

General actions of extended objects and volume-preserving diffeomorphism

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We consider actions that are general functions of the world sheet/world volume metric and the induced metric for extended objects embedded in spacetime as Riemannian manifolds, area-metric manifolds, and volume-metric manifolds. For strings on a Riemannian spacetime, we consider general actions respecting volume-preserving diffeomorphisms (VPD), general diffeomorphisms, and diffeomorphisms with Weyl symmetry, respectively. Well-known Schild, Nambu-Goto, and Polyakov actions are included as special cases. We reach two main conclusions: (1) When actions are functions of both the worldsheet metric and induced metrics, all nontrivial self-consistent actions are classically equivalent. (2) As a physical constraint on the classical action, VPD symmetry is as strong as the full diffeomorphism symmetry. The discussion is then extended to strings in spacetime manifolds equipped with the area or volume metrics. Then, we further consider higher-dimensional extended objects in spacetime defined with area or volume metrics, and show the equivalence between the generalized Schild actions and the generalized Nambu-Goto action. We prove a general theorem on VPD that explains this equivalence. Incidentally, while only the area metric is needed to define the string world sheet action, we show that the Polyakov action with an area-metric perturbation cannot describe critical strings without other interaction terms.

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I. INTRODUCTION

At the classical level, the world sheet theory of a bosonic string can be equivalently described by the Nambu-Goto action, the Schild action, and the Polyakov action as (with the overall coefficients omitted)

$$S_{\text{NG}} \equiv \int d^2\xi (\sqrt{\gamma}), \quad (1.1)$$

$$S_S \equiv \int d^2\xi \left(\frac{1}{4} \epsilon^{ac} \epsilon^{bd} \gamma_{ab} \gamma_{cd} \right) = \int d^2\xi \left(\frac{1}{2} \gamma \right), \quad (1.2)$$

$$S_P \equiv \int d^2\xi \left(\frac{1}{2} \sqrt{h} h^{ab} \gamma_{ab} \right), \quad (1.3)$$

where the ξ^a ($a, b, c, d = 0, 1$) are the world sheet coordinates, and ϵ^{ab} is the two-dimensional Levi-Civita symbol. The induced metric γ_{ab} is defined by

$$\gamma_{ab} \equiv g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu, \quad (1.4)$$

where $g_{\mu\nu}(X)$ is a generic spacetime metric, whose (absolute) determinant is denoted by γ , and h_{ab} is the world sheet metric with h being (the absolute value of) its determinant.

The Nambu-Goto action (1.1) is invariant under the world sheet diffeomorphisms. The Schild action is invariant only under the world sheet volume-preserving diffeomorphisms (VPDs). The Polyakov action enjoys both the full world sheet diffeomorphism and the Weyl symmetry that scales the world sheet metric h_{ab} . The conformal gauge allows us to define the quantum theory of strings. The classical equivalence of all three actions can be easily shown [1,2]. This raises a natural question: *are there other actions classically equivalent to these three?*

In this paper, we study the most general forms of the string world sheet action for each choice of the world sheet gauge symmetry mentioned above. We discuss conditions under which these actions are classically equivalent.

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The above actions are related to the area of the string world sheet induced from the background (pseudo-)Riemannian metric. It is natural to generalize the notion of a (pseudo-)Riemannian metric to that of an area metric. We shall also study general forms of the string world sheet actions with this extension and examine their equivalence.

Let us briefly recall the notion of an area metric. As the relativistic action of a point particle is the mass times the worldline length, the Nambu-Goto action of a string is the tension times the world sheet area. On a (pseudo-)Riemannian manifold, we define length using a metric $g_{\mu\nu}$ via the line element

$$ds^2 = g_{\mu\nu} dX^\mu \otimes_{\text{sym}} dX^\nu, \quad (1.5)$$

where dX^μ is a one-form on the manifold and $\otimes_{\text{sym}}$ is the symmetrized tensor product. The area can then be defined as a notion induced from that of the length.

Mathematically, one can define the area element dA directly as an inner product on the space of two-forms [3–9] as

$$dA^2 = G_{\mu\nu\rho\lambda} (dX^\mu \wedge dX^\nu) \otimes_{\text{sym}} (dX^\rho \wedge dX^\lambda), \quad (1.6)$$

where $G_{\mu\nu\rho\lambda}$ is called the area metric. An area metric may or may not induce the notion of length [7,9]. However, one can define the string world sheet area even in the absence of the notion of length.

In this work, we also take the first step toward the quantum theory of strings in an area-metric background. We consider the ordinary Polyakov action perturbed by an area-metric deformation. Unfortunately, we find that such perturbations do not define critical strings.

Beyond discussion of strings, we shall also consider generic extended objects with d -dimensional world volume as generalizations of strings, in a background geometry defined with a volume metric, as a generalization of the area metric.

In the discussions on the classical equivalence among different actions, we notice that VPDs play a special role. We found an interesting general theorem that is very useful in these discussions.

The plan of this paper is as follows: in Secs. II and III, we consider the most general world sheet actions that respect the following three types of symmetries: (1) VPDs, (2) general diffeomorphisms (Diffs), and (3) diffeomorphism with Weyl symmetry (Diff-Weyl). We show that they are all classically equivalent. They can be viewed, respectively, as generalizations of (1) the Schild action (1.2), (2) the Nambu-Goto action (1.1), and (3) the Polyakov action (1.3). In Sec. IV, we replace the Riemannian metric by the area metric in the Nambu-Goto, the Schild, and the Polyakov actions and prove their classical equivalence. In Sec. V, we consider the most general actions for

these three types of symmetries in area-metric backgrounds. In Sec. VI, we study the quantum theory of the Polyakov action perturbed by the area metric, which was proposed in Ref. [9]. We checked that this perturbation does not define a critical string. It is therefore not clear whether string theories can be consistently defined in area-metric backgrounds. In Sec. VII, we provide some comments on a similar setup for higher-dimensional objects. Specifically, we consider actions that are functions of the world volume metric and the induced metric on both Riemannian and volume-metric manifolds. At the end of the section, we provide a theorem on VPD that can be applied to prove the classical equivalence between the Schild(-like) action and the Nambu-Goto action in arbitrary dimensions on both Riemannian and volume-metric manifolds. The conclusion is given in Sec. VIII.

II. WORLD SHEET ACTIONS ON RIEMANNIAN MANIFOLDS

In this section, we aim to find the most general string world sheet actions that depend only on the induced metric γ_{ab} and the world sheet metric h_{ab} . That is, we assume that the world sheet fields X^μ only enter the action through γ_{ab} (1.4); the only other field that can appear in the action is the world sheet metric h_{ab} , which plays the role of an auxiliary field.¹ This assumption already covers all three types of string actions (1.1)–(1.3).

In Sec. II A, we first discuss the basic idea to construct the general form of Lagrangians which are functions of γ_{ab} and h_{ab} without their derivatives. Then, we consider world sheet Lagrangians that respect different types of symmetries: (1) VPDs, (2) Diffs, and (3) Diff-Weyl.

After integrating out the auxiliary world sheet metric h_{ab} , we find that the latter two classes of Lagrangians reduce to the Nambu-Goto Lagrangian, while the Lagrangian with the VPD symmetry reduces to Lagrangians of the form

$$\mathcal{L}_{\text{GS}} = F(\sqrt{\gamma}), \quad (2.1)$$

where F is an arbitrary function, and we call it the generalized Schild Lagrangian. The classical equivalence between the generalized Schild actions and the Nambu-Goto action will be shown in Sec. III.

A. General world sheet actions

Denoting the matrix form of γ_{ab} as Γ and the matrix form of h_{ab} as H , invariance under VPD demands that the world sheet Lagrangian is a function of the following invariants: $\gamma = \det \Gamma$, $h = \det H$, and $a_n = \text{tr}((\Gamma H^{-1})^n)$ ($n = \pm 1, \pm 2, \dots$). Assuming that the Lagrangian does not contain derivatives of γ_{ab} and h_{ab} , a Lagrangian

¹Fermionic fields are ignored in this work.

invariant under the VPD symmetry is in general of the form $f(\gamma, h, a_1, \dots)$.

For invariants constructed from ΓH^{-1} , since ΓH^{-1} is a 2×2 matrix, it is characterized by two independent invariants (e.g., trace and determinant). All other invariants a_n can be identified with functions of them. For example, take

$$a_1 = \text{tr}(\Gamma H^{-1}) = \gamma_{ab} h^{ba}, \quad (2.2)$$

$$a_2 = \text{tr}((\Gamma H^{-1})^2) = \gamma_{ab} h^{bc} \gamma_{cd} h^{da}, \quad (2.3)$$

as the only independent variables for all invariants constructed from ΓH^{-1} . We have

$$a_1 = \lambda_1 + \lambda_2, \quad (2.4)$$

$$a_2 = (\lambda_1)^2 + (\lambda_2)^2, \quad (2.5)$$

where λ_1 and λ_2 are the two eigenvalues of ΓH^{-1} . Then, $a_{-1} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$ can be written in terms of a_1 and a_2 as $\frac{2a_1}{a_2 - (a_1)^2}$.

Once we have the characteristics of ΓH^{-1} , the determinants γ and h are no longer independent. Indeed, we have

$$\gamma h^{-1} = \lambda_1 \lambda_2 = \frac{a_1^2 - a_2}{2}. \quad (2.6)$$

Therefore, any VPD-invariant Lagrangian can be written as a function of three variables: γ , a_1 , and a_2 .

The Lagrangians with VPD, Diff, and Diff-Weyl symmetries are given, respectively, by

$$\mathcal{L}_{\text{VPD}} = f(\sqrt{\gamma}, a_1, a_2), \quad (2.7)$$

$$\mathcal{L}_{\text{Diff}} = \sqrt{\gamma} f(a_1, a_2), \quad (2.8)$$

$$\mathcal{L}_{\text{Diff-Weyl}} = \sqrt{\gamma} f\left(\frac{a_2}{a_1^2}\right), \quad (2.9)$$

where the functions f are generic functions of their arguments. For example, the Schild action (1.2) is a special case of $\mathcal{L}_{\text{VPD}}$, the Nambu-Goto action (1.1) is a special case of $\mathcal{L}_{\text{Diff}}$, and the Polyakov action (1.3) is a special case of $\mathcal{L}_{\text{Diff-Weyl}}$. The generalized Polyakov action defined by [9]

$$\mathcal{L}_{\text{GP}} = F\left(\sqrt{h} \frac{1}{2} h^{ab} \gamma_{ab}\right) = F\left(\sqrt{h} \frac{a}{2}\right) \quad (2.10)$$

is a subclass of the Diff-Weyl-invariant actions (2.9).

In this and the next sections, we will prove the classical equivalence of these Lagrangians with the Nambu-Goto theory. The strategy is to derive the equations of motion for the auxiliary field h_{ab} in terms of γ_{ab} and then substitute its solution back into the Lagrangian.

Using the definitions of a_1 (2.2) and a_2 (2.3), we find

$$\frac{\partial a_1}{\partial h^{ij}} = \gamma_{ji}, \quad \frac{\partial a_2}{\partial h^{ij}} = 2\gamma_{ja} h^{ab} \gamma_{bi}. \quad (2.11)$$

For any Lagrangian $\mathcal{L}$ among Eqs. (2.7)–(2.9), the equation of motion for h^{ij} is

$$\frac{\partial \mathcal{L}}{\partial h^{ij}} = \frac{\partial \mathcal{L}}{\partial a_1} \frac{\partial a_1}{\partial h^{ij}} + \frac{\partial \mathcal{L}}{\partial a_2} \frac{\partial a_2}{\partial h^{ij}} = \frac{\partial \mathcal{L}}{\partial a_1} \gamma_{ji} + 2 \frac{\partial \mathcal{L}}{\partial a_2} \gamma_{ja} h^{ab} \gamma_{bi} = 0. \quad (2.12)$$

Multiplying both sides by $h_{kc} \gamma^{cj}$ yields

$$\frac{\partial \mathcal{L}}{\partial a_1} h_{ki} + 2 \frac{\partial \mathcal{L}}{\partial a_2} \gamma_{ki} = 0. \quad (2.13)$$

After dividing $\partial \mathcal{L} / \partial a_1$ on both sides, we find that the world sheet metric h_{ab} and the induced metric γ_{ab} differ only by an overall factor ϕ , as follows:

$$h_{ab} = \phi \gamma_{ab}, \quad (2.14)$$

with the proportionality factor given by

$$\phi = -2 \frac{\partial \mathcal{L}}{\partial a_2} \left(\frac{\partial \mathcal{L}}{\partial a_1} \right)^{-1} \Big|_{a_1=2\phi^{-1}, a_2=2\phi^{-2}}. \quad (2.15)$$

Here, the values of a_1 and a_2 are determined in terms of ϕ from Eqs. (2.2), (2.3), and (2.14) as

$$a_1 = 2\phi^{-1}, \quad a_2 = 2\phi^{-2}. \quad (2.16)$$

Thus, ϕ can be, in principle, solved from Eq. (2.15), and h_{ab} is then determined by Eq. (2.14) in terms of γ_{ab} . In the following, we proceed to discuss each case in Eqs. (2.7)–(2.9) with the assumption that they can be solved with $\phi \neq 0$. In Sec. II B, we consider two exceptional cases where Eq. (2.15) has no solution for ϕ , as well as an example where a solution of ϕ exists.

1. VPD-invariant actions

In the case of a VPD-invariant Lagrangian $\mathcal{L}_{\text{VPD}}$ (2.7), Eq. (2.15) is equivalent to

$$\phi = -2 \frac{\partial f(\sqrt{\gamma}, a_1, a_2)}{\partial a_2} \left(\frac{\partial f(\sqrt{\gamma}, a_1, a_2)}{\partial a_1} \right)^{-1} \Big|_{a_1=2\phi^{-1}, a_2=2\phi^{-2}}, \quad (2.17)$$

which may determine ϕ as a function of $\sqrt{\gamma}$. Substituting it back into Eq. (2.7), the Lagrangian becomes

$$\mathcal{L}_{\text{VPD}} = f\left(\sqrt{\gamma}, 2\phi^{-1}(\sqrt{\gamma}), 2\phi^{-2}(\sqrt{\gamma})\right), \quad (2.18)$$

which is in the form of the generalized Schild Lagrangian $\mathcal{L}_{\text{GS}}$ (2.1). Therefore, generic VPD-invariant Lagrangians are equivalent to generalized Schild Lagrangians.

2. Diffeomorphism-invariant actions

As for the diffeomorphism-invariant Lagrangian $\mathcal{L}_{\text{Diff}}$ (2.8), Eq. (2.15) states that

$$\phi = -2 \frac{\partial f(a_1, a_2)}{\partial a_2} \left(\frac{\partial f(a_1, a_2)}{\partial a_1} \right)^{-1} \Big|_{a_1=2\phi^{-1}, a_2=2\phi^{-2}}. \quad (2.19)$$

Unlike Eq. (2.17) for the case of VPD symmetry, this equation fixes ϕ to a constant in the absence of $\sqrt{\gamma}$. The resulting Lagrangian is therefore just the Nambu-Goto Lagrangian

$$\mathcal{L}_{\text{NG}} = c\sqrt{\gamma}, \quad (2.20)$$

with the constant $c \equiv f(a_1(\phi), a_2(\phi))$ and ϕ given by the constant solution of Eq. (2.19).

3. Diffeomorphism-Weyl-invariant actions

For Lagrangians $\mathcal{L}_{\text{Diff-Weyl}}$ (2.9) invariant under Diff-Weyl, Eq. (2.15) is trivially satisfied as follows:

$$\begin{aligned} \phi &= -2 \frac{\partial f(\frac{a_2}{a_1})}{\partial a_2} \left(\frac{\partial f(\frac{a_2}{a_1})}{\partial a_1} \right)^{-1} \Big|_{a_1=2\phi^{-1}, a_2=2\phi^{-2}} \\ &= \frac{a_1}{a_2} \Big|_{a_1=2\phi^{-1}, a_2=2\phi^{-2}} = \phi. \end{aligned} \quad (2.21)$$

That is, ϕ remains arbitrary. This situation is the same as the Polyakov action, where the equation of motion for h_{ab} only demands conformal equivalence between h_{ab} and γ_{ab} without fixing the proportionality factor ϕ .

Although ϕ remains unfixed, the argument of the function f in the Lagrangian is $\frac{a_2}{a_1} = \frac{1}{2}$ (after substituting $a_1 = 2\phi^{-1}$ and $a_2 = 2\phi^{-2}$). Hence, the resulting Lagrangian is

$$\mathcal{L}_{\text{Diff-Weyl}} = f\left(\frac{1}{2}\right)\sqrt{\gamma}, \quad (2.22)$$

which is again the Nambu-Goto Lagrangian. Hence, both Diff-invariant Lagrangians $\mathcal{L}_{\text{Diff}}$ and Diff-Weyl-invariant Lagrangians $\mathcal{L}_{\text{Diff-Weyl}}$ are equivalent to the Nambu-Goto action. In Sec. III, we discuss the equivalence between Eqs. (2.1) and (2.20).

B. Examples

In this subsection, we explicitly solve the proportionality factor ϕ from Eq. (2.15) for concrete examples of Lagrangians and illustrate two possibilities: (i) Eq. (2.15)

admits a real, nonzero solution for ϕ , and (ii) Eq. (2.15) is inconsistent and does not admit any solution for ϕ .

1. Polyakov + cosmological constant term

As the first example, we add a cosmological constant term $\lambda\sqrt{h}$ to the Polyakov Lagrangian, as follows:

$$\mathcal{L} = \frac{1}{2} \sqrt{h} h^{ab} \gamma_{ab} + \lambda \sqrt{h}. \quad (2.23)$$

Using Eqs. (2.2) and (2.6), we rewrite the Lagrangian in the form of Eq. (2.8) with the function f given by

$$f(a_1, a_2) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{a_1^2 - a_2}} \left(a_1 + \frac{\lambda}{2} \right), \quad (2.24)$$

and Eq. (2.19) becomes

$$\phi = \frac{\phi^{-1} + 2\lambda}{\phi^{-2} - 2\lambda\phi^{-1}}. \quad (2.25)$$

This equation is equivalent to $\phi^{-1} - 2\lambda = \phi^{-1} + 2\lambda$, which is self-inconsistent unless $\lambda = 0$. Therefore, the cosmological constant deformation of the Polyakov action does not define a classical system with consistent solutions to its equations of motion.

2. Polyakov + (Polyakov)²

Next, we add a term quadratic in the Polyakov density, as follows:

$$\mathcal{L} = \frac{1}{2} \sqrt{h} h^{ab} \gamma_{ab} + \frac{\mu}{4} \sqrt{h} (h^{ab} \gamma_{ab})^2. \quad (2.26)$$

This Lagrangian can be rewritten in the form of Eq. (2.8), and the explicit form of the function f is

$$f(a_1, a_2) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{a_1^2 - a_2}} \left(a_1 + \frac{\mu}{2} a_1^2 \right). \quad (2.27)$$

Equation (2.19) is then equivalent to

$$\phi^{-1} \mu = 0. \quad (2.28)$$

Assuming that $\mu \neq 0$, this implies that $\phi^{-1} = 0$. Since this implies $h^{ab} \gamma_{ab} = 0$, the Lagrangian becomes trivial for γ_{ab} . At the same time, for finite h_{ab} , Eq. (2.14) implies $\gamma_{ab} = 0$. This suggests $X^\mu = \text{const}$ for all μ , corresponding to a pointlike string. Hence, the Polyakov squared term by itself is also not considered a consistent modification to the Polyakov Lagrangian for a string.

3. Polyakov + cosmological constant term + (Polyakov)²

Finally, we introduce the previous two deformations simultaneously:

$$\mathcal{L} = \frac{1}{2} \sqrt{h} h^{ab} \gamma_{ab} + \lambda \sqrt{h} + \frac{\mu}{4} \sqrt{h} (h^{ab} \gamma_{ab})^2. \quad (2.29)$$

Rewriting the Lagrangian with f in form of Eq. (2.8), we have

$$f(a_1, a_2) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{a_1^2 - a_2}} \left(\frac{\lambda}{2} + a_1 + \frac{\mu}{2} a_1^2 \right). \quad (2.30)$$

The constraint on ϕ (2.19) becomes

$$-\frac{1}{2} \phi^2 \lambda + \mu = 0. \quad (2.31)$$

Thus, for $\lambda \neq 0$ and $\mu/\lambda > 0$ (so that $\phi^2 > 0$), Eq. (2.15) admits real, nonzero solutions for ϕ ($\phi = \pm \sqrt{\frac{2\mu}{\lambda}}$), and the relation $h_{ab} = \phi \gamma_{ab}$ is well defined. In this case, the deformation is consistent.

III. EQUIVALENCES OF (GENERALIZED) SCHILD AND NAMBU-GOTO ACTIONS

In Sec. II, we classified general world sheet actions built from the world sheet metric h_{ab} and the induced metric γ_{ab} , leading to the three symmetry classes (2.7)–(2.9), which are, respectively, invariant under the symmetries of VPD, Diff, and Diff-Weyl. In Secs. II A 2 and II A 3, we have shown that diffeomorphism is sufficient to imply the equivalence with the Nambu-Goto action, regardless of whether there is the Weyl symmetry. But actions with VPD symmetry are only shown to be equivalent to the generalized Schild action in Sec. II A 1. In this section, we take the next step and show that the generalized Schild action is equivalent to the Nambu-Goto action, so that VPD-invariant actions are also equivalent to the Nambu-Goto action. That is, VPD invariance already suffices to ensure equivalence with the Nambu-Goto action.

In Sec. III A, we establish the classical equivalence between the Schild action and the Nambu-Goto action for an arbitrary spacetime metric $g_{\mu\nu}$. We also determine the overall coefficient of the Schild action by demanding its stress-energy tensor to match the string tension specified by the Nambu-Goto action.

In Sec. III B, we show that the generalized Schild action (2.1) is equivalent to the Schild action, and thus it is also equivalent to the Nambu-Goto action.

A. Equivalence of Schild and Nambu-Goto actions

We now prove the classical equivalence between the Schild and the Nambu-Goto theories with an arbitrary spacetime metric $g_{\mu\nu}$. This is achieved by showing that their

equations of motion are equivalent when we take a special gauge ($\sqrt{\gamma} = \text{const}$) for the Nambu-Goto theory. Then, we match their stress-energy tensors to see how the tension is encoded in the Schild action.

1. Classical equivalence

We start from the Schild and the Nambu-Goto Lagrangians:

$$\mathcal{L}_S = a\gamma, \quad (3.1)$$

$$\mathcal{L}_{\text{NG}} = T\sqrt{\gamma}, \quad (3.2)$$

where a is a real constant, T is the string tension and γ is (the absolute value of) the determinant of γ_{ab} defined in Eq. (1.4). Their equations of motion with respect to X^μ are

$$0 = \epsilon^{ab} \partial_a (X^\nu \partial_b \sigma_{\mu\nu}), \quad (3.3)$$

$$0 = \epsilon^{ab} \partial_a \left(\frac{1}{\sqrt{\gamma}} X^\nu \partial_b \sigma_{\mu\nu} \right), \quad (3.4)$$

respectively, where

$$\sigma^{\mu\nu} = \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu. \quad (3.5)$$

Here, we have used

$$\gamma = g_{\mu\rho} g_{\nu\sigma} \sigma^{\mu\nu} \sigma^{\rho\sigma} \quad (3.6)$$

and defined

$$\sigma_{\mu\nu} = g_{\mu\rho} g_{\nu\lambda} \sigma^{\rho\lambda}. \quad (3.7)$$

We shall denote $\gamma = \sigma^2$, as a shorthand for Eq. (3.6).

These two equations of motion appear to be different, since ∂_a in Eq. (3.4) acts on $\frac{1}{\sqrt{\gamma}}$. However, in the Nambu-Goto theory, since $\sqrt{\gamma}$ is a scalar density on the world sheet, we can choose $\sqrt{\gamma}$ to be a world sheet constant as a gauge-fixing condition for the diffeomorphism symmetry. The residual gauge symmetry is VPD.

Hence, the equation of motion (3.4) of the Nambu-Goto theory together with the gauge-fixing condition $\sqrt{\gamma} = \text{const}$ implies the equation of motion (3.3) of the Schild theory. To prove that the Schild theory is equivalent to the Nambu-Goto theory, we show that the gauge-fixing condition is implied by its equation of motion (3.3).

We multiply $\partial_c X^\mu$ to Eq. (3.3) and get

$$\begin{aligned} 0 &= \partial_c X^\mu \epsilon^{ab} \partial_a (X^\nu \partial_b \sigma_{\mu\nu}) \\ &= \frac{1}{4} \epsilon^{ab} \epsilon_{ca} \sigma^{\mu\nu} \partial_b \sigma_{\mu\nu} \\ &= -\frac{1}{8} \partial_c \gamma. \end{aligned} \quad (3.8)$$

In the second equality, we have used the fact that the indices μ, ν in $\partial_c X^\mu \partial_a X^\nu$ is (totally) antisymmetrized due to the antisymmetrized tensor $\sigma_{\mu\nu}$, together with the identity

$$\partial_c X^\mu \partial_a X^\nu - \partial_c X^\nu \partial_a X^\mu = \frac{1}{2} \epsilon_{ca} \sigma^{\mu\nu}. \quad (3.9)$$

In the Schild theory, the relation $\sqrt{\gamma} = \text{const}$ is not a gauge-fixing condition, but rather it is dynamically fixed (by the equation of motion) everywhere on the same connected world sheet once its initial condition is specified somewhere on the same world sheet. Thus, on any connected world sheet, there is no physical difference between the Schild theory and the Nambu-Goto theory.

2. String tension in Schild action

Next, we discuss how to define tension from the Schild action by matching the stress-energy tensors of the Schild action and the Nambu-Goto action. To do so, we rewrite the actions so that they explicitly depend on $g_{\mu\nu}$, as follows:

$$S_{\text{Schild}} = a \int d^2 \xi (g^{\mu\rho}(x) g^{\nu\sigma}(x) \sigma_{\mu\nu} \sigma_{\rho\sigma}) \quad (3.10)$$

$$S_{\text{NG}} = T \int d^2 \xi \sqrt{g^{\mu\rho}(x) g^{\nu\sigma}(x) \sigma_{\mu\nu} \sigma_{\rho\sigma}}. \quad (3.11)$$

Their stress-energy tensors are

$$\begin{aligned} T_{\alpha\beta}^{(\text{Schild})} &= \frac{\delta S_{\text{Schild}}}{\delta g^{\alpha\beta}(y)} \\ &= 2a \int d^2 \xi (\delta^{(D)}(x-y) g^{\nu\sigma} \sigma_{\alpha\nu} \sigma_{\beta\sigma}) \\ T_{\alpha\beta}^{(\text{NG})} &= \frac{\delta S_{\text{NG}}}{\delta g^{\alpha\beta}(y)} \\ &= T \int d^2 \xi \left(\frac{\delta^{(D)}(x-y) g^{\nu\sigma} \sigma_{\alpha\nu} \sigma_{\beta\sigma}}{\sqrt{\sigma^2}} \right), \end{aligned} \quad (3.12)$$

where $\delta^{(D)}(x-y)$ is the δ function in D -dimensional spacetime. Equating the integrands in Eq. (3.12) yields

$$T = 2a \sqrt{\sigma^2}. \quad (3.13)$$

Since both a and T are constants, this relation is only meaningful when σ^2 is constant, which is true on each connected world sheet in the Schild theory as shown in Eq. (3.8). Then, the tension is given by the Schild parameter a and the constant world sheet measure $\sqrt{\sigma^2}$ through Eq. (3.13).

B. Equivalence between generalized Schild and Nambu-Goto actions

We have demonstrated in Sec. III A that the Schild action is equivalent to the Nambu-Goto action. Now we show that the generalized Schild action (2.1) is equivalent to the Schild action, and thus it is equivalent to the Nambu-Goto action.

Starting from the generalized Schild action (2.1), we have the equation of motion for X^μ , as follows:

$$\begin{aligned} 0 &= \epsilon^{ab} \partial_b X^\nu \partial_a (F'(\sigma^2) \sigma_{\mu\nu}) \\ &= \epsilon^{ab} \partial_b X^\nu (F''(\sigma^2) (\partial_a \sigma^2) \sigma_{\mu\nu} + F'(\sigma^2) \partial_a \sigma_{\mu\nu}). \end{aligned} \quad (3.14)$$

Multiplying it by $\partial_c X^\mu$ and using the identity (3.9), we obtain

$$\partial_c \sigma^2 (F'(\sigma^2) + 2\sigma^2 F''(\sigma^2)) = 0. \quad (3.15)$$

For generic F , the term $F'(\sigma^2) + 2\sigma^2 F''(\sigma^2)$ does not vanish identically. The equation above implies that

$$\partial_c \sigma^2 = 0 \quad (3.16)$$

identically on the world sheet (see Appendix for a more detailed proof.) Hence, σ^2 is a world sheet constant. For nonzero $F'(\sigma^2)$, the $F'(\sigma^2)$ in Eq. (3.14) can be moved outside the derivative. It is then straightforward to see that the equation of motion (3.14) of the generalized Schild action matches the equation of motion (3.3) for the Schild action.

Since we have shown that the Schild theory is equivalent to the Nambu-Goto theory in Sec. III A, the generalized Schild action (2.1) is also equivalent to the Nambu-Goto action.² Together with the result in Sec. II A 1 that VPD-invariant actions are equivalent to the generalized Schild actions, we have now completed the proof that all the actions (2.7)–(2.9) are equivalent to the Nambu-Goto action (3.2).

IV. THE NAMBU-GOTO AND SCHILD ACTIONS ON AREA-METRIC MANIFOLDS

In previous sections, we established the classical equivalence among the Schild, the generalized Schild, and the Nambu-Goto actions on Riemannian target manifolds. In this section, we extend this analysis to target manifolds equipped with an area metric $G_{\mu\nu\rho\lambda}$.

Note that, as we will show in Sec. V, any nontrivial dependence on the auxiliary area world sheet degrees of freedom is forbidden by Weyl symmetry. The only possible Polyakov-like action (those with both diffeomorphisms and

²There are cases where the term $F'(\sigma^2) + 2\sigma^2 F''(\sigma^2)$ vanishes identically. However, this is only possible if $F(\sigma^2) = A\sqrt{\sigma^2} + B$, which is nothing but the Nambu-Goto action.

Weyl symmetry) is the one discussed in Sec. VI. Hence, we only include the Nambu-Goto and Schild actions in this section.

An area metric, as a generalization of the ordinary metric, is defined as Eq. (1.6):

$$dA^2 = G_{\mu\nu\rho\lambda} da^{\mu\nu} da^{\rho\lambda}, \quad (4.1)$$

where dA is the infinitesimal area element, $G_{\mu\nu\rho\sigma}$ is the area metric, and $da^{\mu\nu}$ is a two-form on the manifold.

From its definition, we see that $G_{\mu\nu\rho\sigma}$ must have the following symmetries:

$$G_{\mu\nu\rho\lambda} = -G_{\nu\mu\rho\lambda} = -G_{\mu\nu\lambda\rho} = G_{\rho\lambda\mu\nu}. \quad (4.2)$$

In a D -dimensional spacetime, it has $\frac{D(D-1)(D^2-D+2)}{8}$ degrees of freedom. By comparison, an ordinary metric has $\frac{D(D+1)}{2}$ degrees of freedom. An area metric contains more degrees of freedom for $D > 3$.

Since this class of objects has more degrees of freedom than ordinary metrics when $D > 3$, an area metric in target space ($D \geq 4$) in general cannot be captured by a Riemannian metric. In the following, we demonstrate how the Nambu-Goto and Schild actions can be generalized to those defined for area-metric backgrounds; we also discuss a Polyakov-like auxiliary-field formulation. We will also prove their classical equivalence.

In Sec. IV A, we introduce the area-metric analog of the Nambu-Goto action and derive its equation of motion. On area-metric manifolds, we consider the Schild action with VPD symmetry in Sec. IV B and a diffeomorphism-invariant action with an auxiliary world sheet field as the counterpart of the Polyakov action in Sec. IV C. We will prove their equivalence to the Nambu-Goto action on area-metric manifolds.

In Sec. V, we will discuss more general actions on area-metric manifolds.

A. Nambu-Goto action on area-metric manifolds

Due to the geometrical meaning of the Nambu-Goto action, it is straightforward to define the analog of the Nambu-Goto action for an area-metric background to be given by the world sheet area as

$$S_{\text{NG}}^A = \int d^2\xi \sqrt{\frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}}, \quad (4.3)$$

where $G_{\mu\nu\rho\lambda}$ is the area metric, $\sigma^{\mu\nu} = \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu$ (3.5), and the Greek (Roman) letters are spacetime (world sheet) indices. This action reduces to the usual Nambu-Goto action on Riemannian spacetime when the area metric agrees with

$$G_{\mu\nu\rho\lambda} = g_{\mu\rho} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\rho} \quad (4.4)$$

for a given Riemannian metric $g_{\mu\nu}$. We will refer to Eq. (4.3) as the Nambu-Goto action for the area metric G .

B. The Schild action on area-metric manifolds

Since the Schild Lagrangian (1.2) can be obtained by squaring the Nambu-Goto Lagrangian (1.1), we define the area-metric generalization of the Schild action as

$$S_S^A = \int d^2\xi \frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}. \quad (4.5)$$

It respects the VPD symmetry. We now show that S_S^A is classically equivalent to the area-metric Nambu-Goto action S_{NG}^A in Eq. (4.3), for an arbitrary background area metric $G_{\mu\nu\rho\lambda}$. The proof follows essentially the same line as that for Riemannian manifolds.

The equation of motion for X^α obtained from S_S^A (4.5) is

$$0 = \frac{1}{4} \partial_\alpha G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda} - \partial_\alpha (G_{\alpha\nu\rho\lambda} \epsilon^{ab} \partial_b X^\nu \sigma^{\rho\lambda}). \quad (4.6)$$

After multiplying $\partial_c X^\alpha$ on both sides of Eq. (4.6) and using Eq. (3.9), we obtain

$$0 = \partial_c (\sigma_A^2), \quad (4.7)$$

where

$$\sigma_A^2 \equiv \frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}. \quad (4.8)$$

We get from Eq. (4.7) that

$$\sigma_A^2 = \text{constant}. \quad (4.9)$$

This is in complete analogy with the original Schild action (1.2), whose equation of motion implies that $\sigma^2 = g_{\mu\rho} g_{\nu\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}$ is a world sheet constant.

For the case of S_{NG}^A (4.3), the equation of motion for X^α is given by

$$0 = \frac{1}{2\sqrt{\sigma_A^2}} \partial_\alpha G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda} - \partial_\alpha \left(\frac{1}{\sqrt{\sigma_A^2}} G_{\alpha\nu\rho\lambda} \epsilon^{ab} \partial_b X^\nu \sigma^{\rho\lambda} \right). \quad (4.10)$$

Since $\sqrt{\sigma_A^2}$ is a scalar density on the world sheet, we can fix it to a constant via part of the diffeomorphisms, reducing the symmetry to VPD. This allows us to factor out $1/\sqrt{\sigma_A^2}$ from ∂_α in the second term. After multiplying Eq. (4.10) by $\sqrt{\sigma_A^2}$, the equation becomes

$$0 = \frac{1}{2} \partial_\alpha G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda} - \partial_\alpha (G_{\alpha\nu\rho\lambda} \epsilon^{ab} \partial_b X^\nu \sigma^{\rho\lambda}), \quad (4.11)$$

which is the same as Eq. (4.6), the equation of motion for X^μ from the area Schild action S_S^A . Thus, we have proven the classical equivalence for any area metric between the Schild action and the Nambu-Goto action.

C. Action with auxiliary field on area-metric manifolds

We intend to construct an action analogous to the Polyakov action, which is an action involving not only the induced metric γ_{ab} but also an auxiliary world sheet metric h_{ab} . While the spacetime metric is now promoted from $g_{\mu\nu}$ to $G_{\mu\nu\rho\lambda}$, the auxiliary world sheet metric h_{ab} should also be promoted to an area metric $\mathcal{H}_{abcd}$. However, in two dimensions, $\mathcal{H}_{abcd}$ has only one independent component $\mathcal{H}_{0101}$, which we simply denote as $\mathcal{H}$. Using this auxiliary field $\mathcal{H}$, we can remove the square root in Eq. (4.3) with the full diffeomorphism preserved. It turns out that it is impossible to enhance the diffeomorphism to have Weyl symmetry or scale symmetry.

Analogous to the einbein action for particles,

$$S_e = \frac{1}{2} \int d\tau (e^{-1} \dot{X}^2 + em^2), \quad (4.12)$$

we consider

$$S_e^A = \frac{1}{2} \int d^2\xi \sqrt{\mathcal{H}} \left[\mathcal{H}^{-1} \left(\frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda} \right) + T^2 \right]. \quad (4.13)$$

Here, we regard $\sqrt{\mathcal{H}}$ as the analog of the einbein e , $\frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}$ as that of $\dot{X}^2$ and the tension T as that of the mass m . Since $\sqrt{\mathcal{H}}$ transforms as a scalar density under world sheet diffeomorphism, this action is invariant under world sheet diffeomorphism.

To demonstrate its equivalence to the Nambu-Goto action (4.3) for the area metric $G_{\mu\nu\rho\lambda}$, we integrate out the world sheet area metric $\mathcal{H}$. By varying the action (4.13) with respect to $\sqrt{\mathcal{H}}$, we get the equation of motion for $\mathcal{H}$, as follows:

$$0 = \frac{-1}{(\sqrt{\mathcal{H}})^2} \left(\frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda} \right) + T^2, \quad (4.14)$$

which is solved by

$$\sqrt{\mathcal{H}} = \frac{1}{T} \sqrt{\frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}}. \quad (4.15)$$

Substituting this back into Eq. (4.13) gives

$$S_e^A = T \int d^2\xi \sqrt{\frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}}, \quad (4.16)$$

which is exactly the Nambu-Goto action on area-metric manifolds (4.3) with tension T .

V. GENERAL WORLD SHEET ACTIONS ON AREA-METRIC MANIFOLDS

In this section, we extend the discussion in Secs. II and III from a Riemannian metric background to an area-metric background. We show that, in the area-metric case, the most general VPD-invariant actions and diffeomorphism-invariant actions are both classically equivalent to the area Nambu-Goto action (4.3).

Let $G_{\mu\nu\rho\lambda}$ be the area metric in spacetime, ξ^a the world sheet coordinates, and $X^\mu(\xi)$ the embedding functions. The induced area metric on the world sheet is

$$\gamma_{abcd} \equiv G_{\mu\nu\rho\lambda} \partial_a X^\mu \partial_b X^\nu \partial_c X^\rho \partial_d X^\lambda = \epsilon_{ab} \epsilon_{cd} \sigma_A^2, \quad (5.1)$$

where $\sigma^{\mu\nu}$ and σ_A^2 are defined in Eqs. (3.5) and (4.8), respectively. For convenience, we repeat the definitions here:

$$\sigma_A^2 \equiv \frac{1}{4} G_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}, \quad \sigma^{\mu\nu} \equiv \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu.$$

Since $\sigma^{\mu\nu}$ contains a single ϵ^{ab} and the area metric $G_{\mu\nu\rho\lambda}$ is a background tensor, the quantity σ_A^2 transforms under world sheet diffeomorphism as a scalar density of the same weight (weight 2) as the auxiliary field $\mathcal{H}$ defined in Sec. IV C.

Therefore, the most general VPD-invariant actions and diffeomorphism-invariant actions built from an area metric are, respectively,

$$S_{\text{VPD}}^A = \int d^2\xi F(\mathcal{H}, \sigma_A^2), \quad (5.2)$$

$$S_{\text{Diff}}^A = \int d^2\xi \sqrt{\mathcal{H}} F\left(\frac{\sigma_A^2}{\mathcal{H}}\right), \quad (5.3)$$

where F is an arbitrary function.

In analogy to the ordinary case of Weyl transformation $h_{ab} \rightarrow e^\varphi h_{ab}$, we define Weyl scaling for the world sheet area metric $\mathcal{H}_{abcd}$ as

$$\mathcal{H}_{abcd} \rightarrow e^{2\varphi} \mathcal{H}_{abcd}, \quad (5.4)$$

which is equivalent to $\mathcal{H} \rightarrow e^{2\varphi} \mathcal{H}$. The factor 2 in the exponent is chosen so that this transformation is consistent with the Weyl transformation when the world sheet area metric is defined by a world sheet metric h_{ab} as

$$\mathcal{H}_{abcd} = h_{ac} h_{bd} - h_{ad} h_{bc}. \quad (5.5)$$

Under this Weyl scaling, σ_A^2 is invariant. Weyl symmetry implies that the Lagrangian is independent of $\mathcal{H}$, while the Weyl symmetry acts trivially on σ_A^2 in the action.

A. Diffeomorphism-invariant area actions

We first consider the diffeomorphism-invariant action (5.3). To eliminate $\mathcal{H}$ from Eq. (5.3), we use the equation of motion for $\mathcal{H}$, as follows:

$$\frac{1}{2\sqrt{\mathcal{H}}}(F(y) - 2yF'(y)) = 0, \quad y \equiv \frac{\sigma_A^2}{\mathcal{H}}, \quad (5.6)$$

where F' denotes the derivative of F with respect to its argument.

In the following, we demonstrate how the $\mathcal{H}$ -dependence of Eq. (5.3) can be removed using Eq. (5.6). First, we multiply $\partial_a y$ on both sides and get

$$\frac{F^3(y)}{2\sqrt{\mathcal{H}}} \partial_a \left(\frac{y}{F^2(y)} \right) = 0. \quad (5.7)$$

For $F(y) \neq 0$, the term $\frac{y}{F^2(y)}$ must be a world sheet constant,³ say C^{-2} . Since the term being world sheet constant C^{-2} means that

$$\sqrt{\mathcal{H}} F \left(\frac{\sigma_A^2}{\mathcal{H}} \right) = C \sqrt{\sigma_A^2}, \quad (5.8)$$

we substitute Eq. (5.8) into Eq. (5.3), and get

$$S_{A,\text{Diff}} = C \int d^2\xi \sqrt{\sigma_A^2}, \quad (5.9)$$

which is exactly the Nambu-Goto action on area-metric manifolds (4.3) with tension C .

There are two classes of solutions, depending on whether y is a constant on the world sheet. If y is not a constant, we can solve Eq. (5.6) as a differential equation of F as a function of y by

$$F(y) = C\sqrt{y} = C\sqrt{\frac{\sigma_A^2}{\mathcal{H}}}, \quad (5.10)$$

for an arbitrary constant C . The action $S_{A,\text{Diff}}^A$ (5.3) is thus

$$S_{A,\text{Diff}}^A = C \int d^2\xi \sqrt{\sigma_A^2}, \quad (5.11)$$

which coincides with the area Nambu-Goto action (4.3) with tension C .

³There can be some $\frac{\sigma_A^2}{\mathcal{H}} = y_0$ such that $F(y_0) = 0$. This implies $F'(y_0) = 0$ from Eq. (5.6). Then, they ensure that the equation of motion for X^μ is satisfied automatically. In this case, the only equation we have is $\frac{\sigma_A^2}{\mathcal{H}} = y_0$. Then, any X^μ is allowed, as long as we set $\mathcal{H}$ to be $\frac{\sigma_A^2}{y_0}$, which is not a physically interesting solution.

The other class of solutions to Eq. (5.6) corresponds to its algebraic solutions $\{y_i\}$. Then, we can solve Eq. (5.6) by setting $\mathcal{H} = \sigma_A^2/y_i$. The resulting action is

$$S_{\text{Diff}}^A = \int d^2\xi \sqrt{\mathcal{H}} F(y_i) = \frac{F(y_i)}{\sqrt{y_i}} \int d^2\xi \sqrt{\sigma_A^2}. \quad (5.12)$$

This is the area Nambu-Goto action (4.3) with tension $F(y_i)/\sqrt{y_i}$. Hence, except singular cases when $F(y_i) = 0$ or $y_i = 0$, any diffeomorphism-invariant area action of the form (5.3) is classically equivalent to the area Nambu-Goto theory.

B. VPD-invariant area actions

We next study the VPD-invariant action S_{VPD}^A (5.2). Varying it with respect to $\mathcal{H}$ yields

$$\frac{\partial F(\mathcal{H}, \sigma_A^2)}{\partial \mathcal{H}} = 0. \quad (5.13)$$

This implies that F must in fact be independent of $\mathcal{H}$, and thus

$$F(\mathcal{H}, \sigma_A^2) = f(\sigma_A^2) \quad (5.14)$$

for some arbitrary function f . The action (5.2) therefore reduces to

$$S_{\text{GS}}^A \equiv \int d^2\xi f(\sigma_A^2), \quad (5.15)$$

which we call the generalized area Schild action.

To show that S_{GS}^A (5.15) is classically equivalent to the area Nambu-Goto action (4.3), we vary S_{GS}^A with respect to X^α . This gives

$$0 = \frac{1}{4} f'(\sigma_A^2) (\partial_\alpha G_{\mu\nu\rho\lambda}) \sigma^{\mu\nu} \sigma^{\rho\lambda} - \partial_\alpha (f'(\sigma_A^2) G_{\alpha\nu\rho\lambda} \epsilon^{ab} \partial_b X^\nu \sigma^{\rho\lambda}), \quad (5.16)$$

where f' denotes the derivative of f with respect to its argument σ_A^2 .

Multiplying Eq. (5.16) by $\partial_c X^\alpha$ and using the identity (3.9), we obtain

$$0 = \frac{1}{4} f'(\sigma_A^2) (\partial_c G_{\mu\nu\rho\lambda}) \sigma^{\mu\nu} \sigma^{\rho\lambda} - \epsilon^{ab} \partial_c X^\alpha \partial_b X^\nu \partial_\alpha (f'(\sigma_A^2) G_{\alpha\nu\rho\lambda} \sigma^{\rho\lambda}) \quad (5.17)$$

$$= \frac{1}{4} f'(\sigma_A^2) (\partial_c G_{\mu\nu\rho\lambda}) \sigma^{\mu\nu} \sigma^{\rho\lambda} - \epsilon^{ab} \frac{1}{2} \epsilon_{cb} \sigma^{\alpha\mu} \partial_\alpha (f'(\sigma_A^2) G_{\alpha\nu\rho\lambda} \sigma^{\rho\lambda}). \quad (5.18)$$

This can be rewritten as

$$0 = \frac{1}{4} f'(\sigma_A^2) (\partial_c G_{\mu\nu\rho\lambda}) \sigma^{\mu\nu} \sigma^{\rho\lambda} - \frac{1}{2} \sigma^{\alpha\beta} \partial_c (f'(\sigma_A^2) G_{\alpha\nu\rho\lambda} \sigma^{\rho\lambda}) \quad (5.19)$$

$$= -\frac{1}{4 f'(\sigma_A^2)} \partial_c (f'^2(\sigma_A^2) \sigma_A^2), \quad (5.20)$$

or, equivalently,

$$0 = (\partial_c \sigma_A^2) \partial_{\sigma_A^2} (f'^2(\sigma_A^2) \sigma_A^2). \quad (5.21)$$

For the case of the area Nambu-Goto action with $f(\sigma_A^2) = T \sqrt{\sigma_A^2}$, the equation above is satisfied identically. For other choices of f , the factor $\partial_{\sigma_A^2} (f'^2(\sigma_A^2) \sigma_A^2)$ does not vanish identically, so Eq. (5.21) implies

$$\partial_c (\sigma_A^2) = 0. \quad (5.22)$$

Thus, σ_A^2 must be constant on the world sheet (see Appendix for a detailed discussion). Consequently, $f'(\sigma_A^2)$ has to be a world sheet constant as well. We can therefore take $f'(\sigma_A^2)$ outside the derivative in the second term of Eq. (5.16). The resulting equation of motion has the same form as that derived from S_S^A (4.5), which was shown in Sec. IV B to be equivalent to the area Nambu-Goto theory.

We conclude that all area actions are classically equivalent to the area Nambu-Goto action.

VI. AREA-METRIC DEFORMATION OF BACKGROUND METRIC

The Polyakov action on a Riemannian background describes a critical string, but it is not clear whether this statement holds on an area-metric manifold.

To address this question, we consider a small area-metric deformation of an ordinary metric $g_{\mu\nu}$, parametrized by

$$G_{\mu\nu\rho\lambda} = g_{\mu\rho} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\rho} + a_{\mu\nu\rho\lambda}, \quad (6.1)$$

where $a_{\mu\nu\rho\lambda}$ is treated as a perturbation. The following action, proposed in Ref. [9], generalizes the Polyakov action:

$$S_{\text{GP}} = \int d^2 \xi \mathcal{L}_{\text{GP}}, \quad \mathcal{L}_{\text{GP}} \equiv \sqrt{\left(\frac{\sqrt{-h}}{2} h^{ab} \gamma_{ab} \right)^2 + \frac{1}{4} a_{\mu\nu\rho\lambda} \sigma^{\mu\nu} \sigma^{\rho\lambda}}, \quad (6.2)$$

where h_{ab} and γ_{ab} are the world sheet and induced metrics defined in Sec. I. The action (6.2) is invariant under two-dimensional diffeomorphisms and Weyl scaling $h_{ab} \rightarrow e^\phi h_{ab}$.

A. Classical equivalence to area Nambu-Goto

We first show that the action (6.2) is classically equivalent to the area Nambu-Goto action (4.3). Varying S_{GP} with respect to h_{ab} gives

$$0 = \frac{1}{\mathcal{L}_{\text{GP}}} \frac{1}{2} \left(-\frac{1}{2} \sqrt{-h} h^{cd} \gamma_{cd} h_{ab} + \sqrt{-h} \gamma_{ab} \right). \quad (6.3)$$

This is the same relation as in the ordinary Polyakov theory, and implies

$$\frac{h_{ab}}{\sqrt{-h}} = \frac{\gamma_{ab}}{\sqrt{\gamma}}. \quad (6.4)$$

Substituting this solution back to Eq. (6.2) reproduces the area Nambu-Goto action (4.3). Thus, at the classical level, the generalized Polyakov action (6.2) and the area Nambu-Goto theory are equivalent.

B. Quantum consistency and primary operator condition

We now investigate the quantum consistency. In the conformal gauge $h_{ab} = \eta_{ab}$, the leading order of Eq. (6.2) reduces to the ordinary Polyakov action (1.3) of the bosonic string,

$$S = \frac{1}{2} \int d^2 \xi \eta^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \quad (6.5)$$

which defines a conformal field theory. A deformation of the action preserves the conformal symmetry at the quantum level if the corresponding operator is a primary field of weight (1, 1).

As pointed out in Ref. [9], when $a_{\mu\nu\rho\lambda}$ is treated as a perturbation in an area metric (6.1), the action (6.2) can be treated as a perturbation of Eq. (6.5) by a fractional term

$$\frac{a_{\mu\nu\rho\lambda} \epsilon^{ab} \epsilon^{cd} \partial_a X^\mu \partial_b X^\nu \partial_c X^\rho \partial_d X^\lambda}{g_{\alpha\beta} \partial_a X^\alpha \partial_b X^\beta} \quad (6.6)$$

at the leading order of $a_{\mu\nu\rho\lambda}$.

We now test whether this operator is a (1, 1)-primary operator on the flat background

$$g_{\alpha\beta} = \eta_{\alpha\beta}. \quad (6.7)$$

Using complex coordinates $(z, \bar{z})$ on the world sheet, the operator (6.6) becomes

$$\mathcal{O} \equiv \frac{1}{\eta_{\alpha\beta} \partial_z X^\alpha \partial_{\bar{z}} X^\beta} a_{\mu\nu\rho\lambda} \partial_z X^\mu \partial_{\bar{z}} X^\nu \partial_z X^\rho \partial_{\bar{z}} X^\lambda. \quad (6.8)$$

In general, consider a normal-ordered operator of the form

$$\mathcal{O} = :F(\partial_z X^\mu): \quad (6.9)$$

and the holomorphic energy momentum tensor $T(z)$ of the free boson theory. Their operator product expansion takes the form

$$\begin{aligned} T(z)\mathcal{O}(w) &= \frac{-1}{2} \frac{1}{(z-w)^4} : \frac{\partial^2 F}{\partial(\partial_w X^\mu) \partial(\partial_w X_\mu)} \\ &: - \frac{-1}{(z-w)^2} : \partial_w X^\mu \frac{\partial F}{\partial(\partial_w X^\mu)} : + \dots, \end{aligned}$$

where $F = F(\partial_w X^\lambda)$, and the ellipsis are terms that are regular when $z \rightarrow w$. For $\mathcal{O}$ to be a primary with conformal dimension Δ , we must have

$$\eta_{\mu\nu} \frac{\partial^2 F}{\partial(\partial_z X_\mu) \partial(\partial_z X_\nu)} = 0, \quad (6.10)$$

$$\partial_z X^\mu \frac{\partial F}{\partial(\partial_z X^\mu)} = \Delta F. \quad (6.11)$$

In our case, we write these conditions in terms of

$$u^\mu = \partial_z X^\mu, \quad v^\mu = \partial_{\bar{z}} X^\mu, \quad (6.12)$$

and define

$$F(u, v) \equiv \frac{K_{\mu\nu} u^\mu v^\nu}{u \cdot v}, \quad (6.13)$$

where

$$K_{\mu\nu} \equiv a_{\mu\rho\nu\lambda} v^\rho v^\lambda, \quad u \cdot v \equiv \eta_{\mu\nu} u^\mu v^\nu. \quad (6.14)$$

The condition (6.11) is easy to check by counting the power of u^μ in Eq. (6.13). That is, there is one u^μ in the denominator and two in the numerator, so $\Delta = 1$.

The nontrivial constraint is condition (6.10), which is equivalent to

$$0 = \eta^{\mu\nu} \frac{\partial^2 F}{\partial u^\mu \partial u^\nu}. \quad (6.15)$$

A straightforward calculation gives

$$\eta^{\mu\nu} \frac{\partial^2 F(u, v)}{\partial u^\mu \partial u^\nu} = \frac{\eta^{\mu\nu} K_{\mu\nu}}{u \cdot v} - 2 \frac{K_{\mu\nu} u^\mu v^\nu}{(u \cdot v)^2} + \frac{K_{\mu\nu} u^\mu v^\nu \eta_{\alpha\beta} v^\alpha v^\beta}{(u \cdot v)^3}. \quad (6.16)$$

Since $a_{\mu\nu\rho\lambda} = -a_{\nu\mu\rho\lambda}$, we have

$$K_{\mu\nu} v^\mu = a_{\mu\rho\nu\lambda} v^\mu v^\rho v^\lambda = 0, \quad (6.17)$$

and Eq. (6.16) simplifies to

$$\eta^{\mu\nu} \frac{\partial^2 F(u, v)}{\partial u^\mu \partial u^\nu} = \frac{\eta^{\mu\nu} K_{\mu\nu}}{u \cdot v} + \frac{K_{\mu\nu} u^\mu v^\nu \eta_{\alpha\beta} v^\alpha v^\beta}{(u \cdot v)^3}. \quad (6.18)$$

Therefore, the condition (6.15) is equivalent to

$$a_{\mu\rho\nu\lambda} \eta^{\mu\nu} v^\rho v^\lambda (u \cdot v)^2 + a_{\mu\rho\nu\lambda} u^\mu v^\rho v^\lambda v^\lambda v^2 = 0. \quad (6.19)$$

We now show that the only solution to Eq. (6.19) is the trivial one $a_{\mu\nu\rho\lambda} = 0$.

We define a Gaussian average of a function A by

$$\langle A \rangle \equiv \int d^D u (A e^{\frac{1}{2} \eta_{\mu\nu} u^\mu u^\nu}). \quad (6.20)$$

Taking the average of Eq. (6.19) over u , all explicit dependence on u^μ can be expressed in terms of $\langle u^\mu u^\nu \rangle \propto \eta^{\mu\nu}$. The averaged equation becomes

$$a_{\mu\rho\nu\lambda} \eta^{\mu\nu} v^\rho v^\lambda v^2 + a_{\mu\rho\nu\lambda} \eta^{\mu\nu} v^\rho v^\lambda v^2 = 2a_{\mu\rho\nu\lambda} \eta^{\mu\nu} v^\rho v^\lambda v^2 = 0. \quad (6.21)$$

Therefore, $a_{\mu\rho\nu\lambda}$ must vanish. There is no nontrivial area-metric deformation (6.6) that preserves the conformal symmetry by itself.

We emphasize that this argument only applies in the perturbative regime around the flat background in the absence of other background fields.

VII. HIGHER DIMENSIONAL EXTENDED OBJECTS

So far we have focused on string theories, which have two-dimensional world sheets, and established their classical equivalence on both Riemannian and area-metric manifolds. In this section, we extend the analysis to higher-dimensional extended objects and introduce a generalization of an area metric, namely, a volume metric.

For higher-dimensional extended objects in a Riemannian spacetime, the Nambu-Goto action is naturally generalized to

$$\mathcal{L}_{\text{NG}} = T \sqrt{\gamma}, \quad (7.1)$$

where γ is the determinant of the induced metric γ_{ab} on d -dimensional world volume. More explicitly, we have

$$\begin{aligned} \gamma &= \frac{1}{d!} \epsilon^{a_1 a_2 \dots a_d} \epsilon^{b_1 b_2 \dots b_d} \gamma_{a_1 b_1} \gamma_{a_2 b_2} \dots \gamma_{a_d b_d} \\ &= \frac{1}{d!} g_{\mu_1 \nu_1} g_{\mu_2 \nu_2} \dots g_{\mu_d \nu_d} \sigma^{\mu_1 \mu_2 \dots \mu_d} \sigma^{\nu_1 \nu_2 \dots \nu_d}, \end{aligned} \quad (7.2)$$

where γ_{ab} is defined by

$$\gamma_{ab} \equiv g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \quad (a, b = 0, 1, \dots, d-1), \quad (7.3)$$

and

$$\sigma^{\mu_1 \mu_2 \dots \mu_d} \equiv \epsilon^{a_1 a_2 \dots a_d} \partial_{a_1} X^{\mu_1} \partial_{a_2} X^{\mu_2} \dots \partial_{a_d} X^{\mu_d}, \quad (7.4)$$

which can be expressed as the d -dimensional Nambu bracket $\{X^{\mu_1}, X^{\mu_2}, \dots, X^{\mu_d}\}$. The Lagrangian (7.1) can thus be rewritten as

$$\mathcal{L}_{\text{NG}} = T \sqrt{\{X^{\mu_1}, \dots, X^{\mu_d}\}^2}. \quad (7.5)$$

We define the Schild Lagrangian for higher-dimensional objects as

$$\mathcal{L}_S = a \{X^{\mu_1}, \dots, X^{\mu_d}\}^2. \quad (7.6)$$

We shall consider in this section these actions on Riemannian manifolds as well as their generalizations to manifolds equipped with volume metrics.

Similar to the definition of the area metric (1.6), we define the volume metric through the volume element $d(\text{vol})$ as

$$d(\text{vol})^2 \equiv V_{[\mu_1 \mu_2 \dots \mu_d][\nu_1 \nu_2 \dots \nu_d]} (dX^{\mu_1} \wedge dX^{\mu_2} \wedge \dots \wedge dX^{\mu_d}) \otimes_{\text{sym}} (dX^{\nu_1} \wedge dX^{\nu_2} \wedge \dots \wedge dX^{\nu_d}), \quad (7.7)$$

where $V_{[\mu_1 \mu_2 \dots \mu_d][\nu_1 \nu_2 \dots \nu_d]}$ is the volume metric.

We will extend our discussion of the world sheet metric in string actions to higher-dimensional extended objects on Riemannian manifolds in Sec. VII A. In Sec. VII B, we define the volume-metric Nambu-Goto action and the volume-metric Schild actions, and discuss how they are equivalent. In Sec. VII C, we prove a theorem about the VPD symmetry, give some examples, and then discuss a common feature of the generalizations of the Nambu-Goto action.

A. World volume metric for higher-dimensional extended objects on Riemannian manifolds

We now extend the discussion in Sec. II A to d -dimensional world volumes. As in Sec. II A, we denote Γ and H to be the matrix form of induced metric γ_{ab} and world volume metric h_{ab} , and they are now of size $d \times d$. The VPD-invariant Lagrangian density has the form

$$\mathcal{L} = f(\sqrt{\gamma}, a_1, \dots, a_d), \quad (7.8)$$

where $a_1, \dots, a_d$ are the invariants built from the matrix ΓH^{-1} , which are defined as $a_k = \text{tr}((\Gamma H^{-1})^k)$ with $k = 1, \dots, d$.

Using the identity

$$\frac{\partial a_k}{\partial h^{ij}} = k \gamma_{ja_1} h^{a_1 b_2} \gamma_{b_2 a_2} h^{a_2 b_3} \dots \gamma_{b_{k-1} a_{k-1}} h^{a_{k-1} b_k} \gamma_{b_k i}, \quad (7.9)$$

where h^{ab} is the inverse of h_{ab} , the equation of motion for h^{ij} reads

$$\frac{\partial \mathcal{L}}{\partial h^{ij}} = \sum_{k=1}^d k f_k \gamma_{ja_1} h^{a_1 b_2} \gamma_{b_2 a_2} h^{a_2 b_3} \dots \gamma_{b_{k-1} a_{k-1}} h^{a_{k-1} b_k} \gamma_{b_k i} \equiv 0, \quad (7.10)$$

where $f_k \equiv \partial f / \partial a_k$. After multiplying $h^{ib_{k+1}}$ on both sides of the last equality, we get

$$0 = \sum_{k=1}^d k f_k A^k, \quad (7.11)$$

where $A = \Gamma H^{-1}$. Since the solution to this equation depends on the exact form of f , we cannot solve it explicitly. However, physically speaking, since the induced metric should be full rank for extended objects, the matrix A is invertible generically. This implies three things:

- (1) In the case of $d = 2$, after multiplying A^{-1} , Eq. (7.11) becomes

$$0 = f_1 I + 2 f_2 A, \quad (7.12)$$

where I is the two-dimensional identity. This equation implies $A = \frac{-f_1}{2f_2} I$, meaning that the eigenvalues of A must be identical. Also, this expression aligns with the result in Sec. II A, see Eq. (2.15), and all the discussions in previous sections apply.

- (2) The case of $d \geq 3$ is totally different from that of $d = 2$. In such a case, we have more than two terms in the equation, and multiplying A^{-1} does not help to solve the equation. Instead, if we assume that the matrix A is diagonalizable, meaning

$$A = P \Lambda P^{-1}, \quad (7.13)$$

where Λ is a diagonal matrix and P is some invertible matrix. The d eigenvalues λ_i ($i = 1, \dots, d$) in Λ satisfy Eq. (7.11), that is,

$$0 = \sum_{k=1}^d k f_k (\sqrt{\gamma}, a_1(\lambda_1, \dots, \lambda_d), \dots, a_d(\lambda_1, \dots, \lambda_d)) \lambda_i^k, \quad (7.14)$$

which is a system of d coupled equations of λ_i s. Generically, we have a discrete solution set for the system $S = \{(\lambda_1^{(\alpha)}(\sqrt{\gamma}), \dots, \lambda_d^{(\alpha)}(\sqrt{\gamma}))\}$ where α labels the solutions. When we substitute them into our Lagrangian (7.8), we can have multiple possible Lagrangians, as follows:

$$\mathcal{L}^{(\alpha)} = f\left(\sqrt{\gamma}, a_1^{(\alpha)}(\sqrt{\gamma}), \dots, a_d^{(\alpha)}(\sqrt{\gamma})\right), \quad (7.15)$$

where $a_n^{(\alpha)}(\sqrt{\gamma}) = \sum_{k=1}^d (\lambda_k^{(\alpha)}(\sqrt{\gamma}))^n$ for $n = 1, \dots, d$. For any α , the resulting Lagrangians are functions of $\sqrt{\gamma}$.

At any rate, after eliminating the auxiliary world volume metric h_{ab} , the resulting Lagrangian is some function of $\sqrt{\gamma}$. In Sec. VII C 2, we will show that all of them are equivalent to the Nambu-Goto action with a dynamically determined tension. Before that, we generalize the discussion above to manifolds equipped with volume metric in the next section.

B. Higher-dimensional extended objects on volume-metric manifolds

As in the area-metric case, we can generalize $g_{\mu_1\nu_1}g_{\mu_2\nu_2}\dots g_{\mu_d\nu_d}$ in Eq. (7.2) to the d -dimensional volume metric $V_{[\mu_1\mu_2\dots\mu_d][\nu_1\nu_2\dots\nu_d]}$, where indices in the bracket $[\dots]$ are totally antisymmetric among themselves, and the exchange of the two d -tuples $([\mu_1\mu_2\dots\mu_d] \leftrightarrow [\nu_1\nu_2\dots\nu_d])$. Then, we have the Nambu-Goto action on a volume-metric background as

$$S_{\text{NG}}^V = T \int d^d\xi \sqrt{\gamma_v}, \quad (7.16)$$

with

$$\gamma_v \equiv V_{[\mu_1\mu_2\dots\mu_d][\nu_1\nu_2\dots\nu_d]} \sigma^{\mu_1\mu_2\dots\mu_d} \sigma^{\nu_1\nu_2\dots\nu_d} \equiv \sigma_v^2, \quad (7.17)$$

and $\sigma^{\mu_1\mu_2\dots\mu_d}$ is defined by Eq. (7.4). The counterpart of the generalized Schild action on a volume-metric background is given by

$$\mathcal{L}_{\text{GS}}^V = F(\gamma_v) \quad (7.18)$$

with an arbitrary function F .

Below, we show its equivalence to the volume-metric Schild theory:

$$\mathcal{L}_S^V \equiv \gamma_v = \sigma_v^2. \quad (7.19)$$

The equation of motion with respect to X^{μ_1} for Eq. (7.18) is

$$\begin{aligned} 0 &= \epsilon^{a_1 a_2 \dots a_d} \partial_{a_1} \{ (\partial_{a_2} X^{\mu_2} \dots \partial_{a_d} X^{\mu_d}) F'(\gamma_v) \sigma_{\mu_1 \mu_2 \dots \mu_d} \} \\ &= \epsilon^{a_1 a_2 \dots a_d} (\partial_{a_2} X^{\mu_2} \dots \partial_{a_d} X^{\mu_d}) \partial_{a_1} \{ F'(\gamma_v) \sigma_{\mu_1 \mu_2 \dots \mu_d} \} \\ &= \epsilon^{a_1 a_2 \dots a_d} (\partial_{a_2} X^{\mu_2} \dots \partial_{a_d} X^{\mu_d}) \\ &\quad \times [(\partial_{a_1} \sigma_v^2) F''(\gamma_v) \sigma_{\mu_1 \mu_2 \dots \mu_d} + F'(\gamma_v) (\partial_{a_1} \sigma_{\mu_1 \mu_2 \dots \mu_d})], \end{aligned} \quad (7.20)$$

where F' and F'' denote derivatives with respect to σ_v^2 .

Multiplying Eq. (7.20) by $\partial_b X^{\mu_1}$ and using the d -dimensional analog of the identity (3.9),

$$\partial_{a_1} X^{\mu_1} \partial_{a_2} X^{\mu_2} \dots \partial_{a_d} X^{\mu_d} = \epsilon_{a_1 a_2 \dots a_d} \sigma^{\mu_1 \mu_2 \dots \mu_d}, \quad (7.21)$$

where the indices $\mu_1, \dots, \mu_d$ on the lhs are totally antisymmetrized, we obtain

$$0 = (\partial_b \gamma_v) (2\gamma_v F''(\gamma_v) + F'(\gamma_v)). \quad (7.22)$$

For the volume-metric Nambu-Goto action (7.16), $F(\gamma_v) = T\sqrt{\gamma_v}$, the equation above is trivially satisfied for any γ_v because $(2\gamma_v F''(\gamma_v) + F'(\gamma_v)) = 0$ identically. (The equivalence of the Nambu-Goto action with the Schild action will be proved in Sec. VII C 2.)

For a generic F , since $(2\gamma_v F''(\gamma_v) + F'(\gamma_v))$ does not vanish identically, the equation of motion implies

$$(\partial_b \gamma_v) = \partial_b (\sigma_v^2) = 0. \quad (7.23)$$

(See Appendix for a detailed proof.) Thus, $\gamma_v = \sigma_v^2$ is constant on the world volume, meaning that $F'(\gamma_v)$ is also a constant on the world volume. This allows us to extract $F'(\gamma_v)$ from the first line of Eq. (7.20). It becomes

$$0 = \epsilon^{a_1 a_2 \dots a_d} \partial_{a_1} \{ (\partial_{a_2} X^{\mu_2} \dots \partial_{a_d} X^{\mu_d}) \sigma_{\mu_1 \mu_2 \dots \mu_d} \}, \quad (7.24)$$

which is nothing but the equation of motion of the volume-metric Schild theory as long as F is not independent of γ_v .

Here, we have proved the equivalence between the volume-metric generalized Schild action and the volume-metric Schild action. This contains the Riemannian manifolds as special cases, since the line metric is a special case of the volume metric. In Sec. VII C 2, we provide a proof to show the equivalence between the (volume-metric) generalized Schild action and the (volume-metric) Nambu-Goto action.

C. A theorem on VPD symmetry

Theorem 7.1. For an arbitrary action of the form

$$S = S[\phi; \rho], \quad (7.25)$$

where ϕ represents any collection of dynamical scalar, vector, or tensor fields, while ρ is a background scalar density field, if the action is invariant under simultaneous diffeomorphism transformations of both ϕ and ρ , the quantity $\frac{\delta S}{\delta \rho}$ is a spacetime constant when ϕ satisfies the equations of motion.

Proof. The proof goes as follows. By definition, the scalar density ρ transforms as

$$\delta_v \rho = \partial_\mu (\rho v^\mu) \quad (7.26)$$

under a diffeomorphism v^μ . On the other hand, the action S transforms as

$$0 = \int d^D x \left(\frac{\delta S}{\delta \phi(x)} \delta_v \phi(x) + \frac{\delta S}{\delta \rho(x)} \partial_\mu (v^\mu \rho) \right). \quad (7.27)$$

By taking $\phi(x)$ on shell, the first term vanishes. Via integration by parts, the second term becomes

$$0 = \int d^D x \left(-\partial_\mu \left(\frac{\delta S}{\delta \rho(x)} \right) v^\mu \rho \right), \quad (7.28)$$

which implies $\partial_\mu \left(\frac{\delta S}{\delta \rho(x)} \right) = 0$, and therefore $\frac{\delta S}{\delta \rho(x)}$ is a space-time constant. ■

1. Example: Unimodular gravity and cosmological constant

It is well known that the cosmological constant in unimodular gravity arises as an initial condition of the dynamical equations of motion [10,11]. This can be viewed as a result of the application of Theorem 7.1.

Consider the action

$$S[\phi; \rho] = \int d^D x (\sqrt{g} R + \lambda(\sqrt{g} - \rho)), \quad (7.29)$$

where $\phi = \{g_{\mu\nu}, \lambda\}$ are dynamical fields and ρ is the background scalar density field. It becomes the action for unimodular gravity when the auxiliary field λ is integrated out.

By definition, both the metric $g_{\mu\nu}$ and the scalar λ are dynamical fields. In particular, the field λ can be any spacetime function. The field ρ is, on the other hand, a scalar density of a given background.

Theorem 7.1 states that since $\delta S / \delta \rho = -\lambda$, λ is forced to be a constant, i.e.,

$$\lambda = C, \quad (7.30)$$

when the metric $g_{\mu\nu}$ satisfies the equations of motion. This makes λ a cosmological constant, although its value is to be determined by the initial condition.

This example suggests that the cosmological constant may arise as a result of the equations of motion when the theory respects VPD; we do not need to put in by hand a constant in the theory from the beginning.

2. Universal proof for constant area/volume density

From our experience in previous examples, we notice that the key to showing the equivalence between the generalized Schild and Nambu-Goto actions is the constant area or volume density σ^2 , e.g., Eqs. (3.8), (3.16), (4.7), (5.22), and (7.23). Below, we provide a universal proof

using Theorem 7.1 to reproduce these results. This applies to extended objects of any dimensions on manifolds with any metrics.

Starting with a general form of action on a d -dimensional world volume,

$$\mathcal{S} = \int d^d \xi (\rho F(\rho^{-1} \sqrt{\gamma})) \quad (7.31)$$

where ρ is the background scalar density, and $\sqrt{\gamma}$ is the induced volume form for arbitrary generalized metrics. Theorem 7.1 demands

$$C = F(\rho^{-1} \sqrt{\gamma}) - \frac{\sqrt{\gamma}}{\rho} F'(\rho^{-1} \sqrt{\gamma}), \quad (7.32)$$

where C is some world volume constant.⁴ Upon fixing the background ρ to 1, the equation becomes⁵

$$C = F(\sqrt{\gamma}) - \sqrt{\gamma} F'(\sqrt{\gamma}). \quad (7.33)$$

For generic F , Eq. (7.33) implies that $\sqrt{\gamma}$ is a constant on the world volume. Therefore, the Nambu-Goto and the Schild actions⁶ are equivalent, and the constant tension is dynamically determined. Note that this result holds not only for the Riemannian metric, but also for the area metric and volume metrics.

VIII. CONCLUSION

In the first part of the paper (Secs. II and III), we have studied general string actions on Riemannian manifolds that are functions of world sheet metric h_{ab} and induced metric γ_{ab} . This includes three well-known string actions: the Schild, the Nambu-Goto, and the Polyakov actions. They have different symmetries: VPD, Diff, and Diff-Weyl, respectively. Hence, we consider the general forms of actions that respect these symmetries and show their classical equivalence. Also, we discuss how tension can be defined from the Schild action by matching its stress-energy tensor to the that of the Nambu-Goto.

In the second part (Secs. IV–VI), we have extended the analysis from Riemannian manifolds to area-metric manifolds. For VPD and diffeomorphisms, we have constructed the corresponding area-metric analogs of the Schild action

⁴Since Theorem 7.1 holds on world volumes as well, we choose to make C world volume constant instead of spacetime constant.

⁵This is the same procedure as in the ordinary Polyakov theory [12]. That is, after gauge fixing the auxiliary world sheet metric h_{ab} , the equation of motion for h_{ab} becomes the Virasoro constraint, and the action becomes free bosons on two-dimensional flat space.

⁶For $d \geq 3$, the Schild action is defined by the Nambu bracket squared.

and the Nambu-Goto action⁷ and have shown that they are classically equivalent. We also have shown that the area-metric type perturbation around the ordinary Polyakov action cannot describe critical strings.

In Sec. VII, we have generalized the discussion to d -dimensional world volumes on manifolds with a volume metric, the higher-dimensional analog of an area metric. Using a general theorem on VPD in Sec. VII C, we show that on d -dimensional Riemannian and volume-metric manifolds, the Schild actions are classically equivalent to the Nambu-Goto action with the tension determined dynamically as an integration constant. As long as we regard our Universe as a single connected world volume, this implies that the Schild theories are classically equivalent to the Nambu-Goto theory in any dimension and on any Riemannian and volume-metric backgrounds.

Through our argument, there are only two natural directions for genuine generalizations. First, one considers actions with explicit derivatives of h_{ab} and γ_{ab} . In such case, the analysis above can be regarded as their low-energy limit. We did not discuss this case here, but we expect they will provide a nontrivial modification to the theory. Second, as we discussed in Sec. VI, when the (line) metric is generalized to an area metric, the modification is nontrivial in that the modified theory can no longer describe critical strings unless further modifications are introduced. Hence, it would be interesting to investigate systematically whether consistent (possibly nonperturbative) string or brane theories can be defined on such generalized metric backgrounds, e.g., through AdS/CFT correspondence [13].

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⁷Note that the only action that is diffeomorphism and Weyl invariant is the area Nambu-Goto action (4.3) when there is only auxiliary world sheet area metric $\mathcal{H}_{abcd}$ but no auxiliary world sheet line metric h_{ab} .

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: DETAILS ON SOLVING

$$\partial_a(\sigma^2)(2\sigma^2 F''(\sigma^2) + F'(\sigma^2)) = 0$$

In this Appendix, we discuss in detail how one solves equations of the type

$$\partial_a(\sigma^2)(2\sigma^2 F''(\sigma^2) + F'(\sigma^2)) = 0, \quad (\text{A1})$$

where σ^2 is a function on the world sheet/world volume and ∂_a is the derivative on it.

Generically, $\partial_a \sigma^2$ has to vanish. However, there may be specific points on the world volume on which $2\sigma^2 F''(\sigma^2) + F'(\sigma^2)$ vanishes. We assume that the solution set S for σ^2 to $2\sigma^2 F''(\sigma^2) + F'(\sigma^2) = 0$ is discrete. Suppose that there is a point ξ_*^a on the world volume that simultaneously satisfies

$$\partial_a \sigma^2 \neq 0, \quad (\text{A2})$$

$$\sigma^2 = \sigma_*^2 \in S. \quad (\text{A3})$$

Assuming that σ^2 is a smooth function around a sufficiently small neighborhood of ξ_*^a , these conditions still remain. However, this means $\partial_a \sigma^2 = 0$ around the point ξ_*^a . Hence, when the set of solutions of $2\sigma^2 F''(\sigma^2) + F'(\sigma^2) = 0$ is discrete, $\partial_a \sigma^2$ must vanish.

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