

Superluminal modes in a quantum field simulator for cosmology from analog trans-Planckian physics

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The quantum-field-theoretic description for the U(1)-Goldstone boson of a scalar Bose-Einstein condensate with time-dependent contact interactions is developed beyond the acoustic approximation in accordance with Bogoliubov theory. The resulting effective action is mapped to a relativistic quantum field theory on a dispersive (or rainbow) cosmological spacetime which has a superluminal Corley-Jacobson dispersion relation. Time-dependent changes of the s-wave scattering length to quantum-simulate cosmological particle production are accompanied by a time-dependent healing length that can be interpreted as an analog Planck length in the comoving frame. Nonadiabatic transitions acquire a dispersive character, which is thoroughly discussed. The framework is applied to exponentially expanding or power-law contracting $(2 + 1)$ -dimensional spacetimes which are known to produce scale-invariant cosmological power spectra. The sensitivity of these scenarios to the time dependence of the Bogoliubov dispersion is investigated: We find a violation of scale invariance via analytically trackable trans-Planckian damping effects if the cutoff scale is not well separated from the horizon-crossing scale. In case of the exponential expansion, these damping effects remarkably settle and converge to another scale-invariant plateau in the far ultraviolet regime where nonadiabatic transitions are suppressed by the high dispersion. The developed framework enables quantitative access to more drastic analog cosmological scenarios with improved predictability in the ultraviolet regime that ultimately may lead to the observation of a scale-invariant cosmological power spectrum in the laboratory.

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I. INTRODUCTION

Effective field theories (EFTs) for (linearized) perturbations in condensed matter systems exhibit structural analogies to quantum field theories on curved spacetimes (QFTCS) that can be utilized to perform analog quantum simulations of hallmark phenomena such as Hawking radiation [1,2], cosmological particle production [3], black-hole superradiance [4,5], or the dynamical Casimir-effect [6,7] (consider Refs. [8–11] for recent reviews on this field of research).

What happens when the aforementioned phenomena exceed the range of validity of the EFT from which they are derived is a nontrivial question. For instance, the formalism of QFTCS that underlies both Hawking radiation and inflationary particle production [12–14] (or alternative bouncing scenarios [15–17]) is expected to break

down on scales close to or below the Planck length, the so-called *trans-Planckian* scales, where quantum-gravitational effects need to be taken into account. However, the outcome of both processes (asymptotic thermal radiation in the black-hole system or a scale-invariant spectrum in case of primordial cosmology) relies on modes that become strongly gravitationally blueshifted into the trans-Planckian regime when traced back to their origin (the event horizon in the black-hole case or the initial causally connected patch in the cosmological case) which led to concerns regarding their viability [18,19]. Nevertheless, recent measurements of the Cosmological Microwave Background [20] still confirm an almost scale-invariant spectrum of cosmological perturbations and therefore the cosmological EFT.

A phenomenological approach to analyze these so-called *trans-Planckian problems* (reviewed among others in Ref. [21]) consists of imposing a modified dispersion relation (MDR) which deviates from linearity above some cutoff momentum to model Lorentz-invariance violations (LIVs) which characteristically arise in quantum gravity models such as Hořava-Lifshitz gravity [22–26] or Einstein-Æther gravity [27,28] (the tight constraints on such LIV are discussed in Ref. [29]). Inspired by the subluminal dispersion relation of his condensed-matter analog [30], Unruh carried

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out a pioneering numerical study [31] that was investigated analytically and generalized in Ref. [32]. These developments led Corley and Jacobson to analyze the issue via the lowest-order (quartic) expansion of a subluminal dispersion relation [33] and later on the corresponding superluminal form [34] (consider Ref. [35] for a review on these early developments).

For these models, the trans-Planckian problem of cosmology was analyzed by Martin and Brandenberger [36–40] and Niemeyer and Parentani [41,42], who found a preservation of scale-invariance if the Hubble horizon is well separated from the cutoff scale [42–44] (or equivalently adiabatic evolution in the ultraviolet [21,36,42,45]). Similar concepts also showed a robustness of the Hawking effect for radiation from black holes if the surface gravity is well separated from the cutoff scale [46–48]. However, there are also physically plausible constructions that violate these criteria [40,46,49] such that the MDR-approach does not seem to entirely resolve the original issues which arguably only a full theory of quantum gravity with phenomenological prowess could achieve.

In recent years, the condensed matter systems that inspired the MDR approach [43] found success as simulators for dynamics in curved spacetime [50–64], such that the lessons and techniques obtained from dispersive field theory are still applied to this day [65–78] where considerable interest also stems from their potential as tabletop testing grounds for quantum gravity phenomenology [23–25,79–82]. As a result of the MDR in the analog quantum simulators, the emergent spacetime itself is modified, which leads to a so-called rainbow spacetime [83–85] whose geometry and symmetries were discussed in further detail [86]. Recently, a study on analog trans-Planckian cosmology using MDRs occurring in dipolar Bose-Einstein condensates has been conducted [78] with an emphasis on the minimal length principle [87–92].

Motivated by these considerations, we generalize the theoretical description of the quantum field simulator for cosmology introduced in Refs. [61,93] from the acoustic regime [94] to the dispersive regime by accounting for the full Bogoliubov dispersion relation (which is analog to a first-order expansion of a generic superluminal MDR). Our theoretical findings were previously compared to experimental results by the Oberthaler group [62].

The remainder of this paper is organized as follows. In Sec. II, we derive an effective field theory for the U(1) Goldstone boson and its conjugate momentum in full Bogoliubov theory under the assumption of a time-dependent, homogeneous background. In Sec. III, the effective field theory is interpreted as a quantum field theory on a dispersive, cosmological spacetime, and properties of the quantum mode evolution are discussed in terms of a dispersive, parametric oscillator. We furthermore contextualize the time-dependent Bogoliubov dispersion relation with common modifications of dispersion relations

in the literature including a quantitative perspective on (non)adiabaticity. In Sec. IV, we theoretically analyze the implications of the Bogoliubov dispersion relation on the emergence of scale invariance via an analog exponential expansion or power-law contraction. In Sec. V, we introduce laboratory boundary conditions and examine out-of-equilibrium field correlations in context of *in situ* density measurements. The impact of these boundary conditions on scale-invariant power spectra is discussed, and previous experimental scenarios [62] are analyzed via the concepts developed in this work. We formulate some conclusions in Sec. VI.

We use the following abbreviation for real-space and momentum-space integrals:

$$\int_{t,x} \equiv \int dt d^2\mathbf{x}, \quad \int_k \equiv \int \frac{d^2\mathbf{k}}{(2\pi)^2}, \quad \int_k \equiv \int_0^\infty \frac{dk k}{2\pi}. \quad (1)$$

II. BEYOND THE ACOUSTIC APPROXIMATION

In this section, we extend the field-theory methodology of Ref. [93] beyond the acoustic approximation. For a comparison to full Bogoliubov theory in the operator formalism, which is oftentimes used in the literature, consider Appendix A.

A. Condensate background

An effectively two-dimensional Bose-Einstein condensate with repulsive contact interactions can be described in the dilute regime by the effective action [95]

$$\Gamma[\Phi] = \int dt d^2\mathbf{x} \left\{ \hbar \Phi^*(t, \mathbf{x}) \left[i \frac{\partial}{\partial t} - V_{\text{ext}}(t, \mathbf{x}) \right] \Phi(t, \mathbf{x}) - \frac{\hbar^2}{2m} \nabla \Phi^*(t, \mathbf{x}) \nabla \Phi(t, \mathbf{x}) - \frac{\lambda(t)}{2} [\Phi^*(t, \mathbf{x}) \Phi(t, \mathbf{x})]^2 \right\}. \quad (2)$$

It is convenient to split the field Φ into a background part ϕ_0 and a fluctuating part parametrized by two real fields ϕ_1 and ϕ_2 , such that

$$\Phi(t, \mathbf{r}) = \phi_0(t, \mathbf{r}) + \frac{1}{\sqrt{2}} [\phi_1(t, \mathbf{r}) + i\phi_2(t, \mathbf{r})]. \quad (3)$$

The background field is specified as a solution to the Gross-Pitaevskii equation [96],

$$i\hbar \frac{\partial}{\partial t} \phi_0 = \left(-\frac{\hbar^2}{2m} \nabla^2 + V + \lambda |\phi_0|^2 \right) \phi_0, \quad (4)$$

and corresponds to a condensate mean field, which is discussed in further detail in Ref. [93]. In terms of the Madelung representation [97],

$$\phi_0(t, \mathbf{r}) = \sqrt{n_0(t, \mathbf{r})} \exp\{i\theta_0(t, \mathbf{r})\}, \quad (5)$$

the Gross-Pitaevskii equation (4) is equivalent to a pair of hydrodynamic equations involving the background particle number density $n_0(t, \mathbf{r}) = |\phi_0(t, \mathbf{r})|^2$ and the background phase $\theta_0(t, \mathbf{r})$. More concretely, one finds the continuity equation

$$0 = \partial_t n_0 + \nabla(n_0 \mathbf{v}) \quad (6)$$

and the Euler equation

$$0 = \hbar \partial_t \theta_0 + V + \lambda n_0 + \frac{m}{2} \mathbf{v}^2 + q, \quad (7)$$

where we introduced the superfluid velocity via

$$\mathbf{v} = \frac{\hbar}{m} \nabla \theta_0 \quad (8)$$

and consider the quantum pressure term

$$q = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n_0}}{\sqrt{n_0}} \quad (9)$$

in (7), which allow us to be beyond the acoustic approximation. Note that q in Eq. (9) is of second order in $\hbar$, as well as in spatial derivatives, such that it is expected to be subleading for sufficiently smooth density.

B. Effective theory for fluctuations

Expanding the effective action $\Gamma[\Phi]$ up to second order in the fluctuating fields, $\Gamma[\Phi] = \Gamma_0[\phi_0] + \Gamma_2[\phi_1, \phi_2]$ (see more details in Ref. [93]), one finds

$$\begin{aligned} \Gamma_2 = \int_{t,x} \left\{ \frac{1}{2} \phi_2 \left(\frac{\hbar^2}{2m} \nabla^2 + q \right) \phi_2 \right. \\ \left. + \frac{1}{2} \phi_1 \left(-2\lambda n_0 + \frac{\hbar^2}{2m} \nabla^2 + q \right) \phi_1 \right. \\ \left. - \hbar \phi_1 \left[\partial_t \phi_2 + (\mathbf{v} \cdot \nabla) \phi_2 + \frac{1}{2} (\nabla \cdot \mathbf{v}) \phi_2 \right] \right\}, \quad (10) \end{aligned}$$

upon utilizing the Euler equation (7).

Since we want to calculate the dispersive extension to the analog cosmological metric calculated in Ref. [93], we restrict to a stationary background flow, $\mathbf{v} = 0$. Furthermore, we assume a homogeneous background density, $n_0(r) = \bar{n}_0$. This substantially simplifies the inclusion of dispersive effects since the quantum pressure still vanishes at the background level, $q = 0$. Invoking these assumptions, the fluctuating fields ϕ_1 and ϕ_2 are described by the quadratic action

$$\begin{aligned} \Gamma_2 = \int_{t,x} \left\{ \phi_2 \hbar \partial_t \phi_1 + \frac{\hbar^2}{4m} \phi_2 \nabla^2 \phi_2 \right. \\ \left. + \frac{1}{2} \phi_1 \left(\frac{\hbar^2}{2m} \nabla^2 - 2\lambda n_0 \right) \phi_1 \right\} \quad (11) \end{aligned}$$

and evolve according to the linearized equations of motion

$$2\lambda n_0 \phi_1 - \frac{\hbar^2}{2m} \nabla^2 \phi_1 + \hbar \partial_t \phi_2 = 0, \quad (12)$$

and

$$\hbar \partial_t \phi_1 + \frac{\hbar^2}{2m} \nabla^2 \phi_2 = 0. \quad (13)$$

C. Quantization and mode dynamics

The conjugate momentum fields of ϕ_1 and ϕ_2 are obtained as

$$\pi_1(t, \mathbf{x}) \equiv \frac{\delta \Gamma_2}{\delta \dot{\phi}_1(t, \mathbf{x})} = -\hbar \phi_2(t, \mathbf{x}), \quad (14)$$

$$\pi_2(t, \mathbf{x}) \equiv \frac{\delta \Gamma_2}{\delta \dot{\phi}_2(t, \mathbf{x})} = \hbar \phi_1(t, \mathbf{x}). \quad (15)$$

Hence, ϕ_1 and ϕ_2 are conjugate field variables. With the splitting in Eq. (3) and for $\phi_0 \in \mathbb{R}$, they can be interpreted as density and phase fluctuations (see more details in Appendix A). Therefore, we can quantize both fields in terms of a single set of annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$. To that end, we employ the mode expansions

$$\begin{aligned} \phi_1(t, \mathbf{x}) &= \int_k [\hat{a}_k w_k(t) e^{ik \cdot \mathbf{x}} + \hat{a}_k^\dagger w_k^*(t) e^{-ik \cdot \mathbf{x}}], \\ \phi_2(t, \mathbf{x}) &= \int_k [\hat{a}_k v_k(t) e^{ik \cdot \mathbf{x}} + \hat{a}_k^\dagger v_k^*(t) e^{-ik \cdot \mathbf{x}}] \quad (16) \end{aligned}$$

and demand the operators to obey the bosonic commutation relations

$$\begin{aligned} [\hat{a}_k, \hat{a}_k^\dagger] &= \delta(\mathbf{k} - \mathbf{k}'), \\ [\hat{a}_k, \hat{a}_{k'}] &= [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0, \quad (17) \end{aligned}$$

which in turn enable the equal-time canonical commutation relations

$$[\phi_2(t, \mathbf{x}), \pi_2(t, \mathbf{y})] = i\hbar \delta(\mathbf{x} - \mathbf{y}), \quad (18)$$

$$[\phi_2(t, \mathbf{x}), \phi_2(t, \mathbf{y})] = [\pi_2(t, \mathbf{x}), \pi_2(t, \mathbf{y})] = 0, \quad (19)$$

if the normalization constraint,

$$w_k(t)v_k^*(t) - v_k(t)w_k^*(t) = i, \quad (20)$$

is fulfilled. This constraint is preserved by the evolution equations

$$\hbar \partial_t w_k(t) = E_k v_k(t), \quad (21)$$

$$-\hbar \partial_t v_k(t) = \frac{\epsilon_k^2(t)}{E_k} w_k(t), \quad (22)$$

where we introduced

$$\epsilon_k(t) \equiv \hbar \omega_k(t), \quad E_k \equiv \frac{\hbar^2 k^2}{2m}, \quad (23)$$

with the Bogoliubov dispersion relation

$$\omega_k^2(t) \equiv c_s^2(t)k^2 + \left(\frac{\hbar k^2}{2m}\right)^2, \quad (24)$$

and the squared speed of sound

$$c_s^2(t) \equiv \frac{\lambda(t)n_0}{m}. \quad (25)$$

For our further discussion, the phase velocity of Bogoliubov modes

$$c_{\text{ph}}(t, k) = \frac{\omega_k(t)}{k} = c_s(t) \sqrt{1 + \frac{1}{2} k^2 \xi^2(t)} \quad (26)$$

and the healing length

$$\xi(t) = \frac{\hbar}{\sqrt{2} m c_s(t)} \quad (27)$$

will play a central role. The latter is a characteristic length scale below which excitations lose their collective character and rather represent individual atoms. Throughout all numerical evaluations of dispersive effects in this work, we assume the atomic mass in Eq. (27) to be given by ^{39}K (corresponding to the Bose-Einstein condensate (BEC) used in Ref. [62]).

III. DISPERSIVE EFFECTS IN EMERGENT SPACETIMES

The concepts developed so far provide a bona fide framework to study the evolution and correlations of the nonrelativistic quantum fields in the BEC. There exists, however, an alternative geometric formulation that bears a strong resemblance to relativistic quantum field theory which shall be laid out in the following.

A. Dispersive metric

In the dynamical system (21) and (22), one can eliminate the mode $w_k(t)$ in favor of the time-derivative $\dot{v}_k(t)$, resulting in

$$\ddot{v}_k(t) - 2 \frac{\dot{c}_{\text{ph}}(t, k)}{c_{\text{ph}}(t, k)} \dot{v}_k(t) + c_{\text{ph}}^2(t, k) k^2 v_k(t) = 0. \quad (28)$$

This mode elimination can be realized on the level of the effective action (11) by integrating out ϕ_1 in Fourier space. One finds

$$\Gamma_2 = \frac{\hbar^2}{4m} \int dt \int_k \left\{ \frac{1}{c_{\text{ph}}^2(t, k)} \partial_t \phi_2(t, -\mathbf{k}) \partial_t \phi_2(t, \mathbf{k}) - k^2 \phi_2(t, -\mathbf{k}) \phi_2(t, \mathbf{k}) \right\} \quad (29)$$

as the effective action of phase fluctuations in the BEC, whose velocity and momentum only depend on the absolute value $k = |\mathbf{k}|$ due to the spatially homogeneous background.

Introducing the rescaled field $\phi = \phi_2/\sqrt{2m}$ and the operator

$$\tilde{\partial}_\mu = \begin{pmatrix} \partial_t \\ -i\mathbf{k} \end{pmatrix}, \quad (30)$$

this effective action assumes a form resembling a relativistic theory,

$$\Gamma_2 = -\frac{\hbar^2}{2} \int dt \int_k \sqrt{g(t, k)} g^{\mu\nu}(t, k) \tilde{\partial}_\mu \phi(t, -\mathbf{k}) \tilde{\partial}_\nu \phi(t, \mathbf{k}), \quad (31)$$

with the dispersive metric

$$g_{\mu\nu}(t, k) = \begin{pmatrix} -1 & 0 \\ 0 & c_{\text{ph}}^{-2}(t, k) \mathbb{1}_2 \end{pmatrix}, \quad \sqrt{g(t, k)} = c_{\text{ph}}^2(t, k). \quad (32)$$

Note that, due to the wave number dependence of the metric, the effective action (31) describes a nonlocal theory as one can show explicitly by transforming back to real space. The wave-number-dependent, geometric structure can be interpreted by tracing wavefronts of constant phase that propagate according to the Bogoliubov dispersion relation $\omega^2(t) = c_{\text{ph}}^2(k, t)k^2$, which translates into a null condition,

$$g_{\mu\nu}(t, k) k^\mu k^\nu = 0, \quad (33)$$

for the 4-momentum $k^\mu \equiv (\omega, \mathbf{k})^T$. Note that Eq. (33) is in general *not* invariant under (analog) Lorentz transformations, $k^\mu \rightarrow \Lambda_\nu^\mu k^\nu$, such that the laboratory system defines a preferred frame of reference. On scales where $k\xi/\sqrt{2} \ll 1$, however, we have $c_{\text{ph}}(t, k) = c_s(t) + \mathcal{O}(k^2 \xi^2)$ such that Eq. (33) becomes Lorentz invariant on these scales. In that sense, the healing length ξ functions as an analog Planck

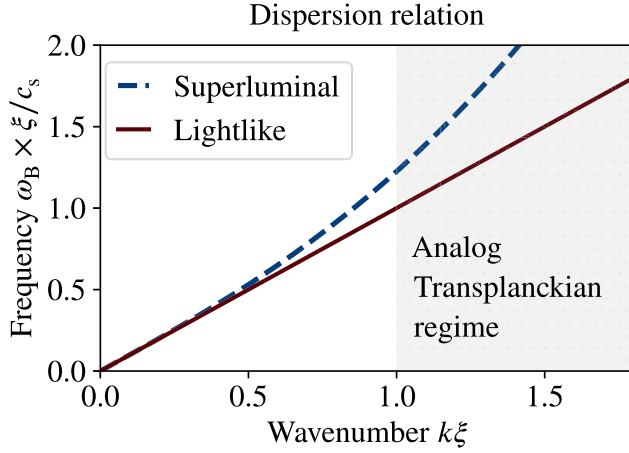


FIG. 1. Modified dispersion relation in the quantum field simulator (dashed line) compared to the nondispersive lightlike case (solid line).

length [98] below which Lorentz-invariance is broken and replaced by a nonlocal theory with a superluminal (or supersonic) dispersion relation (see Fig. 1 for a visualization). The described analog spacetime structure is also known as rainbow metric [66] and has been shown to admit a formulation in terms of a Finsler geometry [86] (see Ref. [85] for a study on geodesics in such geometries).

B. Quantization on dispersive, cosmological spacetime

The metric (32) can be brought into a form that is suitable to describe a cosmological spacetime (i.e., a dynamic, homogenous and isotropic space) but with a wave-number-dependent scale factor

$$a_k(t) = c_{\text{ph}}^{-1}(t, k), \quad (34)$$

with which we will work in the following, bearing in mind that all statements can be equivalently made from (or translated to) the perspective of Bogoliubov excitations in a BEC with time-dependent contact interactions.

The form of Eq. (28) is equivalent to a mode equation of a massless Klein-Gordon field, which is minimally coupled to such a spacetime evolving dynamically with scale factor a_k where each mode experiences a different expansion (or contraction) rate. In accordance with Eq. (34), we can write the dispersive scale factor as

$$a_k(t) = \frac{a(t)}{\sqrt{1 + \frac{1}{2}k^2\xi^2(t)}}, \quad (35)$$

where

$$a(t) = \lim_{k \rightarrow 0} a_k(t) = c_s^{-1}(t) \quad (36)$$

is the scale factor of the acoustic metric derived in Ref. [93].

For completeness, let us also mention that the conjugate momentum of the relativistic field $\phi = \phi_2/\sqrt{2m}$ is related to ϕ_1 via

$$\pi(t, \mathbf{k}) = \frac{\delta\Gamma_2}{\delta\dot{\phi}(t, \mathbf{k})} = \hbar^2 a_k^2 \dot{\phi}(t, -\mathbf{k}) = -\sqrt{2m}\hbar\phi_1(t, -\mathbf{k}), \quad (37)$$

as one would expect. The quantization of ϕ_1 and ϕ_2 is thus equivalent to quantizing ϕ and π according to the momentum-space equal-time commutation relation

$$[\phi(t, \mathbf{k}), \pi(t, \mathbf{k}')] = i\hbar\delta(\mathbf{k} + \mathbf{k}'), \quad (38)$$

with the mode v_k being subjected to the canonical normalization constraint

$$\hbar a_k^2 (v_k \dot{v}_k^* - \dot{v}_k v_k^*) = i, \quad (39)$$

which is equivalent to the normalization condition (20) upon mode elimination and a generalization of the usual Wronskian determinant of cosmological scalar fields [93] to the case of a dispersive spacetime described by the scale factor $a_k(t)$.

C. Dispersive parametric oscillator

The goal of this section is to transform the mode equation (28) into a parametric oscillator for convenience. To that end, we rescale

$$v_k(t) = [a(t)\mathcal{J}_k(t)]^{-1/2}\psi_k(t), \quad (40)$$

where

$$\mathcal{J}_k(t) \equiv \frac{c_s^2(t)}{c_{\text{ph}}^2(k, t)} = \left(1 + \frac{1}{2}k^2\xi^2(t)\right)^{-1} \quad (41)$$

arises as a consequence of the modified canonical volume form in Eq. (31) and captures the reduction in number of phase-space degrees of freedom [92].

Furthermore, we introduce conformal time $d\eta = dt/a(t)$ with respect to the bare scale factor $a(t)$ such that η is still equivalent to the sound horizon in the BEC [94]. The mode equation (28) then takes the form

$$\psi_k''(\eta) + \Omega_B^2(k, \eta)\psi_k(\eta) = 0, \quad (42)$$

with the effective Bogoliubov frequency

$$\Omega_B^2(k, \eta) = k^2/\mathcal{J}_k(\eta) + m_{\text{eff}}^2(k, \eta) \quad (43)$$

and the effective mass

$$m_{\text{eff}}^2(k, \eta) = -\frac{1}{2} \frac{[a(\eta)\mathcal{J}_k(\eta)]''}{a(\eta)\mathcal{J}_k(\eta)} + \frac{1}{4} \left(\frac{[a(\eta)\mathcal{J}_k(\eta)]'}{a(\eta)\mathcal{J}_k(\eta)} \right)^2 \quad (44)$$

(see Appendix B for a detailed derivation).

To clarify the notion of an effective mass, consider that Eq. (43) has the form of an energy-momentum relation, $E^2 = k^2 + m^2$, but with the effective mass being generated by the time dependence of $\mathcal{J}_k a$ and one has $k^2 \rightarrow k^2/\mathcal{J}_k$ because of the dispersion.

The splitting of terms in the effective frequency (43) is also motivated from the effective mass being responsible for nonadiabatic transitions [99], whereas the modes can adiabatically adjust to $k^2/\mathcal{J}_k(\eta)$. The magnitude of the former depends on the rate of change in spacetime (or the laboratory parameters in the simulator) and becomes adiabatically suppressed in the high-momentum regime $k^2/\mathcal{J}_k(\eta) \gg m_{\text{eff}}^2$ as we will show with explicit examples later on.

Moreover, the magnitude of nonadiabatic transition is wave-number dependent in the first place since the cutoff scale (or comoving analog Planck length) $\xi(\eta)$ is time dependent. More explicitly, the effective mass has the form

$$m_{\text{eff}}^2(k, \eta) = -\frac{1}{4} \frac{3k^4 \xi^4(\eta)/4 - 5k^2 \xi^2(\eta) - 1}{[1 + k^2 \xi^2(\eta)/2]^2} \left(\frac{a'(\eta)}{a(\eta)} \right)^2 - \frac{1}{2} \frac{1 - k^2 \xi^2(\eta)/2}{1 + k^2 \xi^2(\eta)/2} \left(\frac{a''(\eta)}{a(\eta)} \right), \quad (45)$$

where we have used the time dependence of $\mathcal{J}_k(\eta)$. Put differently, if ξ would be time independent, the effective mass (B5) would be wave-number-independent such that the dispersive effects only influence the mode by its instantaneous eigenenergy in Eq. (42) as is also the case for the *ad hoc* modifications studied in Refs. [36–42]. The physical reason underlying the time dependence of $\xi(\eta)$ [and thus $\mathcal{J}_k(\eta)$] is the time-dependent atomic interactions in the BEC. This places the BEC automatically in the frame comoving with the analog cosmos, which is also the preferred frame in which Eq. (33) holds and where the Lorentz-invariance-violating physics need to be evaluated.

As a consistency check, consider the infrared and ultraviolet limit: in the infrared limit where $\mathcal{J}_k$ becomes time-independent, one finds the standard result [94],

$$\lim_{k \rightarrow 0} m_{\text{eff}}^2(k, \eta) = -\frac{1}{2} \frac{a''(\eta)}{a(\eta)} + \frac{1}{4} \left(\frac{a'(\eta)}{a(\eta)} \right)^2. \quad (46)$$

In the ultraviolet limit, the mode equation is a harmonic oscillator with a quadratic, nonrelativistic dispersion where the effective mass is adiabatically suppressed, as one would expect (see details in Appendix B).

D. Time-dependent dispersion and nonadiabaticity

In this section, we analyze the significance of the dispersive effects and contextualize with the literature [36–42]. Therein, it is common to write the effective frequency as

$$\Omega_F^2(k, \eta) = a^2(\eta) F^2(k/a(\eta)) + m_{\text{eff}}^2(k=0, \eta), \quad (47)$$

where the form function F specifies the modification of the dispersion relation as a function of the physical wave number k/a (i.e., the one that is subjected to redshift by an expanding space) and is required to become linear in the nondispersive regime. In fact, F can be considered as an effective physical wave number, which is specified as a function of the bare physical wave number k/a [100]. For example, F can be a quartic polynomial that is either curved subluminally or superluminally, as is the case for the Corley-Jacobson dispersion relation [33,34], or a hyperbolic tangent in the case of Unruh [31]. In the context of the mode equation in this work (42), we would therefore introduce

$$F_B^2(k, a(\eta)) = \left(\frac{k}{a(\eta)} \right)^2 + \frac{\xi_0^2}{2a_0^2} k^4, \quad (48)$$

such that

$$\Omega_B^2(k, \eta) = a^2(\eta) F_B^2(k, a) + m_{\text{eff}}^2(k, \eta). \quad (49)$$

At this point, we want to highlight two key differences of the effective frequency (43) relative to form (47) that stem from the time dependence of $\xi(\eta)$:

- (i) The effective mass in Eq. (43) (and thus the probability for nonadiabatic transitions [99]) is wave number dependent as we discussed in Sec. III.
- (ii) The instantaneous mode eigenenergy $k^2 + \frac{1}{2} \xi^2(\eta) k^4$ is time dependent since $\xi(\eta) = \xi_0 a(\eta)/a_0$. As an immediate consequence, the function F_B defined in Eq. (48) cannot be written as a function of the (bare) physical wave number k/a alone.

As a comparison, consider the case where the healing length would be constant in time, $\xi = \xi_0$, leading to

$$F_{B,0}^2(k/a(\eta)) = \left(\frac{k}{a(\eta)} \right)^2 + \frac{a_0^2 \xi_0^2}{2} \left(\frac{k}{a(\eta)} \right)^4, \quad (50)$$

which agrees with the superluminal Corley-Jacobson dispersion relation [37,39,42] and can be written as a polynomial function solely depending on the physical momentum k/a .

For a quantitative analysis of these effects, let us consider the nonadiabaticity parameter [21]

$$\alpha_F(k, \eta) = \left| \frac{3}{4} \left(\frac{\Omega'_F(k, \eta)}{\Omega_F(k, \eta)} \right)^2 - \frac{1}{2} \frac{\Omega''_F(k, \eta)}{\Omega_F^3(k, \eta)} \right|^{1/2}, \quad (51)$$

which, when $\alpha_F \ll 1$, indicates that the mode function adiabatically adjusts to the effective frequency and remains in a Wentzel-Kramers-Brillouin (WKB)-form (which is also known as the adiabatic vacuum; see Appendix C for details). In contrast, when $\alpha_F \gg 1$, nonadiabatic transitions are highly probable, and particle production is strong (consider Ref. [99] for a link of these concepts to the adiabatic theorem of quantum mechanics).

As an exemplary case study, let us consider a linearly expanding scenario (to be discussed in further detail in Sec. V) and compute the nonadiabaticity parameter (51) for the time-dependent Bogoliubov dispersion (43). As a proper reference, let us perform the same task for a time-independent Bogoliubov dispersion,

$$\Omega_{B,0}^2(k, \eta) = a^2(\eta) F_{B,0}^2(k/a) + m_{\text{eff}}^2(k=0, \eta), \quad (52)$$

where we impose the standard Friedmann-Lemaître-Robertson-Walker (FLRW) effective mass in $(2+1)$ dimensions, which is given by $m_{\text{eff}}^2(k=0, \eta)$ [cf. Eq. (45)] and use a dispersion relation of the form (50). To have a non-dispersive reference, we also consider

$$\Omega_{\text{Ac}}^2(k, \eta) = k^2 + m_{\text{eff}}^2(k=0, \eta), \quad (53)$$

which is the familiar nondispersive FLRW case.

In Fig. 2, we show the nonadiabaticity parameter for the time-dependent Bogoliubov case (upper image) and take its ratio to the time-independent Bogoliubov case (center image) and acoustic case (lower image). In the Bogoliubov case, we observe throatlike trajectories of high (low) nonadiabaticity in blue (red). Relative to the time-independent case, there are lines of high nonadiabaticity in the ultraviolet which move toward lower wave numbers due to redshift, which can be explained via the dynamics of $\xi(\eta)$. At the critical (final) wave number $k\xi_f = 1$, the Bogoliubov dispersion relation becomes adiabatic relative to the nondispersive case as one can conclude from the central throat that appears at $\eta/\eta_f \gtrsim 0.5$.

IV. SCALE INVARIANCE

In this section, we analyze the perspectives of approaching the trans-Planckian problem of cosmology in analog BEC-Cosmology simulators with a time-dependent scattering length. The following discussion is purely theoretical and can be understood as a first step, using the straightforward scenarios of an exponential expansion and a power-law contraction, to gain intuition and experience for future scenarios where one has to stay within current experimental capabilities which requires innovative solutions. In this section, we set $\hbar = 1$ in accordance with the literature.

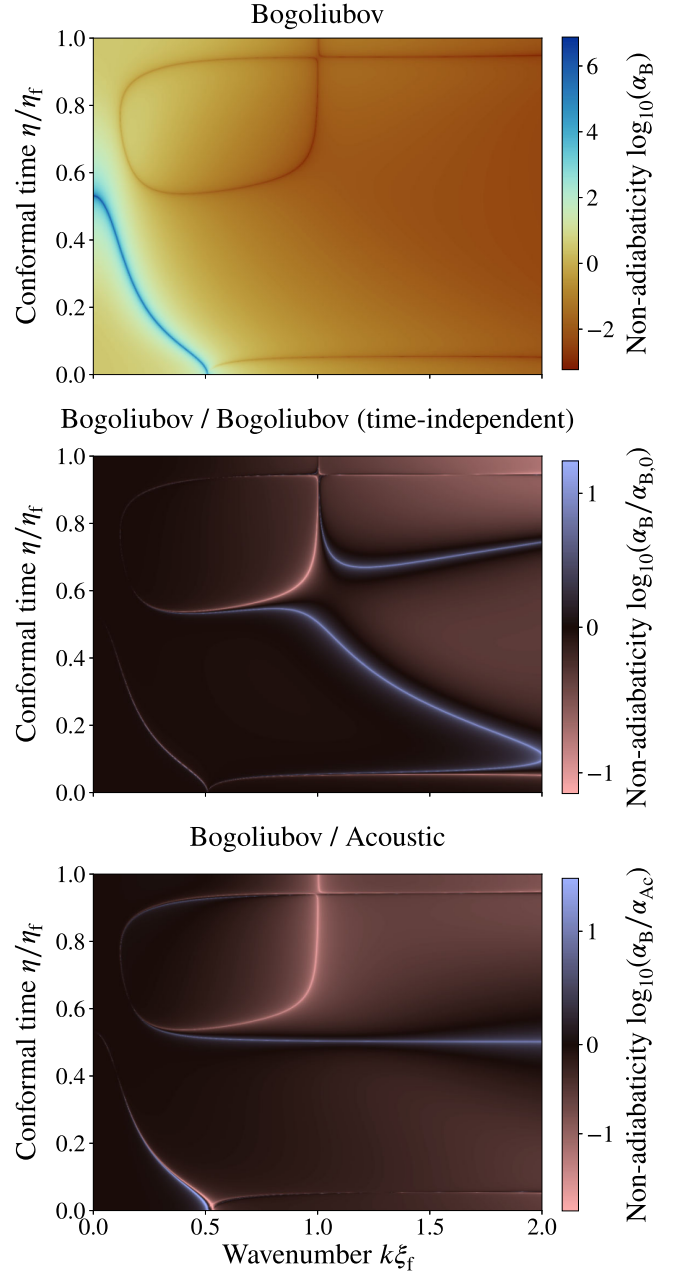


FIG. 2. Nonadiabaticity for time-dependent Bogoliubov dispersion (absolute values, upper image) compared to the time-independent case (relative values, central image) and the nondispersive (acoustic) case (relative values, lower image) as a function of conformal time η/η_f and wave numbers $k\xi_f$ in case of the linearly expanding scenario discussed in Sec. V. Blue regions in the upper image show nonadiabaticity to be highly localized in time along a single throatlike trajectory. As expected, one has an increasing adiabaticity toward the ultraviolet regime at all times. Dark regions in the central and lower image indicate similarity regarding (non)adiabaticity, whereas blue regions demonstrate a trend toward the Bogoliubov dispersion. To numerically evaluate α including the singular contributions to the effective mass (to be discussed in Sec. V), we smoothed the delta distributions via the Lorentzian profile $\delta_\epsilon(x) = (\epsilon/\pi)/(x^2 + \epsilon^2)$ where $\epsilon = 0.1$ models a relative switching sensitivity of 10%.

A. Exponential expansion

We first describe how a scale-invariant spectrum of quantum fluctuations emerges in a nondispersive theory from an exponentially growing scale factor, $a(t) = a_i e^{Ht}$ which has to be identified with the reciprocal sound speed in the analog simulator. The passage of time is described by a growing (sound-)particle horizon (the sound cone) η , which is defined as

$$\eta(t) = \int_0^t \frac{dt}{a(t)} = \eta_0(1 - e^{-Ht}), \quad \eta_0 = \frac{1}{a_i H}, \quad (54)$$

and asymptotically grows from $\eta_i = 0$ toward

$$\eta_f = \eta_0(1 - e^{-N}), \quad (55)$$

where we introduced the number of e -foldings $N = \ln(a_f/a_i)$. Quantizing a nondispersive, massless scalar field that is minimally coupled to a flat $(2+1)$ -dimensional FLRW spacetime leads to the mode equation

$$\psi_k''(\eta) + \left(k^2 - \frac{3}{4(\eta - \eta_0)^2}\right)\psi_k(\eta) = 0, \quad (56)$$

where the scale factor reads

$$a(\eta) = a_i(1 - \eta/\eta_0)^{-1} \quad (57)$$

as a function of conformal time. The boundary condition is chosen as an instantaneous vacuum state at the initial time $\eta_i = 0$ [101],

$$\psi_k(\eta_i) = 1/\sqrt{2k}, \quad \psi_k'(\eta_i) = -ik\psi_k(\eta_i), \quad (58)$$

which leads to the canonically normalized mode solution

$$\psi_k(\eta) = \frac{i\pi}{4} \frac{1}{\sqrt{2k}} \sqrt{\frac{\eta - \eta_0}{\eta_0}} \times (rJ_1[k(\eta_0 - \eta)] + sY_1[k(\eta_0 - \eta)]), \quad (59)$$

where J_1 , Y_1 are Bessel functions and the choice of the coefficients

$$r = 2k\eta_0 Y_0(k\eta_0) - (1 + 2ik\eta_0)Y_1(k\eta_0), \quad (60)$$

$$s = -2k\eta_0 J_0(k\eta_0) + (1 + 2ik\eta_0)J_1(k\eta_0) \quad (61)$$

ensures an instantaneous vacuum at $\eta = 0$. In the limit of many e -folds, $N \rightarrow \infty$, this corresponds to the Bunch-Davies state.

Of central interest is the dimensionless power spectrum of field excitations which in two spatial dimensions is introduced as [102]

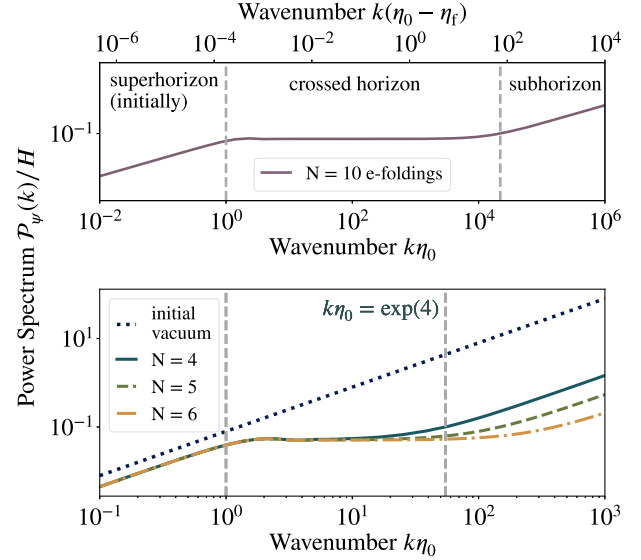


FIG. 3. Upper image: power spectrum $\mathcal{P}_\psi/H$ after an exponential expansion by $N = 10$ e -foldings compared to the initial vacuum spectrum. A constant (scale-invariant) value $\mathcal{P}_\psi/H = 1/2\pi^2$ occurs for modes $1 \ll k\eta_0 \ll e^N$, where the boundaries become more accurate as N increases. In the limit $\eta_0 \rightarrow \infty$, the left boundary is shifted to arbitrary large scales, resulting in the well-known Bunch-Davies spectrum which would furthermore extend to arbitrary small scales in the limit $\eta_f \rightarrow \eta_0$ (or $N \rightarrow \infty$). Lower image: time dependence of the power spectrum as parametrized by the number of e -foldings N . The scale-invariant plateau grows with time toward smaller comoving scales.

$$P_\psi(k, \eta) = \frac{1}{2\pi} \frac{k^2}{a(\eta)} |\psi_k(\eta)|^2. \quad (62)$$

At the final time, $\eta_f = \eta_0(1 - e^{-N})$, it takes the form

$$\frac{P_\psi(k, \eta_f)}{Hk\eta_0} = \frac{\pi}{64} e^{-2N} |rJ_1(k\eta_0 e^{-N}) + sY_1(k\eta_0 e^{-N})|^2, \quad (63)$$

which is a function of the dimensionless combination $k\eta_0$ only. Now, for modes in the regime $1 \ll k\eta_0 \ll e^N$, this spectrum becomes independent of k (scale invariance) as we show in Fig. 3. The physical reason is best understood in the comoving frame: here, one compares the mode wavelength k^{-1} to the (comoving) Hubble horizon $(aH)^{-1} = (\eta_0 - \eta)$, which indicates the distance beyond which two spatial positions recede faster than the speed of light (or sound in the simulator). From the definition of the (sound-)particle horizon (54), one finds that the comoving Hubble horizon exponentially shrinks from its initial value η_0 to its final value $\eta_0 e^{-N}$ as η grows from $\eta_i = 0$ toward $\eta_f = \eta_0(1 - e^{-N})$. Hence, the comoving wavelength k^{-1} of a mode transitions from being smaller than the Hubble horizon (subhorizon),

$$k(\eta_0 - \eta) = k/aH > 1. \quad (64)$$

to being larger than the Hubble horizon (superhorizon),

$$k(\eta_0 - \eta) = k/aH < 1, \quad (65)$$

where it freezes in because it is now too large for causal evolution. In terms of the wave number in the physical (expanding) frame, $k_{\text{phys}} = ka$, we can write

$$k(\eta_0 - \eta) = k_{\text{phys}}/H, \quad (66)$$

upon using Eqs. (54) and (57). Thus, the time evolution of the mode can be entirely absorbed into the physical scale k_{phys} [103]. Put differently, $k(\eta_0 - \eta)$ is large at early times (or equivalently small scales) and small at late times (or equivalently large scales), and the time evolution of a mode merely consists of transitioning from subhorizon to superhorizon scales around the turning point $k(\eta_0 - \eta) = k_{\text{phys}}H = 1$.

Each mode crosses the Hubble horizon at a different value of the (sound-)particle horizon η_k that fulfills $k(\eta_0 - \eta_k) = 1$; hence, modes which have crossed the horizon within the time window $\eta \in [\eta_i, \eta_f]$ satisfy $k(\eta_0 - \eta_i) = k\eta_0 > 1$ and

$$k(\eta_0 - \eta_f) = k\eta_0 e^{-N} < 1. \quad (67)$$

For modes which fall well into this regime, $k\eta_0 e^{-N} \ll 1$, we find at leading order [104]

$$\begin{aligned} P_\psi(k, \eta_f) &= \frac{H}{16\pi k\eta_0} [(J_1(k\eta_0) - 2k\eta_0 J_0(k\eta_0))^2 + (2k\eta_0)^2 J_1^2(k\eta_0)]. \end{aligned} \quad (68)$$

If we demand the modes to be significantly subhorizon initially, $k\eta_0 \gg 1$, we can use the large-argument asymptotics of the Bessel functions to find the constant value

$$P_\psi(k, \eta_f) = H/2\pi^2. \quad (69)$$

If the modes are only weakly subhorizon initially, $k\eta_0 \gtrsim 1$, there will be residual oscillations in k , as can be seen from Eq. (68), which is shown in Fig. 3. The magnitude of the described transition process and thus the size of the scale-invariant regime depend on the number of e -foldings N (as we show in Fig. 3). In the limiting case $\eta_f \rightarrow \eta_0$ (or $N \rightarrow \infty$), the scale-invariant regime would extend to arbitrary small scales.

Figure 3 suggests that e -foldings $N \in \{4, 5, 6\}$ suffice to realize scale invariance within 1 order of magnitude in k . We can constrain the remaining parameters such that this magnitude falls within the μm regime where the quantum field simulation takes place. In terms of an initial sound speed $c_{s,i}$ and an expansion time Δt , we have

$$\eta_0 = \frac{c_{s,i}}{H} = c_{s,i} \Delta t / N, \quad (70)$$

such that typical expansion times are required to be

$$\Delta t = 20 \text{ ms} \left(\frac{\eta_0}{10 \mu\text{m}} \right) \left(\frac{2.5 \mu\text{m/ms}}{c_{s,i}} \right) \left(\frac{N}{5} \right), \quad (71)$$

where the reference value of the initial sound speed $c_{s,i}$ represents relatively high initial scattering lengths [61,62]. The reference value of η_0 stems from demanding that scale invariance should occur for $k \gtrsim 0.5 \mu\text{m}^{-1}$ where an empirical factor of 5 is read off from Fig. 3 to effectively realize $k \gg 1/\eta_0 = 0.1 \mu\text{m}^{-1}$.

B. Contraction

It is well known that scale-invariant power spectra can also be achieved via contracting scenarios [15]. In two spatial dimensions, we find this to be realized for the power-law contraction

$$a(t) = a_i(1 - t/\tau)^{3/4} \quad (72)$$

that also been independently analyzed in the context of dipolar BECs [78]. Here, τ is a constant that corresponds to the time where $a(t)$ would reach zero. In the contracting scenario, the Hubble parameter $H(t) = \dot{a}(t)/a(t)$ decreases according to

$$H(t) = -\frac{3}{4(\tau - t)}, \quad (73)$$

instead of remaining constant as it was the case for the exponential expansion. We again define the (sound-)particle horizon via the integral

$$\eta(t) = \int_0^t \frac{ds}{a(s)} = \eta_0 [1 - (1 - t/\tau)^{1/4}], \quad \eta_0 = \frac{4\tau}{a_i}, \quad (74)$$

that asymptotes η_0 . As a function of η , one has

$$a(\eta) = a_i \left(1 - \frac{\eta}{\eta_0} \right)^3, \quad H(\eta) = -\frac{3}{4\tau} \left(1 - \frac{\eta}{\eta_0} \right)^{-4}, \quad (75)$$

which generates the effective mass $3/4(\eta_0 - \eta)^2$, leading to the same mode equation (56) as an exponential expansion. Choosing an instantaneous vacuum initially [given through Eq. (58)], we thus find the mode solution (59). For the power spectrum (62), we have to consider that, in terms of the cosmological parameters given in Eqs. (75), (73), and (74), the condition where the modes freeze out, $k(\eta_0 - \eta) = 1$, only translates into a horizon-crossing interpretation up to a factor of 3, $3k/aH = 1$. Also, the comoving Hubble horizon $(aH)^{-1}$ does not shrink but rather grows according to

$$[a(t)H(t)]^{-1} = a_i^{-1}(1 - t/\tau)^{-3/4}. \quad (76)$$

Nevertheless, we can compute the power spectrum (62) at final times $\eta_f = \eta_0(1 - e^{-N/3})$, leading to

$$P_\psi(k, \eta_f) = \frac{\pi k \eta_0}{64 \cdot 4\tau} e^{2N/3} |rJ_1(k\eta_0 e^{-N/3}) + sY_1(k\eta_0 e^{-N/3})|^2, \quad (77)$$

which is structurally similar to (63). For $k\eta_0 \ll e^{N/3}$, we find accordingly

$$P_\psi(k, \eta_f) = \frac{1}{4\tau} \frac{1}{16\pi k \eta_0} |rJ_1(k\eta_0 e^{-N/3}) + sY_1(k\eta_0 e^{-N/3})|^2, \quad (78)$$

which asymptotes to the constant value $P_\psi(k, \eta_f) \approx (8\pi^2\tau)^{-1}$ for $k\eta_0 \gg 1$. In summary, both the exponential expansion and the power-law contraction (72) lead to a scale-invariant spectrum,

$$P_\psi(k, \eta_f) \approx (2\pi^2 a_i \eta_0)^{-1}, \quad (79)$$

where η_0 is given by Eq. (54) (in case of an exponential expansion) or Eq. (74) (in case of the power-law contraction). However, the scale-invariant plateau is narrower than for an exponential expansion by the same strength, since the plateau is limited from above by the criterium $k\eta_0 \lesssim e^{N/3}$ (instead of $k\eta_0 \lesssim e^N$ in context of the exponential expansion). Consequently, a scale-invariant spectrum within 1 order of magnitude requires $N \simeq 15$, which is significantly larger than in the expanding case. Demanding the scale-invariant regime to be confined to micrometer scales requires the contraction timescale to be of the order

$$\tau = 2 \text{ ms} \left(\frac{\eta_0}{10 \text{ } \mu\text{m}} \right) \left(\frac{1.25 \text{ } \mu\text{m/ms}}{c_{s,i}} \right), \quad (80)$$

which is much shorter than the timescale of the exponential expansion (71). Note that N does not enter Eq. (80) but rather enters when computing the contraction duration $\Delta t = \tau/(1 - e^{-4N/3})$ that basically equals τ for sufficiently large N . Here, we chose a smaller reference speed of sound in accordance with relatively low initial scattering lengths required for an analog contraction.

C. Dispersive effects

Clearly, the presence of superluminal (or supersonic) modes can critically affect the horizon-crossing conditions and thus the emergence of scale invariance. In the following, we analyze the mode equation (42) for three classes of effective frequencies:

- (i) The time-dependent Bogoliubov dispersion relation with Ω_B^2 defined by Eq. (43).
- (ii) The time-independent Bogoliubov dispersion relation with $\Omega_{B,0}^2$ defined by Eq. (52).
- (iii) The nondispersive case Ω_{Ac}^2 defined by Eq. (53).

1. Exponential expansion

In Fig. 4, we show how these three cases modify the power spectrum created by an exponential expansion with reference values (71) using numerical simulations of the mode evolution. The time-independent Bogoliubov dispersion (which is equivalent to the superluminal Corley-Jacobson case [34]) does respect scale invariance as is well known [37–39,42]. The time-dependent Bogoliubov case deviates from the nondispersive case already in the mildly infrared regime $k\eta_0 \lesssim 0.1$, which we interpret as a dispersive reduction of nonadiabatic transitions (i.e., the wave-number dependency of the effective mass). The steep decline in

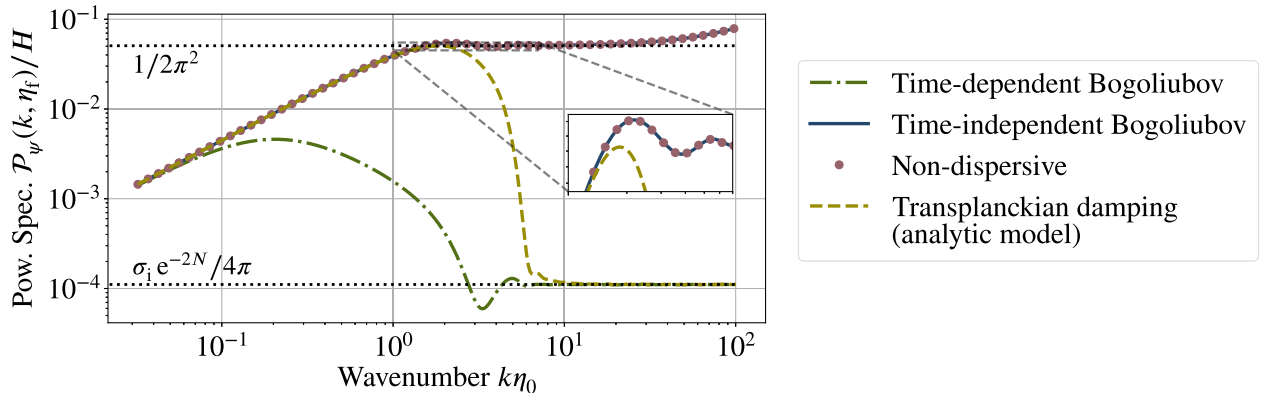


FIG. 4. Sensitivity of scale-invariant power spectra produced by an exponential expansion to the dispersive effects described in cases (i)–(iii) using the reference values (71) that can be combined into the initial scale-separation parameter $\sigma_i = 30.68$ (more details in the main text and in the context of Fig. 6). The time-independent dispersion preserves the scale-invariant value $P_\psi/H = 1/2\pi^2$ whereas the time-dependent case violates it on moderate ultraviolet scales due to trans-Planckian damping but converges to another plateau in the far ultraviolet given by $P_\psi/H = \sigma_i e^{-2N}/2\pi^2$.

power has been described as *trans-Planckian damping* in Ref. [92] (see also Ref. [78] for a discussion in context of dipolar BECs) and was attributed to a reduced number of degrees of freedom at high energies that can be traced back to the canonical volume form in Eq. (31) [and is captured by the function $\mathcal{J}_k$ defined in Eq. (41)].

As an analytic model for the trans-Planckian damping, we consult an auxiliary scenario where in the effective frequency (43) the nondispersive effective mass $m_{\text{eff}}^2(k=0, \eta)$ is used [cf. Eq. (46)] while keeping the time dependence in $k^2/\mathcal{J}_k(\eta)$. This leads to the mode equation

$$\psi_k''(\eta) + \left(k^2 - \frac{\nu^2(k) - 1/4}{(\eta - \eta_0)^2} \right) \psi_k(\eta) = 0, \quad (81)$$

which can be solved analytically by Bessel functions that have the wave-number-dependent order

$$\nu^2(k) = 1 - (k\eta_0)^4/\sigma_i^2, \quad \sigma_i = \sqrt{2}\eta_0/\xi_i, \quad (82)$$

where we choose ν to have a positive real part throughout the following. Here, we introduced the initial scale-separation parameter σ_i in terms of the initial healing length ξ_i . It describes how small the initial (analog) Planck length $\xi_i/\sqrt{2}$ is relative to the horizon-crossing scale η_0 such that the dispersive effects are expected to be weak if $\sigma_i \gg 1$ (more details in the following subsection). The auxiliary reference case can be classified into two regimes:

- (a) *Trans-Planckian damping (mild ultraviolet)*. For $(k\eta_0)^4 \leq 1/\sigma_i$, one has $\nu(k) \in \mathbb{R}$ and finds the power spectrum

$$\begin{aligned} P_\psi(k, \eta_f) = & \frac{H}{32\pi} \Gamma^2(\nu(k)) \left(\frac{k\eta_0}{2} \right)^{1-2\nu(k)} \frac{e^{-2N(1-\nu(k))}}{\sqrt{1 + (k\eta_0\sigma_i)^2}} \\ & \times \{ [(-1 + 2\nu(k))J_{\nu(k)}(k\eta_0) \\ & - 2k\eta_0 J_{\nu(k)-1}(k\eta_0)]^2 \\ & + (2\omega_{\text{in}}\eta_0)^2 J_{\nu(k)}(k\eta_0)^2 \}, \end{aligned} \quad (83)$$

for $k\eta_0 \ll e^N$, which converges to the nondispersive result (68) in the infrared regime, $k\xi_i \ll 1$, where $\nu(k)$ approaches unity. Here, the trans-Planckian damping arises from the factor $(k\eta_0)^{-2\nu(k)}$ in Eq. (83).

- (b) *Trans-Planckian damping (far ultraviolet)*: In the far ultraviolet, $(k\eta_0)^4 \gg 1/\sigma_i$, the power spectrum associated to both time-dependent and time-independent Bogoliubov dispersion remarkably converge to a plateau as well. In this regime, nonadiabatic transitions do not occur anymore, and the nonrelativistic, instantaneous mode eigenenergy dominates the mode equation. This limit can be computed analytically, resulting in

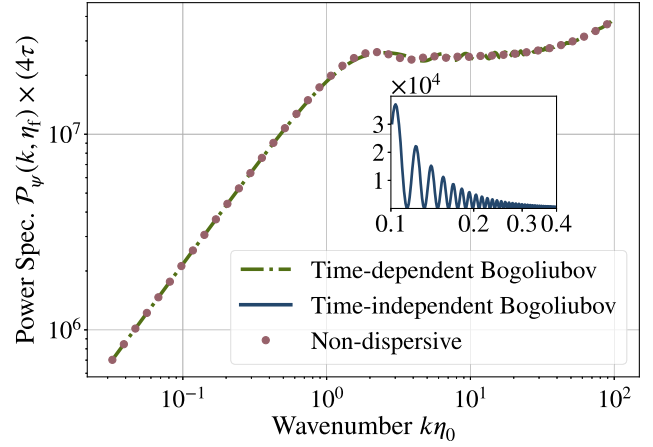


FIG. 5. Sensitivity of scale-invariant power spectra produced by a power-law contraction to the dispersive effects described in cases (i)–(iii) using the reference values (80). The inset shows the time-independent Bogoliubov case (ii) where the spectrum is orders of magnitude lower and exhibits an oscillatory pattern.

$$P_\psi(k, \eta_f) \approx \frac{H\sigma_i}{4\pi} e^{-2N}, \quad (84)$$

upon expressing the mode for $\nu(k) \in i\mathbb{R}$ in terms of Dunsters Bessel functions of purely imaginary order [105,106] and performing a uniform asymptotic expansion in $|\nu(k)| \gg 1$ and $k\eta_0 \gg 1$ [details to be found in Appendix D; note that Eq. (83) becomes numerically unstable for purely imaginary $\nu(k)$].

2. Contraction

In Fig. 5, we show the sensitivity of the power-law contraction with reference values (80) to the dispersive effects described in cases (i)–(iii). Therein, the situation is reversed in the sense that the time-dependent dispersion weakly alters the nondispersive power spectrum by small oscillations, whereas the time-independent case leads to a substantially damped signal of completely different form (as shown in the inset of Fig. 5). This reversal can be understood as follows: per definition, the analog comoving Hubble horizon $(aH)^{-1}$ indicates the boundary in space from which space recedes faster than the speed of sound (in comoving coordinates). As discussed in Ref. [42], whether a violation of scale invariance occurs can be efficiently reduced to whether the dispersion relation is significantly modified in the regime where the horizon-crossing occurs. More concretely, this can be quantified via the so-called scale-separation parameter [42,91,107], which we define as

$$\sigma = \begin{cases} k_C/aH & \text{expansion,} \\ 3k_C/a|H| & \text{contraction,} \end{cases} \quad (85)$$

where k_C is the comoving wave number well below which the dispersion relation is linear. For instance, for the

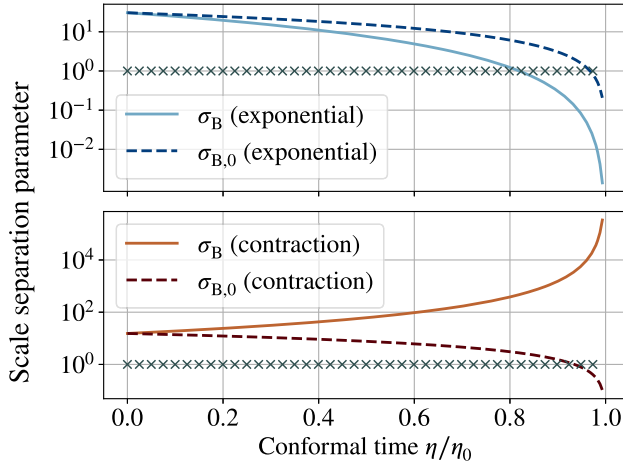


FIG. 6. Scale separation parameters for the time-dependent and time-independent Bogoliubov dispersion relation in case of the exponential expansion ($N = 5$, upper image) and the power-law contraction ($N = 10$, lower image) which are associated to Figs. 4 and 5, respectively. The initial scale-separation parameter is $\sigma_i = 30.68$ for the exponential expansion and $\sigma_i = 15.34$ for the power-law contraction.

effective comoving wave numbers F_B and F_{CJ} , we find

$$\sigma_B(\eta) = \frac{\sqrt{2}}{\xi(\eta)} [a(\eta)H(\eta)]^{-1}, \quad \sigma_{B,0}(\eta) = \frac{\sqrt{2}}{\xi_i} [a(\eta)H(\eta)]^{-1}. \quad (86)$$

Specifically for the expanding and contracting scenarios discussed in this section, one finds

$$\sigma_B(\eta) = \sigma_i \begin{cases} (1 - \eta/\eta_0)^2 & \text{expansion} \\ (1 - \eta/\eta_0)^{-2} & \text{contraction} \end{cases}, \quad (87)$$

and

$$\sigma_{B,0}(\eta) = \sigma_i \begin{cases} (1 - \eta/\eta_0) & \text{expansion} \\ (1 - \eta/\eta_0) & \text{contraction} \end{cases}, \quad (88)$$

where the initial scale-separation parameter $\sigma_i = \sqrt{2}\eta_0/\xi_i$ was already used in context of Eq. (82) and is set by the chosen parameter configurations that are respectively given by Eq. (71) (expanding case) or Eq. (80) (contracting case). The evolution in time of both scale-separation parameters is shown in Fig. 6 for both the expanding (cf. Fig. 4) and the contracting case (cf. Fig. 5). In the expanding case, the time-dependent Bogoliubov dispersion violates scale-separation at later times and therefore small scales, whereas the time-independent dispersion respects scale separation for most times. In the contracting case, this behavior is reversed such that the time-independent dispersion leads to a violation of scale invariance. In contrast, the

time-dependent Bogoliubov case approaches scale invariance at late times.

V. INFLUENCE OF LABORATORY BOUNDARY CONDITIONS

In the preceding discussion of scale invariance, we assumed an initial instantaneous vacuum state and traced the mode evolution up to some final instant in time where we evaluated the power spectrum. Therein, we did not account for boundary conditions that would be present in a laboratory setting such as an initial switch-on process or a final readout protocol of experimentally accessible field correlations. The introduction and analysis of both is the subject of this section.

A. Nonequilibrium statistical functions

In the quantum field simulator, the nonequilibrium dynamics are initiated through a specific temporal profile of $\lambda(t)$ for $t \in [t_i, t_f]$, whereas λ is static initially ($t < t_i$) and finally ($t > t_f$) (consider Fig. 7 for a schematic visualization). This protocol leads to pair production of Bogoliubov excitations which are associated with well-defined vacuum states for $t < t_i$ and $t > t_f$. More details on this process can be found in Refs. [61,62,93,94]. Evaluated in the initial vacuum state, the equal-time statistical functions of the fluctuating fields ϕ_1 and ϕ_2 take the form

$$\frac{1}{2} \langle \{\phi_1(t, \mathbf{x}), \phi_1(t, \mathbf{x}')\} \rangle_c = \int_k J_0(kL) |w_k(t)|^2, \quad (89)$$

$$\frac{1}{2} \langle \{\phi_2(t, \mathbf{x}), \phi_2(t, \mathbf{x}')\} \rangle_c = \int_k J_0(kL) |v_k(t)|^2, \quad (90)$$

where we already carried out the angular integral in Fourier space, due to the isotropy of the correlations. Because of homogeneity, the geometric kernel, given in terms of the cylindrical Bessel function $J_0(kL)$, only depends on the distance $L = |\mathbf{x} - \mathbf{x}'|$. In the Schrödinger picture, the time dependence of the state is encoded in the mode functions. In particular, the pair-production process resulting from the out-of-equilibrium quantum dynamics can be captured by means of the Bogoliubov transformation

$$v_k(t) = \alpha_k^* u_k(t) - \beta_k u_k^*(t), \quad (91)$$

where u_k is a positive frequency plane wave in the final region,

$$u_k(t \geq t_f) = \sqrt{\frac{mc_k(t_f)}{\hbar k}} e^{-i\omega_k(t_f)t}. \quad (92)$$

According to Eq. (22) which relates w_k and v_k , one consequently has

$$w_k(t) = -i \frac{E_k}{\epsilon_k(t_f)} [\alpha_k^* u_k(t) + \beta_k u_k^*(t)], \quad (93)$$

where the prefactor in Eq. (92) ensures the canonical normalization constraint (20) to be fulfilled. Evaluating the mode functions in the final region, one finds

$$\frac{|w_k(t)|^2}{k} = \frac{E_k}{\epsilon_k(t_f)} \left(\frac{1}{2} + |\beta_k|^2 - \text{Re}[\alpha_k \beta_k e^{2i\hbar\omega_k(t_f)t}] \right), \quad (94)$$

$$\frac{|v_k(t)|^2}{k} = \frac{\epsilon_k(t_f)}{E_k} \left(\frac{1}{2} + |\beta_k|^2 + \text{Re}[\alpha_k \beta_k e^{2i\hbar\omega_k(t_f)t}] \right), \quad (95)$$

with

$$\frac{\epsilon_k}{E_k} = \frac{\sqrt{2}}{k\xi} \sqrt{1 + \frac{1}{2}k^2\xi^2}. \quad (96)$$

The phase factor in Eqs. (94) and (95) indicates coherent oscillations in the final region $t \geq t_f$ [58,61,62,108], which stems from interference effects within a produced particle pair. In the literature [58,109], the left-hand-side of Eqs. (94) and (95) is also known as the structure factor. Indeed, we may equivalently write

$$\frac{1}{2} \langle \{ \phi_a(t, \mathbf{x}), \phi_b(t, \mathbf{x}') \} \rangle_c = \int_k J_0(kL) \mathcal{S}_{ab}(t, k), \quad (97)$$

with $\mathcal{S}_{ab}(t, k)$ as the structure factor. If the quasiparticles are in thermal equilibrium, the structure factors becomes independent of time and take the form

$$\mathcal{S}_{ab}(k) = \frac{1}{2} \coth\left(\frac{1}{2} \frac{\hbar\omega(k)}{k_B T}\right) \begin{pmatrix} E_k/\epsilon_k & 0 \\ 0 & \epsilon_k/E_k \end{pmatrix}, \quad (98)$$

as we show in Appendix F (see also Ref. [110] and supplementary material of Ref. [53]). Therefore, if the initial state is thermal, the prefactors of Eqs. (94) and (95) have to be adjusted according to the thermal structure factor (98).

B. Density correlations

Let us consider the density contrast relative to a homogeneous background, which is defined as

$$\delta_c(t, \mathbf{x}) = \frac{n(t, \mathbf{x}) - n_0}{n_0}. \quad (99)$$

Its equal-time two-point correlation function is related to the statistical function of ϕ_1 via

$$\begin{aligned} \mathcal{G}_{nn}(t, L) &\equiv \langle \delta_c(t, \mathbf{x}) \delta_c(t, \mathbf{x}') \rangle, \\ &= \frac{1}{n_0} \langle \{ \phi_1(t, \mathbf{x}), \phi_1(t, \mathbf{x}') \} \rangle_c + \mathcal{O}(\phi^3). \end{aligned} \quad (100)$$

In terms of a Fourier-decomposition, one thus has

$$\mathcal{G}_{nn}(t, L) = \frac{2}{n_0} \int_k J_0(kL) |w_k(t)|^2. \quad (101)$$

Employing Eq. (22), we find

$$\begin{aligned} \mathcal{G}_{nn}(t, L) &= \frac{2}{n_0} \int_k J_0(kL) \frac{\hbar^2}{(2m)^2} a_k^4 |\dot{v}_k(t)|^2 \\ &= \frac{\hbar a_f}{mn_0} \int_k J_0(kL) \frac{k S_k}{\sqrt{1 + \frac{1}{2}k^2\xi_f^2}}, \end{aligned} \quad (102)$$

where we used that $|\dot{v}_k|^2 \simeq \omega_k^2(t_f) |v_k|^2$ up to a π -phase shift in the coherent term. Here, we introduced the spectrum

$$S_k(t) = \frac{\hbar}{2m} \frac{k}{c_k} |v_k(t)|^2, \quad (103)$$

as a generalization to the discussion in Ref. [93] to which Eq. (102) is consistent for $k\xi_f \ll 1$ [111]. Moreover, Eq. (102) is consistent with Bogoliubov theory [58,108,112], where one finds a well-known shot noise spectrum in the ultraviolet regime $k\xi_f \gg 1$ (consider Appendix E for a derivation of the shot noise).

C. Singular contributions

In the specific case of the nonequilibrium setup shown in Fig. 7, the analog scale factor is kept static initially and finally,

$$a(t) = a_i \quad \text{for } t \leq t_i, \quad (104)$$

$$a(t) = a_f \quad \text{for } t \geq t_f, \quad (105)$$

and switched on (and off) at t_i (and t_f), leading to singular contribution to the effective mass, as we previously discussed in the acoustic approximation in Ref. [94]. To generalize this discussion to the dispersive case, we insert the laboratory scale factor into Eq. (45) and transform back

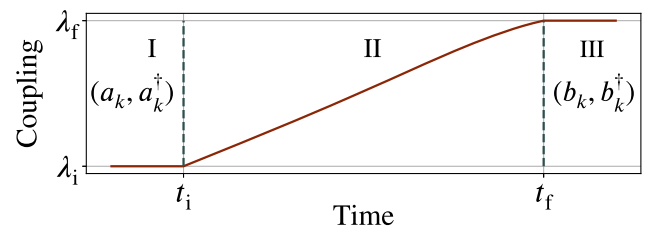


FIG. 7. Nonequilibrium quantum dynamics via a temporal variation of the interatomic coupling constant $\lambda(t)$. In the Heisenberg picture, the process is understood as follows: A vacuum state is defined initially as the state that is annihilated by $\hat{a}_k$; the two-point correlations are read out finally where the state can be described by the Bogoliubov-transformed operator $b_k = \alpha_k^* a_k - \beta_k^* a_{-k}^\dagger$.

to laboratory time, resulting in

$$m_{\text{eff}}^2(k, t) = m_{\text{eff;reg}}^2(k, t) + m_{\text{eff;sing}}^2(k, t), \quad (106)$$

with the regular terms

$$m_{\text{eff;reg}}^2(k, t) = -\frac{1}{4} \frac{1 - 5k^2\xi^2(\eta) + 4k^4\xi^4(\eta)}{[1 + \frac{1}{2}k^2\xi^2(\eta)]^2} \dot{a}^2(t) - \frac{1}{2} \frac{1 - \frac{1}{2}k^2\xi^2(\eta)}{1 + \frac{1}{2}k^2\xi^2(\eta)} \ddot{a}(t)a(t) \quad (107)$$

and the singular contributions

$$m_{\text{eff;sing}}^2(k, t) = -\frac{\mathcal{H}(\eta)}{2} \frac{1 - \frac{1}{2}k^2\xi^2(\eta)}{1 + \frac{1}{2}k^2\xi^2(\eta)} \times [\delta(t(\eta) - t_i) - \delta(t(\eta) - t_f)], \quad (108)$$

where we introduced the conformal Hubble rate $\mathcal{H}(\eta) = a'(\eta)/a(\eta)$. The expression (108) is consistent with our previous result given in Ref. [94] in the acoustic limit. In the presence of dispersion, however, these distributional terms differ for each mode and cross through zero at $k = \sqrt{2}/\xi$.

D. Boundary conditions

Because of the initial and final stasis of space, the canonically normalized mode function obeys the boundary conditions

$$\psi_k(\eta \leq \eta_i) = \frac{1}{\sqrt{2\omega_{\text{in}}(k)}} e^{-i\omega_{\text{in}}(k)\eta} \quad (109)$$

$$\psi_k(\eta \geq \eta_f) = \frac{1}{\sqrt{2\omega_{\text{out}}(k)}} (\alpha_k^* e^{-i\omega_{\text{out}}(k)\eta} - \beta_k e^{i\omega_{\text{out}}(k)\eta}) \quad (110)$$

with the Bogoliubov coefficients α_k, β_k as well as the incoming and outgoing frequencies,

$$\omega_{\text{in}}(k) = k\sqrt{1 + \frac{1}{2}k^2\xi_i^2}, \quad \omega_{\text{out}}(k) = k\sqrt{1 + \frac{1}{2}k^2\xi_f^2}, \quad (111)$$

where ξ_i and ξ_f are the initial and final values of the healing length, respectively. In general, the choice of the initial vacuum is ambiguous in dispersive theories, which has been discussed extensively [37]. However, in the BEC-cosmology simulator, the mode $\psi_k(\eta)$ for $\eta \leq \eta_i$ is associated to a unique quantum state that is determined by the initial stasis. These boundary conditions affect the mode function via matching conditions that read

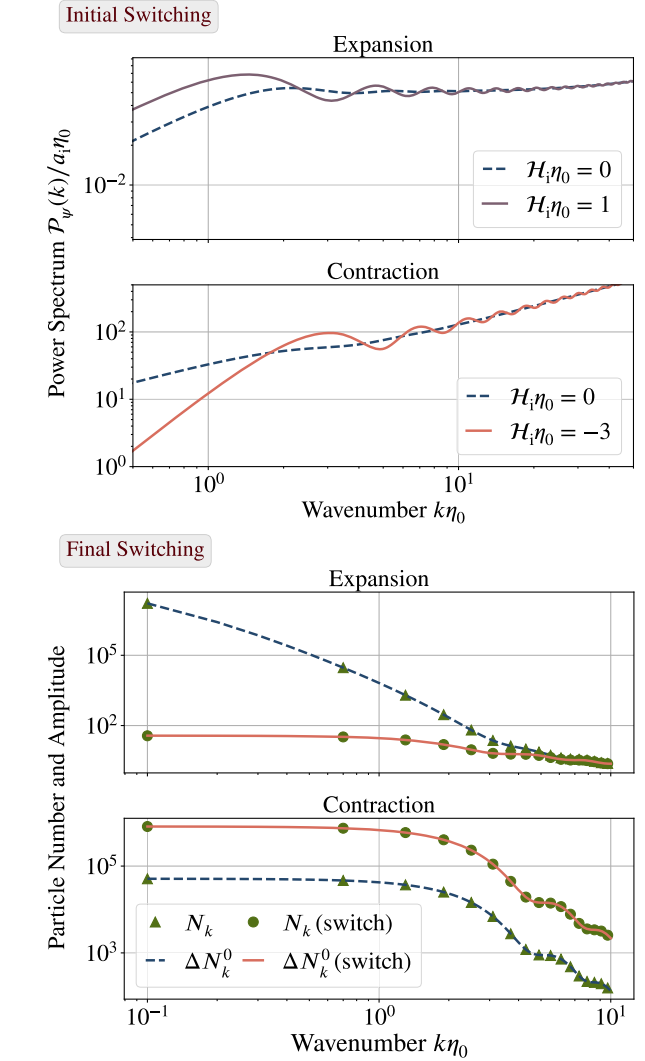


FIG. 8. Influence of an initial switching (upper two panels) and a final switching (lower two panels) on the power spectrum (upper two panels) or the coherent oscillation amplitude (lower two panels) produced by either an exponential expansion and power-law contraction for $N = 5$ according to nondispersive theory.

$$\begin{aligned} \lim_{\eta \searrow \eta_i} \psi_k(\eta) &= \lim_{\eta \nearrow \eta_i} \psi_k(\eta), \\ \lim_{\eta \searrow \eta_i} \psi'_k(\eta) &= \left(-i\omega_i(k) + \frac{\mathcal{H}(\eta_i)}{2} \frac{1 - k^2\xi_i^2/2}{1 + k^2\xi_i^2/2} \right) \lim_{\eta \nearrow \eta_i} \psi_k(\eta) \end{aligned} \quad (112)$$

at $\eta = \eta_i$ and

$$\begin{aligned} \lim_{\eta \nearrow \eta_f} \psi_k(\eta) &= \lim_{\eta \searrow \eta_f} \psi_k(\eta), \\ \lim_{\eta \nearrow \eta_f} \psi'_k(\eta) &= \lim_{\eta \searrow \eta_f} \psi'_k(\eta) + \frac{\mathcal{H}(\eta_f)}{2} \frac{1 - k^2\xi_f^2/2}{1 + k^2\xi_f^2/2} \lim_{\eta \searrow \eta_f} \psi_k(\eta) \end{aligned} \quad (113)$$

at $\eta = \eta_f$, where the extra terms proportional to the conformal Hubble rate $\mathcal{H}$ stem from the singular contributions to the effective mass. The coherent oscillations discussed in context of Eqs. (94) and (95) manifest themselves in the relativistic particle spectrum $S(k)$ introduced in Eq. (103) via the form

$$S_k(\eta) = \frac{1}{2} + N_k + \Delta N_k^0 \cos(2\omega_f(\eta - \eta_f) + \vartheta_k), \quad (114)$$

where the particle number

$$N_k = |\beta_k|^2, \quad (115)$$

the oscillation amplitude

$$\Delta N_k^0 = |\alpha_k \beta_k|, \quad (116)$$

and the phase

$$\vartheta_k = \arg(\alpha_k \beta_k e^{2i\omega_f \eta_f}) \quad (117)$$

follow from the matching conditions (112) and (113), as is extensively discussed, for example, in Refs. [93,94,113].

E. Influence of switching effects on scale invariance

Let us now consider the exponential expansion or power-law contraction to be embedded as a dynamical region II in the sense of Fig. 7. To separate these switching effects from the dispersive effects, we restrict the following analysis to the acoustic (nondispersive) limit. Accounting for an initial

switch-on at a conformal Hubble rate $\mathcal{H}_i \equiv \mathcal{H}(\eta_i)$ via invoking the boundary condition (112) changes the power spectrum on final superhorizon scales into

$$\begin{aligned} \frac{P_\psi(k, \eta_f)}{a_i \eta_0} &= \frac{1}{16\pi k \eta_0} \\ &\times [(J_1(k\eta_0) - 2k\eta_0 J_0(k\eta_0))^2 \\ &+ (2k\eta_0 - \mathcal{H}_i \eta_0)^2 J_1^2(k\eta_0)], \end{aligned} \quad (118)$$

where $\mathcal{H}_i \eta_0 = 1$ (in the expanding case) and $\mathcal{H}_i \eta_0 = -3$ (in the contracting case). The contributions of these finite values to Eq. (118) leads to oscillations instead of a scale-invariant plateau as shown in the upper two panels of Fig. 8. Interestingly, the initial switching reduces power in the infrared regime for a contracting scenario (or desqueezes the state [94]) since $\mathcal{H}_i \eta_0 < 0$ as opposed to the expanding case.

Considering a final switch-off at a conformal Hubble rate $\mathcal{H}_f \equiv \mathcal{H}(\eta_f)$ and computing the particle number N_k as well as the oscillation amplitude ΔN_k^0 , one finds the results shown in the lower two panels of Fig. 8, where $\mathcal{H}_f \equiv e^N \mathcal{H}_i$ (in the expanding case) and $\mathcal{H}_f = e^{N/3} \mathcal{H}_i$ (in the contracting case). Therein, one sees a desqueezing of the state after the expanding case because $\mathcal{H}_f$ enters with a relative minus sign to $\mathcal{H}_i$ in Eqs. (112) and (113) as opposed to the contracting case; note that the switch-off effect is orders of magnitude stronger than the switch-on effect due to the strong blueshift, $\mathcal{H}_f \gg \mathcal{H}_i$.

To realize why the particle number N_k agrees with the amplitude ΔN_k^0 , consider that for a strong particle

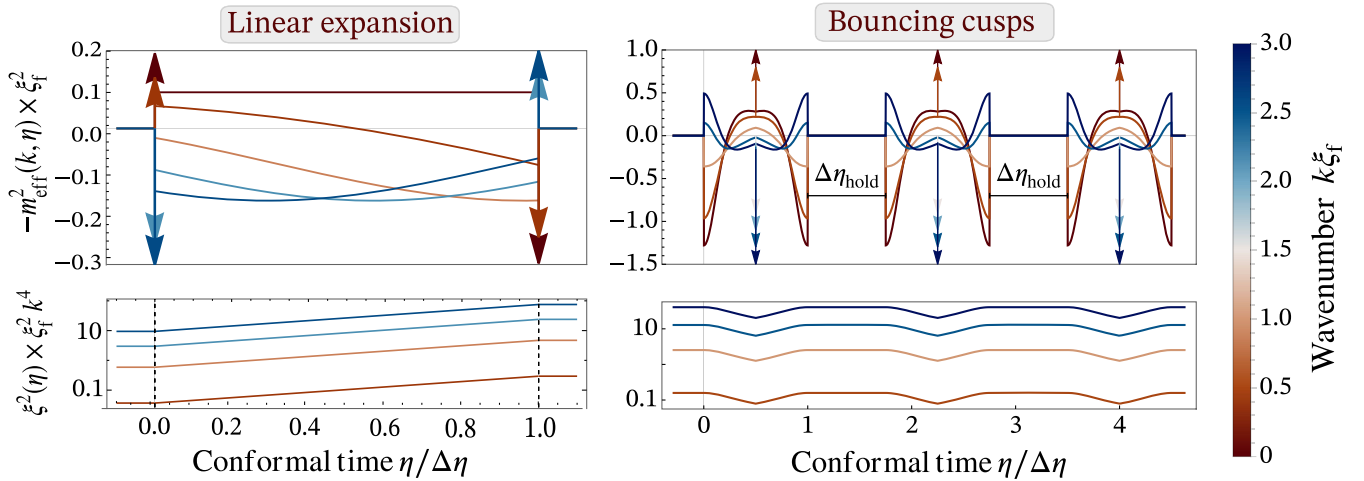


FIG. 9. Effective mass $m_{\text{eff}}^2(k, \eta)$ (top) and the purely dispersive factor $k^4 \xi(\eta)^2$ (bottom) for the linear expansion (left panels) and the bouncing cusp scenario (right panels). Open parameters are set in accordance with the experiment [62]: in the former case, space expands by a ratio of $a_f/a_i = \sqrt{8}$, whereas it bounces by $a_b/a_i = \sqrt{2}$ in the latter case. The initial healing length is set to $\xi_i = 0.12 c_s^i \Delta t$ (for the expanding scenario) and to $\xi_i = 1.6 c_s^i / \omega$ (for the bouncing scenario), where $c_s^i \Delta t$ and c_s^i / ω represent characteristic sound (or particle) horizon scales in terms of the initial sound speed c_s^i , the expansion duration Δt , and the bouncing frequency ω . Distributional contributions are indicated via arrows.

production process where $|\beta_k| \gg 1$ one can use the symplectic property of the Bogoliubov transformation, $|\alpha_k|^2 - |\beta_k|^2 = 1$, to infer that

$$N_k = \Delta N_k^0 \left[1 - \frac{1}{2} |\beta_k|^{-2} + \mathcal{O}(|\beta_k|^{-4}) \right]. \quad (119)$$

F. Deeper dive into experimental realizations

In this final subsection, we utilize the concepts developed in this work to provide a further detailed theoretical description of scenarios that were experimentally realized in Ref. [62].

1. Linear expansion

A linear expansion between initial and final scale-factor values $a_{i,f}$ corresponds to $a(t)/a_i = 1 + H_0 t$ where the constant Hubble rate is $H_0 = (a_f/a_i - 1)/(c_s^i \Delta t)$ in terms of the expansion duration Δt . The resulting effective mass and the purely dispersive term are shown in the left panel of Fig. 9, where one sees that the latter dominates in the dispersive regime whereas the former is approximately constant in the infrared and switches sign toward the ultraviolet. Figure 10 shows the particle number N_k and the oscillation amplitude ΔN_k^0 obtained from a numerical integration of the mode equation (42) under consideration of the matching procedure described in Sec. V D. In the acoustic approximation, zero crossings of the amplitude occur for modes which satisfy $\sqrt{k^2 - H_0^2/4} = n\pi/\Delta\eta$ where n is a positive integer and $\Delta\eta = (c_s^i \Delta t) \ln(a_f/a_i)/(a_f/a_i - 1)$ is the final particle horizon. This was explained in terms of a scattering picture in Refs. [62,94]. Taking into account the Bogoliubov dispersion, these quantum recurrences of the initial state become minima which are narrower spaced since

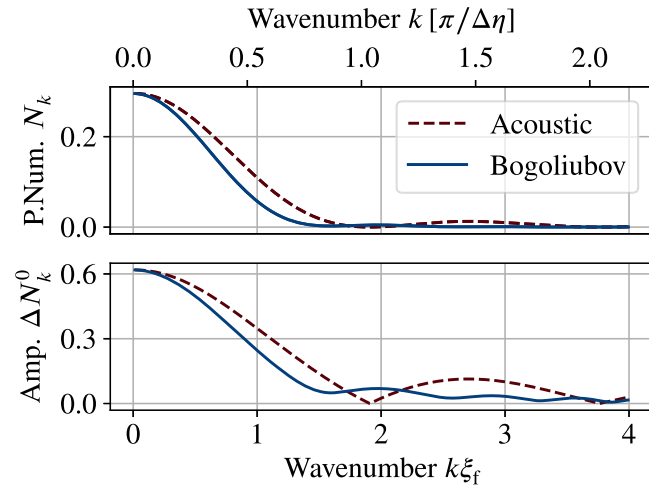


FIG. 10. Particle number (upper image) and oscillation amplitude (lower image) after a linear expansion scenario.

the phase velocity c_k is larger than the speed of sound c_s , which also leads to a weaker signal in total.

2. Bouncing cusps

To increase the magnitude of the particle production signal, one can consider a cosmological contraction followed by an expansion by a ratio $a_b/a_i < 1$ via the scale factor

$$a_{\text{cusp}}(t) = [(a_b/a_i)^2 - |\cos(\omega_0 t)|]^{-1/2}, \quad (120)$$

with a cusp singularity at the turning point. This has the advantage that the probability of nonadiabatic transitions is boosted by the cusp. Periodically continuing such bouncing cusps with intervals of constant sound speed $c_{s,i} = 1/a_i$ in between (as displayed in Fig. 11) creates a periodic pattern in the effective mass with periodicity $\Delta\eta_{\text{hold}} = c_{s,i} \Delta t_{\text{hold}}$ where Δt_{hold} is the duration of the holding intervals. The

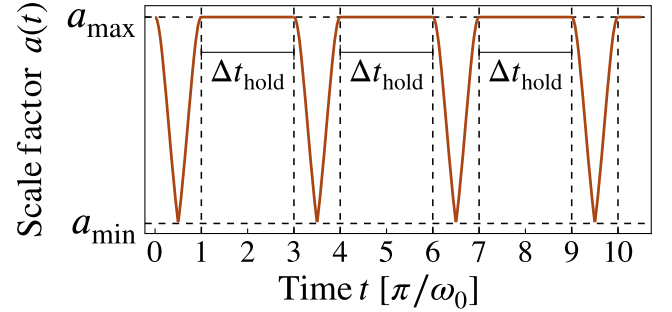


FIG. 11. Temporal profile of a cusp-modulated scale-factor with $a_b/a_i = \sqrt{2}$ for $\Delta t_{\text{hold}} = 2\Delta t_{\text{cusp}}$.

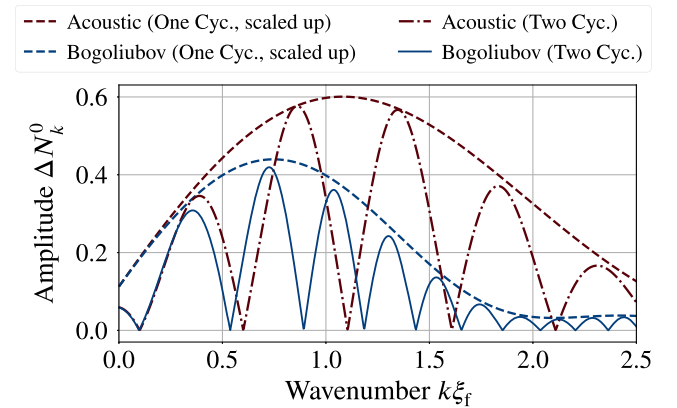


FIG. 12. Coherent oscillation amplitude after a single (solid line) and two cycles (dashed line) in both acoustic (green curves) and Bogoliubov theory (red curves). Note that the signal actually grows with the amount of cycles, we merely scaled up the single cycle amplitudes by a factor of 2.5 for better visibility of the diffractive pattern.

dynamical evolution of modes is then equivalent to a propagation of waves through a dispersive medium whose elementary cells have substructure of size $\Delta\eta = \int_0^{\pi/\omega_0} dt/a_{\text{cusp}}(t)$, as is shown in the right panel of Fig. 9. Since particle production can be described as reflection in that medium [62,94], a pattern of minima and maxima develops as a result of Bragg diffraction of reflected waves which is visualized in Fig. 12. Therein, it is clearly visible that the single-cycle amplitude sets an envelope for the multicycle amplitudes which is related to the Fourier convolution theorem and a characteristic feature of Bragg diffraction.

VI. CONCLUSION

We have derived an effective field theory for the U(1)-Goldstone boson of a scalar Bose-Einstein condensate with time-dependent contact interactions that is consistent with Bogoliubov theory and mapped it to a superluminal, non-local quantum field theory on a dispersive, curved spacetime whose geometry is set by the phase velocity of Bogoliubov excitations and can be classified as a rainbow geometry. Assuming a spatially homogeneous background density profile enabled us to neglect quantum pressure terms, thereby considerably facilitating the mapping at the expense of a spatially flat emergent spacetime. Nevertheless, these developments could also be used as an approximation to the spatially curved case (with an inhomogeneous background density) for modes whose wavelength is smaller than the curvature radius (which is also known as the eikonal approximation [66] in analogy to the small-wavelength limit of geometric optics).

In the developed theory, effective Lorentz invariance is restored on scales larger than the healing length which serves as an analog of the Planck length, with a modified dispersion relation assumed for the trans-Planckian regime. We continued by exploring this framework in a cosmological setup where the (dispersive) scale factor is identified with the inverse phase velocity and the comoving frame is the laboratory frame. The mode equation was transformed into a parametric oscillator form that features wave-number-dependent nonadiabatic transitions. The latter originate from a dynamical, analog Planck length in the comoving frame or, equivalently, time-dependent interatomic interactions. This scenario is different from *ad hoc* approaches to trans-Planckian regimes that only modify the instantaneous eigenenergy of a mode.

Subsequently, we discussed aspects of the cosmological trans-Planckian problem, approached from the perspective of the BEC-analog simulator. To prepare the discussion, we compared the time-dependent dispersion (with the dynamical analog Planck length) to the time independent and the nondispersive case regarding nonadiabaticity. We continued by describing how a scale-invariant power spectrum of fluctuations can emerge from an exponential expansion or power-law contraction in the BEC and estimated the

required parameters to produce such a spectrum within 1 order of magnitude in the micrometer regime, which resulted in a required number of four to six *e*-foldings for an expansion time of about 20 ms (in the expanding case), whereas the contracting case demands an equivalent of 15 *e*-foldings and a contraction time scale on the order of 2 ms; both scenarios clearly lie beyond current experimental capabilities. It followed an investigation of the sensitivity of scale invariance under the time-dependent Bogoliubov dispersion with a comparison to the time-independent case (that has been discussed extensively in the literature [37–39,42]). The time-dependent case violates scale invariance in the exponentially expanding scenario and showcases trans-Planckian damping [92] (see also Ref. [78] for a discussion in context of a dipolar BEC) that can be traced back to a reduced density of analog relativistic states at high energies. The time-independent case on the other hand preserves scale invariance since the analog Planck scale and the horizon scale are well separated [42,91,107] at all times (in contrast to the time-dependent case). Similar arguments explain the reversed behavior in the contracting case.

In a final section, we introduced laboratory boundary conditions in time as typical for a BEC experiment. Relationships between the nonequilibrium statistical two-point functions of the canonically normalized, relativistic field variables and experimentally observable *in situ* density correlations were obtained including Bogoliubov dispersion. We furthermore studied the impact of switching protocols on the scale-invariant power spectra, where switch-on processes lead to residual acoustic oscillations in the power spectrum whose frequency is determined by the switch-on rate. Depending on a contracting or an expanding scenario, the switch-off process can lead to desqueezing and typically is more influential than the switch-on process due to the strong blueshift.

Finally, we applied the developed concepts to previously realized experimental scenarios [62], thereby showcasing the occurrence of Bragg reflection and a suppression of the particle-production signal relative to the acoustic case, which surprisingly occurs already at scales well above the healing length which needs to be taken into account when, for example, making statements about two-mode quantum entanglement close to the dispersive regime.

With the present work, we have contributed to the general discussion of the trans-Planckian problem of cosmology via an analysis of dispersive effects with a time-dependent UV cutoff in the comoving frame that is motivated from the perspective of a BEC-cosmology simulator with time-dependent contact interactions. This type of time dependence dynamically disfavors scale separation between the Hubble horizon and the cutoff scale in the expanding case but dynamically favors it in the contracting case, such that the scale invariance of only the latter is robust to such dispersive effects. We

furthermore conclude that trans-Planckian damping effects induced by this time dependence can settle to a another scale-invariant regime at much lower power in the far UV spectrum. We also would like to note that trans-Planckian damping occurred despite the absence of a minimal length principle (in the context of which it is studied, for example, in Refs. [78,92]).

The extended framework provides quantitative access to stronger particle production scenarios whose spectrum can be measured deeper into the ultraviolet regime which enables a path toward constructing innovative scenarios that lie within experimental capabilities and overcome the described effects that obstruct scale invariance. This may ultimately lead to an experimental observation of a scale-invariant power spectrum of an analog relativistic field.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: COMPARISON TO BOGOLIUBOV THEORY

In this Appendix, we compare the field-theory formalism of Sec. II to the more common Bogoliubov theory in the operator formalism. First, it will be important to realize that the fluctuating fields ϕ_1, ϕ_2 are related to density and phase fluctuations $\delta n, \delta\theta$ which are the variables commonly used in the literature (a nonexhaustive list is Refs. [66,81,108,115]) and can be introduced here via

$$\Phi(t, \mathbf{r}) = [n_0(t, \mathbf{r}) + \delta n(t, \mathbf{r})] e^{i[\theta_0(t, \mathbf{r}) + \delta S(t, \mathbf{r})]}. \quad (\text{A1})$$

At leading order in both fluctuations, one finds

$$\phi_1(t, \mathbf{r}) = \delta n(t, \mathbf{r}) / \sqrt{2n_0(t, \mathbf{r})}, \quad (\text{A2})$$

$$\phi_2(t, \mathbf{r}) = \delta\theta(t, \mathbf{r}) \sqrt{2n_0(t, \mathbf{r})}, \quad (\text{A3})$$

after a local gauge transformation $\Phi \rightarrow e^{-i\theta_0(t, \mathbf{r})} \Phi$. However, since phase fluctuations become ill defined in the limit $n_0 \rightarrow 0$, where Eq. (A3) becomes singular, we work with the fields ϕ_1 and ϕ_2 in the main text. In the following, we closely follow Ref. [108].

1. Atomic density correlations

At linear order, the total atomic density operator is expanded into a background and fluctuating part,

$$\hat{\rho}(t, \mathbf{x}) = \rho_0 + \sqrt{\rho_0}(\delta\hat{\phi} + \delta\hat{\phi}^\dagger). \quad (\text{A4})$$

The connected equal-time density correlator,

$$G^{\rho\rho}(t, \mathbf{x}; t', \mathbf{x}') = \langle \hat{\rho}(t, \mathbf{x}) \hat{\rho}(t', \mathbf{x}') \rangle - \langle \hat{\rho}(t, \mathbf{x}) \rangle \langle \hat{\rho}(t', \mathbf{x}') \rangle, \quad (\text{A5})$$

can be expanded in Fourier space,

$$\begin{aligned} G^{\rho\rho}(t, \mathbf{k}; t', \mathbf{k}') &= \int d\mathbf{x} d\mathbf{x}' e^{-i(\mathbf{k}\mathbf{x} + \mathbf{k}'\mathbf{x}')} G^{\rho\rho}(t, \mathbf{x}; t', \mathbf{x}') \\ &= \langle \delta\hat{\rho}_{\mathbf{k}}(t) \delta\hat{\rho}_{\mathbf{k}'}^\dagger(t') \rangle, \end{aligned} \quad (\text{A6})$$

where

$$\delta\hat{\rho}_{\mathbf{k}}(t) = \int d\mathbf{x} e^{-i\mathbf{k}\mathbf{x}} \delta\hat{\rho}(t, \mathbf{x}). \quad (\text{A7})$$

Since $\hat{\rho}(t, \mathbf{x})$ is a Hermitian operator, one has

$$\delta\hat{\rho}_{\mathbf{k}}(t) = \delta\hat{\rho}_{-\mathbf{k}}^\dagger(t). \quad (\text{A8})$$

In terms of perturbative operators, we have

$$\delta\hat{\rho}_{\mathbf{k}}(t) = \sqrt{N}(\delta\hat{\phi}_{\mathbf{k}}(t) + \delta\hat{\phi}_{-\mathbf{k}}^\dagger(t)), \quad (\text{A9})$$

where $\sqrt{N}$ ensures the fulfillment of the equal-time commutation relations

$$[\delta\hat{\phi}_{\mathbf{k}}(t), \delta\hat{\phi}_{\mathbf{k}'}^\dagger(t)] = \delta_{\mathbf{k}\mathbf{k}'}. \quad (\text{A10})$$

As a result of both $\mathbf{k}$ and $-\mathbf{k}$ excitations appearing in $\delta\hat{\rho}(\mathbf{k})$, it is indistinguishable to destroy a phonon of momentum $\mathbf{k}$ or to create one with momentum $-\mathbf{k}$. Furthermore, these two excitations necessarily interfere when performing density measurements.

2. Collective excitations

The Hamiltonian of the linear perturbations is diagonalized by the Bogoliubov transformation

$$\delta\phi_{\mathbf{k}} = u_{\mathbf{k}} \hat{\phi}_{\mathbf{k}} + v_{-\mathbf{k}}^* \hat{\phi}_{-\mathbf{k}}^\dagger, \quad (\text{A11})$$

where we introduce the quasiparticle operators $\hat{\phi}_{\mathbf{k}}, \hat{\phi}_{\mathbf{k}}^\dagger$. The Bogoliubov coefficients can be parametrized as

$$u_k = \cosh \chi_k, \quad v_k = \sinh \chi_k, \quad (\text{A12}) \quad \text{correlations are}$$

with

$$\coth(2\chi_k) = -1 - k^2 \xi^2, \quad (\text{A13})$$

where

$$\frac{\hbar^2}{2m\xi^2} = \lambda n_0 = mc_s^2. \quad (\text{A14})$$

In the following, we restrict the analysis to a stationary and homogeneous system. Additionally, we assume that the phonon states are statistically homogeneous. As a consequence, the correlations only depend the spatial distance $|\mathbf{x} - \mathbf{x}'|$ or absolute values $k = |\mathbf{k}|$ in Fourier space. Moreover, one only has correlations between k and $-k$.

a. Density fluctuations in the phonon basis

In terms of the quasiparticle operators, the density-fluctuation operator reads

$$\delta \hat{\rho}_k = \sqrt{N}(u_k + v_k)(\hat{\phi}_k + \hat{\phi}_{-k}^\dagger). \quad (\text{A15})$$

In the quasiparticle ground state, the density correlations are

$$G_{\text{vac}}^{\rho\rho}(k) = N(u_k + v_k)^2, \quad (\text{A16})$$

whereas in an excited phonon state, one has

$$G_{\text{exc}}^{\rho\rho}(k) = G_{\text{vac}}^{(2)}(k)(1 + n_k + n_{-k} + 2\text{Re}[c_k e^{-i\Omega_k t}]), \quad (\text{A17})$$

where

$$n_{\pm k} = \langle \hat{\phi}_{\pm k}^\dagger \hat{\phi}_{\pm k} \rangle, \quad c_k = \langle \hat{\phi}_k \hat{\phi}_{-k} \rangle \quad (\text{A18})$$

represent the occupation numbers of and the coherence between k and $-k$ excitations.

b. Phase fluctuations in the phonon basis

Phase fluctuations can be introduced in the atomic basis via the operator

$$\delta \hat{\phi}(t, \mathbf{x}) = \frac{\delta \hat{\rho}}{2\sqrt{\rho_0}} + i\sqrt{\rho_0} \delta \hat{\theta}. \quad (\text{A19})$$

In terms of the quasiparticle operators, one has

$$\delta \hat{\theta}_k = \frac{\sqrt{N}}{2i\rho_0}(u_k - v_k)(\hat{\phi}_k - \hat{\phi}_{-k}^\dagger), \quad (\text{A20})$$

such that $\delta \hat{\phi}$ is the proper conjugate variable to the density fluctuation $\delta \hat{n}_k$. In the quasiparticle ground state, the phase

$$G_{\text{vac}}^{\theta\theta}(k) = \frac{N}{4\rho_0^2}(u_k - v_k)^2, \quad (\text{A21})$$

whereas for an excited state,

$$G_{\text{exc}}^{\theta\theta}(k) = G_{\text{vac}}^{\theta\theta}(k)(1 + n_k + n_{-k} - 2\text{Re}[c_k e^{-i\Omega_k t}]). \quad (\text{A22})$$

This correlator contains no additional information on the phonon pair state as the coherent term is merely π -shifted. The vacuum normalization is the respective inverse of the density normalization factor as it should be for proper conjugate variables. These considerations follow from the useful identities

$$(u_k + v_k)^2 = \frac{k\xi}{\sqrt{2}} \frac{1}{\sqrt{1 + \frac{1}{2}(k\xi)^2}} = \frac{E_k}{\epsilon(k)},$$

$$(u_k - v_k)^2 = \frac{\sqrt{2}}{k\xi} \sqrt{1 + \frac{1}{2}(k\xi)^2} = \frac{\epsilon(k)}{E_k}, \quad (\text{A23})$$

where

$$\epsilon(k) = c_s k \sqrt{1 + \frac{1}{2}k^2 \xi^2}, \quad E_k = \frac{\hbar^2 k^2}{2m}. \quad (\text{A24})$$

APPENDIX B: DERIVATION OF DISPERSIVE EFFECTIVE MASS

The goal of this Appendix is to absorb the influence of the time-dependent and dispersive cosmological background into an effective mass, by transforming away the Hubble friction term in Eq. (28). To that end, we rescale

$$v_k(t) = f_k(t)\psi_k(t) \quad (\text{B1})$$

and introduce conformal time $d\eta = dt/a(t)$ with respect to the bare scale factor $a(t) = 1/c_s(t)$ such that η is still equivalent to the sound horizon in the BEC [94]. The mode equation (28) then takes the form

$$\psi_k'' + 2\left(\frac{f_k'}{f_k} + \frac{a_k'}{a_k} - \frac{1}{2}\frac{a_k'}{a_k}\right)\psi_k' + \left[\frac{f_k''}{f_k} - 2\left(\frac{1}{2}\frac{a_k'}{a_k} - \frac{a_k'}{a_k}\right)\frac{f_k'}{f_k}\right]\psi_k = -k^2 \frac{a^2}{a_k^2} \psi_k. \quad (\text{B2})$$

To have a vanishing Hubble friction term, the scaling function is specified to fulfill

$$\frac{f_k'(\eta)}{f_k(\eta)} + \frac{a_k'(\eta)}{a_k(\eta)} = \frac{1}{2} \frac{a'(\eta)}{a(\eta)}, \quad (\text{B3})$$

such that the mode equation becomes

$$\psi_k''(\eta) + \left[k^2 + m_{\text{eff}}^2(k, \eta) + \frac{1}{2} k^4 \xi^2(\eta) \right] \psi_k(\eta) = 0, \quad (\text{B4})$$

with the effective mass

$$m_{\text{eff}}^2(k, \eta) = \frac{d}{d\eta} \frac{f'_k(\eta)}{f_k(\eta)} - \left(\frac{f'_k(\eta)}{f_k(\eta)} \right)^2. \quad (\text{B5})$$

The specific form of the function f_k can be obtained by considering Eq. (35). We can then rewrite Eq. (B3) as

$$\frac{f'_k(\eta)}{f_k(\eta)} = \left(\frac{1}{2} - \mathcal{J}_k(\eta) \right) \frac{a'(\eta)}{a(\eta)}, \quad (\text{B6})$$

which can be integrated to

$$f_k(\eta) = [a(\eta) \mathcal{J}_k(\eta)]^{-1/2} \quad (\text{B7})$$

and is the $(2+1)$ -dimensional version of the mode rescaling employed in Ref. [92].

1. Acoustic limit

As a consistency check, we find in the infrared regime, $k \ll k_\xi$, where the dispersion relation (24) is linear and thus $a_k = a + \mathcal{O}((k/k_\xi)^2)$, that

$$m_{\text{eff}}^2(k, \eta) = \frac{1}{4} \left(\frac{a'(\eta)}{a(\eta)} \right)^2 - \frac{1}{2} \frac{a''(\eta)}{a(\eta)} + \mathcal{O}\left(\frac{k^2}{k_\xi^2}\right), \quad (\text{B8})$$

which coincides with the analysis [94].

2. Ultraviolet limit

In the ultraviolet limit, $k \gg \sqrt{2}/\xi$, the elementary excitations obey a nonrelativistic, quadratic dispersion relation and represent individual atoms that are resolved on small scales $k^{-1} \ll \xi/\sqrt{2}$. As it is not possible anymore to project phonons out of the condensate basis, analog particle production should cease. Indeed, the phase velocity becomes

$$c_k = \frac{\hbar k}{2m} + \mathcal{O}(\hbar^2 k^2 / 4m^2 c_s^2(t)). \quad (\text{B9})$$

Then, the mode equation (28) is a time-independent harmonic oscillator,

$$\ddot{v}_k(t) + \frac{\hbar^2 k^4}{(2m)^2} v_k(t) = 0, \quad (\text{B10})$$

with nonrelativistic dispersion and no time-dependent frequency, such that quanta can not be excited. The ultraviolet limit of the effective mass is

$$\lim_{k \rightarrow \infty} m_{\text{eff}}^2(k, \eta) = -\frac{3}{4} \left(\frac{a'(\eta)}{a(\eta)} \right)^2 + \frac{1}{2} \frac{a''(\eta)}{a(\eta)}, \quad (\text{B11})$$

which is adiabatically suppressed by the purely dispersive term $k^4/k_\xi^2(\eta)$ in the ultraviolet limit (as visible in Fig. 9).

APPENDIX C: ADIABATIC VACUUM VIA WKB APPROXIMATION

We recall the well-known WKB solution of a parametric oscillator with frequency $\Omega(k, \eta)$ that reads

$$\psi_k^{\text{WKB}}(\eta) = \frac{1}{\sqrt{2W_k(\eta)}} \exp \left[-i \int_{\eta_i}^{\eta} W_k(\eta') d\eta' \right], \quad (\text{C1})$$

where the prefactor ensures canonical normalization and the quantity W_k satisfies the nonlinear differential equation [116]

$$W_k^2(\eta) = \Omega^2(k, \eta) - \frac{1}{2} \left[\frac{W_k''(\eta)}{W_k(\eta)} - \frac{3}{2} \left(\frac{W_k'(\eta)}{W_k(\eta)} \right)^2 \right]. \quad (\text{C2})$$

This equation can be solved iteratively, resulting in an adiabatic expansion of the mode function where each iteration is referred to as an *adiabatic order* [116]. The vacuum associated to a mode function at a certain adiabatic order is called the *adiabatic vacuum*. At zeroth adiabatic order, one has $W_k^2(\eta) = \Omega^2(k, \eta)$, and the mode *adiabatically adjusts* to the time-dependent background (an adjustment local in time). The zeroth-order adiabatic approximation is valid as long as both the conditions

$$\left| \frac{\Omega'(k, \eta)}{\Omega(k, \eta)} \right| \ll |\Omega(k, \eta)|, \quad \left| \frac{\Omega''(k, \eta)}{\Omega(k, \eta)} \right| \ll \Omega^2(k, \eta) \quad (\text{C3})$$

are fulfilled [117].

APPENDIX D: ANALYTIC MODEL FOR TRANS-PLANCKIAN DAMPING

Neglecting the wave number dependence of the effective mass but keeping the time dependence of the healing length, the mode equation induced by an exponentially expanding spacetime reads

$$\psi_k''(\eta) + \left(k^2 - \frac{\nu(k)^2 - 1/4}{(\eta - \eta_0)^2} \right) \psi_k(\eta) = 0 \quad (\text{D1})$$

with

$$\nu(k)^2 = 1 - (k\eta_0)^4 / \sigma_i^2, \quad (\text{D2})$$

whose square root we take to have a positive real part in the following. We also did introduce the initial scale-separation parameter

$$\sigma_i = \sqrt{2}\eta_0/\xi_i \quad (\text{D3})$$

to conveniently analyze the spectrum as a function of $k\eta_0$. The general solution to the mode equation can be written as

$$\psi_k(\eta) = \frac{i\pi}{4} \frac{1}{\sqrt{2\omega_i}} \sqrt{\frac{\eta_0 - \eta}{\eta_0}} \times (rJ_{\nu(k)}[k(\eta_0 - \eta)] + sY_{\nu(k)}[k(\eta_0 - \eta)]). \quad (\text{D4})$$

The boundary conditions

$$\psi_k(0) = 1/\sqrt{2\omega_{\text{in}}}, \quad \psi'_k(0) = -i\omega_{\text{in}}\psi_k(0), \quad (\text{D5})$$

are solved by

$$r = (1 - 2\nu(k) - 2i\omega_i\eta_0)Y_{\nu(k)}(k\eta_0) + 2k\eta_0Y_{\nu(k)-1}(k\eta_0) \quad (\text{D6})$$

and

$$s = (-1 + 2\nu(k) + 2i\omega_i\eta_0)J_{\nu(k)}(k\eta_0) - 2k\eta_0J_{\nu(k)-1}(k\eta_0). \quad (\text{D7})$$

The power spectrum is

$$\frac{P_\psi(k, \eta)}{Hk\eta_0} = \frac{\pi}{64} \frac{e^{-2N}}{\sqrt{1 + \frac{1}{2}(k\xi_i)^2}} \quad (\text{D8})$$

$$\times |rJ_{\nu(k)}[k(\eta_0 - \eta)] + sY_{\nu(k)}[k(\eta_0 - \eta)]|^2. \quad (\text{D9})$$

Let us again evaluate it at final times $\eta_f = \eta_0(1 - e^{-N})$ for modes with $1 \ll k\eta_0 \ll e^N$.

1. Mild ultraviolet regime

As long as $(k\eta_0)^4 < 1/\sigma$, we have $\nu(k) \in \mathbb{R}$ and therefore can approximate

$$Y_{\nu(k)}(k\eta_0 e^{-N}) \approx -\frac{\Gamma(\nu(k))}{\pi} \left(\frac{k\eta_0 e^{-N}}{2}\right)^{-\nu(k)} \quad (\text{D10})$$

to find

$$P_\psi(k, \eta_f) = \frac{H}{32\pi} \Gamma^2(\nu(k)) \left(\frac{k\eta_0}{2}\right)^{1-2\nu(k)} \frac{e^{-2N(1-\nu(k))}}{\sqrt{1 + (k\eta_0\sigma)^2}} \times \{[(-1 + 2\nu(k))J_{\nu(k)}(k\eta_0) - 2k\eta_0J_{\nu(k)-1}(k\eta_0)]^2 + (2\omega_i\eta_0)^2 J_{\nu(k)}^2(k\eta_0)\}, \quad (\text{D11})$$

which converges to the nondispersive result (68) in the infrared regime, $k\xi_i \ll 1$, where $\nu(k) \rightarrow 1$.

2. Far ultraviolet regime

For $(k\eta_0)^4 > 1/\sigma_i$, the order $\nu(k)$ becomes purely imaginary, $\nu(k) \in i\mathbb{R}$. This requires us to write the mode solution in a different basis,

$$\psi_k(\eta) = \frac{i\pi}{4} \frac{1}{\sqrt{2\omega_i}} \sqrt{\frac{\eta_0 - \eta}{\eta_0}} \times (rF_{|\nu(k)|}[k(\eta_0 - \eta)] + sG_{|\nu(k)|}[k(\eta_0 - \eta)]), \quad (\text{D12})$$

with the coefficients

$$r = -(1 + 2i\omega_i\eta_0)G_{|\nu(k)|}(k\eta_0) - 2k\eta_0\partial_x G_{|\nu(k)|}(x)|_{x=k\eta_0}, \quad (\text{D13})$$

$$s = (1 + 2i\omega_i\eta_0)F_{|\nu(k)|}(k\eta_0) + 2k\eta_0\partial_x F_{|\nu(k)|}(x)|_{x=k\eta_0} \quad (\text{D14})$$

and the basis-functions

$$F_\nu(x) = \frac{1}{2 \cosh(\pi\nu/2)} (J_{i\nu}(x) + J_{-i\nu}(x)) \quad (\text{D15})$$

and

$$G_\nu(x) = \frac{1}{2i \sinh(\pi\nu/2)} (J_{i\nu}(x) - J_{-i\nu}(x)) = \frac{1}{2 \cosh(\pi\nu/2)} (Y_{i\nu}(x) + Y_{-i\nu}(x)), \quad (\text{D16})$$

which are numerically stable alternatives to the unmodified Bessel-functions of purely imaginary order (see Ref. [105] for the construction by Dunster). They are real for positive arguments and have similar small-argument asymptotics

$$F_\nu(x) = \sqrt{\frac{2 \tanh(\pi\nu/2)}{\pi\nu}} \cos(\nu \ln(x/2) - \arg \Gamma(1 + i\nu)), \quad (\text{D17})$$

$$G_\nu(x) = \sqrt{\frac{2 \coth(\pi\nu/2)}{\pi\nu}} \sin(\nu \ln(x/2) - \arg \Gamma(1 + i\nu)) \quad (\text{D18})$$

as $x \rightarrow 0^+$ and large-argument asymptotics

$$F_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4), \quad (\text{D19})$$

$$G_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin(x - \pi/4) \quad (\text{D20})$$

as $x \rightarrow +\infty$. Furthermore, their Wronskian,

$$\partial_x F_\nu(x) G_\nu(x) - \partial_x G_\nu(x) F_\nu(x) = 2/\pi x, \quad (\text{D21})$$

is identical to the Wronskian of the unmodified Bessel functions. In the regime $k\eta_0 \ll e^N$, we can use the small-argument asymptotics given by Eqs. (D17) and (D18) to

write the power spectrum as

$$\begin{aligned} \frac{P_\psi(k, \eta_f)}{Hk\eta_0} &= \frac{e^{-2N}}{64|\nu(k)|} \frac{1}{\sqrt{1 + (k\eta_0\sigma)^2}} \\ &\times \left| \tanh\left(\frac{1}{2}\pi|\nu(k)|\right) r \cos(\varphi_k) \right. \\ &\quad \left. + \coth\left(\frac{1}{2}\pi|\nu(k)|\right) s \sin(\varphi_k) \right|^2, \end{aligned} \quad (\text{D22})$$

where [105]

$$\varphi_k = |\nu(k)| \ln\left(\frac{1}{2}e^{-N}k\eta_0\right) - \arg \Gamma(1 + i|\nu(k)|). \quad (\text{D23})$$

For modes that are well superhorizon finally, $k\eta_0 \gg 1$, the order $|\nu(k)| \approx (k\eta_0)^2/\sigma_i$ becomes large such that we can further simplify

$$P_\psi(k, \eta_f) = \frac{H\sigma_i e^{-2N}}{64|\nu(k)|} |r \cos(\varphi_k) + s \sin(\varphi_k)|^2. \quad (\text{D24})$$

To approximate the coefficients r and s , we have to take into account that *both* the argument $k\eta_0$ and the order $|\nu(k)|$ of the remaining Bessel functions becomes large. This kind of situation can be handled by the uniform asymptotic expansion that has been derived recently by Dunster [106]. According to Dunster, one has

$$|J_{i|\nu|}(|\nu|x)| \sim \frac{\exp\left(\frac{1}{2}|\nu|\pi + \mathcal{O}(|\nu|^{-2})\right)}{\sqrt{2\pi|\nu|}(x^2 + 1)^{1/4}}, \quad (\text{D25})$$

and

$$\arg J_{i|\nu|}(|\nu|x) \sim |\nu|\rho(x) - \frac{\pi}{4} + \mathcal{O}(x^{-3}), \quad (\text{D26})$$

for $|\nu| \gg 1$, uniformly in $x \in (0, \infty)$, where

$$\rho(x) = \sqrt{x^2 + 1} - \ln\left(\frac{1 + \sqrt{x^2 + 1}}{x}\right) \quad (\text{D27})$$

and $\sim$ denotes asymptotic equivalence. We directly infer that

$$F_\nu(\nu x) \sim \sqrt{\frac{2}{\pi\nu}}(x^2 + 1)^{-1/4} \cos\left(\nu\rho(x) - \frac{\pi}{4}\right), \quad (\text{D28})$$

$$G_\nu(\nu x) \sim \sqrt{\frac{2}{\pi\nu}}(x^2 + 1)^{-1/4} \sin\left(\nu\rho(x) - \frac{\pi}{4}\right), \quad (\text{D29})$$

according to the definitions (D15) and (D16). Applied to the present case, this means that we can take

$$F_{(k\eta_0)^2/\sigma_i}(k\eta_0) \sim \sqrt{\frac{2}{\pi\sigma}}(k\eta_0)^{-1} \cos(\mu_k), \quad (\text{D30})$$

$$G_{(k\eta_0)^2/\sigma_i}(k\eta_0) \sim \sqrt{\frac{2}{\pi\sigma}}(k\eta_0)^{-1} \sin(-\mu_k), \quad (\text{D31})$$

where

$$\mu_k = \frac{(k\eta_0)^2}{\sigma_i} \ln(2k\eta_0), \quad (\text{D32})$$

to approximate the coefficients r and s at leading order in $1/(k\eta_0)$. As a result, we have

$$r \sim \sqrt{\frac{2}{\pi\sigma_i}}(2ik\eta_0) \sin(\mu_k), \quad s \sim \sqrt{\frac{2}{\pi\sigma_i}}(2ik\eta_0) \cos(\mu_k), \quad (\text{D33})$$

and therefore find the constant value

$$P_\psi(k, \eta_f) = \frac{H\sigma_i}{4\pi} e^{-2N}, \quad (\text{D34})$$

where we averaged over the rapidly oscillating phases μ_k and φ_k .

APPENDIX E: DENSITY CORRELATIONS: ACOUSTIC AND ULTRAVIOLET LIMITS

1. Infrared (acoustic) limit

The relation (102) between the density-density correlations $\mathcal{G}_{nn}$ and the power spectrum $S(k)$ reduces to

$$G_{nn}(t, L) = \frac{\hbar a_f}{mn_0} \int_k J_0(kL) k S(k), \quad (\text{E1})$$

which coincides with the acoustic result [93].

2. Ultraviolet limit: Shot noise

As a consistency check, let us apply this limit to Eq. (102) for the density correlation function. As a consequence of absent quasiparticles, the limit of the excitation spectrum assumes the value $S(k) \simeq 1/2$ and becomes static. From the integrand of Eq. (102), we can read off that

$$\langle \delta_c(\mathbf{k}) \delta_c(\mathbf{k}') \rangle = \frac{\hbar a_f}{mn_0} \frac{k S(k)}{\sqrt{1 + \frac{1}{2}k^2 \xi_f^2}} \delta(\mathbf{k} + \mathbf{k}'), \quad (\text{E2})$$

which in the limit of large momenta $k\xi_f \gg 1$ and $k'\xi_f \gg 1$ can be approximated to

$$\langle \delta_c(\mathbf{k}) \delta_c(\mathbf{k}') \rangle \simeq \frac{1}{n_0} \delta(\mathbf{k} + \mathbf{k}') \quad (\text{E3})$$

or in terms of connected density fluctuations

$$\langle n(\mathbf{k}) n(\mathbf{k}') \rangle_c = \frac{N}{V} \delta(\mathbf{k} + \mathbf{k}'), \quad (\text{E4})$$

where we set $n_0 = N/V$, with N as the particle number and V as the volume of the atomic cloud. Thus, the density fluctuations obey a shot noise spectrum $\Delta N \simeq \sqrt{N}$ in the ultraviolet regime, as expected from considering that for high wave numbers the kinetic energies of the atoms surpass the interaction energies by far, resulting in an equivalence to an ideal Bose gas. To realize why one expects a shot noise spectrum in the ultraviolet limit of the latter, let us consider a homogeneous, ideal Bose gas contained in a volume V . The effective bosonic field Ψ is quantized via the mode expansion [118]

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}}(\mathbf{x}) \hat{a}_{\mathbf{k}}, \quad \phi_{\mathbf{k}}(\mathbf{x}) = \exp\{i\mathbf{k} \cdot \mathbf{x}\} / \sqrt{V}, \quad (\text{E5})$$

where one has the well-known commutation relations

$$\begin{aligned} [\Phi(\mathbf{x}), \Phi^\dagger(\mathbf{y})] &= \delta(\mathbf{x} - \mathbf{y}), \\ [\Phi(\mathbf{x}), \Phi(\mathbf{y})] &= [\Phi^\dagger(\mathbf{x}), \Phi^\dagger(\mathbf{y})] = 0. \end{aligned} \quad (\text{E6})$$

The two-point density correlations in the atomic gas can be computed from the expectation values

$$\begin{aligned} \langle n(\mathbf{x}) n(\mathbf{y}) \rangle_c &= \langle n(\mathbf{x}) n(\mathbf{y}) \rangle - \langle n(\mathbf{x}) \rangle \langle n(\mathbf{y}) \rangle, \\ \langle n(\mathbf{x}) n(\mathbf{y}) \rangle &= \langle \Phi^\dagger(\mathbf{x}) \Phi^\dagger(\mathbf{y}) \Phi(\mathbf{y}) \Phi(\mathbf{x}) \rangle, \\ \langle n(\mathbf{x}) \rangle &= \langle \Phi^\dagger(\mathbf{x}) \Phi(\mathbf{x}) \rangle, \end{aligned} \quad (\text{E7})$$

which are taken in the state that is annihilated by $\hat{a}_{\mathbf{k}}$ from Eq. (E5). For the connected density two-point function, one finds [118]

$$\langle n(\mathbf{x}) n(\mathbf{y}) \rangle_c = \sum_{\mathbf{k}, \mathbf{k}'} \langle \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}'} \hat{a}_{\mathbf{k}} \rangle \phi_{\mathbf{k}}^*(\mathbf{x}) \phi_{\mathbf{k}'}(\mathbf{x}) \phi_{\mathbf{k}'}^*(\mathbf{y}) \phi_{\mathbf{k}}(\mathbf{y}), \quad (\text{E8})$$

where one has

$$\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} = \hat{n}_{\mathbf{k}}, \quad \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger = \hat{n}_{\mathbf{k}} + 1, \quad (\text{E9})$$

in terms of the number-density operator $\hat{n}_{\mathbf{k}}$. Assuming that the system is in the noncondensed phase enables us to replace $\sum_{\mathbf{k}} \rightarrow V \int_{\mathbf{k}}$ for a sufficiently large volume V (the condensed case is, for example, discussed in Refs. [119,120]). We then find [118]

$$\langle n(\mathbf{x}) n(\mathbf{y}) \rangle_c = \frac{N}{V} \delta(\mathbf{x} - \mathbf{y}) + \left| \int_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \langle \hat{n}_{\mathbf{k}} \rangle \right|^2, \quad (\text{E10})$$

where we set $\langle n(\mathbf{x}) \rangle = N/V$. Taking a Fourier transform, we find

$$\langle n(\mathbf{k}) n(\mathbf{k}') \rangle_c = \frac{N}{V} \delta(\mathbf{k} + \mathbf{k}') + \int_{\mathbf{q}} \langle \hat{n}_{\mathbf{q}} \rangle \langle \hat{n}_{\mathbf{k}+\mathbf{q}} \rangle \delta(\mathbf{k} + \mathbf{k}'), \quad (\text{E11})$$

where the first term represents shot noise and the second term represents autocorrelations. If the Bose gas is in thermal equilibrium, $\langle \hat{n}_{\mathbf{q}+\mathbf{k}} \rangle$ is the Bose-Einstein distribution, which decays exponentially as $|\mathbf{k}| \rightarrow \infty$, such that only the shot-noise term remains in the ultraviolet limit.

In this section, we focussed on the case of an ideal gas, which is appropriate in the ultraviolet limit. The nonideal case can be found, for example, in Refs. [120,121] and references therein.

APPENDIX F: THERMAL STRUCTURE FACTOR

An introduction to the spectral methods employed in this section can be found in chapter IV of Ref. [110], where also the following calculation is carried out in a similar manner. The connection between the full quantum field of the condensate Φ and the full condensate density n is given by the relation

$$\begin{aligned} n(t, u, \varphi) &= |\Phi(t, u, \varphi)|^2 \\ &= n_0(u) + \sqrt{2n_0(u)} \phi_1(t, u, \varphi) \\ &\quad + \frac{1}{2} \phi_1^2(t, u, \varphi) + \frac{1}{2} \phi_2^2(t, u, \varphi), \end{aligned} \quad (\text{F1})$$

where we considered the background field $\phi_0 \equiv \sqrt{n_0}$, ϕ_1 and ϕ_2 to be real after some local $U(1)$ gauge transformation. When we take the expectation value of the full condensate density $\langle n(t, u, \varphi) \rangle$, the second term in Eq. (F1) drops out, because it is linear in fluctuations.

Let us now determine the spectral density $\Delta_{ab}^\rho(\hbar\omega, \mathbf{k})$ in the present situation for the fields ϕ_1 and ϕ_2 . In momentum space, the quadratic action becomes

$$\begin{aligned} \Gamma[\phi_1, \phi_2] &= \int \frac{d\omega}{2\pi} \frac{d^2k}{(2\pi)^2} \left\{ -\frac{1}{2} \begin{pmatrix} \tilde{\phi}_1 & \tilde{\phi}_2 \end{pmatrix} \tilde{G}^{-1}(\omega, k) \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} \right\}, \end{aligned} \quad (\text{F2})$$

with the inverse propagator

$$\tilde{G}_{ab}^{-1}(\hbar\omega, \mathbf{k}) = \begin{pmatrix} \frac{\hbar^2 k^2}{2m} + 2\lambda n_0 & -i\hbar\omega \\ i\hbar\omega & \frac{\hbar^2 k^2}{2m} \end{pmatrix} \quad (\text{F3})$$

such that

$$\tilde{G}_{ab}(\hbar\omega, \mathbf{k}) = \frac{1}{-\hbar^2\omega^2 + \hbar^2\omega_B^2(k)} \begin{pmatrix} \frac{\hbar^2 k^2}{2m} & i\hbar\omega \\ -i\hbar\omega & \frac{\hbar^2 k^2}{2m} + 2\lambda n_0 \end{pmatrix}, \quad (\text{F4})$$

where we introduced the celebrated Bogoliubov dispersion relation

$$\hbar\omega_B(k) = \pm \sqrt{\left(\frac{\hbar^2 k^2}{2m} + 2\lambda n_0\right) \frac{\hbar^2 k^2}{2m}} \quad (\text{F5})$$

as the pole of the propagator,

$$\det \tilde{G}^{-1}(\hbar\omega_B, \mathbf{k}) = 0. \quad (\text{F6})$$

The spectral function follows from this as

$$\Delta_{ab}^\rho(\hbar\omega, \mathbf{k}) = -i\tilde{G}_{ab}(\hbar\omega + i\epsilon, \mathbf{k}) + i\tilde{G}_{ab}(\hbar\omega - i\epsilon, \mathbf{k}). \quad (\text{F7})$$

To evaluate this, one needs the identity

$$\frac{1}{x \pm i\epsilon} = \mp i\pi\delta(x) + \text{P.V.}\left(\frac{1}{x}\right) \quad (\text{F8})$$

and $-(\hbar\omega \pm i\epsilon)^2 = -\omega^2 \mp i\epsilon\text{sign}(\omega)$, which results in the spectral function

$$\begin{aligned} \Delta_{ab}^\rho(\hbar\omega, \mathbf{k}) &= 2\pi\delta(\hbar^2\omega^2 - \hbar^2\omega_B^2)\text{sign}(\omega) \\ &\times \begin{pmatrix} \frac{\hbar^2 k^2}{2m} & i\hbar\omega \\ -i\hbar\omega & \frac{\hbar^2 k^2}{2m} + 2\lambda n_0 \end{pmatrix}. \end{aligned} \quad (\text{F9})$$

One can now determine the static structure factor as

$$\begin{aligned} S_{ab}(\mathbf{k}) &= \hbar \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\frac{1}{2} + n_B(\hbar\omega) \right] \Delta_{ab}^\rho(\hbar\omega, \mathbf{k}) \\ &= \frac{\hbar}{2} \int_{-\infty}^{\infty} d\omega \coth(|\hbar\omega|/2) \delta(\hbar^2\omega^2 - \hbar^2\omega_B^2(k)) \\ &\times \text{sign}(\hbar\omega) \begin{pmatrix} \frac{\hbar^2 k^2}{2m} & i\hbar\omega \\ -i\hbar\omega & \frac{\hbar^2 k^2}{2m} + 2\lambda n_0 \end{pmatrix}, \end{aligned} \quad (\text{F10})$$

where we used

$$\begin{aligned} \frac{1}{2} + n_B(\hbar\omega) &= \left[\frac{1}{2} + n_B(|\hbar\omega|) \right] \text{sign}(\hbar\omega) \\ &= \frac{1}{2} \coth\left(\frac{|\hbar\omega|}{2}\right) \text{sign}(\hbar\omega). \end{aligned} \quad (\text{F11})$$

One further observes that the off-diagonal terms $i\hbar\omega$ do not contribute to the static structure factor, because they are odds with respect to $\hbar\omega \rightarrow -\hbar\omega$ while the rest of the integral is even. Otherwise, one can perform the frequency integration using the Dirac deltas with zero crossings are $\hbar\omega = \pm\epsilon_k$, which gives

$$S_{ab}(\mathbf{k}) = \frac{1}{2} \coth\left(\frac{|\hbar\omega|}{2}\right) \begin{pmatrix} E_k/\epsilon_k & 0 \\ 0 & \epsilon_k/E_k \end{pmatrix}, \quad (\text{F12})$$

with

$$\epsilon_k = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2\lambda n_0 \right)}, \quad E_k = \frac{\hbar^2 k^2}{2m}. \quad (\text{F13})$$

In terms of the healing length $\xi = \hbar/\sqrt{2m\lambda n_0}$ and the speed of sound $c = \sqrt{\lambda n_0/m}$, one can introduce the phase velocity $c_k = c\sqrt{1 + \xi^2|\mathbf{k}|^2/2}$, such that the Bogoliubov energy can be written as

$$\epsilon_k = \hbar c_k |\mathbf{k}| = \hbar c |\mathbf{k}| \sqrt{1 + \xi^2|\mathbf{k}|^2/2}, \quad (\text{F14})$$

and also

$$\frac{E_k}{\epsilon_k} = \frac{\xi k}{\sqrt{2}} \frac{1}{\sqrt{1 + \frac{1}{2}k^2\xi^2}}, \quad \frac{\epsilon_k}{E_k} = \frac{\sqrt{2}}{k\xi} \sqrt{1 + \frac{1}{2}k^2\xi^2}. \quad (\text{F15})$$

The components of the matrix are identical to expressions obtained from invoking a Bogoliubov transformation between the atom and the phonon basis as one can see from Eqs. (A16), (A21), and (A23).

The leading terms in the acoustic approximations as well as their corrections are now clear to see. In particular, we find for the Fourier transform of the density correlation function

$$\begin{aligned} S_{nn}(\mathbf{k}) &= 2n_0 S_{11}(\mathbf{k}) \\ &= \sqrt{2}n_0 \left[\frac{1}{2} + n_B(E_k) \right] \frac{\xi|\mathbf{k}|}{\sqrt{1 + \frac{\xi^2|\mathbf{k}|^2}{2}}}. \end{aligned} \quad (\text{F16})$$

- [1] S. W. Hawking, *Nature (London)* **248**, 30 (1974).
- [2] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
- [3] L. Parker, *Phys. Rev.* **183**, 1057 (1969).
- [4] Y. B. Zel'Dovich, *JETP Lett.* **14**, 180 (1971), <https://ui.adsabs.harvard.edu/abs/1971JETPL..14..180Z>.
- [5] W. G. Unruh, *Phys. Rev. D* **10**, 3194 (1974).
- [6] G. T. Moore, *J. Math. Phys. (N.Y.)* **11**, 2679 (1970).
- [7] S. A. Fulling and P. C. W. Davies, *Proc. R. Soc. A* **348**, 393 (1976).
- [8] M. J. Jacquet, T. Boulier, F. Claude, A. Maître, E. Cancellieri, C. Adrados, A. Amo, S. Pigeon, Q. Glorieux, A. Bramati, and E. Giacobino, *Phil. Trans. R. Soc. A* **378**, 20190225 (2020).
- [9] C. Barcelo, S. Liberati, and M. Visser, [arXiv:gr-qc/0505065v4](https://arxiv.org/abs/gr-qc/0505065v4).
- [10] R. Schützhold, *Prog. Part. Nucl. Phys.* **145**, 104198 (2025).
- [11] T. Grass, D. Bercioux, U. Bhattacharya, M. Lewenstein, H. S. Nguyen, and C. Weitenberg, *Rev. Mod. Phys.* **97**, 011001 (2025).
- [12] A. A. Starobinsky, *JETP Lett. (USSR)* **30**, 682 (1979), <https://www.osti.gov/biblio/5336557>.
- [13] V. F. Mukhanov and G. V. Chibisov, *JETP Lett.* **33**, 532 (1981), <https://ui.adsabs.harvard.edu/abs/1981JETPL..33..532M>.
- [14] V. Mukhanov, H. Feldman, and R. Brandenberger, *Phys. Rep.* **215**, 203 (1992).
- [15] D. Wands, *Phys. Rev. D* **60**, 023507 (1999).
- [16] F. Finelli and R. Brandenberger, *Phys. Rev. D* **65**, 103522 (2002).
- [17] R. H. Brandenberger, [arXiv:1206.4196](https://arxiv.org/abs/1206.4196).
- [18] T. Jacobson, *Phys. Rev. D* **44**, 1731 (1991).
- [19] R. H. Brandenberger, *Inflationary cosmology: Progress and problems* (Springer Netherlands, Dordrecht, 2000), pp. 169–211.
- [20] Y. Akrami *et al.* (Planck Collaboration), *Astron. Astrophys.* **641**, A10 (2020).
- [21] R. H. Brandenberger and J. Martin, *Classical Quantum Gravity* **30**, 113001 (2013).
- [22] P. Hořava, *Phys. Rev. D* **79**, 084008 (2009).
- [23] M. Visser, *Phys. Rev. D* **80**, 025011 (2009).
- [24] T. P. Sotiriou, M. Visser, and S. Weinert, *Phys. Rev. Lett.* **102**, 251601 (2009).
- [25] G. E. Volovik, *JETP Lett.* **89**, 525 (2009).
- [26] T. P. Sotiriou, *J. Phys. Conf. Ser.* **283**, 012034 (2011).
- [27] T. Jacobson and D. Mattingly, *Phys. Rev. D* **64**, 024028 (2001).
- [28] B. Cropp, S. Liberati, A. Mohd, and M. Visser, *Phys. Rev. D* **89**, 064061 (2014).
- [29] S. Liberati, *Classical Quantum Gravity* **30**, 133001 (2013).
- [30] W. G. Unruh, *Phys. Rev. Lett.* **46**, 1351 (1981).
- [31] W. G. Unruh, *Phys. Rev. D* **51**, 2827 (1995).
- [32] R. Brout, S. Massar, R. Parentani, and P. Spindel, *Phys. Rev. D* **52**, 4559 (1995).
- [33] S. Corley and T. Jacobson, *Phys. Rev. D* **54**, 1568 (1996).
- [34] S. Corley, *Phys. Rev. D* **57**, 6280 (1998).
- [35] T. Jacobson, *Prog. Theor. Phys. Suppl.* **136**, 1 (1999).
- [36] J. Martin and R. H. Brandenberger, in *9th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (MG 9)* (2000), pp. 2001–2002, [arXiv:astro-ph/0012031](https://arxiv.org/abs/astro-ph/0012031).
- [37] J. Martin and R. H. Brandenberger, *Phys. Rev. D* **63**, 123501 (2001).
- [38] R. H. Brandenberger and J. Martin, *Mod. Phys. Lett. A* **16**, 999 (2001).
- [39] J. Martin and R. H. Brandenberger, *Phys. Rev. D* **65**, 103514 (2002).
- [40] R. H. Brandenberger, S. E. Jorás, and J. Martin, *Phys. Rev. D* **66**, 083514 (2002).
- [41] J. C. Niemeyer, *Phys. Rev. D* **63**, 123502 (2001).
- [42] J. C. Niemeyer and R. Parentani, *Phys. Rev. D* **64**, 101301 (2001).
- [43] R. Parentani, *Int. J. Mod. Phys. A* **17**, 2721 (2002).
- [44] J. Macher and R. Parentani, *Phys. Rev. D* **78**, 043522 (2008).
- [45] A. A. Starobinsky, *JETP Lett.* **73**, 371 (2001).
- [46] W. G. Unruh and R. Schützhold, *Phys. Rev. D* **71**, 024028 (2005).
- [47] J. Macher and R. Parentani, *Phys. Rev. D* **79**, 124008 (2009).
- [48] A. Coutant and R. Parentani, *Phys. Rev. D* **90**, 121501 (2014).
- [49] R. H. Brandenberger and J. Martin, *Int. J. Mod. Phys. A* **17**, 3663 (2002).
- [50] F. Belgiorno, S. L. Cacciatori, M. Clerici, V. Gorini, G. Ortenzi, L. Rizzi, E. Rubino, V. G. Sala, and D. Faccio, *Phys. Rev. Lett.* **105**, 203901 (2010).
- [51] S. Weinfurter, E. W. Tedford, M. C. J. Penrice, W. G. Unruh, and G. A. Lawrence, *Phys. Rev. Lett.* **106**, 021302 (2011).
- [52] J.-C. Jaskula, G. B. Partridge, M. Bonneau, R. Lopes, J. Ruaudel, D. Boiron, and C. I. Westbrook, *Phys. Rev. Lett.* **109**, 220401 (2012).
- [53] C.-L. Hung, V. Gurarie, and C. Chin, *Science* **341**, 1213 (2013).
- [54] J. Steinhauer, *Nat. Phys.* **12**, 959 (2016).
- [55] T. Torres, S. Patrick, A. Coutant, M. Richartz, E. W. Tedford, and S. Weinfurter, *Nat. Phys.* **13**, 833 (2017).
- [56] S. Eckel, A. Kumar, T. Jacobson, I. B. Spielman, and G. K. Campbell, *Phys. Rev. X* **8**, 021021 (2018).
- [57] M. Wittemer, F. Hakelberg, P. Kiefer, J.-P. Schröder, C. Fey, R. Schützhold, U. Warring, and T. Schaetz, *Phys. Rev. Lett.* **123**, 180502 (2019).
- [58] C.-A. Chen, S. Khlebnikov, and C.-L. Hung, *Phys. Rev. Lett.* **127**, 060404 (2021).
- [59] M. C. Braidotti, R. Prizia, C. Maitland, F. Marino, A. Prain, I. Starshynov, N. Westerberg, E. M. Wright, and D. Faccio, *Phys. Rev. Lett.* **128**, 013901 (2022).
- [60] J. Steinhauer, M. Abuzarli, T. Aladjidi, T. Bienaimé, C. Piekarski, W. Liu, E. Giacobino, A. Bramati, and Q. Glorieux, *Nat. Commun.* **13**, 2890 (2022).
- [61] C. Viermann, M. Sparr, N. Liebster, M. Hans, E. Kath, Á. Parra-López, M. Tolosa-Simeón, N. Sánchez-Kuntz, T. Haas, H. Strobel, S. Floerchinger, and M. K. Oberthaler, *Nature (London)* **611**, 260 (2022).
- [62] M. Sparr, E. Kath, N. Liebster, J. Duchene, C. F. Schmidt, M. Tolosa-Simeón, A. Parra-López, S. Floerchinger, H. Strobel, and M. K. Oberthaler, *Phys. Rev. Lett.* **133**, 260201 (2024).

- [63] V. Gondret, R. Dias, C. Lamirault, L. Camier, A. Micheli, C. Leprince, Q. Marolleau, S. Robertson, D. Boiron, and C. I. Westbrook, *C.R. Phys.*, **25**, 1 (2024), online first.
- [64] P. Švančara, P. Smaniotto, L. Solidoro, J. F. MacDonald, S. Patrick, R. Gregory, C. F. Barenghi, and S. Weinfurter, *Nature (London)* **628**, 66 (2024).
- [65] S. Weinfurter, A. White, and M. Visser, *Phys. Rev. D* **76**, 124008 (2007).
- [66] S. Weinfurter, P. Jain, M. Visser, and C. W. Gardiner, *Classical Quantum Gravity* **26**, 065012 (2009).
- [67] J. Macher and R. Parentani, *Phys. Rev. A* **80**, 043601 (2009).
- [68] J. Macher and R. Parentani, *Phys. Rev. D* **79**, 124008 (2009).
- [69] C. Barceló, L. J. Garay, and G. Jannes, *Phys. Rev. D* **79**, 024016 (2009).
- [70] S. Finazzi and R. Parentani, *Phys. Rev. D* **85**, 124027 (2012).
- [71] S. Finazzi and R. Parentani, *J. Phys. Conf. Ser.* **314**, 012030 (2011).
- [72] A. Coutant, R. Parentani, and S. Finazzi, *Phys. Rev. D* **85**, 024021 (2012).
- [73] A. Coutant and S. E. C. Weinfurter, *Phys. Rev. D* **97**, 025006 (2017).
- [74] S.-Y. Chä and U. R. Fischer, *Phys. Rev. Lett.* **118**, 130404 (2017).
- [75] C. C. Holanda Ribeiro and U. R. Fischer, *Phys. Rev. D* **107**, L121502 (2023).
- [76] F. Del Porro, M. Herrero-Valea, S. Liberati, and M. Schneider, *J. High Energy Phys.* **12** (2023) 094.
- [77] F. Del Porro, S. Liberati, and M. Schneider, *C.R. Phys.* **25**, 1 (2025).
- [78] S. M. Chandran and U. R. Fischer, *Eur. Phys. J. C* **85**, 1476 (2025).
- [79] T. Jacobson, S. Liberati, and D. Mattingly, Quantum gravity phenomenology and lorentz violation, in *Particle Physics and the Universe* (Springer-Verlag, Berlin, 2005), pp. 83–98.
- [80] S. Liberati, M. Visser, and S. Weinfurter, *Classical Quantum Gravity* **23**, 3129 (2006).
- [81] S. Weinfurter, S. Liberati, and M. Visser, *J. Phys. A* **39**, 6807 (2006).
- [82] G. Amelino-Camelia, *Living Rev. Relativity* **16**, 5 (2013).
- [83] J. Magueijo and L. Smolin, *Classical Quantum Gravity* **21**, 1725 (2004).
- [84] Y. Ling, *J. Cosmol. Astropart. Phys.* **08** (2007) 017.
- [85] I. P. Lobo, N. Loret, and F. Nettel, *Eur. Phys. J. C* **77**, 451 (2017).
- [86] F. Girelli, S. Liberati, and L. Sindoni, *Phys. Rev. D* **75**, 064015 (2007).
- [87] A. Kempf, G. Mangano, and R. B. Mann, *Phys. Rev. D* **52**, 1108 (1995).
- [88] L. J. GARAY, *Int. J. Mod. Phys. A* **10**, 145 (1995).
- [89] R. Brout, C. Gabriel, M. Lubo, and P. Spindel, *Phys. Rev. D* **59**, 044005 (1999).
- [90] A. Kempf, *Phys. Rev. D* **63**, 083514 (2001).
- [91] A. Kempf and J. C. Niemeyer, *Phys. Rev. D* **64**, 103501 (2001).
- [92] S. Hassan and M. S. Sloth, *Nucl. Phys. B* **674**, 434 (2003).
- [93] M. Tolosa-Simeón, A. Parra-López, N. Sánchez-Kuntz, T. Haas, C. Viernann, M. Sparr, N. Liebster, M. Hans, E. Kath, H. Strobel, M. K. Oberthaler, and S. Floerchinger, *Phys. Rev. A* **106**, 033313 (2022).
- [94] C. F. Schmidt, A. Parra-López, M. Tolosa-Simeón, M. Sparr, E. Kath, N. Liebster, J. Duchene, H. Strobel, M. K. Oberthaler, and S. Floerchinger, *Phys. Rev. D* **110**, 123523 (2024).
- [95] S. Floerchinger and C. Wetterich, *Phys. Rev. A* **77**, 053603 (2008).
- [96] E. M. Lifshitz and L. Pitaevskii, *Statistical Physics Part 2* (Pergamon Press, Oxford, 1980).
- [97] E. Madelung, *Z. Phys.* **40**, 322 (1927).
- [98] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, New York, 2009).
- [99] S. Massar and R. Parentani, *Nucl. Phys.* **513**, 375 (1997).
- [100] Note that the additional factor of a converts the effective physical wavenumber $F(k/a)$ into an effective comoving wavenumber $a(\eta)F(k/a)$ (i.e. the wavenumber in the rest-frame of the cosmological background evolution).
- [101] This choice corresponds to the ground state in the interacting BEC. In particular, the unique existence of that state places the initial superhorizon modes in a vacuum state. Note that in the cosmological target system, the nature of the vacuum on initial superhorizon scales is generally unknown [103].
- [102] Where the pre-factor accounts for momentum space isotropy and that the conformal factor in $(2+1)$ dimensions is given by $\sqrt{a}$.
- [103] V. Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity* (Cambridge University Press, Cambridge, England, 2007).
- [104] We neglect the term involving J_1 in Eq. (63) and approximate $Y_1(x) \approx -2/\pi x$ for $x \ll 1$.
- [105] T. M. Dunster, *SIAM J. Math. Anal.* **21**, 995 (1990).
- [106] T. M. Dunster, [arXiv:2412.12595](https://arxiv.org/abs/2412.12595).
- [107] J. C. Niemeyer, R. Parentani, and D. Campo, *Phys. Rev. D* **66**, 083510 (2002).
- [108] S. Robertson, F. Michel, and R. Parentani, *Phys. Rev. D* **96**, 045012 (2017).
- [109] C. Duval and N. Cherroret, *Phys. Rev. A* **107**, 043305 (2023).
- [110] E. Lifshitz and L. Pitaevskii, *Statistical Physics: Theory of the Condensed State*, Course of Theoretical Physics No. Bd. 9 (Butterworth-Heinemann, London, 2013).
- [111] Note that Eq. (102) corrects a typo in equation (S2) of the supplementary material of [62].
- [112] J. Esteve, J.-B. Trebbia, T. Schumm, A. Aspect, C. I. Westbrook, and I. Bouchoule, *Phys. Rev. Lett.* **96**, 130403 (2006).
- [113] N. Sánchez-Kuntz, A. Parra-López, M. Tolosa-Simeón, T. Haas, and S. Floerchinger, *Phys. Rev. D* **105**, 105020 (2022).
- [114] F. Crameri, Scientific colour maps (2021).
- [115] P. Jain, S. Weinfurter, M. Visser, and C. W. Gardiner, *Phys. Rev. A* **76**, 033616 (2007).
- [116] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1982).
- [117] L. H. Ford, *Rep. Prog. Phys.* **84**, 116901 (2021).

-
- [118] L. D. Landau and E. Lifshitz, in *Statistical Physics (Third Edition)*, 3rd ed., edited by L. D. Landau and E. Lifshitz (Butterworth-Heinemann, Oxford, 1980), pp. 333–400.
- [119] R. M. Ziff, G. E. Uhlenbeck, and M. Kac, *Phys. Rep.* **32**, 169 (1977).
- [120] M. Naraschewski and R. J. Glauber, *Phys. Rev. A* **59**, 4595 (1999).
- [121] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity* (Oxford University Press, Oxford, 2016).