

Topological charge asymmetry in a $\mathbb{CP}^N$ skyrmion-fermion coupled system

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Topology plays a central role in classifying solitonic configurations in field theories, providing robustness and a nonperturbative label, the so-called topological charge Q . In soliton-fermion coupled systems, the relation between the topological charge and the number of zero modes is well established through the index theorem. However, the physical consequences of the sign of the topological charge have remained largely unexplored. In this work, we study fermions in $2 + 1$ dimensions coupled to skyrmions with target space $\mathbb{CP}^N$, particularly focusing on the backreactions of the fermions and on the sign of the topological charge. We obtain the solutions in a self-consistent manner, which exhibit an asymmetry with respect to the topological charge $\pm Q$ especially in the strong coupling regimes. This asymmetry is caused by the fermionic eigenvalue problem inherent in the self-consistent formulation. Although the Lagrangian is symmetric under $Q \rightarrow -Q$, the coupled equations for the skyrmions and anti-skyrmions become inequivalent once fermionic backreaction is taken into account. We demonstrate the mechanism in $\mathbb{CP}^1$ and $\mathbb{CP}^2$ skyrmions, but the analysis is directly extendable for the general $\mathbb{CP}^N$.

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I. INTRODUCTION

Topological solitons are localized, nonperturbative field configurations that appear in a wide class of nonlinear systems [1–5]. They arise in diverse physical contexts, ranging from vortices and spin textures in condensed matter to nonperturbative excitations in relativistic field theories, such as monopoles and instantons in gauge theories and solitonic models of baryons in effective descriptions of quantum chromodynamics (QCD). In general, topological solitons can be formulated as mappings from physical space to a target manifold, and the corresponding homotopy classes define an integer-valued quantity, the topological charge Q . This quantity is invariant under continuous deformations of the field and thereby ensures the stability of localized configurations.

The topological charge Q not only guarantees the stability of solitons but also plays a crucial role in the localization of

fermions in soliton-fermion coupled systems [6–16]. The Dirac operator admits normalizable zero energy eigenstates, so-called zero modes, and the net number of such modes corresponds to the topological charge of the background soliton [17–19]. This correspondence is supported by the Atiyah-Patodi-Singer index theorem, providing a connection between a topological quantity and analytical spectra of matters [20,21]. This feature can be investigated in more detail through the spectral flow analysis as illustrated in Fig. 1, where the rearrangement of the fermion spectra under continuous deformation of the background field illustrates how fermionic energy levels cross the zero energy line [9,12,13,22–27].

In recent years, increasing attention has been paid to going beyond the background approximation by explicitly incorporating the backreaction of fermions on the soliton [28–39]. In this approach, the Dirac equation and the soliton field equation are solved simultaneously or self-consistently, allowing one to investigate how fermion localization affects the soliton configuration itself. It has been demonstrated that the presence of fermionic zero modes or bound states induce a remarkable deformation of the soliton profile. This self-consistent treatment makes it possible to uncover properties of solitons that are invisible in the background approximation.

Despite these developments, an important issue about the sign of topological charge Q remains unresolved.

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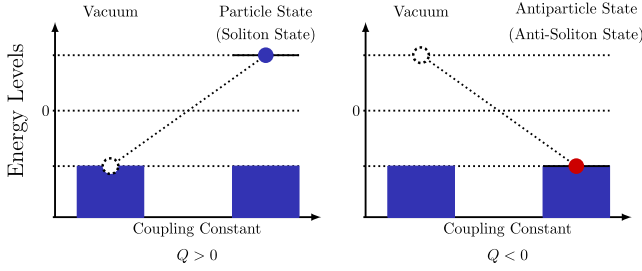


FIG. 1. Schematic illustration of a spectral flow with solitons ($Q > 0$) and antisolitons ($Q < 0$).

The previous studies have focused on configurations with either positive or negative topological charge. From a physical viewpoint, this raises a fundamental and still insufficiently explored question: can the sign of the topological charge have observable consequences beyond its conventional role as a classification label for soliton and antisoliton sectors? If the sign of Q are to influence how a soliton responds to external fields or self-consistent interactions, it would indicate that the sign itself carries physical information. Although the energy of a soliton and an antisoliton degenerates in many cases, not all systems possess this degeneracy; for example, it is lifted in chiral magnets [40–43] and in QCD with a strong magnetic field [44]. Once the backreaction from fermions is included, similar inequivalence between solitons ($Q > 0$) and antisolitons ($Q < 0$) is expected to appear. Since the fermion deforms the soliton configuration, the whole system is completely different in the sign of Q . In this paper, we refer to the inequivalence as topological charge asymmetry.¹

In this work, we address this issue by studying the fermion backreaction on a skyrmion with the target space $\mathbb{CP}^N$ in $2 + 1$ dimensions, which we simply call $\mathbb{CP}^N$ skyrmion [45–52] as an example. The skyrmions were originally proposed by Skyrme in the early 1960s as a nonlinear theory of the pion field, and they provided an effective model of baryons by identifying the baryon number with the topological charge [53,54]. The configurations considered in the present work are lower-dimensional analogs of these skyrmions, commonly referred to as baby skyrmions. While retaining the essential topological features, they allow for a more transparent analysis of fermion-soliton interactions. When $\mathbb{CP}^N$ skyrmions possess rotational symmetry, the windings and the boundary conditions can be chosen independently, and their combination determines the sign of the topological charge. Consequently, the sign of the topological charge can be reversed by changing the winding orientation alone,

¹Although such an inequivalence has been referred to as a soliton-antisoliton asymmetry [44], we use the term “topological charge asymmetry” to highlight that the effect originates from the fermionic spectral properties associated with the sign of the topological charge as shown at the end of this paper.

without modifying the boundary condition. This feature is crucial for our purpose, as it enables a controlled comparison between configurations with opposite topological charge under the same boundary conditions, which is essential for identifying intrinsic topological charge asymmetries induced by fermionic backreaction. In practice, this separation is conveniently realized within a rotationally symmetric configuration. Moreover, fermions coupled to two-dimensional skyrmions provide effective models for Dirac fermions interacting with topological textures or defects in condensed matter systems.

In the present work, we focus on $N = 1, 2$, treating the $\mathbb{CP}^1$ skyrmions as trivially embedding configurations within the $\mathbb{CP}^2$ framework. We find that the topological charge asymmetry appears in both cases, indicating that the effect is not tied to special features of a particular target space, but rather originates from the generic interplay between fermions, topology, and backreaction.

This paper is organized as follows. In Sec. II, the $\mathbb{CP}^N$ skyrmion-fermion coupled model is introduced. Moreover, we write down equations of motion with rotationally symmetric ansatz. Then, we also discuss how the $\mathbb{CP}^1$ skyrmion can be realized as a specific configuration in the $\mathbb{CP}^2$ model. In Sec. III, we explain the boundary conditions, and the numerical computation method. Typical numerical results, which reveal the topological charge asymmetry, are presented in Sec. IV. In Sec. V, the mechanism of the asymmetry is discussed. We conclude with a summary and outlook in Sec. VI.

II. THE MODEL AND ANSATZ

We consider the $\mathbb{CP}^N$ skyrmion-Dirac fermion coupled system in $2 + 1$ dimensions in the form

$$\mathcal{L} = \bar{\Psi}(i\hat{\gamma}^\mu \partial_\mu - m\hat{\Sigma})\Psi + g^{-2} \left[\frac{M^2}{2} \text{Tr}(\partial_\mu \Sigma \partial^\mu \Sigma) + e^2 \text{Tr}([\partial_\mu \Sigma, \partial_\nu \Sigma]^2) - V \right], \quad (1)$$

with $M^2, e^2, g^{-2} > 0$. Here, Ψ represents a spinor multiplet and Σ is a Hermitian matrix-valued field² defined as

$$\Sigma = I_{N+1} - 2Z \otimes Z^\dagger \quad (2)$$

with the $(N + 1) \times (N + 1)$ unit matrix I_{N+1} and the $(N + 1)$ -component complex unit vector Z . To act the spinor, $\hat{\Sigma}$ and $\hat{\gamma}^\mu$ are defined by

$$\hat{\Sigma} \equiv I_2 \otimes \Sigma, \quad \hat{\gamma}^\mu \equiv \gamma^\mu \otimes I_{N+1} \quad (3)$$

where γ^μ are the gamma matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with the Minkowski metric $\eta^{\mu\nu} = \text{diag}(1, -1, -1)$. The two-dimensional versions of

²The field Σ has a linear relation with one of the $\text{SU}(N + 1)$ color field $\mathbf{n}$, such that $\mathbf{n} = \frac{N-1}{N+1} I_{N+1} - \Sigma$.

the gamma matrices are defined by the usual Pauli matrices σ_i ($i = 1, 2, 3$) as $\gamma^0 = \sigma_3$, $\gamma^1 = -i\sigma_1$ and $\gamma^2 = -i\sigma_2$ in the minimal representation. In this work, we employ the potential term V of the form

$$V = \frac{\mu^2}{16} [\text{Tr}(I_{N+1} - \Sigma_\infty \Sigma)]^2 \quad (4)$$

where $\mu^2 > 0$ and Σ_∞ is the vacuum value of Σ at the spatial infinity. This potential is square of the so-called old-baby potential. We employ this because the ordinary old-baby potential does not support CP^N ($N > 1$) soliton solutions [52].

There are five parameters in the coupled Lagrangian (1), m , g , M , e and μ . The length scale of the parameters and fields are

$$\begin{aligned} \Psi &: [L^{-1}], & U &: [L^0] \\ m &: [L^{-1}], & g &: [L^0], & M &: [L^{-\frac{1}{2}}], & e &: [L^{\frac{1}{2}}], & \mu &: [L^{-\frac{3}{2}}]. \end{aligned}$$

Suitably rescaling the coordinates, the fields and parameters

$$x^\mu \rightarrow \frac{e}{M} x^\mu, \quad \Psi \rightarrow \frac{M}{e} \Psi, \quad \mu \rightarrow \frac{M^2}{e} \mu, \quad m \rightarrow \frac{M}{e} m,$$

the rescaled Lagrangian becomes

$$\begin{aligned} \mathcal{L}' &= \bar{\Psi}(i\hat{\gamma}^\mu \partial_\mu - m\hat{\Sigma})\Psi \\ &+ g^{-2} \left[\frac{1}{2} \text{Tr}(\partial_\mu \Sigma \partial^\mu \Sigma) + \text{Tr}([\partial_\mu \Sigma, \partial_\nu \Sigma]^2) - V \right], \end{aligned} \quad (5)$$

where we redefine the coupling constant g as $g^{-1} \rightarrow g^{-1}/\sqrt{Me}$. Note that all the parameters (m, g, μ) and fields (Ψ, Σ) are dimensionless. In this paper, we fix $\mu = 1$ for simplicity.

The field Σ is invariant under the local gauge transformation $Z \rightarrow e^{i\theta} Z$ with $\theta \in \mathbb{R}$. Due to the phase redundancy and the normalization condition $Z^\dagger Z = 1$, the field Z takes its value on $\text{CP}^N \simeq \text{SU}(N+1)/(\text{SU}(N) \times \text{U}(1))$. In this representation, the stabilizer subgroup $\text{SU}(N) \times \text{U}(1)$ describes the isotropy subgroup of a point on the target space. The nontrivial topology of CP^N arises from the $\text{U}(1)$ factor in this stabilizer: the relevant homotopy relation,

$$\pi_2(\text{CP}^N) \simeq \pi_1(\text{SU}(N) \times \text{U}(1))_{\text{SU}(N+1)} \simeq \pi_1(\text{U}(1)) \simeq \mathbb{Z}, \quad (6)$$

shows that the second homotopy group of the target space CP^N is inherited from the loop structure of this $\text{U}(1)$ fiber. Thus there is a topological index, so-called topological charge Q , which is given by

$$Q = -\frac{i}{2\pi} \oint_{S^1} Z^\dagger dZ. \quad (7)$$

This integer labels the skyrmion, and corresponds to the number of fermionic zero modes. This topological charge is determined by the $\text{U}(1)$ factor of the stabilizer subgroup,

which also governs the topology in the simpler case of $\text{CP}^1 \simeq \text{SU}(2)/\text{U}(1)$.

In order to obtain the field equations, we rewrite the Lagrangian (1) in terms of $Z = (Z_1, Z_2, \dots, Z_N, Z_{N+1})^T$ where T represents transposition of the vector. In this article, we choose the vacuum as $\Sigma_\infty = \text{diag}(1, 1, \dots, 1, -1)$ which corresponds to $Z_\infty \propto (0, 0, \dots, 0, 1)^T$. Then, decomposing the Lagrangian into the fermionic part $\mathcal{L}_f$ and bosonic part $\mathcal{L}_b$, i.e., $\mathcal{L} = \mathcal{L}_f + g^{-2} \mathcal{L}_b$, one can write them in terms of Z as follows:

$$\mathcal{L}_f = \bar{\Psi}[i\hat{\gamma}^\mu \partial_\mu - mI_2 \otimes (I_{N+1} - 2Z \otimes Z^\dagger)]\Psi, \quad (8)$$

$$\begin{aligned} \mathcal{L}_b &= (D_\mu Z)^\dagger D^\mu Z + 16((D_\mu Z)^\dagger D_\nu Z)^2 \\ &- 8((D_\mu Z)^\dagger D^\mu Z)^2 - 8|(D_\mu Z)^\dagger D_\nu Z|^2 \\ &- \frac{\mu^2}{4} (1 - |Z_{N+1}|^2)^2, \end{aligned} \quad (9)$$

where $D_\mu \equiv \partial_\mu - Z^\dagger \partial_\mu Z$ is the covariant derivative. Based on the variational principle with the nonlinear constraint $Z^\dagger Z = 1$, we obtain the equations of motion of the form

$$(i\gamma_{\alpha\beta}^\mu \delta_{ab} \partial_\mu - m\delta_{\alpha\beta} \delta_{ab} + 2m\delta_{\alpha\beta} Z_a Z_b^*) \Psi_{\beta,b} = 0, \quad (10)$$

and

$$\begin{aligned} &(D_\mu D^\mu - Z^\dagger D_\mu D^\mu Z) Z_a + 32D_\mu \{[(D^\mu Z)^\dagger D^\nu Z] D_\nu Z_a\} \\ &- 32Z^\dagger D_\mu \{[(D^\mu Z)^\dagger D^\nu Z]\} D_\nu Z_a \\ &- 16D_\mu \{[(D_\nu Z)^\dagger D^\nu Z] D^\mu Z_a + [(D^\nu Z)^\dagger D^\mu Z] D_\nu Z_a\} \\ &+ 16Z^\dagger D_\mu \{[(D_\nu Z)^\dagger D^\nu Z] D^\mu Z + [(D^\nu Z)^\dagger D^\mu Z] D_\nu Z\} Z_a \\ &- \frac{mg^2}{2} [\Psi_{\alpha,b}^* Z_b \gamma_{\alpha\beta}^0 \Psi_{\beta,a} - (\Psi_{\alpha,b}^* \gamma_{\alpha\beta}^0 Z_b Z_c^* \Psi_{\beta,c}) Z_a] \\ &- \frac{\mu^2}{2} (1 - |Z_{N+1}|^2) Z_{N+1} \delta_{a,N+1} = 0, \end{aligned} \quad (11)$$

where $\alpha, \beta = 1, 2$ denote the spinor indices and $a, b, c = 1, 2, \dots, N+1$ represent the internal indices.

A. Rotationally symmetric ansatz and reduced field equations

In this paper, we concentrate on rotationally symmetric static CP^2 skyrmions. Because of the overall phase redundancy of the Z field, such configurations can generally be parametrized as³

³While CP^2 solitons are often constructed using local coordinates (u_1, u_2) [46], we parametrize the homogeneous coordinate Z . When one uses the local coordinates, the boundary values at the origin and infinity involve divergence, requiring additional redefinitions for numerical computations [52]. To save this effort, we adopt the expression in terms of Z , but essentially the same thing can be done using local coordinates.

$$Z = \begin{pmatrix} \cos F(r) \\ \sin F(r) \cos G(r) e^{in_1 \varphi} \\ \sin F(r) \sin G(r) e^{in_2 \varphi} \end{pmatrix}. \quad (12)$$

Here, we assume the profile functions $F(r)$ and $G(r)$ are monotonic functions of the radial coordinate r , n_1 and n_2 are integers, and φ is the angular coordinate. The rotationally symmetric $\mathbb{CP}^2$ ansatz introduced above naturally includes $\mathbb{CP}^1$ configurations as a special subclass. In order to make this explicit, we describe how a $\mathbb{CP}^1$ embedding is realized within the present parametrization. Here we set $n_1 = 0$ and $G(r) = \frac{\pi}{2}$, i.e., $Z_2 = 0$. Then we obtain a $\mathbb{CP}^1$ configuration $Z^{\mathbb{CP}^1}$ as

$$Z^{\mathbb{CP}^1} = \begin{pmatrix} \cos F(r) \\ 0 \\ \sin F(r) e^{in_2 \varphi} \end{pmatrix}. \quad (13)$$

This embedding is used as a convenient reference when comparing $\mathbb{CP}^1$ and $\mathbb{CP}^2$ configurations.

The topological charge can be written as an integral of a total derivative. From Eqs. (7) and (12), we obtain

$$Q = \int_0^\infty dr \partial_r \Omega, \quad (14)$$

$$\Omega \equiv \sin^2 F (n_1 \cos^2 G + n_2 \sin^2 G). \quad (15)$$

As a result, the topological charge Q is determined by the boundary values, $\Omega(0)$ and $\Omega(\infty)$.

Substituting the ansatz (12) into Eq. (9), we obtain the static energy density $\mathcal{E}_b = -\mathcal{L}_b$

$$\begin{aligned} \mathcal{E}_b &= (F')^2 + \sin^2 F (G')^2 \\ &+ \frac{1}{r^2} \left[\frac{\Omega^2}{\tan^2 F} + (n_1 - n_2)^2 \sin^2 F \sin^2 G \cos^2 G \right] \\ &+ \frac{48}{r^2} \left[\frac{\Omega}{\tan F} F' - (n_1 - n_2) \sin^2 F \sin G \cos G G' \right]^2 \\ &+ \frac{\mu^2}{4} (1 - \sin^2 F \sin^2 G)^2, \end{aligned} \quad (16)$$

where the derivative with respect to r is denoted by $\frac{d}{dr} \equiv '$. Formally, the Euler-Lagrange equation of Eq. (16) can be written as

$$\begin{aligned} c_1 F'' + c_2 G'' + c_3 (F')^2 + c_4 (G')^2 \\ + c_5 F' G' + \frac{c_6}{r} F' + \frac{c_7}{r} G' + c_8 = 0, \end{aligned} \quad (17)$$

$$\begin{aligned} c_9 G'' + c_{10} F'' + c_{11} (G')^2 + c_{12} (F')^2 \\ + c_{13} G' F' + \frac{c_{14}}{r} G' + \frac{c_{15}}{r} F' + c_{16} = 0, \end{aligned} \quad (18)$$

where the coefficients $c_i (i = 1, 2, \dots, 16)$ are complicated trigonometric polynomials and their explicit forms are given in Appendix A.

Since the rotationally symmetric ansatz for skyrmion is introduced in this section, the fermion field Ψ is also parametrized with the rotational symmetry. Recalling that the skyrmions are static, we impose that the fermion field Ψ has the stationary form $\Psi = \psi e^{-i\epsilon t}$. Then, the Dirac equation (10) becomes

$$\mathcal{H}\psi = \epsilon\psi, \quad \mathcal{H} = \hat{\gamma}^0 (-i\hat{\gamma}^j \partial_j + m\hat{\Sigma}) \quad (19)$$

where $\mathcal{H}$ is the Dirac Hamiltonian and ϵ is the energy eigenvalue. The corresponding eigenfunction of the Dirac Hamiltonian $\mathcal{H}$ can be written as

$$\psi = \mathcal{N} \begin{pmatrix} u_1(r) e^{i\ell\varphi} \\ d_1(r) e^{i(\ell+n_1)\varphi} \\ s_1(r) e^{i(\ell+n_2)\varphi} \\ u_2(r) e^{i(\ell+1)\varphi} \\ d_2(r) e^{i(\ell+n_1+1)\varphi} \\ s_2(r) e^{i(\ell+n_2+1)\varphi} \end{pmatrix} \quad (20)$$

where u_i , d_i and s_i ($i = 1, 2$) are the radial functions of spinor components, $\ell \in \mathbb{Z}$, and $\mathcal{N}$ is the normalization factor which is defined by

$$\begin{aligned} \int d^2x \psi^\dagger \psi &= 2\pi \mathcal{N}^2 \int_0^\infty dr r (u_1^2 + d_1^2 + s_1^2 + u_2^2 + d_2^2 + s_2^2) \\ &= 1. \end{aligned} \quad (21)$$

Note that the Lagrangian $\mathcal{L}_f + g^{-2} \mathcal{L}_b$ on our ansatz (12) and (20) is invariant under a sign flip of the windings, $n_1 \rightarrow -n_1$ and $n_2 \rightarrow -n_2$ which corresponds to a sign reversal of the topological charge Q .

In this parametrization, the Dirac Hamiltonian $\mathcal{H}$ is commuted with an extended spin operator $\mathcal{K}$, a so-called grand-spin operator, which is defined by

$$\begin{aligned} \mathcal{K} &= -i\partial_\varphi I_2 \otimes I_3 + \frac{1}{2} \sigma_3 \otimes I_3 + \frac{n_1}{2} I_2 \otimes \lambda_3 \\ &- \frac{n_1 - 2n_2}{2\sqrt{3}} I_2 \otimes \lambda_8 \end{aligned} \quad (22)$$

where λ_3 and λ_8 represent the third and eighth components of Gell-Mann matrices, respectively. Since this operator satisfies $\mathcal{K}\mathcal{H}\mathcal{K}^{-1} = \mathcal{H}$, the solutions of Eq. (19) are labeled by the eigenvalue κ , which is given by

$$\mathcal{K}\psi = \kappa\psi, \quad (23)$$

$$\kappa = \ell + \frac{1}{2} + \frac{n_1 + n_2}{3}. \quad (24)$$

Plugging the ansatz (12) and (20) into the equations of the system (11) and (19), we can obtain some ordinary differential equations(ODEs)

$$\begin{aligned}
& c_1 F'' + c_2 G'' + c_3 (F')^2 + c_4 (G')^2 + c_5 F' G' + \frac{c_6}{r} F' + \frac{c_7}{r} G' + c_8 \\
& - \frac{mg^2}{2} r [-(u_1^2 - u_2^2) \sin 2F + (u_1 d_1 - u_2 d_2) \cos 2F \cos G + (u_1 s_1 - u_2 s_2) \cos 2F \sin G] \\
& - \frac{mg^2}{2} r [(u_1 d_1 - u_2 d_2) \cos 2F \cos G + (d_1^2 - d_2^2) \sin 2F \cos^2 G + (d_1 s_1 - d_2 s_2) \sin 2F \sin G \cos G] \\
& - \frac{mg^2}{2} r [(u_1 s_1 - u_2 s_2) \cos 2F \sin G + (d_1 s_1 - d_2 s_2) \sin 2F \sin G \cos G + (s_1^2 - s_2^2) \sin 2F \sin^2 G] = 0, \quad (25)
\end{aligned}$$

$$\begin{aligned}
& c_9 G'' + c_{10} F'' + c_{11} (G')^2 + c_{12} (F')^2 + c_{13} G' F' + \frac{c_{14}}{r} G' + \frac{c_{15}}{r} F' + c_{16} \\
& - \frac{mg^2}{2} r [-(u_1 d_1 - u_2 d_2) \sin F \cos F \sin G + (u_1 s_1 - u_2 s_2) \sin F \cos F \cos G] \\
& - \frac{mg^2}{2} r [-(u_1 d_1 - u_2 d_2) \sin F \cos F \sin G - (d_1^2 - d_2^2) \sin^2 F \sin 2G + (d_1 s_1 - d_2 s_2) \sin^2 F \cos 2G] \\
& - \frac{mg^2}{2} r [(u_1 s_1 - u_2 s_2) \sin F \cos F \cos G + (d_1 s_1 - d_2 s_2) \sin^2 F \cos 2G + (s_1^2 - s_2^2) \sin^2 F \sin 2G] = 0, \quad (26)
\end{aligned}$$

$$m(1 - 2 \cos^2 F)u_1 + m(-2 \sin F \cos F \cos G)d_1 + m(-2 \sin F \cos F \sin G)s_1 + \left(-\partial_r - \frac{\ell + 1}{r}\right)u_2 = \varepsilon u_1, \quad (27)$$

$$m(-2 \sin F \cos F \cos G)u_1 + m(1 - 2 \sin^2 F \cos^2 G)d_1 + m(-2 \sin^2 F \sin G \cos G)s_1 + \left(-\partial_r - \frac{\ell + n_1 + 1}{r}\right)d_2 = \varepsilon d_1, \quad (28)$$

$$m(-2 \sin F \cos F \sin G)u_1 + m(-2 \sin^2 F \sin G \cos G)d_1 + m(1 - 2 \sin^2 F \sin^2 G)s_1 + \left(-\partial_r - \frac{\ell + n_2 + 1}{r}\right)s_2 = \varepsilon s_1, \quad (29)$$

$$m(1 - 2 \cos^2 F)u_2 + m(-2 \sin F \cos F \cos G)d_2 + m(-2 \sin F \cos F \sin G)s_2 + \left(-\partial_r + \frac{\ell}{r}\right)u_1 = -\varepsilon u_2, \quad (30)$$

$$m(-2 \sin F \cos F \cos G)u_2 + m(1 - 2 \sin^2 F \cos^2 G)d_2 + m(-2 \sin^2 F \sin G \cos G)s_2 + \left(-\partial_r + \frac{\ell + n_1}{r}\right)d_1 = -\varepsilon d_2, \quad (31)$$

$$m(-2 \sin F \cos F \sin G)u_2 + m(-2 \sin^2 F \sin G \cos G)d_2 + m(1 - 2 \sin^2 F \sin^2 G)s_2 + \left(-\partial_r + \frac{\ell + n_2}{r}\right)s_1 = -\varepsilon s_2. \quad (32)$$

Solutions of these equations give the stationary point of the total energy functional

$$E_{\text{tot}} = \int d^2x (\psi^\dagger \mathcal{H} \psi + g^{-2} \mathcal{E}_b). \quad (33)$$

In order to obtain the solutions numerically, some appropriate boundary conditions, at the origin and the spatial infinity, are required.

III. NUMERICAL SCHEME

In this section, we describe the numerical scheme we used to solve the coupled ODEs (25)–(32) self-consistently. The Subsec. III A and III B explain the boundary conditions for the skyrmion field and fermion field respectively. The Subsec. III C illustrates the numerical method.

A. Boundary conditions for profile functions

As we wrote above, we impose $Z_\infty \propto (0, 0, 1)^T$. It implies that F and G must satisfy

$$\begin{aligned} F(\infty) &= \left(\frac{1}{2} + p\right)\pi, \\ G(\infty) &= \left(\frac{1}{2} + q\right)\pi(p, q \in \mathbb{Z}). \end{aligned} \quad (34)$$

The $\mathbb{CP}^2$ skyrmions can be constructed with the following boundary conditions at the origin:

$$F(0) = p\pi, \quad G(0) = q\pi(p, q \in \mathbb{Z}) \quad (35)$$

For simplicity, we set $p = q = 0$, and therefore the boundary conditions we impose for the $\mathbb{CP}^2$ skyrmions are

$$F(0) = G(0) = 0, \quad F(\infty) = G(\infty) = \frac{\pi}{2}. \quad (36)$$

Then, it follows from Eqs. (14) and (15) that the topological charge is

$$Q = \Omega(\infty) - \Omega(0) = n_2. \quad (37)$$

In this case, n_1 is not a topological quantity. Note that for the $\mathbb{CP}^1$ embedding configuration, we set $G(r) = \pi/2$, instead of $G(0) = 0$. One can see that the topological charge of such $\mathbb{CP}^1$ skyrmions is also given by $Q = n_2$.

B. Boundary conditions for spinors

Here, we perform the asymptotic analysis for the Dirac equation. Taking into account the boundary condition $F(0)$ and $G(0)$ given in Eq. (35), the Dirac equation at the vicinity of origin becomes

$$(m + \varepsilon)u_1 + \left(\partial_r + \frac{\ell + 1}{r}\right)u_2 = 0, \quad (38)$$

$$(-m + \varepsilon)d_1 + \left(\partial_r + \frac{\ell + n_1 + 1}{r}\right)d_2 = 0, \quad (39)$$

$$(-m + \varepsilon)s_1 + \left(\partial_r + \frac{\ell + n_2 + 1}{r}\right)s_2 = 0, \quad (40)$$

$$(m - \varepsilon)u_2 + \left(\partial_r - \frac{\ell}{r}\right)u_1 = 0, \quad (41)$$

$$(-m - \varepsilon)d_2 + \left(\partial_r - \frac{\ell + n_1}{r}\right)d_1 = 0, \quad (42)$$

$$(-m - \varepsilon)s_2 + \left(\partial_r - \frac{\ell + n_2}{r}\right)s_1 = 0. \quad (43)$$

Assuming that each spinor component can be expanded around the origin in a power series, we can find relations between the boundary values of the spinor components and the angular momentum ℓ of the form

$$\ell u_1(0) = 0, (\ell + 1)u_2(0) = 0, \quad (44)$$

$$(\ell + n_1)d_1(0) = 0, (\ell + n_1 + 1)d_2(0) = 0, \quad (45)$$

$$(\ell + n_2)s_1(0) = 0, (\ell + n_2 + 1)s_2(0) = 0. \quad (46)$$

For the embedded $\mathbb{CP}^1$ case, we set $G(r) = \frac{\pi}{2}$ and $n_1 = 0$. Then we have the Dirac equations

$$\begin{aligned} m(1 - 2\cos^2 F)u_1 + m(-2\sin F \cos F)s_1 \\ + \left(-\partial_r - \frac{\ell + 1}{r}\right)u_2 = \varepsilon u_1, \end{aligned} \quad (47)$$

$$md_1 + \left(-\partial_r - \frac{\ell + 1}{r}\right)d_2 = \varepsilon d_1, \quad (48)$$

$$\begin{aligned} m(-2\sin F \cos F)u_1 + m(1 - 2\sin^2 F)s_1 \\ + \left(-\partial_r - \frac{\ell + n_2 + 1}{r}\right)s_2 = \varepsilon s_1, \end{aligned} \quad (49)$$

$$\begin{aligned} m(1 - 2\cos^2 F)u_2 + m(-2\sin F \cos F)s_2 \\ + \left(-\partial_r + \frac{\ell}{r}\right)u_1 = -\varepsilon u_2, \end{aligned} \quad (50)$$

$$md_2 + \left(-\partial_r + \frac{\ell}{r}\right)d_1 = -\varepsilon d_2, \quad (51)$$

$$\begin{aligned} m(-2\sin F \cos F)u_2 + m(1 - 2\sin^2 F)s_2 \\ + \left(-\partial_r + \frac{\ell + n_2}{r}\right)s_1 = -\varepsilon s_2. \end{aligned} \quad (52)$$

These equations show that d_1 and d_2 represent plane wave solutions in this case. Since these free components may destroy the numerical calculation, because the corresponding bosonic Z_2 component vanishes, we set $d_1(r) = d_2(r) = 0$. Based on this argument, we can get similar boundary conditions,

$$\ell u_1(0) = 0, \quad (\ell + 1)u_2(0) = 0, \quad (53)$$

$$(\ell + n_2)s_1(0) = 0, \quad (\ell + n_2 + 1)s_2(0) = 0. \quad (54)$$

At the spatial infinity, the Dirac equations approach

$$(-m + \varepsilon)u_1 + \left(\partial_r + \frac{\ell + 1}{r}\right)u_2 = 0, \quad (55)$$

$$(-m + \varepsilon)d_1 + \left(\partial_r + \frac{\ell + n_1 + 1}{r}\right)d_2 = 0, \quad (56)$$

$$(m + \varepsilon)s_1 + \left(\partial_r + \frac{\ell + n_2 + 1}{r}\right)s_2 = 0, \quad (57)$$

$$(-m - \varepsilon)u_2 + \left(\partial_r - \frac{\ell}{r}\right)u_1 = 0, \quad (58)$$

$$(-m - \varepsilon)d_2 + \left(\partial_r - \frac{\ell + n_1}{r}\right)d_1 = 0, \quad (59)$$

$$(m - \varepsilon)s_2 + \left(\partial_r - \frac{\ell + n_2}{r}\right)s_1 = 0. \quad (60)$$

These equations have analytical solutions written by the standard Bessel function. Using these solutions and the appropriate boundary condition for the Bessel function, one can construct the spinor basis in a cylinder of radius R as shown in Refs. [9,12,13,26,27]. The explicit form of the plane wave basis is presented in Appendix B.

C. Numerical methods

Here, we provide a quick overview of the numerical procedure we use to solve the coupled ODEs (25)–(32). Instead of solving the Dirac equation (27)–(30) directly, we solve the following secular equation, which corresponds to the eigenequation (19). The secular equation for the energy eigenvalue ε is given by

$$\det(H - \varepsilon \tilde{I}) = 0, \quad (61)$$

where the matrix element of H and $\tilde{I}$ is defined as

$$H_{k^{(p)}, k^{(q)}} \equiv \int d^2x \Phi_\kappa^{(p)}(k^{(p)}, x)^\dagger \mathcal{H} \Phi_\kappa^{(q)}(k^{(q)}, x), \quad (62)$$

$$\tilde{I}_{k^{(p)}, k^{(q)}} \equiv \int d^2x \Phi_\kappa^{(p)}(k^{(p)}, x)^\dagger \Phi_\kappa^{(q)}(k^{(q)}, x), \quad (63)$$

$(p, q = u, d, s),$

where $\Phi_\kappa^{(q)}$ is the orthonormal basis with the quantum number κ and $k^{(q)}$ is the discretized wave number associated with the boundary condition of the Bessel function. (For details, see Refs. [9,12,13,26,27] and Appendix B).

By solving the secular equation (61) numerically, we can obtain some energy spectra $\{\varepsilon\}$ and corresponding eigenspinors $\{\psi\}$ for each κ . The eigenproblem (61) is solved by a standard matrix diagonalization solver “dsyev” in the LAPACK, Math Kernel Library. This simulation uses the computational domain-size $R = 25.0$, and the number of discrete wave numbers is chosen as 512, which gives sufficient convergence.

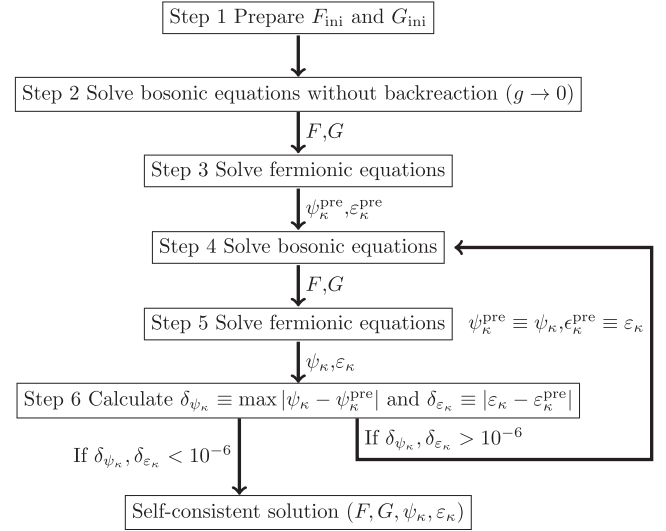


FIG. 2. Schematic illustration of the self-consistent analysis.

In order to solve the ODEs (25) and (26) for the skyrmion field, we use the Newton-Raphson method with 512 mesh points. The computational error is of order 10^{-6} . Since the skyrmion localizes around the origin, we impose the boundary condition $F(R) = G(R) = \frac{\pi}{2}$ at $r = R$ instead of the spatial infinity.

We solve the coupled equations (25)–(32) by combining the above two numerical calculation methods. The numerical procedure consists of the following steps (see Fig. 2).

- (1) Prepare a suitable initial configuration F_{ini} and G_{ini} .⁴
- (2) Solve the ODEs for F and G without backreaction (17) and (18) by the Newton-Raphson method with the initial configurations F_{ini} and G_{ini} .
- (3) Solve the secular equation (61) with the matrix diagonalization method, using the skyrmion configuration obtained in Step 2, as the background field.
- (4) Solve the skyrmion field equations (25) and (26) by the Newton-Raphson method using the fermion configuration $\{\psi_\kappa, \varepsilon_\kappa\}$ with specific κ obtained in Step 3 as the background field.
- (5) Solve the secular equation (61) with the matrix diagonalization method, using the skyrmion configuration obtained in Step 4 as the background field.
- (6) Calculate the relative errors between the fermion configurations obtained in Step 3 and Step 5. If the relative error is sufficiently small (we employ a criterion of order 10^{-6}), the configuration can be regarded as a solution in a self-consistent manner.

Otherwise, repeat Steps 4 and 5 until the relative errors between fermion configurations in successive iterations satisfy the convergence criterion.

Note that, in this paper, we consider the coupled system with a single fermion mode. One can generalize this

⁴We choose an analytical solution of the nonlinear sigma model $\mathcal{L} = \frac{M^2}{2} \text{Tr}(\partial_\mu \Sigma \partial^\mu \Sigma)$ as the initial condition [1,55–58].

TABLE I. Sets of the winding numbers (n_1, n_2) that we consider in this paper. The integer n_2 equals the topological charge.

n_1	0	0	0	0	1	-1
n_2	1	-1	3	-3	3	-3

analysis toward more general situations that multifermion modes couple to the skyrmion. The detail is explained in the next section.

IV. NUMERICAL RESULTS

Here we present our typical numerical solutions. Throughout this paper, we consider six combinations (n_1, n_2) as shown in Table I, where n_2 is equal to the topological charge. The cases of $n_1 = 0$ correspond to the embedded $\mathbb{CP}^1$ cases with $G(r) = \pi/2$. On the other hand, the genuine $\mathbb{CP}^2$ skyrmions possess $n_1 \neq 0$. In this paper, we consider two genuine $\mathbb{CP}^2$ cases with $(n_1, n_2) = (\pm 1, \pm 3)$, which gives the simplest genuine $\mathbb{CP}^2$ skyrmions in our model.⁵ In order to compare the $\mathbb{CP}^1$ cases with the $\mathbb{CP}^2$ cases, we also consider the simplest embedding cases, $(n_1, n_2) = (0, \pm 1)$, and the $\mathbb{CP}^1$ skyrmions possessing the same topological charge with the genuine $\mathbb{CP}^2$ skyrmions, i.e., $(n_1, n_2) = (0, \pm 3)$.

As for the fermion solutions, we define the fermion density ρ_κ which is given by

$$\rho_\kappa(r) = \frac{1}{2\pi} \int_0^\infty d\varphi \psi_\kappa^\dagger \psi_\kappa. \quad (64)$$

Thanks to the index theorem, the fermion energy spectra $\{\varepsilon\}$ possess a number of zero-crossing modes corresponding to the topological charge of skyrmions as shown in Fig. 3 (see also Refs. [9,26,27]). Since our primary interest is to clarify the qualitative difference between cases with positive and negative topological charges, we extract the essence of the asymmetry by choosing a single zero mode, i.e., a configuration in which only a single fermionic zero mode is occupied. The choices of quantum number κ for the variety of (n_1, n_2) are shown in Table II. Note that the $|Q|$ smallest values of $|\kappa|$ correspond to the zero modes. For $|Q| > 1$, the skyrmion is no longer concentrated at the origin but develops a ring-shaped profile peaked at a finite radius [2,59]. To align the fermion localization with this structure, it is natural to couple the skyrmion to a zero mode whose radial profile also peaks away from the origin. Since the zero modes with larger quantum number $|\kappa|$ localize at larger radii, we choose the mode with the highest $|\kappa|$, which

⁵Although static bosonic solutions with $|Q| = 2$ can formally be constructed, it has been shown that such configurations exhibit pathological behavior once time dependence is considered, leading to divergent energies when interpreted as vortices [52].

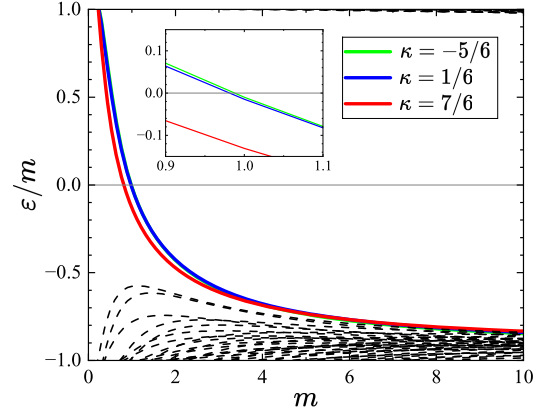


FIG. 3. Example of the spectral flow on a background skyrmion with $(n_1, n_2) = (-1, -3)$.

achieves the best spatial overlap with the ringlike skyrmion core and therefore maximizes the backreaction.

Moreover, from the spectral flow in the background approximation (see Fig. 3), one expects that different zero modes produce qualitatively similar backreaction effects. We have explicitly verified that this expectation holds: occupying different single zero mode leads to essentially the same deformation of the skyrmion configuration. If all zero modes are occupied, their contributions simply add up in the coupled equations (25) and (26) through interaction terms of the same structure, without inducing any additional interactions among different zero modes. As a result, the fermionic backreaction differs only by an overall multiplicative factor proportional to the number of occupied modes. For this reason, restricting to a single occupied zero mode is sufficient to capture the qualitative features of the fermion-induced deformation.

Our results in Fig. 4 show that increasing the Yukawa coupling constant m , the fermion densities ρ_κ begin to localize. In the weak coupling regime, ρ_κ spreads out the space like plane waves. In the strong coupling regime, however, ρ_κ is strongly localized around the origin. This is a consequence of the index theorem.

One can see the difference between positive topological charge cases and negative charge cases in the static energy density $g^{-2}\mathcal{E}_b$ of the skyrmion. In Fig. 5, the energy density $g^{-2}\mathcal{E}_b$ is presented for each parameter set (n_1, n_2) . Our results in Fig. 5 show that increasing the Yukawa coupling constant m , the energy density $g^{-2}\mathcal{E}_b$ is deformed due to the backreaction. However, the deformation pattern varies depending on the sign of topological charge Q .

TABLE II. Choices of the grand spin quantum number κ for each combination of (n_1, n_2) .

(n_1, n_2)	(0, 1)	(0, -1)	(0, 3)	(0, -3)	(1, 3)	(-1, -3)
κ	-1/6	1/6	-3/2	3/2	-7/6	7/6

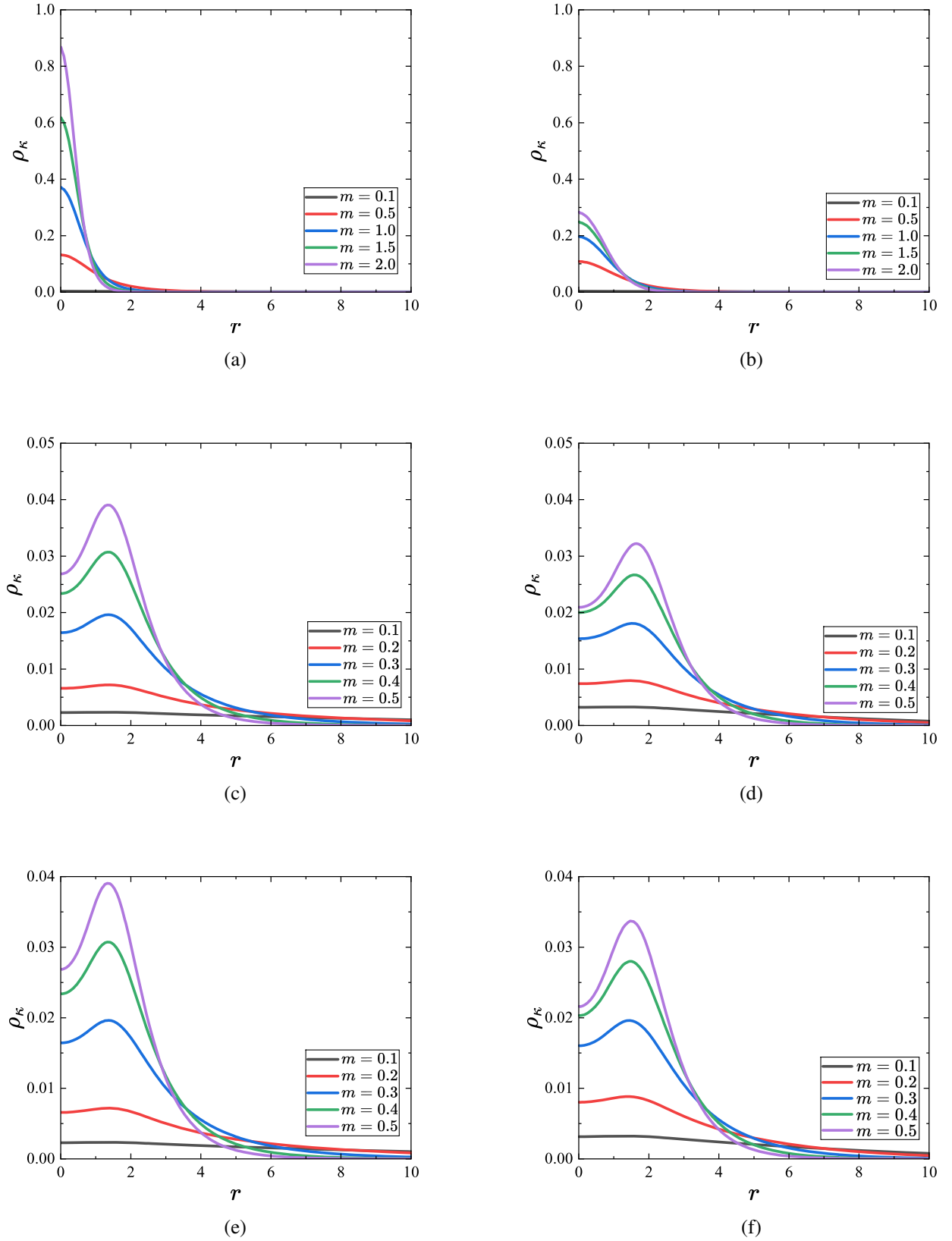


FIG. 4. Typical fermion density ρ_κ for several values of the Yukawa coupling constant m , with $g^{-2} = 0.9$. For each sector, results for the same set of coupling constant m are shown to allow a direct comparison between $Q > 0$ and $Q < 0$. (a) CP^1 case with $Q = 1$ ($n_1 = 0, n_2 = 1$). (b) CP^1 case with $Q = -1$ ($n_1 = 0, n_2 = -1$). (c) CP^1 case with $Q = 3$ ($n_1 = 0, n_2 = 3$). (d) CP^1 case with $Q = -3$ ($n_1 = 0, n_2 = -3$). (e) CP^2 case with $Q = 3$ ($n_1 = 1, n_2 = 3$). (f) CP^2 case with $Q = -3$ ($n_1 = -1, n_2 = -3$).

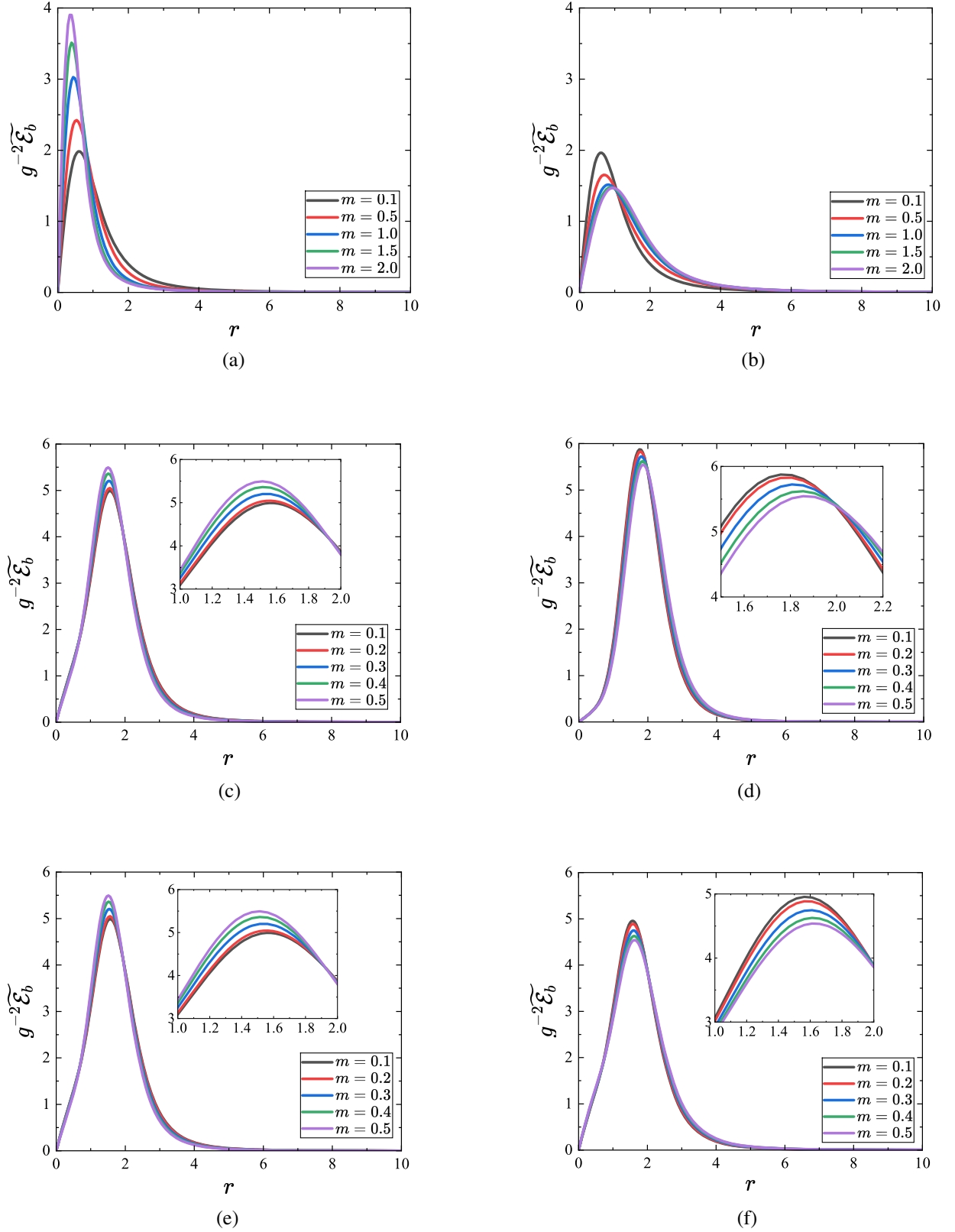


FIG. 5. Typical static skyrmion energy density $g^{-2}\mathcal{E}_b$ weighted by the Jacobian $2\pi r$, $g^{-2}\widetilde{\mathcal{E}}_b \equiv 2\pi r g^{-2}\mathcal{E}_b$ for several values of the Yukawa coupling constant m , with $g^{-2} = 0.9$. The same set of coupling values is used for both $Q > 0$ and $Q < 0$ to directly compare fermion-induced deformations. (a) CP^1 case with $Q = 1$ ($n_1 = 0, n_2 = 1$). (b) CP^1 case with $Q = -1$ ($n_1 = 0, n_2 = -1$). (c) CP^1 case with $Q = 3$ ($n_1 = 0, n_2 = 3$). (d) CP^1 case with $Q = -3$ ($n_1 = 0, n_2 = -3$). (e) CP^2 case with $Q = 3$ ($n_1 = 1, n_2 = 3$). (f) CP^2 case with $Q = -3$ ($n_1 = -1, n_2 = -3$).

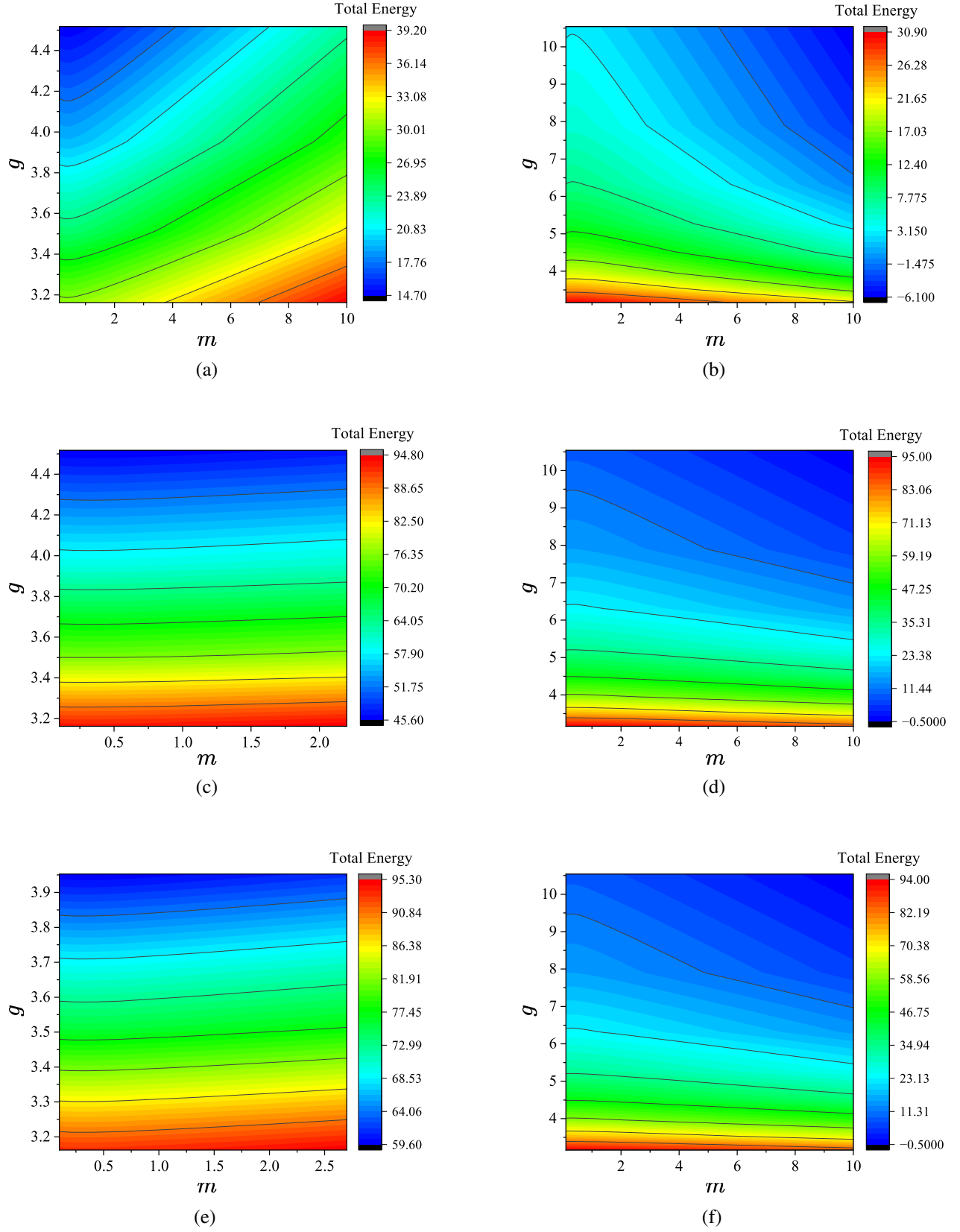


FIG. 6. Total energy (33) in the parameter space (m, g) . The upper four panels Figs. (a)–(d) represent the embedded CP¹ cases, and the lower two panels Figs. (e)–(f) are genuine CP² cases. (a) CP¹ case with $Q = 1$ ($n_1 = 0, n_2 = 1$). (b) CP¹ case with $Q = -1$ ($n_1 = 0, n_2 = -1$). (c) CP¹ case with $Q = 3$ ($n_1 = 0, n_2 = 3$). (d) CP¹ case with $Q = -3$ ($n_1 = 0, n_2 = -3$). (e) CP² case with $Q = 3$ ($n_1 = 1, n_2 = 3$). (f) CP² case with $Q = -3$ ($n_1 = -1, n_2 = -3$).

For $Q > 0$ cases (see left panels in Fig. 5), the peak of $g^{-2}\mathcal{E}_b$ moves toward the origin, and for $Q < 0$ cases (see right panels in Fig. 5) it moves away from the origin. Furthermore, this difference in the deformation pattern can be interpreted as follows. For $Q > 0$ cases, the backreaction acts attractively, causing the energy density $g^{-2}\mathcal{E}_b$ to concentrate at the origin. For $Q < 0$ cases, the backreaction acts repulsively, displacing $g^{-2}\mathcal{E}_b$ from the origin. This is an aspect of the asymmetry.

One can also find the asymmetry in the total energy of the coupled system (33) in the parameter space (m, g) . The parameter dependence of the total energy is presented in Fig. 6. Our results exhibit a clear asymmetry of the system. For $Q > 0$ cases (see left panels in Fig. 6), lower energy configurations are located in regions where m is small and g is large. This indicates that the skyrmions with $Q > 0$ possess lower energy when the backreaction from fermions is weaker. This behavior can simply be explained by the fact that increasing g lowers the static energy via $g^{-2} \int d^2x \mathcal{E}_b$. On the other hand, for $Q < 0$ cases (see right panels in Fig. 6), lower energy configurations are located in regions where both m and g are large. This means that the energy of skyrmions with $Q < 0$ decrease when the backreaction is stronger. Moreover, the skyrmion energy is comparable to the energy-scale ($\sim m$) of fermions.

V. MECHANISM OF THE TOPOLOGICAL CHARGE ASYMMETRY

We now discuss the origin of the asymmetry between configurations with topological charges Q and $-Q$, even though the underlying Lagrangian is symmetric under the sign reversal of the topological charge.

Formally, the classical action is defined as a functional of the fields. The topological charge Q is not an independent variable at this level, but rather a global label that classifies distinct sectors. In the absence of topological terms that explicitly couple to Q , the action itself does not distinguish between configurations with opposite topological charges. Note that one can check that the reduced Lagrangian with our ansatz (12) and (20) is also symmetric under the transformation $n_1 \rightarrow -n_1$ and $n_2 \rightarrow -n_2$, namely $Q \rightarrow -Q$.

This symmetry is also preserved at the level of the background approximation. For a fixed bosonic background Z and the complex-conjugation Z^* , the Dirac Hamiltonian $\mathcal{H}[Z]$ satisfies

$$\mathcal{S}\mathcal{H}[Z]\mathcal{S}^{-1} = -\mathcal{H}[Z^*], \quad (65)$$

where the operator $\mathcal{S}$ is defined by $\mathcal{S} \equiv \sigma_1 \otimes I_3 K$ with the complex-conjugation operator K . This relation implies that the fermionic spectrum in the background Z is mapped to that in the background Z^* with all eigenvalues reversed in sign with the counterpart Dirac equation

$$\mathcal{H}[Z^*](\mathcal{S}\psi) = -\varepsilon(\mathcal{S}\psi). \quad (66)$$

Within our rotationally symmetric ansatz, the conjugation $Z \rightarrow Z^*$ corresponds to $n_1 \rightarrow -n_1$ and $n_2 \rightarrow -n_2$, and consequently $Q \rightarrow -Q$. Then, under the sign reversal of the topological charge $Q \rightarrow -Q$, corresponding to $Z \rightarrow Z^*$, $\psi \rightarrow \mathcal{S}\psi$ and $\varepsilon \rightarrow -\varepsilon$, the reduced Lagrangian

$$\mathcal{L}_f = \psi^\dagger(\varepsilon - \mathcal{H}[Z])\psi \quad (67)$$

is transformed as

$$\begin{aligned} \mathcal{L}'_f &= (\mathcal{S}\psi)^\dagger(-\varepsilon - \mathcal{H}[Z^*])(\mathcal{S}\psi) \\ &= -\mathcal{L}_f. \end{aligned} \quad (68)$$

This change does not affect the classical theory.

The situation changes once the fermionic degrees of freedom are constrained by the Dirac eigenvalue problem and incorporated self-consistently into the bosonic equations. Under the sign reversal of the topological charge Q corresponding to $Z \rightarrow Z^*$ and $\psi \rightarrow \mathcal{S}\psi$, the total energy (33) is transformed as

$$E_{\text{tot}} \rightarrow E'_{\text{tot}} = \int d^2x (-\psi^\dagger \mathcal{H} \psi + g^{-2} \mathcal{E}_b), \quad (69)$$

where the static energy density $\mathcal{E}_b$ is symmetric. As a consequence, the reduced total energy functional is no longer invariant under the sign reversal of the topological charge. Therefore, although the Lagrangian is symmetric under $Q \rightarrow -Q$, the solutions obtained after imposing the fermionic eigenvalue problem become inequivalent for Q and $-Q$. This treatment leads to a loss of the $Q \rightarrow -Q$ symmetry, which provides the mechanism underlying the topological charge asymmetry observed in our numerical solutions.

Note that, in general, the complex-conjugate $Z \rightarrow Z^*$ does not necessarily imply $Q \rightarrow -Q$. Therefore, the above discussion relies on the specific ansatz adopted in this work (12). Nevertheless, the extension of this argument to $\mathbb{CP}^N$ with $N > 2$ is straightforward. As long as the ansatz is chosen such that each component of Z factorizes into a radial profile and an angular phase carrying the winding number, complex conjugation reverses the corresponding windings and thus implements $Q \rightarrow -Q$ within the restricted class of configurations. Under these conditions, the mechanism discussed above generically allows a topological charge asymmetry to emerge in the numerical solutions.

VI. SUMMARY AND OUTLOOK

In this work, we investigate the $\mathbb{CP}^N$ skyrmion-fermion coupled system under the assumption of rotational symmetry, focusing on how the system responds to changes in

the sign of the topological charge Q through the backreaction. We find that the system exhibits a distinct asymmetry with respect to the sign of Q . In particular, an asymmetry is found in the skyrmion's energy density $g^{-2}\mathcal{E}_b$, which is shifted toward the origin for $Q > 0$ and outward for $Q < 0$. These features can be interpreted as reflecting an aspect of the backreaction: it acts attractively for $Q > 0$ and repulsively for $Q < 0$. Moreover, the total energy also exhibits a nontrivial asymmetry: for $Q > 0$, the configuration with weaker fermion-skyrmion interaction is energetically favored, whereas for $Q < 0$, the more strongly coupled configuration is favored. This suggests that, in the backreacting fermion-soliton system, the stability condition itself may depend asymmetrically on the sign of Q .

We clarify the mechanism underlying the topological charge asymmetry observed in our $\mathbb{CP}^N$ skyrmion-fermion system. At the level of the Lagrangian, including both bosonic and fermionic sectors, the theory is symmetric under $Q \rightarrow -Q$. This symmetry persists in the background approximation, where the fermion spectrum is reversed under complex conjugation of the skyrmion field. However, once the fermionic degrees of freedom are determined through the Dirac eigenvalue problem and incorporated self-consistently into the bosonic equations, the fermionic contribution changes sign while the bosonic sector remains invariant. Consequently, the self-consistent equations for Q

and $-Q$ become inequivalent, allowing a topological charge asymmetry to emerge.

As a future direction, it is important to analyze the dependence of the system on the potential parameter μ , and to explore the existence and properties of solutions in the limit without a potential term. In addition, the asymmetry in the total energy with respect to the sign of Q suggests a possible difference in the stability of the solutions. A linear stability analysis around these configurations would provide further insight into the topological charge asymmetry.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: THE COEFFICIENTS IN THE FIELD EQUATIONS

The coefficients $c_i (i = 1, 2, \dots, 16)$ which appear in Eqs. (17) and (18) [and also the coupled equations (25) and (26)] are trigonometric polynomials that are defined as follows:

$$c_1 = 4\{\cos 4F[(n_1 - n_2)\cos 2G + n_1 + n_2]^2 + (n_1 - n_2)^2 \cos 2F \sin^2 2G\} + 8[(n_2^2 - n_1^2)\cos 2G - (n_1^2 + n_2^2)] - 2r^2, \quad (\text{A1})$$

$$c_2 = 24(n_1 - n_2) \sin^3 F \cos F \sin 2G[(n_1 - n_2)\cos 2G + n_1 + n_2], \quad (\text{A2})$$

$$c_3 = 2\{(n_1 - n_2)^2 \sin 2F(\cos 4G - 1) - 4 \sin 4F[(n_1 - n_2)\cos 2G + n_1 + n_2]^2\}, \quad (\text{A3})$$

$$\begin{aligned} c_4 = & 2 \sin F \cos F \cos 2F \{3(n_1 - n_2) \sin^2 F [4(n_1 + n_2) \cos 2G + (n_1 - n_2) \cos 4G] - 5n_1^2 + 18n_1 n_2 - 5n_2^2\} \\ & + 2 \sin F \cos F (n_1 - n_2) \sin^2 F [28(n_1 + n_2) \cos 2G + 9(n_1 - n_2) \cos 4G] \\ & - \frac{3}{2} \sin F \cos F (3n_1^2 + 2n_1 n_2 + 3n_2^2) \cos 4F + \frac{1}{2} \sin F \cos F (29n_1^2 - 66n_1 n_2 + 29n_2^2) + 2r^2 \sin F \cos F, \end{aligned} \quad (\text{A4})$$

$$c_5 = 16(n_1 - n_2)[2(n_1 + n_2) \sin^2 2F \sin 2G + (n_1 - n_2) \sin^2 F (2 \cos 2F + 1) \sin 4G], \quad (\text{A5})$$

$$c_6 = -4\{\cos 4F[(n_1 - n_2)\cos 2G + n_1 + n_2]^2 + (n_1 - n_2)^2 \cos 2F \sin^2 2G\} - 8[(n_2^2 - n_1^2)\cos 2G - (n_1^2 + n_2^2)] - 2r^2, \quad (\text{A6})$$

$$c_7 = -24(n_1 - n_2) \sin^3 F \cos F \sin 2G[(n_1 - n_2)\cos 2G + n_1 + n_2], \quad (\text{A7})$$

$$c_8 = \frac{1}{16} \sin 4F [4(n_1 - n_2)(n_1 + n_2) \cos 2G + 3n_1^2 + 2n_1 n_2 + 3n_2^2] - \frac{1}{8} (n_1 - n_2)^2 (4 \sin^3 F \cos F \cos 4G - \sin 2F) + \frac{\mu^2}{64} r^2 [8 \sin^3 F \cos F \cos 4G + 2 \sin 2F (4 \cos 2G - 5) + \sin 4F (4 \cos 2G - 3)], \quad (\text{A8})$$

$$c_9 = -2 \sin^4 F [8(n_1^2 - n_2^2) \cos^2 F \cos 2G + (n_1 - n_2)^2 (\cos 2F - 7) \cos 4G] - 2 \sin^4 F [(3n_1^2 + 2n_1 n_2 + 3n_2^2) \cos 2F + 11n_1^2 - 14n_1 n_2 + 11n_2^2] - 2r^2 \sin^2 F, \quad (\text{A9})$$

$$c_{10} = 24(n_1 - n_2) \sin^3 F \cos F \sin 2G [(n_1 - n_2) \cos 2G + n_1 + n_2], \quad (\text{A10})$$

$$c_{11} = 4(n_1 - n_2) \sin^4 F [4(n_1 + n_2) \cos^2 F \sin 2G + (n_1 - n_2) (\cos 2F - 7) \sin 4G], \quad (\text{A11})$$

$$c_{12} = 4(n_1 - n_2) \sin^2 F [2(n_1 + n_2) (2 \cos 2F - 1) \sin 2G + (n_1 - n_2) (2 \cos 2F + 1) \sin 4G], \quad (\text{A12})$$

$$c_{13} = -24 \sin^3 F \cos F \cos 2F [(n_1 - n_2) \cos 2G + n_1 + n_2]^2 - 16 \sin^3 F \cos F (n_1 - n_2) (n_1 + n_2) \cos 2G + 60 \sin^3 F \cos F (n_1 - n_2)^2 \cos 4G + 4 \sin^3 F \cos F (-19n_1^2 + 30n_1 n_2 - 19n_2^2) - 2r^2 \sin 2F, \quad (\text{A13})$$

$$c_{14} = 2(n_1 - n_2) \sin^4 F [8(n_1 + n_2) \cos^2 F \cos 2G + (n_1 - n_2) (\cos 2F - 7) \cos 4G] - 4 \sin^2 F [(3n_1^2 + 2n_1 n_2 + 3n_2^2) \cos 4F + 16(n_1 - n_2)^2 \cos 2F - 19n_1^2 + 30n_1 n_2 - 19n_2^2 + 4r^2], \quad (\text{A14})$$

$$c_{15} = -24(n_1 - n_2) \sin^3 F \cos F \sin 2G [(n_1 - n_2) \cos 2G + n_1 + n_2], \quad (\text{A15})$$

$$c_{16} = -\frac{1}{2} (n_1 - n_2) \sin^2 F [(n_2 - n_1) \sin^2 F \sin 4G + 2(n_1 + n_2) \cos^2 F \sin 2G] - \frac{\mu^2}{8} r^2 \sin^2 F [\sin^2 F \sin 4G + (\cos 2F + 3) \sin 2G]. \quad (\text{A16})$$

Note that these coefficients themselves are invariant under $n_1 \rightarrow -n_1$ and $n_2 \rightarrow -n_2$, namely $Q \rightarrow -Q$.

APPENDIX B: THE PLANE WAVE BASIS

In this section, we provide the plane wave basis $\Phi_\kappa^{(p)}$, which we use to solve the secular equation (61) for the $\mathbb{CP}^2$ case [9]. The basis is the eigenvector of the Hamiltonian of the massive free fermions $\mathcal{H}^0 = \hat{\gamma}^0 (-i\hat{\gamma}^j \partial_j + m\hat{\Sigma}_\infty)$. The explicit form can be summarized as follows:

$$\Phi_\kappa^{(u)} = \mathcal{N}_i^{(u)} \begin{pmatrix} \zeta_+^{(u)} J_{\kappa - \frac{1}{2} - \frac{n_1 + n_2}{3}}(k_i^{(u)} r) \exp[i(\kappa - \frac{1}{2} - \frac{n_1 + n_2}{3})\varphi] \\ \zeta_-^{(u)} J_{\kappa + \frac{1}{2} - \frac{n_1 + n_2}{3}}(k_i^{(u)} r) \exp[i(\kappa + \frac{1}{2} - \frac{n_1 + n_2}{3})\varphi] \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (\text{B1})$$

$$\Phi_\kappa^{(d)} = \mathcal{N}_i^{(d)} \begin{pmatrix} \zeta_+^{(d)} J_{\kappa - \frac{1}{2} - \frac{-2n_1 + n_2}{3}}(k_i^{(d)} r) \exp[i(\kappa - \frac{1}{2} - \frac{-2n_1 + n_2}{3})\varphi] \\ \zeta_-^{(d)} J_{\kappa + \frac{1}{2} - \frac{-2n_1 + n_2}{3}}(k_i^{(d)} r) \exp[i(\kappa + \frac{1}{2} - \frac{-2n_1 + n_2}{3})\varphi] \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (\text{B2})$$

$$\Phi_\kappa^{(s)} = \mathcal{N}_i^{(s)} \begin{pmatrix} \zeta_-^{(s)} J_{\kappa - \frac{1}{2} - \frac{n_1 - 2n_2}{3}}(k_i^{(s)} r) \exp[i(\kappa - \frac{1}{2} - \frac{n_1 - 2n_2}{3})\varphi] \\ \zeta_+^{(s)} J_{\kappa + \frac{1}{2} - \frac{n_1 - 2n_2}{3}}(k_i^{(s)} r) \exp[i(\kappa + \frac{1}{2} - \frac{n_1 - 2n_2}{3})\varphi] \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (\text{B3})$$

where $J_{l^{(p)}}(k_i^{(p)} r)$ are standard Bessel functions with the integer order $l^{(p)}$. The coefficients ζ are given by

$$\zeta_{\pm}^{(p)} = \begin{cases} 1 & (\omega_i^{(p)} > 0) \\ \frac{k_i^{(p)}}{|\omega_i^{(p)}| + m} & (\omega_i^{(p)} < 0) \end{cases},$$

$$\zeta_{\pm}^{(p)} = \begin{cases} -\frac{k_i^{(p)}}{\omega_i^{(p)} + m} & (\omega_i^{(p)} > 0) \\ 1 & (\omega_i^{(p)} < 0) \end{cases} \quad (\text{B4})$$

where $\omega_i^{(p)} = \pm \sqrt{(k_i^{(p)})^2 + m^2}$ are the vacuum energy eigenvalues.

We can construct the bases in a cylinder of radius R . This radius is ideally infinite, however for numerical computation, it must take a finite value. Imposing boundary conditions

$$J_{l^{(p)}}(k_i^{(p)}R) = 0 \quad (\text{B5})$$

for $l^{(p)} = \kappa - \frac{1}{2} - \frac{n_1 + n_2}{3}, \kappa - \frac{1}{2} - \frac{-2n_1 + n_2}{3}$, or $\kappa - \frac{1}{2} - \frac{n_1 - 2n_2}{3}$, we obtain some sets of discretized wave numbers $k_i^{(p)}$, $p = (u, d, s)$. The orthogonality conditions are given by

$$\begin{aligned} & \int_0^R dr r J_{l^{(p)}}(k_i^{(p)}r) J_{l^{(p)}}(k_j^{(p)}r) \\ &= \int_0^R dr r J_{l^{(p)} \pm 1}(k_i^{(p)}r) J_{l^{(p)} \pm 1}(k_j^{(p)}r) \\ &= \delta_{ij} \frac{R^2}{2} [J_{l^{(p)} \pm 1}(k_i^{(p)}R)]^2. \end{aligned} \quad (\text{B6})$$

Then, the normalization constants $\mathcal{N}_i^{(p)}$ are determined as

$$\mathcal{N}_i^{(p)} = \left[\frac{2\pi R^2 |\omega_i^{(p)}|}{|\omega_i^{(p)}| + m} (J_{l^{(p)}}(k_i^{(p)}R))^2 \right]^{-\frac{1}{2}}. \quad (\text{B7})$$

In this setup, the plane waves form an orthonormal basis.

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