within the second and third index of the epsilon tensor, which do not give a sign due to the antisymmetry of the Riemann tensor. From that, we obtain

$$K^a l^b K^c p^d{}_e R_{abcd} \left(\frac{1}{4} \ell^2 \epsilon^{ef} \epsilon^{gh} p^i{}_f p^j{}_h \beta_{ij} + q^{eg} \right) = O(k, f), \quad (D14)$$

and since the term in (...) is close to q^{ab} for a weakly coupled theory, it is invertible. It follows that

$$K^a l^b K^c p^d{}_e R_{abcd} = O(k, f), \quad (D15)$$

i.e., (A6a) holds. ■

We now use these results on $\mathcal{H}^+$ for the evaluation of the tensors appearing in the definition of Q in the following Lemma D.3. To state this compactly, and for later, it is

convenient to introduce an inner product $\langle \cdot, \cdot \rangle_h$ on tensor fields $T_{a_1 \dots a_r}$ projected by $p^a{}_b$. The inner product uses h_{ab} and its inverse q^{ab} [see (23), (24)] and is given by

$$\langle T, S \rangle_h := q^{a_1 b_1} \dots q^{a_r b_r} S_{a_1 \dots a_r} T_{b_1 \dots b_r}. \quad (D16)$$

We denote by

$$\|T\|_h^2 = \langle T, T \rangle_h \quad (D17)$$

the corresponding norm. By abuse of notation, we also write

$$k^2 = \|k\|_h^2, \quad (D18)$$

as before to shorten some formulas

Lemma D.3. We have

$$\det(Cg^{-1})(du)_a (du)_b (C^{-1})^{ab} = f + k^2 + \ell^2 \left[-\frac{1}{2} \langle k \mathcal{L}_l f - Df, D\beta \rangle_h + f \text{tr}_h D^2 \beta + \frac{1}{2} f \mathcal{L}_l f \mathcal{L}_l \beta + O(k^2, kf, f^2) \right], \quad (D19a)$$

$$(du)_a (du)_b P^{ab} = -(f + k^2) + \ell^2 [-(f + k^2) \alpha X + O(k^2, kf, f^2)], \quad (D19b)$$

on $\mathcal{H}^+$, where $\text{tr}_h D^2 \beta = q^{ab} D_a D_b \beta$.

Proof. By performing all contractions (see Appendix C) and using the results of Lemma D.2, we find

$$\begin{aligned}
& \det(Cg^{-1})(du)_a(du)_b(C^{-1})^{ab} \\
&= \frac{1}{2}\epsilon^{ab}\epsilon^{cd}(-f + \ell^2 K^e K^f \beta_{ef})(h_{ac} + \ell^2 \beta_{ac})(h_{bd} + \ell^2 \beta_{bd}) + \frac{1}{3!}\epsilon^{ab}\epsilon^{cd}(-2k_a k_c h_{bd} + k_a k_d h_{bc} + k_c k_b h_{ad} - 2k_d k_b h_{ac}) \\
&\quad + \ell^2 O(k^2, kf, f^2) \\
&= f + k^2 + \ell^2[-K^a K^b \beta_{ab} + f q^{ab} \beta_{ab} + O(k^2, kf, f^2)] + \ell^4 O(k, f) \\
&= f + k^2 + \ell^2 \left[-\frac{1}{2} q^{ab}(k_a \mathcal{L}_l f - D_a f) D_b \beta + q^{ab} D_a D_b \beta + \frac{1}{2} f \mathcal{L}_l f \mathcal{L}_l \beta + O(k^2, kf, f^2) \right] + \ell^4 O(k, f), \tag{D20}
\end{aligned}$$

where we have defined $\epsilon^{ab} = \epsilon^{AB}(\partial_A)^a(\partial_B)^b$. For the second part of the lemma, we first note that

$$(du)_a \nabla^a \Phi = (f + k^2) l^a \nabla_a \Phi - q^{ab} k_b \nabla_a \Phi = O(k, f), \tag{D21}$$

using that K^a is Killing. We immediately obtain

$$\begin{aligned}
(du)_a(du)_b P^{ab} &= -(1 + \ell^2 \alpha X)(du)_a(du)_b g^{ab} + \ell^2 \alpha (du)_a \nabla^a \Phi (du)_b \nabla^b \Phi \\
&= -(f + k^2)(1 + \ell^2 \alpha X) + \ell^2 O(k, f)^2 \\
&= -(f + k^2) + \ell^2(-(f + k^2)\alpha X + O(k^2, kf, f^2)), \tag{D22}
\end{aligned}$$

which completes the proof. ■

We do the same for contractions involving p^a_b :

Lemma D.4. We have

$$\det(Cg^{-1})p^a_b(du)_c(C^{-1})^{bc} = -q^{ab}k_b + \ell^2 O(k, f), \tag{D23a}$$

$$p^a_b(du)_c P^{bc} = q^{ab}k_b + \ell^2 O(k, f), \tag{D23b}$$

on $\mathcal{H}^+$.

Proof. As before, we use the formulas (D20) and (D21) to compute C^{-1} . From Appendix B, we see that all terms involving $p^a_b l^c C_{ac}$ do not contribute to the ℓ^0 part of $p^a_b(du)_c \text{adj}(C)^{bc}$. Moreover, each term contains at least one factor of $K^a K^b C_{ab}$ or $p^a_b K^c C_{ac}$. Thus,

$$\begin{aligned}
\det(Cg^{-1})p^a_b(du)_c(C^{-1})^{bc} &= \frac{1}{3!}\epsilon^{ab}\epsilon^{cd}(3k_c h_{bd} - 3k_d h_{bc}) + \ell^2 O(k, f) \\
&= -q^{ab}k_b + \ell^2 O(k, f). \tag{D24}
\end{aligned}$$

For the second part of the lemma, we have already seen that $(du)_a \nabla^a \Phi = O(k, f)$, which yields

$$\begin{aligned}
p^a_b(du)_c P^{bc} &= q^{ab}k_b + \ell^2(\alpha X q^{ab}k_b + \alpha p^a_b \nabla^b \Phi O(k, f)) \\
&= q^{ab}k_b + \ell^2 O(k, f). \tag{D25}
\end{aligned}$$

■

The second term in the definition of Q is evaluated as follows:

Lemma D.5. We have

$$\begin{aligned}
& (du)_c(du)_d(du)_e(du)_f(2C_{a_1 a_2} C_{b_1 b_2} - C_{a_1 b_1} C_{a_2 b_2}) C^{a_1 c d a_2} C^{b_1 e f b_2} \\
&= \frac{1}{4} \ell^4 (\beta'(\Phi))^2 \left\| \frac{1}{2} \mathcal{L}_k h \mathcal{L}_l f + k \otimes D \mathcal{L}_l f \right\|_h^2 + \ell^4 O(k^3, kf, f^2) + \ell^6 O(k^2, kf, f^2), \tag{D26a}
\end{aligned}$$

$$(du)_c(du)_d(du)_e p^f_g (2C_{a_1 a_2} C_{b_1 b_2} - C_{a_1 b_1} C_{a_2 b_2}) C^{a_1(c d | a_2} C^{b_1 | e g) b_2} = \ell^4 O(k^2, kf, f^2), \quad (D26b)$$

on $\mathcal{H}^+$, where $\|\cdot\|_h$ is the tensor norm associated to h .

Proof. First note that, from Lemma D.2,

$$K^a p^b_c K^d p^e_f R_{abde} = -\frac{1}{4} \mathcal{L}_k h_{cf} \mathcal{L}_l f - \frac{1}{2} k_c D_f \mathcal{L}_l f + O(k^2, f) = T_{cf} + O(k^2, f), \quad (D27)$$

as well as $K^a K^b \beta_{ab}, p^a_b K^c \beta_{ac} = O(k, f)$, so that

$$K^a K^b C_{ab} = -f + \ell^2 O(k, f). \quad (D28)$$

The full expression for the above contraction can be found in the Appendix C, where the last term is the important one. As all other terms are at least $O(k^3, kf, f^2) + \ell^2 O(k^2, kf, f^2)$, it remains to show that this term equals $16\|T\|_h^2 + O(k^3, kf, f^2) + \ell^2 O(k^2, kf, f^2)$. Note that for this purpose, we only have to consider the ℓ^0 terms in $p^a_b p^c_d p^e_f p^g_h (2C_{ac} C_{eg} - C_{ae} C_{cg})$ since the above expression must be multiplied by ℓ^4 ,

$$\begin{aligned} & 16 p^{a_1}_{c_1} p^{a_2}_{c_2} p^{b_1}_{d_1} p^{b_2}_{d_2} (2C_{a_1 a_2} C_{b_1 b_2} - C_{a_1 b_1} C_{a_2 b_2}) \epsilon^{c_1 e_1} \epsilon^{c_2 e_2} K^{n_1} p^{g_1}_{e_1} K^{n_2} p^{g_2}_{e_2} R_{n_1 g_1 n_2 g_2} \\ & \quad \times \epsilon^{d_1 f_1} \epsilon^{d_2 f_2} K^{m_1} p^{h_1}_{f_1} K^{m_2} p^{h_2}_{f_2} R_{m_1 h_1 m_2 h_2} \\ & = 16 (2h_{c_1 c_2} h_{d_1 d_2} - h_{c_1 d_1} h_{c_2 d_2}) \epsilon^{c_1 e_1} \epsilon^{c_2 e_2} \epsilon^{d_1 f_1} \epsilon^{d_2 f_2} T_{e_1 e_2} T_{f_1 f_2} + O(k^3, kf, f^2) + \ell^2 O(k^2, kf, f^2) \\ & = 16 q^{e_1 f_1} q^{e_2 f_2} T_{e_1 e_2} T_{f_1 f_2} + O(k^3, kf, f^2) + \ell^2 O(k^2, kf, f^2) \\ & = 16 \left\| -\frac{1}{4} \mathcal{L}_k h \mathcal{L}_l f - \frac{1}{2} k \otimes D \mathcal{L}_l f \right\|_h^2 + O(k^3, kf, f^2) + \ell^2 O(k^2, kf, f^2). \end{aligned} \quad (D29)$$

The claim follows when multiplying by $\frac{1}{16} \ell^4 (\beta')^2$. For the second relation, it suffices to check that each term contains at least two of the following factors:

$$\begin{aligned} & K^a K^b C_{ab}, p^a_b K^c C_{ac}, K^a p^b_c K^d p^e_f R_{abde}, \\ & K^a p^b_c p^d_e p^f_g R_{abdf}, K^a l^b K^c p^d_e R_{abcd}. \end{aligned} \quad (D30)$$

Again, the expression for a specific permutation can be found in the Appendix C, and for the remaining permutations, this result can be verified analogously. ■

We are now in the position to write (D2a) and (D2b) in a useful form:

Lemma D.6. We have

$$\begin{aligned} & \left[f + k^2 + \ell^2 \left(\frac{1}{2} \langle -k \mathcal{L}_l f + Df, D\beta \rangle_h + f \text{tr}_h DD\beta + \frac{1}{2} f \mathcal{L}_l f \mathcal{L}_l \beta + O(k^2, kf, f^2) \right) + \ell^4 O(k, f) \right] \\ & \quad \times [f + k^2 + \ell^2 ((f + k^2) \alpha X + O(k^2, kf, f^2))] \\ & = \frac{1}{4} \ell^4 (\beta')^2 \left\| \frac{1}{2} \mathcal{L}_k h \mathcal{L}_l f + k \otimes D \mathcal{L}_l f \right\|_h^2 + \ell^4 O(k^3, kf, f^2) + \ell^6 O(k^2, kf, f^2), \end{aligned} \quad (D31)$$

on $\mathcal{H}^+$ and

$$\begin{aligned} & \frac{1}{2} \left[f + k^2 + \ell^2 \left(\frac{1}{2} \langle -k \mathcal{L}_l f + Df, D\beta \rangle_h + f \text{tr}_h DD\beta + \frac{1}{2} f \mathcal{L}_l f \mathcal{L}_l \beta + O(k^2, kf, f^2) \right) + \ell^4 O(k, f) \right] [q^{ab} k_b + \ell^2 O(k, f)] \\ & \quad + \frac{1}{2} [f + k^2 + \ell^2 ((f + k^2) \alpha X + O(k^2, kf, f^2))] [q^{ab} k_b + \ell^2 O(k, f)] \\ & = c \phi^a + \ell^4 O(k^2, kf, f^2), \end{aligned} \quad (D32)$$

on $\mathcal{H}^+$.

Proof. (D31) is essentially the relation

$$\begin{aligned} & (du)_a(du)_b(du)_c(du)_d Q^{abcd} \\ &= (du)_a(du)_b(du)_c(du)_d (\det(Cg^{-1})(C^{-1})^{ab}P^{cd} + (2C_{a_1a_2}C_{b_1b_2} - C_{a_1b_1}C_{a_2b_2})C^{a_1aba_2}C^{b_1cdb_2}) = 0, \end{aligned} \quad (D33)$$

which clearly implies the first part of the lemma when using the expressions for $(du)_a(du)_b(C^{-1})^{ab}$, $(du)_a(du)_bP^{ab}$, etc. found in Lemmas D.3 and D.5 and multiplying by (-1) on both sides, whereas (D32) is

$$\begin{aligned} & (du)_a(du)_b(du)_c P^d_e Q^{abce} \\ &= (du)_a(du)_b(du)_c P^d_e \left(\frac{1}{2} \det(Cg^{-1}) [(C^{-1})^{ea}P^{bc} + (C^{-1})^{ab}P^{ec}] + (2C_{a_1a_2}C_{b_1b_2} - C_{a_1b_1}C_{a_2b_2})C^{a_1(ab|a_2}C^{b_1|ce)b_2} \right) \\ &= c\phi^d, \end{aligned} \quad (D34)$$

and using Lemma D.3, D.4, and D.5.

3. Inductive proof of Theorem A

By (i), we may expand all tensor fields such as $\Phi, g_{ab}, h_{ab}, q^{ab}, p^a_b, c$ as power series in ℓ as in Sec. III C. The coefficients in these expansions are analytic tensor fields on $\mathcal{M}$.

The zeroth order $g_{(0)ab}$ and $\Phi_{(0)}$ are solutions to the EoMs of ordinary Einstein-scalar theory. In this case, $\mathcal{H}^+$ is characteristic if and only if it is null. Thus, on $\mathcal{H}^+$, $f_{(0)} = k_{(0)a} = 0$ as well as $\mathcal{L}_l f_{(0)} = 2\kappa, D_a \Phi_{(0)} = 0$, since $\mathcal{H}^+$ is a nondegenerate Killing horizon for $\ell = 0$ by (ii).

Now we go inductively order by order in ℓ through (D31) and (D32) to successively redefine the Killing vector field $K^a \rightarrow K^a - \ell^j \omega_{(j)} \phi^a$ in every order ℓ^j such that, with respect to the gGNCs associated with the new K^a , $f_{(j)} = k_{(j)a} = 0$ on $\mathcal{H}^+$ in all orders of ℓ . To do so, we substitute all series expansions of all quantities into (D31) and (D32) and compare the coefficients corresponding to a specific order in ℓ of both sides.

To begin, (D32) in order ℓ^0 and ℓ^1 trivially implies $c_{(0)} = c_{(1)} = 0$ on $\mathcal{H}^+$, whereas from (D31) in order ℓ^2 we obtain $(f + k^2)_{(1)} = 0 = f_{(1)}$ on $\mathcal{H}^+$, which we can use to infer from (D32) at order ℓ^2 that $c_{(2)} = 0$ on $\mathcal{H}^+$. At order ℓ^3 , (D32) reads

$$(f + k^2)_{(2)} k^a_{(1)} = c_{(3)} \phi^a, \quad (D35)$$

where here and in the following, we set

$$k^a := q^{ab} k_b, \quad (D36)$$

contrary to our usual convention that indices are always raised with g^{ab} . From (D31) in order ℓ^4 , we obtain $(f + k^2)_{(2)} = 0$ on $\mathcal{H}^+$, and therefore, $c_{(3)} = 0$ on $\mathcal{H}^+$.

a. Induction start: Base case

Having dealt with these relatively straightforward consequences, we can now start the induction proper. As already outlined in Sec. III B, it is possible to prove the vanishing of f, k_a, c on the horizon inductively. The $n = 0$ case corresponds to $f_{(j)} = 0$ for all $j \leq 3$, $k_{(j)a} = 0$ for all $j \leq 1$, $c_{(j)} = 0$ for all $j \leq 5$ after a suitable readjustment of K^a . As we see, the key equations for the proof of these statements are given in the following lemma:

Lemma D.7. Define $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{h_{(0)}}$ and $\| \cdot \|^2 = \langle \cdot, \cdot \rangle$. Then, we have

$$\begin{aligned} & \frac{1}{4} (\beta'_{(0)})^2 \kappa^2 \| \mathcal{L}_{k_{(1)}} h_{(0)} \|^2 \\ &= [(f + k^2)_{(3)} + \kappa \langle k_{(1)}, D\beta_{(0)} \rangle] (f + k^2)_{(3)}, \end{aligned} \quad (D37a)$$

$$\left[(f + k^2)_{(3)} + \frac{1}{2} \kappa \langle k_{(1)}, D\beta_{(0)} \rangle \right] k^a_{(1)} = c_{(4)} \phi^a, \quad (D37b)$$

on $\mathcal{H}^+$. ■

Proof. Consider (D31) in order ℓ^6 and (D32) in order ℓ^4 . Whenever $(\beta')^2$ or $\| \cdot \|_h$ appear in order ≥ 1 , the tensor norm always gives combinations of inner products between $(\frac{1}{2} \mathcal{L}_k h \mathcal{L}_l f + k \otimes D \partial_u f)_{(j_1)}$ and $(\frac{1}{2} \mathcal{L}_k h \mathcal{L}_l f + k \otimes D \mathcal{L}_l f)_{(j_2)}$, where either $j_1 = 0, j_2 = 1$ or $j_1 = 1, j_2 = 0$. But $O(k, f)_{(0)} = 0$, so that such terms vanish. Therefore, the right side of (D31) reads

$$\begin{aligned} & \frac{1}{4} (\beta'^2)_{(0)} \left\| \frac{1}{2} \mathcal{L}_{k_{(1)}} h_{(0)} \mathcal{L}_l f_{(0)} + k_{(1)} \otimes D \mathcal{L}_l f_{(0)} \right\|^2 \\ &+ O(k^3, kf, f^2)_{(2)} + O(k^2, kf, f^2)_{(0)}. \end{aligned} \quad (D38)$$

Note that $\mathcal{L}_l f_{(0)} = 2\kappa$ on $\mathcal{H}^+$ implies $D_a \mathcal{L}_l f_{(0)} = 0$ on $\mathcal{H}^+$. The left side of (D31) yields

$$\left[(f + k^2)_{(3)} + \frac{1}{2} \langle k_{(1)} \mathcal{L}_t f_{(0)}, D\beta_{(0)} \rangle \right] (f + k^2)_{(3)}, \quad (\text{D39})$$

which implies (D37a). By an analogous argument, the left side of (D32) gives

$$\begin{aligned} & \frac{1}{2} k_{(1)}^a [(f + k^2)_{(3)} + \kappa \langle k_{(1)}, D\beta_{(0)} \rangle] \\ & + \frac{1}{2} k_{(1)}^a (f + k^2)_{(3)}, \end{aligned} \quad (\text{D40})$$

proving (D37b). ■

Lemma D.8. The quantity

$$\zeta := (f + k^2)_{(3)} + \frac{1}{2} \kappa \langle k_{(1)}, D\beta_{(0)} \rangle \quad (\text{D41})$$

vanishes on $\mathcal{H}^+$.

Proof. By Lemma D. 1, we either have $c = 0$ or $c \neq 0$ on all of $\mathcal{H}^+$, so the same applies to $c_{(4)}$ since it is the leading coefficient of c in the expansion in ℓ .

Case (a) $c_{(4)} = 0$: Either $\zeta = 0$ or $k_{(1)}^a = 0$ by (D37b) on some open set, hence everywhere on $\mathcal{H}^+$ by analyticity and rotation symmetry. In both cases there is nothing to show as $k_{(1)}^a = 0$ implies $f_{(2)} = f_{(3)} = 0$ via (D37a).

Case (b) $c_{(4)} \neq 0$: By analyticity, if ζ does not vanish identically on $\hat{H} := H/U(1)$, it can vanish at most at an isolated set of $\hat{H}$. On an open set where $\eta := c_{(4)}^{-1} \zeta$ does not vanish, we have

$$\langle k_{(1)}, D\beta_{(0)} \rangle = \beta'_{(0)} \eta^{-1} \mathcal{L}_\phi \Phi_{(0)} = 0, \quad (\text{D42})$$

and therefore, $\zeta = (f + k^2)_{(3)}$ on this open set, hence everywhere on $\mathcal{H}^+$ by analyticity. On any open set where η does not vanish, (D37a) yields, using $\langle k_{(1)}, D\beta_{(0)} \rangle = 0 = \mathcal{L}_\phi \eta$,

$$\frac{1}{2} c_{(4)}^{-2} (\beta'_{(0)})^2 \kappa^2 \|\phi\|^2 \|D\eta\|^2 = \eta^6. \quad (\text{D43})$$

By analyticity, this relation thereby extends to all of $\mathcal{H}^+$.

For further analysis, it is useful to appeal to the uniformization theorem for the closed Riemannian manifold $(H, h_{(0)ab})$. By (iv), H is analytically diffeomorphic to a 2-sphere, and by (v), ϕ^a generates a $U(1)$ isometry group. Hence, locally we can find conformally spherical coordinates (ϑ, φ) such that $\phi^a = (\partial_\varphi)^a$, and in these coordinates,

$$h_{(0)AB} dx^A dx^B = \Psi [d\vartheta^2 + (\sin \vartheta)^2 d\varphi^2], \quad (\text{D44})$$

such that $\Psi = \Psi(\vartheta) > 0$ is an analytic function on H (see, e.g., [33] for further details). Since, in these coordinates, ϕ^a is a coordinate vector field and an isometry of our solution, $\eta, \Phi_{(0)}$ and $c_{(4)}$ are independent of φ and (D43) gives us

$$\frac{1}{2} (\beta'_{(0)})^2 \kappa^2 c_{(4)}^{-2} (\sin \vartheta)^2 (\eta')^2 = \eta^6, \quad (\text{D45})$$

where $' = d/d\vartheta$. If η is not identically zero, its zeros are isolated by analyticity. Consider two consecutive zeros ϑ_1, ϑ_2 of η [there are at least two such zeros at $0, \pi$ by (D45)]. By the mean value theorem, there exists a $\vartheta_2 \in (\vartheta_1, \vartheta_2)$ such that $\eta'(\vartheta_2) = 0$. By (D45), ϑ_2 therefore is another zero strictly between the two zeros ϑ_1, ϑ_2 , which is a contradiction. Thus, η is identically zero, as is thereby ζ . ■

Lemma D.9. On $\mathcal{H}^+$ we have $k_{(1)}^a = \omega_{(1)} \phi^a$ for some constant $\omega_{(1)}$, and $(f + k^2)_{(3)} = c_{(4)} = 0$.

Proof. By the previous lemma, $\zeta = 0$, and therefore, $c_{(4)} = 0$ on $\mathcal{H}^+$ by (D37b). We now substitute this into (D37a) to obtain¹⁴

$$\frac{1}{2} (\beta'_{(0)})^2 \|\mathcal{L}_{k_{(1)}} h_{(0)}\|^2 = -(\beta'_{(0)})^2 (\mathcal{L}_{k_{(1)}} \Phi_{(0)})^2. \quad (\text{D46})$$

Since both sides have different signs, this equation can only be fulfilled if either $\beta'_{(0)} (= \beta'(\Phi_{(0)})) = 0$, or $\mathcal{L}_{k_{(1)}} h_{(0)ab} = \mathcal{L}_{k_{(1)}} \Phi_{(0)} = 0$ on $\mathcal{H}^+$ or both. The first case is excluded because we are assuming in (vi) that β' is nowhere zero. Consequently, $k_{(1)}^a$ is a Killing vector field of $h_{(0)ab}$, and $(f + k^2)_{(3)} = 0$ on $\mathcal{H}^+$. Since $h_{(0)ab}$ does not have a second Killing field by (v), there exists a constant $\omega_{(1)}$ such that $k_{(1)}^a = \omega_{(1)} \phi^a$ on $\mathcal{H}^+$. ■

Consider now the flow F_t on H generated by the Killing field $-\ell k_{(1)}^a = -\ell \omega_{(1)} \phi^a$, i.e. $x^A(t) = F_t^A(x^B)$ solves the ordinary differential equation,

$$\dot{x}^A(t) = -\ell k_{(1)}^A [x^B(t)], \quad x^A(0) = x^A. \quad (\text{D47})$$

Then, we define the following diffeomorphism ψ in an open neighborhood of $\mathcal{H}^+$ where our gGNCs are well defined:

$$\psi(u, v, x^A) = (u, v, F_v^A(x^B)). \quad (\text{D48})$$

This ℓ dependent analytic diffeomorphism maps $\mathcal{H}^+$ to itself by construction. We furthermore define $\Phi' = \psi^* \Phi$, $g'_{ab} = \psi^* g_{ab}$, $K'^a = \psi^* K^a$.

Lemma D.10. We have $K'^a = K^a - \omega_{(1)} \ell \phi^a$, and g'_{ab} takes Gaussian null form (A1) with new f', k'_a, h'_{ab} that are analytic functions Lie derived by K^a, ϕ^a such that $f'_{(j)} = 0$ for all $j \leq 3$, $k'_{(j)a} = 0$ for all $j \leq 1$ on $\mathcal{H}^+$.

Proof. The proof is by an elementary computation for pull backs, using Lemma D. 9 and the already known facts $f_{(0)} = f_{(1)} = f_{(2)} = k_{(0)a} = 0$ on $\mathcal{H}^+$.

¹⁴Here, we have also cancelled κ^2 on both sides, which is legal since $\kappa > 0$ by (ii).

We now rerun the entire argument with (Φ', g'_{ab}) and K^a , or equivalently, with (Φ, g_{ab}) and K'^a . Taking the second viewpoint, we use our new K'^a to construct our gGNCs (A1), and we identify our spacetimes $\mathcal{M}(\ell)$ by identifying points if their new gGNCs are the same. The tensors associated with the new gGNCs now satisfy $f'_{(j)} = 0$ for all $j \leq 3$, $k'_{(j)a} = 0$ for all $j \leq 1$.

From (D32) in order ℓ^5 , we obtain additionally $c_{(5)} = 0$ on $\mathcal{H}^+$. This concludes the base case. We rename K'^a into K^a to simplify the notation.

b. Induction step: $n \rightarrow n+1$ and proof of Theorem A

Lemma D.11. There exist constants $\omega_{(j)}, j \geq 1$ such that the following is true: If (u, v, x^A) denote the gGNCs associated to the Killing vector field $K^a - \sum_{j \geq 1} \omega_{(j)} \ell^j \phi^a$ with constants $\omega_{(j)}$ and corresponding quantities f, k_a, c , then $f_{(j)}, k_{(j)a}, c_{(j)} = 0$ on $\mathcal{H}^+$ for all j .

Proof (of Theorem A). *A priori*, our construction of the constants $\omega_{(j)}, j \geq 1$ in Lemma D.11 does not guarantee that the series for K^a converges. However, by assumption

(iv), K^a , regarded as a formal power series, is by construction a Killing vector field such that $K_a = g_{ab}K^b$ is characteristic for $\mathcal{H}^+$ in the formal sense (to all orders in ℓ). Since χ^a is also such a vector with *convergent* expansion in ℓ , and since there can at most be one such vector, $K^a = \chi^a$ so the series is convergent *a posteriori*. Since K^a is furthermore null, so is χ^a ; thus, $\mathcal{H}^+$ is a Killing horizon. ■

Remark. Note that this reasoning also gives $\omega_{(j)} = -\Omega_{(j)} \forall j \geq 1$ *a posteriori*.

Proof (of Lemma D.11). Our induction hypothesis is the following: for $n \in \mathbb{N}_0$, assume that

$$f_{(j)}|_{\mathcal{H}^+} = 0 \quad \text{for all } j \leq n+3, \quad (\text{D49a})$$

$$k_{a,(j)}|_{\mathcal{H}^+} = 0 \quad \text{for all } j \leq n+1, \quad (\text{D49b})$$

$$c_{(j)}|_{\mathcal{H}^+} = 0 \quad \text{for all } j \leq 2n+5, \quad (\text{D49c})$$

in gGNCs defined with respect to $K^a - \sum_{j=1}^{n+1} \omega_{(j)} \ell^j \phi^a$. We start by defining the following quantities:

$$A_{(j)} = \left[f + k^2 + \ell^2 \left(\frac{1}{2} \langle -k \mathcal{L}_l f + Df, D\beta \rangle_h + f \text{tr}_h DD\beta + \frac{1}{2} f \mathcal{L}_l f \mathcal{L}_l \beta + O(k^2, kf, f^2) \right) + \ell^4 O(k, f) \right]_{(j)}, \quad (\text{D50a})$$

$$B_{(j)} = [f + k^2 + \ell^2 ((f + k^2) \alpha X + O(k^2, kf, f^2))]_{(j)}, \quad (\text{D50b})$$

$$C_{(j)} = \left[\frac{1}{4} \ell^4 (\beta')^2 \left\| \frac{1}{2} \mathcal{L}_k h \mathcal{L}_l f + k \otimes D \mathcal{L}_l f \right\|_h^2 \right]_{(j)}. \quad (\text{D50c})$$

In the base case $n=0$ which we have established in Sec. D 3 a, we obtained (D37a) from (D31) in order ℓ^6 and (D37b) from (D32) in order ℓ^4 , so one might expect to obtain the desired result for the induction step $n \rightarrow n+1$ by considering (D31) in order $\ell^{2(n+1)+6}$ and (D32) in order $\ell^{2(n+1)+4}$. We have

$$(AB)_{(2n+8)} = A_{(0)} B_{(2n+8)} + A_{(1)} B_{(2n+7)} + \dots + A_{(n+4)} B_{(n+4)} + \dots + A_{(2n+8)} B_{(0)}. \quad (\text{D51})$$

We claim that the important term is $A_{(n+4)} B_{(n+4)}$, so we need to show that all other terms vanish. Note that these terms always contain one factor of $A_{(j)}$ or $B_{(j)}$ for $j < n+4$. Let $j \leq n+3$, then

$$(f + k^2)_{(j)} = f_{(j)} + 2k_{(0)}^a k_{(j)a} + \dots + 2k_{(\lfloor j/2 \rfloor)}^a k_{(\lceil j/2 \rceil)a}, \quad (\text{D52})$$

if j is odd (otherwise exchange $2k_{(\lfloor j/2 \rfloor)}^a k_{(\lceil j/2 \rceil)a}$ with $k_{(j/2)}^a k_{(j/2)a}$). Noting that

$$\left\lfloor \frac{j}{2} \right\rfloor \leq \frac{j}{2} \leq \frac{n}{2} + \frac{3}{2} \leq n+1 \quad \text{for all } n \geq 1, \quad (\text{D53})$$

$$j \leq n+3, \quad (\text{D54})$$

and using the induction hypothesis, we can deduce $(f + k^2)_{(j)} = 0$ for all $j \leq n+3, n \geq 1$ (recalling the base case was already shown in sec. D 3 a). $O(f)$ terms with a prefactor of ℓ^2 always have order $\leq n+3$ in f ; hence, they do vanish, as well as k^2 and kf terms as already seen. In $(\langle k \mathcal{L}_l f, D\beta \rangle_h)_{(j-2)}$, k^a always appears in order $\leq n+1$, so this term gives also zero. The ℓ^4 term and the terms in $B_{(j)}$ for $j \leq n+3$ vanish for the same reason. Hence, we must have $(AB)_{(2n+8)} = A_{(n+4)} B_{(n+4)}$. Within $A_{(n+4)}$, the ℓ^2 terms at least linear in f are zero since they appear in order $\leq n+2 < n+3$. For $(\langle k \mathcal{L}_l f, D\beta \rangle_h)_{(n+2)}$, assume $\mathcal{L}_l f, D\beta$ or h have order ≥ 1 , then k has order $\leq n+2-1 = n+1$ and therefore vanishes by assumption. Thus, all quantities except k must have order 0,

$$(\langle k\mathcal{L}_l f, D\beta \rangle_h)_{(n+2)} = \langle k_{(n+2)}\mathcal{L}_l f_{(0)}, D\beta_{(0)} \rangle. \quad (\text{D55})$$

Moreover, we have already seen that the term $(f + k^2)_{(n+2)}$ vanishes, so that $B_{(n+4)} = (f + k^2)_{(n+4)}$. For the left side, assume that the order of β' is ≥ 1 , then $\|\dots\|_h^2$ has order $\leq 2n + 2$. Thus, each term in $(\|\dots\|_h^2)_{(j)}$ with $j \leq 2n + 2$ contains at least one factor $(\|\dots\|_h)_{(j')}$ with $j' \leq n + 1$, so f and k always appear in order $\leq n + 1$ and those terms vanish. Hence, the only nonzero term in $C_{(2n+8)}$ contains β' and, as can be verified similarly, h_{ab} and f in order 0 and thus k_a in order $n + 2$. The only nontrivial term left is $O(k^3)_{(2n+4)}$, which contains terms with at least one k factor of order $\leq \frac{2}{3}n + \frac{4}{3} \leq n + 1$ for all $n \geq 1$. Thus, $O(k^3)_{(2n+4)} = 0$ and, using again $D\mathcal{L}_l f_{(0)} = 0$ on the horizon, we obtain from (D31)

$$\begin{aligned} & \frac{1}{4}(\beta'_{(0)})^2 \kappa^2 \|\mathcal{L}_{k_{(n+2)}} h_{(0)}\|^2 \\ &= (f + k^2)_{(n+4)} [(f + k^2)_{(n+4)} + \kappa \langle k_{(n+2)}, D\beta_{(0)} \rangle]. \end{aligned} \quad (\text{D56})$$

Now consider the left side of (D32) in order $\ell^{2(n+1)+4}$ and define $E^a = k^a + \ell^2 O(k, f)$. First note that $E^a_{(j)} = 0$ for all $j \leq n + 1$ and $A_{(j)} = 0$ for all $j \leq n + 3$ from before. Therefore, we have

$$\begin{aligned} (AE^a)_{(2n+6)} &= A_{(0)}E^a_{(2n+6)} + \dots + A_{(n+3)}E^a_{(n+3)} \\ &\quad + A_{(n+4)}E^a_{(n+2)} + A_{(n+5)}E^a_{(n+1)} \\ &\quad + \dots + A_{(2n+6)}E^a_{(0)} \\ &= A_{(n+4)}E^a_{(n+2)}, \end{aligned} \quad (\text{D57})$$

and similarly, $(E^a B)_{(2n+6)} = E^a_{(n+2)} B_{(n+4)}$. Also note that $O(k^2, kf, f^2)_{(2n+2)}$ contains terms where k and f appear in order $\leq n + 1$. By assumption, those terms vanish. Thus, (D32) yields, using the expressions for $A_{(n+4)}$ and $B_{(n+4)}$ from before,

$$\left[(f + k^2)_{(n+4)} + \frac{1}{2} \kappa \langle k_{(n+2)}, D\beta_{(0)} \rangle \right] k^a_{(n+2)} = c_{(2n+6)} \phi^a. \quad (\text{D58})$$

(D56) and (D58) have the same form as (D37a) and (D37b), so we can apply the same procedure as before in Sec. D 3 a while exchanging $(f + k^2)_{(3)} \leftrightarrow (f + k^2)_{(n+4)}$, $k^a_{(1)} \leftrightarrow k^a_{(n+2)}$ and $c_{(4)} \leftrightarrow c_{(2n+6)}$ as well as $\omega_{(1)} \ell \leftrightarrow \omega_{(n+2)} \ell^{n+2}$ within the $n = 0$ case. From this procedure, we can directly

deduce $f_{(j)} = 0$ for all $j \leq n + 4$, $k_{(j)a} = 0$ for all $j \leq n + 2$ and $c_{(j)} = 0$ for all $j \leq 2n + 7$ on the horizon for gGNCs defined with respect to the Killing vector field $K^a - \sum_{j=1}^{n+2} \omega_{(j)} \ell^j \phi^a$. ■

APPENDIX E: t - φ REFLECTION SYMMETRY ι

Here, we elucidate the reflection symmetry ι on $\mathcal{H}^+$ referred to in assumption (iii') of Sec. IV A.

We may introduce coordinates (φ, ϑ) on the cut H as in (iii') such that $(\partial_\varphi)^a = \phi^a$. By assumption (iii'), we have $\iota^* \phi^a = -\phi^a$, $\iota^* h_{ab} = h_{ab}$, $\iota^* p^a_b = p^a_b$, $\iota^* k_a = -k_a$, $\iota^* f = f$ on H . In view of $\iota^* \phi^a = -\phi^a$, either $\iota(\varphi, \vartheta) = (-\varphi, \vartheta)$, or $\iota(\varphi, \vartheta) = (-\varphi, \pi - \vartheta)$. The last possibility may be excluded as follows by considering the condition on the complex structure on H in (iii').

The almost complex structure on $H \cong \mathbb{S}^2$ is $J^a_b = q^{ac} \epsilon_{cb}$, where $\epsilon_{ab} = 2\Psi \sin \vartheta (d\vartheta)_{[a} (d\varphi)_{b]}$ and $q^{ab} = \Psi^{-1}[(\sin \vartheta)^{-1}(\partial_\varphi)^a (\partial_\varphi)^b + (\partial_\vartheta)^a (\partial_\vartheta)^b]$ in our coordinates, see (D44). From these formulas, we immediately see that the case $\iota(\varphi, \vartheta) = (-\varphi, \pi - \vartheta)$ is excluded by (40) in (iii').

By Lie transporting the coordinates (φ, ϑ) along $K^a = (\partial_v)^a$, we get a coordinate system (v, φ, ϑ) on $\mathcal{H}^+$, and in this coordinate system, ι is given by

$$\iota(v, \varphi, \vartheta) = (-v, -\varphi, \vartheta). \quad (\text{E1})$$

Since v is a coordinate associated with an asymptotically timelike Killing vector field, we may thus think of ι as a “ t - φ reflection symmetry”. Note that this symmetry only applies to the *pull back* of the metric on $\mathcal{H}^+$ and is different from the symmetry $(t, \varphi) \rightarrow (-t, -\varphi)$ in Boyer–Lindquist coordinates in Kerr, which maps between the future and past horizons.

By exploiting (E1), one may easily see¹⁵

$$k^a \equiv q^{ab} k_b = F \phi^a, \quad (\text{E2})$$

on $\mathcal{H}^+$, for some F compatible with all symmetries ϕ^a, K^a, ι . Indeed, these statements may be demonstrated in a pedestrian manner using the coordinates (v, φ, ϑ) : using that $\iota^* k_a = -k_a$, we see that k_a may not have a ϑ component, and using $\iota^* q^{ab} = q^{ab}$, we likewise see that q^{ab} may not have a mixed $\vartheta\varphi$ component, and no component may depend on (v, φ) .

¹⁵This is the only place where ι is used in this part of the proof.

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