

Henneaux-Teitelboim form of the generalized unimodular gravity action

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We propose an alternative description of generalized unimodular gravity, extending the Henneaux-Teitelboim approach to unimodular gravity. The central feature of this formulation is the consistent incorporation of time reparametrization, which enhances the gauge structure and reveals a spatial nonlocality hidden in the dynamics of the original formulation. We examine the resulting dynamics, emphasizing the effects of spatial nonlocality, and outline the constraint structure. We show that the gauge symmetry in the gravitational sector is extended by a functionally incomplete symmetry, as occurs in the unimodular gravity. However, in contrast to the latter, the resulting action is not fully diffeomorphism invariant.

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I. GENERALIZED UNIMODULAR AND UNIMODULAR GRAVITY THEORIES

Modifications of general relativity remain an active field of research, driven by open problems in cosmology, quantum gravity, and puzzles of dark energy and dark matter. Among these modifications, it is worth noting models that deform Einstein's general relativity (GR) by imposing additional restrictions on the configuration space. A prominent example of restricted gravity is unimodular gravity (UMG) with the local restriction $\sqrt{|g|} = 1$ on the spacetime metric determinant [1–6]. A virtue of this theory is the alternative covariant formulation proposed by Henneaux and Teitelboim [7]. Although UMG preserves the local physical dynamics of general relativity, it constrains the volume of spacetime while leaving the cosmological constant unspecified. Interest in unimodular gravity has noticeably revived over the past two decades, especially in studying its quantum properties and applications to the cosmological-constant problem [8–21].

Generalized unimodular gravity (GUMG) models, introduced by Barvinsky and Kamenshchik [22],

$$S[g, \lambda_\perp] = \int dt dx \left(\sqrt{|g|} {}^dR(g) - \lambda_\perp (N^\perp - F(\sqrt{\gamma})) \right), \quad (1)$$

are the simplest generalizations of unimodular gravity. In the action (1), $g_{\mu\nu}$ is the unrestricted d-dimensional spacetime metric, decomposed into Arnowitt-Deser-Misner components [23,24] as $g_{\mu\nu} dX^\mu dX^\nu = \gamma_{mn} (dx^m + N^m dt) \times (dx^n + N^n dt) - N^{\perp 2} dt dt$, where $N^\perp$ is the lapse function, N^m are shift functions, and γ_{mn} is the induced metric on $t = \text{const}$ hypersurfaces. The first term in (1) is the GR Einstein-Hilbert action and the second term via the Lagrange multiplier mechanism enforces the GUMG restriction condition,

$$N^\perp \cong F(\sqrt{\gamma}), \quad (2)$$

which expresses the lapse function in terms of some positive monotonic function of the square root of the induced-metric determinant $\sqrt{\gamma} \equiv \sqrt{\det \gamma_{mn}}$. The choice of F defines a specific model in the GUMG family.

The observation, which drew attention to GUMG, was that the restriction term in (1) generates an energy-momentum tensor of the cosmological perfect fluid on the right-hand side of the Einstein equations: $T^{\mu\nu} = q u^\mu u^\nu + p (g^{\mu\nu} + u^\mu u^\nu)$, where u^μ is the timelike unit vector field, $g_{\mu\nu} u^\mu u^\nu = -1$, normal to $t = \text{const}$ hypersurfaces. Expressions for energy density, $q = \frac{\lambda_\perp}{2\sqrt{\gamma}}$, and pressure, $p = \frac{\lambda_\perp}{2\sqrt{\gamma}} \frac{FW}{N^\perp}$, on the restriction surface (2) imply the equation of state

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$$p \cong W\varrho, \quad (3)$$

where barotropic equation-of-state parameter

$$W(\sqrt{\gamma}) \equiv \frac{d \ln F(\sqrt{\gamma})}{d \ln \sqrt{\gamma}} \quad (4)$$

is the derived characteristic function that depends on the induced volume density on spatial sections.¹

Beyond the initial motivation as a possible dark energy mechanism, the dynamical and constraint analysis of GUMG [25,26] reveals additional features, including the possibility of an inflationary scenario consistent with observations. The dynamical nature of W is likewise an appealing aspect in cosmological phenomenology [27]. Moreover, scenarios involving cosmological fluids are currently considered among viable options for addressing the Hubble tension problem [28,29].

Unimodular gravity emerges as a special case of the GUMG family, defined by

$$F(\sqrt{\gamma}) = \sqrt{\gamma}^{-1}, \quad W(\sqrt{\gamma}) \equiv -1 \quad [\text{UMG}]. \quad (5)$$

However, this is an exceptional case, whose dynamical content is almost general-relativistic, whereas GUMG describes gravitational dynamics in the presence of an additional physical degree of freedom associated with the cosmological perfect fluid [22,25]. Also, unlike the unimodular case, GUMG models imply a preferred time direction, reflecting the spacetime anisotropy of the perfect fluid.

II. GUMG ANALOG OF THE HENNEAUX-TEITELBOIM ACTION FOR UMG

Henneaux and Teitelboim suggested an alternative formulation of unimodular gravity via the so-called covariant action [7], providing unrestricted diffeomorphism invariance while remaining physically equivalent to the standard UMG setup.

We claim that the analog of the Henneaux-Teitelboim representation for GUMG theories is

$$S_{\text{alt}}[g, \Lambda, \mathcal{V}] = \int dt dx \sqrt{|g|}({}^dR(g) - \Lambda) + \int dt dx \partial_\mu \mathcal{V}^\mu \hat{E}F\sqrt{\gamma}\Lambda, \quad (6)$$

where ${}^dR(g)$ is the scalar curvature for unrestricted space-time metric $g_{\mu\nu}$, $\sqrt{\gamma}$ is the covariant induced volume density

¹To avoid pathologies in the constraint structure, we assume $W(\sqrt{\gamma})$ to be a sign-definite function, which implies monotonicity of $F(\sqrt{\gamma})$. However, the dust case $W \equiv 0$ also fits the description.

on $t = \text{const}$ hypersurfaces, Λ is the so-called cosmological-constant field, $\mathcal{V}^\mu$ is the auxiliary Henneaux-Teitelboim field, and $\hat{E}$ is the spatially nonlocal, nondegenerate operator defined by the kernel

$$E(t|x; x') = \delta(x; x') - \frac{W^{-1}(t, x)W(t, x')}{V} + \frac{1}{V}, \quad (7)$$

where $W(t, x) \equiv W(\sqrt{\gamma}(t, x))$ and $V \equiv \int 1$ is the background spatial volume. Integrals without an explicit measure symbol denote spatial integration

$$\int f(t) \equiv \int_{t=\text{const}} dx f(t, x). \quad (8)$$

In the unimodular case (5) the whole factor $\hat{E}F\sqrt{\gamma}$ becomes unity and the UMG covariant action [7] is restored.

Another convenient form of the action,

$$S_{\text{alt}_0}[g, \Lambda_0, \mathcal{V}] = \int dt dx \sqrt{|g|}({}^dR(g) - \sqrt{\gamma}^{-1}F^{-1}\hat{E}^{-1}\Lambda_0) + \int dt dx \partial_\mu \mathcal{V}^\mu \Lambda_0, \quad (9)$$

is obtained from (6) via nondegenerate redefinition $\Lambda_0 \equiv \hat{E}F\sqrt{\gamma}\Lambda$. The inverse operator $\hat{E}^{-1}$ has the kernel

$$E^{-1}(t|x; x') = \delta(x; x') - \frac{W^{-1}(t, x)}{\int W^{-1}(t)} - \frac{W(t, x')}{\int W(t)} + 2 \frac{W^{-1}(t, x)VW(t, x')}{\int W^{-1}(t) \int W(t)}. \quad (10)$$

The convenience follows from the gauge invariance of the Λ_0 field, which is a spacetime constant on shell.

The origin and consequences of the delocalization will be discussed later. Here we highlight a prominent GUMG subfamily, termed w-GUMG and defined by

$$F(\sqrt{\gamma}) = \sqrt{\gamma}^w, \quad W(\sqrt{\gamma}) \equiv w = \text{const} \quad [\text{w-GUMG}], \quad (11)$$

for which spatial delocalization disappears because $\hat{E}$ and $\hat{E}^{-1}$ become identity operators with kernel $\delta(x, x')$.²

III. TIME PARAMETRIZATION OF GUMG CANONICAL ACTION

In this section, we derive the alternative GUMG action, which simultaneously establishes the equivalence

²The derivation admits a freedom in the form of the delocalization: for any sign-definite $a(\gamma_{mn})$, operators $E_a(t|x; x') = a^{-1}(t, x)\delta(x; x')W(t, x') - a^{-1}(t, x)\frac{1}{V}W(t, x') + \frac{1}{V}$ describe classically equivalent models. The choice (7) is preferable mainly because of the identity limit for w-GUMG.

of the representations. The key step is the consistent time parametrization of the canonical extended action of the theory, which deserves special discussion.

We start with the extended canonical action, previously analyzed for general GUMG in [25] using the Dirac scheme. Here, we briefly revisit this through a concise Hamiltonization. First, we reduce (1) with respect to the $\lambda_\perp$, $N^\perp$, then introduce conjugate momenta π^{mn} only for the spatial metric γ_{mn} . This yields a canonical theory on the phase space (γ_{mn}, π^{mn}) with a Hamiltonian density $F(\sqrt{\gamma})H_\perp(\gamma, \pi)$ and primary constraints $H_n(\gamma, \pi) = 0$, corresponding to spatial diffeomorphisms. The structures $H_\perp$ and H_n are identical to the Hamiltonian and momentum constraints in canonical Einstein gravity [24],

$$H_\perp(\gamma, \pi) = \frac{1}{\sqrt{\gamma}} \left(\pi^{mn} \pi_{mn} - \frac{1}{d-2} \pi^2 \right) - \sqrt{\gamma} R(\gamma), \quad (12)$$

$$H_n(\gamma, \pi) = -2\gamma_{nm} \pi^{mk}{}_{;k}, \quad (13)$$

where $\pi_{mn} \equiv \gamma_{mk} \pi^{kl} \gamma_{ln}$, $\pi \equiv \gamma_{mn} \pi^{mn}$, $R(\gamma)$ is the scalar curvature of $t = \text{const}$ hypersurfaces, and semicolons denote spatial covariant derivatives. It is important to note that in GUMG $H_\perp$ (12) is not a constraint, but a factor in the nontrivial Hamiltonian density $F(\sqrt{\gamma})H_\perp(\gamma, \pi)$.

Dynamics does not preserve primary constraints $H_n = 0$ (for $W \neq 0$)³ and the Dirac consistency procedure reveals secondary constraints

$$(W(\sqrt{\gamma})F(\sqrt{\gamma})H_\perp(\gamma, \pi))_{,m} = 0, \quad (14)$$

which have a form of spatial gradient ($f_{,m} \equiv \partial_m f$). This effectively constrains the average-free component $\overline{WFH}_\perp \equiv WFH_\perp - \overline{WFH}_\perp = 0$ on $t = \text{const}$ hypersurfaces. We denote “average” (homogeneous) and “average-free” (inhomogeneous) components of local functions with overlines and tildes, respectively,⁴

$$\tilde{f}(t) \equiv \int f(t, x) / \int 1, \quad \tilde{f}(t, x) \equiv f(t, x) - \tilde{f}(t). \quad (15)$$

³The dust case, $W \equiv 0$ ($F = \text{const}$), is the special GUMG case in which dynamics preserves all constraints $H_n = 0$, which are all first class. However, the constraint $(b(\sqrt{\gamma})H_\perp)_{,n}$ can be added by hand as a partial gauge fixing. Thus, the following derivation and the final representation are also valid for the $W \equiv 0$ case upon substituting $W(\sqrt{\gamma}) \rightarrow b(\sqrt{\gamma})$ with almost any $|b(\sqrt{\gamma})| > 0$.

⁴Various arguments rely on decomposing quantities on average and average-free parts and on the equivalence of average-free quantities with spatial vector divergencies. Though these are straightforward for compact spaces, the construction extends for spacetimes with noncompact spatial sections [30].

The constraints $H_n = 0$ and $(WFH_\perp)_{,m} = 0$ form a complete set,⁵ so the GUMG concise extended action reads

$$S_E[\gamma, \pi, N, \mu] = \int dt dx (\dot{\gamma}_{mn} \pi^{mn} - FH_\perp - N^n H_n - \mu^m (WFH_\perp)_{,m}). \quad (16)$$

Time parametrization of the field-theoretic canonical action in a mechanical-like form by means of the local fields, which was applied in [7] for UMG, remains valid for the w-GUMG subfamily, characterized by constant $W \equiv w = \text{const}$ and $F \rightarrow F_{(w)} = \sqrt{\gamma}^w$, for which the parametrized action can be written as

$$S_{\text{par}}^{(w)}[\gamma, \pi, \tau^0, \pi_0, \lambda^0, N, \mu] = \int dt dx (\dot{\gamma}_{mn} \pi^{mn} + \dot{\tau}^0 \pi_0 - \lambda^0 (\pi_0 + F_{(w)} H_\perp) - N^n H_n - \mu^m (w F_{(w)} H_\perp)_{,m}). \quad (17)$$

To ensure the equivalence of the original and parametrized representations of general field theory, two conditions must be satisfied. First, the new parametrized Hamiltonian constraint must introduce a new first-class constraint into the extended set. This new constraint is not necessarily the parametrized Hamiltonian itself; it may involve a linear combination with the original constraints. Second, there must exist an *accessible* gauge fixing (e.g., $\tau^0 - t = 0$) for the new first-class constraint, which recovers the original action (16) upon reduction.

These requirements are satisfied in w-GUMG and UMG for the local parametrized action (17). Notably, in w-GUMG ($w \neq -1$) the new first-class constraint introduced through parametrization is $\int \varepsilon P_I \equiv \int \varepsilon (\pi_0 + \overline{F_{(w)} H_\perp}) = \int (\varepsilon \pi_0 + \bar{\varepsilon} F_{(w)} H_\perp)$, which involves only the average component of the Hamiltonian density. (By $\varepsilon, \zeta, \eta, \xi$, etc., we denote local gauge parameters.) This means that the time-reparametrization gauge symmetry introduced in (17) by local parametrization, in the sector of metric fields, manifests as a functionally incomplete symmetry with a homogeneous parameter $\bar{\varepsilon}(t)$. Whereas in the sector of the auxiliary fields, this

⁵We do not provide a separate discussion of the so-called GR branch of GUMG [25], which is the $H_\perp = 0$ submanifold characterized by a rank discontinuity of the matrix of Poisson brackets of constraints. Changes affect the constraint classification: on the GR branch, all constraints become first class and thus there are no dynamical constraints on any of the Lagrange multipliers. Importantly, the GR branch is correctly described by the new parametrized GUMG description.

symmetry remains local. The constraint $\int \varepsilon P_I$ is a linear combination of the local parametrization constraint $\int \varepsilon(\pi_0 + F_{(w)}H_\perp)$ and the secondary constraint $\int (\partial_n \eta^n)(wF_{(w)}H_\perp)$ with $(\partial_n \eta^n) = -\tilde{\varepsilon}w^{-1}$, both of which are second class. This suggests that the local parametrization (17) is valid because the average-free part of the Hamiltonian density was constrained (14) prior to parametrization.⁶

In the general field theory, particularly for GUMG with nonconstant $W(\sqrt{\gamma})$, a mechanical-like parametrization similar to (17) does *not* yield an equivalent theory. The local constraint $\pi_0 + FH_\perp$ fails to introduce a new first-class constraint. From the Poisson brackets

$$\begin{aligned} \left\{ \int \xi^n H_n, \int \eta^m H_m \right\} &= \int (\xi^m \partial_m \eta^n - \eta^m \partial_m \xi^n) H_n, \\ \left\{ \int \xi^n H_n, \int \eta^m T_{,m} \right\} &= \int \xi^n (\partial \eta) T_{,n} + \int (\partial \xi) (\partial \eta) \Omega T, \\ \left\{ \int \xi^n T_{,n}, \int \eta^m T_{,m} \right\} &= \int ((\partial \xi) \overset{\leftrightarrow}{\partial}_n (\partial \eta)) \gamma^{nm} F^2 W^2 H_m, \end{aligned} \quad (18)$$

and

$$\begin{aligned} \left\{ \int \varepsilon(\pi_0 + FH_\perp), \int \varepsilon'(\pi_0 + FH_\perp) \right\} &= \int (\varepsilon \overset{\leftrightarrow}{\partial}_n \varepsilon') F^2 \gamma^{nk} H_k, \\ \left\{ \int \varepsilon(\pi_0 + FH_\perp), \int \eta^n H_n \right\} &= - \int \varepsilon_{,n} \eta^n \frac{1}{W} T + \int \varepsilon (\partial \eta) T, \\ \left\{ \int \varepsilon(\pi_0 + FH_\perp), \int \eta^n T_{,n} \right\} &= - \int (\varepsilon \overset{\leftrightarrow}{\partial}_n (\partial \eta) W) F^2 \gamma^{nk} H_k \\ &\quad - \int \varepsilon (\partial \eta) \Theta \pi T, \end{aligned} \quad (19)$$

where we use shorthand notations $\mathcal{T} \equiv WFH_\perp$, $(\partial \xi) \equiv \partial_n \xi^n$, and $(f \overset{\leftrightarrow}{\partial}_n g) \equiv (f \partial_n g - g \partial_n f)$ and introduce $\Omega(\sqrt{\gamma}) \equiv \frac{d \ln W}{d \ln \sqrt{\gamma}} + W + 1$, $\Theta(\sqrt{\gamma}) \equiv \frac{1}{d-2} \frac{d \ln W}{d \ln \sqrt{\gamma}} \frac{F}{\sqrt{\gamma}}$, it follows that the Poisson bracket matrix for the constraint set $(\pi_0 + FH_\perp, H_n, (WFH_\perp)_{,m})$ has a greater rank than for

⁶The constraint basis of the parametrized w-GUMG can be rearranged to disentangle first-class subspace spanned by P_I and the transverse part of H_n , which generate the canonical gauge algebra of the model. Meanwhile, the average-free part of $F_{(w)}H_\perp$ (encoded in the secondary constraint) and the longitudinal part of H_n form a pair of second-class constraints, which follows from (18) and (19) for $W = w$. In the exceptional UMG case, $w = -1$, the latter pair of constraints becomes first class, restoring full spacetime diffeomorphism invariance in accordance with [7].

the initial set $(H_n, (WFH_\perp)_{,m})$.⁷ To maintain the rank and satisfy the correspondence condition, one may extend all constraints with $O(\tau_n^0)$ terms. This extension results in an infinite series in powers of τ^0 and its spatial derivatives. Furthermore, since the structure functions in (18) and (19) are spatial differential operators, and some constraints are functionally incomplete, the result becomes uncontrollably nonlocal.

However, for field theories in which the weighted average-free part of the Hamiltonian density is constrained in the original theory, the equivalent parametrized action with local auxiliary fields can be constructed in a concise explicit form at the cost of simple spatial nonlocality. Conceptually, the desired representation of the theory is achieved in two steps, each of which clearly preserves the physical equivalence of the representations. First, a universally valid homogeneous time parametrization⁸ is applied to (16) by introducing homogeneous fields $\bar{\tau}^0(t), \bar{\pi}_0(t), \bar{\lambda}^0(t)$. Then, the average-free parts of these fields are added via separate pure-gauge combination $\dot{\bar{\tau}}^0 \bar{\pi}_0 - \dot{\bar{\lambda}}^0 \bar{\pi}_0$. After this, it becomes possible to combine a functionally complete $H_\perp$ term in a constraint, which is necessary to obtain the desired generalization of the Henneaux-Teitelboim representation.

For the general GUMG model, this parametrization yields the equivalent parametrized action in the form

$$\begin{aligned} S_{\text{par}}[\gamma, \pi, \tau^0, \pi_0, \lambda^0, N, \mu] &= \int dt dx \left(\dot{\gamma}_{mn} \pi^{mn} + \dot{\tau}^0 \pi_0 \right. \\ &\quad \left. - \lambda^0 (\pi_0 + \overline{FH}_\perp) - N^n H_n \right. \\ &\quad \left. - \mu^m (WFH_\perp)_{,m} \right), \end{aligned} \quad (20)$$

where auxiliary fields are combined into local fields $\tau^0(t, x), \pi_0(t, x), \lambda^0(t, x)$. It can be verified that the rank of the matrix of Poisson brackets of constraints does not change, the gauge $\tau^0 - t = 0$ is accessible, and the reduction with respect to $(\pi_0 + \overline{FH}_\perp, \tau^0 - t)$ reproduces (16).⁹

⁷Note that on the right-hand sides of (18) and (19), $\mathcal{T} \equiv WFH_\perp$ (or, more precisely, its average component) does not vanish on shell. The exceptional cases with $\Omega \equiv 0$ correspond either to UMG, $W \equiv w = -1$, or to models with pathological constraint structures and are thus excluded from regular GUMG [30].

⁸Homogeneous parametrization in field theory is generally possible because the dynamical and constraint structures in the canonical theory (16) are harmonized by the Dirac consistency procedure: the spatially integrated Hamiltonian density, even when shifted by a commuting element, remains in involution with the complete set of the original constraints.

⁹The rank property may be checked using (19), whose right-hand side vanishes on the constraint surface after substitution $\varepsilon \rightarrow \bar{\varepsilon}$, $\varepsilon' \rightarrow \bar{\varepsilon}'$, caused by the change of the parametrized Hamiltonian constraint. The correspondence property is evident since the canonical action of $\pi_0 + \overline{FH}_\perp$ is obviously transversal to $\tau^0 - t$.

The form of the new first-class constraint, introduced during such parametrization, can be guessed from general principles. This is $P_I = \pi_0 + \overline{FH}_\perp + \overline{U}_0^n H_n$, where $U_0^n(\gamma, \pi)$ is an on-shell solution for the Lagrange multipliers N^n , which in w-GUMG can be chosen equal to zero, but is nontrivial in general GUMG.¹⁰ The new first-class constraint $\int \varepsilon P_I$ and the transverse diffeomorphisms $\int \zeta_{(t)}^n H_n$, $(\partial_k \zeta_{(t)}^k = 0)$, generate the canonical gauge algebra of the model. The secondary constraint and longitudinal spatial diffeomorphisms are second class.

For w-GUMG ($w \neq -1$), this reproduces the constraint structure derived from the local action (17). The newly introduced parametrization symmetry acts homogeneously on the metric phase-space fields, with the reason for this being more transparent in this scheme. Only the average component of $FH_\perp$ from P_I generates gauge transformations, while its average-free part does not.

For the UMG case $W \equiv -1$, parametrization (20) clarifies how the lacking homogeneous mode of $\sqrt{\gamma}^{-1} H_\perp$ enters the game, with the inhomogeneous part of the new symmetry acting entirely within the auxiliary sector. Unlike in GUMG, the inhomogeneous part of the Hamiltonian constraint and the longitudinal part of spatial diffeomorphisms are initially first class, so the homogeneous mode just completes the original gauge symmetry to full diffeomorphism invariance in the metric sector.

This parametrization scheme also clarifies that in GUMG only the homogeneous time-reparametrization transformations are shared by both the metric and the auxiliary fields, while the remaining average-free symmetries act independently in these sectors.

To derive the generalized Henneaux-Teitelboim form of the action from (20), we rearrange its constraint basis. The constraint equations $\pi_0 + \overline{FH}_\perp = 0$, $(WFH_\perp)_{,m} = 0$ are equivalent to the set $\pi_0 + \overline{FH}_\perp + W^{-1}(\overline{WFH}_\perp) = 0$, $(WFH_\perp)_{,m} = 0$. The first constraint in the latter set can be recast as $\pi_0 + \hat{E}FH_\perp = 0$, where the operator $\hat{E}$, with kernel (7), is nondegenerate and acts to the right on the Hamiltonian density $FH_\perp$. This is the origin of the spatial nonlocality in the parametrized GUMG: it is generally not possible to merge functionally incomplete constraints into functionally complete local combinations.

To assemble the dynamical metric dependence into a GR-like combination, the $H_\perp$ must be consolidated into a single constraint. Substituting $FH_\perp = -\hat{E}^{-1}\pi_0$, where $\hat{E}^{-1}$ is defined in (10), into the secondary constraint results in $(W\hat{E}^{-1}\pi_0)_{,m} = 0$. The latter looks awkward but simplifies to $\pi_{0,m} = 0$ [30]. In the new constraint basis, the canonical action becomes

¹⁰The combination $FH_\perp + U_0^n H_n$, which appears in P_I , is the so-called first-class Hamiltonian density [31]. Also we recall that the discussed properties of the constraints are related to the physical space of the theory outside the GR branch [26].

$$S_{par}[\gamma, \pi, \tau^0, \pi_0, \lambda, N, \nu] = \int dt dx (\dot{\gamma}_{mn} \pi^{mn} + \dot{\tau}^0 \pi_0 - \lambda(\pi_0 + \hat{E}FH_\perp) - N^n H_n - \nu^m \pi_{0,m}). \quad (21)$$

The equivalence of (20) and (21) is straightforward, as they are related by a nondegenerate redefinition of the Lagrange multipliers, $\lambda^0 = \lambda - \partial_k \nu^k$, $\partial_k \mu^k = -\lambda \widetilde{W}^{-1}$.

Finally, the effective lapse function $N^\perp$ is recovered via the redefinition $\lambda = N^\perp F^{-1} \hat{E}^{-1}$, which completes the formation of the canonical expression for the Einstein-Hilbert Lagrangian. The Legendre reduction to Lagrangian action proceeds identically to GR, which leads to the action (9), where spacetime metric is $g_{\mu\nu} dX^\mu dX^\nu = \gamma_{mn} (dx^m + N^m dt)(dx^n + N^n dt) - N^{\perp 2} dt dt$, the cosmological-constant field Λ_0 is just the renamed π_0 , and the Henneaux-Teitelboim field is $\mathcal{V}^\mu = (\tau^0, \nu^m)$.

IV. DYNAMICAL PROPERTIES

The variational equations of motion for action $S_{alt}[g, \Lambda, \mathcal{V}]$ (6) with respect to the auxiliary fields $\mathcal{V}^\nu$,

$$\partial_\nu (\hat{E}F\sqrt{\gamma}\Lambda) = \partial_\nu \Lambda_0 \cong 0, \quad (22)$$

generalize the cosmological-constant behavior, implying that Λ_0 is an unfixed on-shell spacetime constant. Operator $\hat{E}$ (7) introduces the spatial nonlocality into the classical behavior of the cosmological-constant field Λ ,

$$\Lambda \cong \sqrt{\gamma}^{-1} F^{-1} \hat{E}^{-1} c = \sqrt{\gamma}^{-1} F^{-1} \frac{W^{-1}}{\overline{W}^{-1}} c, \quad (23)$$

where c is an on-shell constant value of Λ_0 and $\overline{W}^{-1}(t)$ is the average of $W^{-1}(\sqrt{\gamma}(t, x))$ over $t = \text{const}$ hypersurface. For the w-GUMG subfamily (11) $W^{-1}/\overline{W}^{-1} = 1$ and the spatial nonlocality disappears. In the unimodular case (5), Λ coincides with Λ_0 and on shell equals c .

The variational equation with respect to Λ gives

$$\sqrt{|g|} \cong \partial_\mu \mathcal{V}^\mu \hat{E}F\sqrt{\gamma}, \quad (24)$$

with $\hat{E}$ acting to the left.

For GUMG models, unlike the exceptional unimodular case (5), two average-free second-class constraints arise. Differential consequences of the equations of motion impose two constraints on metric components, which in canonical treatment are the Lagrange multipliers at these constraints. First, the lapse function $N^\perp$ satisfies

$$N^\perp \widetilde{F}^{-1} \cong 0. \quad (25)$$

This together with (24) imply $\partial_\mu \widetilde{\mathcal{V}^\mu \hat{E}} \sim (\partial_\mu \widetilde{\mathcal{V}^\mu W^{-1}}) \cong 0$, leading to

$$\partial_\mu \mathcal{V}^\mu \hat{E} \cong \overline{\partial_\mu \mathcal{V}^\mu} = \dot{\tau}^0, \quad (26)$$

and reinstating the GUMG restriction condition (2) as $N^\perp \cong \dot{\tau}^0 F(\sqrt{\gamma})$. Factor $\dot{\tau}^0(t)$ reflects time-reparametrization gauge ambiguity in parametrized GUMG, which acts on the gravitational variables homogeneously and is equal to 1 in the correspondence gauge $\tau^0 - t = 0$. The second constraint concerns Lagrange multipliers for longitudinal spatial diffeomorphisms,

$$\partial_n N^n \cong \dot{\tau}^0 \left(\Omega^{-1} \Theta \pi - \Omega^{-1} \frac{\int \Omega^{-1} \Theta \pi}{\int \Omega^{-1}} \right), \quad (27)$$

where $\Omega(\sqrt{\gamma}) \equiv \frac{d \ln W}{d \ln \sqrt{\gamma}} + W + 1$, $\Theta(\sqrt{\gamma}) \equiv \frac{1}{d-2} \frac{d \ln W}{d \ln \sqrt{\gamma}} \frac{F}{\sqrt{\gamma}}$.

Finally, the variational derivative of $S_{\text{alt}}[g, \Lambda, \mathcal{V}]$ with respect to $g_{\mu\nu}$ leads to the Einstein equations

$${}^d R^{\mu\nu} - \frac{1}{2} {}^d R g^{\mu\nu} \cong \frac{1}{2} T^{\mu\nu}, \quad (28)$$

where the Einstein tensor for spacetime metric $g_{\mu\nu}$ is on the left, and terms proportional to Λ are attributed to the matter sector, whose contribution is encoded on the right-hand side in the energy-momentum tensor $T^{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{mat}}}{\delta g_{\mu\nu}}$. The latter can be cast on shell to

$$T^{\mu\nu} \cong \Lambda n^\mu n^\nu + W \Lambda (g^{\mu\nu} + n^\mu n^\nu), \quad (29)$$

where n^μ is the unit vector field orthogonal to constant-time hypersurfaces. In (29) $\varrho \cong \Lambda$ is the energy density and $p \cong W \Lambda$ is the pressure, reproducing the perfect fluid equation of state (3),

$$p \cong W \varrho. \quad (30)$$

Note that for nonconstant $W(\sqrt{\gamma})$ the on-shell behavior of energy and pressure bears the imprint of the spatial nonlocality due to (23).

V. CONCLUSIONS

In this paper, we introduced an alternative formulation of generalized unimodular gravity, inspired by the Henneaux-Teitelboim action for unimodular gravity. We derived this formulation, addressing issues of consistent parametrization, and highlighted its key dynamical and gauge features, including the emergence of spatial nonlocality. A detailed discussion, including a complete analysis of the gauge structure of the Lagrangian theory, a comprehensive specification of exceptional particular cases, and various conceptual and calculational subtle aspects of the formalism are presented in [30].

The parametrization approach and the alternative action in the generalized Henneaux-Teitelboim form, along with the underlying techniques for dealing with functionally incomplete constraints, can be extended to restricted gravity models with more general restriction conditions. Also, these methods and some results can be directly applied to other field theories where the gradient of the weighted Hamiltonian density vanishes on the constraint surface, which may be generated by analogous restriction of a Lagrange multiplier at Hamiltonian constraint in diffeomorphism-invariant theories.

In conclusion, we remark on the nature of the emerged spatial nonlocality. While one might view the nonlocality, encoded in $\hat{E}$, as an artifact of the “unfortunate” parametrization by local auxiliary fields, we argue that this nonlocality is implicitly present in the physical sector of $W \neq \text{const}$ GUMG models already in the original formulation. In the canonical framework, the integrated Hamiltonian density $\overline{FH}_\perp$ is a gauge-invariant constant of motion, with its on-shell values parametrized by a constant $-c$. The secondary constraint $\widehat{WFH}_\perp$ is a gauge invariant that vanishes on shell. Thus, $\widehat{EFH}_\perp$ is functionally complete (but spatially nonlocal) gauge-invariant constant of motion—a property that the alternative parametrized representation makes explicit via a constraint.

The significance of this observation is further supported by the perspective of the restricted-theory formalism [32,33], which clarifies the connection between the effective actions of the parental and restricted gauge theories. The key point is to provide a convenient parametrization of the difference between the physical spaces of these two theories when a restriction is not a pure gauge fixing [32]. In GUMG the deformation of the physical space is parametrized by the values of the global quantity c . In particular, the physical trajectories of the parental theory (GR) are tied to the constraint surface $H_\perp = 0$, $H_n = 0$, whereas in the restricted theory (GUMG) it extends to $H_\perp = -F^{-1} \hat{E}^{-1} c (= -F^{-1} \frac{W^{-1}}{W^{-1}} c)$, $H_n = 0$.

We expect that the results and methods developed for generalized unimodular gravity will be useful for further study of similar theories and their quantum properties.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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