

Renormalization group running of the parity operator in Lorentz-violating quantum field theory

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In conventional relativistic quantum field theory, the discrete operators C , P , and T are matrix operators with no renormalization scale dependence. However, in a Lorentz-violating theory with a fermion f^μ term in the action, these operators may acquire nontrivial renormalization group behavior. Because the f^μ term may actually be exchanged in the action for an equivalent $c^{\nu\mu}$ term, the scale dependence depends explicitly on the renormalization scheme, even at one-loop order. The scheme dependence means it is always possible to set the scale dependence parameter $1 - X$ to zero, but for analyses of some high-energy electron-photon processes, using a scheme with $X = 0$ —and thus definite scale dependences for C , P , and T —may nonetheless be more convenient.

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I. INTRODUCTION

Special relativity underlies both of the theories that we use to describe fundamental physics: the standard model of particle physics and the general theory of relativity. The Lorentz symmetry of special relativity manifests itself in somewhat different ways in these two regimes, but it is a through line connecting both of them. Given the importance of this symmetry, there has long been significant interest in testing whether it holds exactly or merely approximately. This interest is quite natural. We already know that an apparent, but in reality only approximate, symmetry can have profound implications and provide key insights about where new fundamental physics might be found.

In this millennium, thanks primarily to advances in effective field theory methods, it has been possible to parametrize violations of rotational isotropy and Lorentz boost invariance in a systematic way. This, in turn, has prompted experimental interest in many types of Lorentz violation that had previously flown under scientists' radar. These kinds of renewed experimental searches have not found any compelling evidence that Lorentz symmetry is not exact, but they remain ongoing.

The local effective field theory used for describing Lorentz violation with known standard model fields is prosaically known as the standard model extension

(SME) [1,2]. Because of the close connection between Lorentz invariance and CPT invariance [3], the SME is also capable of describing generically any forms of CPT violation that are stable, unitary, local, and constructed out of the standard model's fermion and boson fields. (Gravity is slightly trickier.) The SME (or at least the particle theory part) is a quantum field theory (QFT), and properly understanding a QFT always entails understanding the characteristics of its $\mathcal{O}(\hbar)$ and higher-loop corrections. The minimal SME constitutes the subtheory of the full SME framework that should be expected to be renormalizable, as it contains only the finite number of local, Hermitian, and gauge-invariant operators that can be constructed up to mass dimension no greater than 4. The minimal SME action looks much like the ordinary standard model action, except that the field operators it contains are not constrained to have all their Lorentz indices fully contracted. This minimal theory is often the most logical framework for analyzing the results of experimental Lorentz and CPT tests. There has been a fair amount of good work on the elucidation of the renormalization properties of the minimal SME, especially at one-loop order [4–12]. However, our understanding of the theory at the quantum level is by no means complete.

II. FERMION LORENTZ VIOLATION

The free fermion sector of the SME contains the operators that control the propagation of spin- $\frac{1}{2}$ particles. In the minimal SME, with only superficially renormalizable operators, the Lagrange density for a single species of fermion is

$$\mathcal{L}_\psi = \bar{\psi}(i\Gamma^\mu\partial_\mu - M)\psi. \quad (1)$$

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The Γ and M terms generalize the usual Dirac Lagrange density's γ and m , by including all possible matrix structures, contracted with the fixed vector and tensor backgrounds that describe the Lorentz violation. These background structures enter in the forms

$$\Gamma^\mu = \gamma^\mu + \Gamma_1^\mu = \gamma^\mu + c^{\nu\mu}\gamma_\nu + d^{\nu\mu}\gamma_5\gamma_\nu + e^\mu + if^\mu\gamma_5 + \frac{1}{2}g^{\lambda\mu}\sigma_{\lambda\nu}, \quad (2)$$

$$M = m + im'\gamma_5 + M_1 = m + im'\gamma_5 + a^\nu\gamma_\nu + b^\nu\gamma_5\gamma_\nu + \frac{1}{2}H^{\mu\nu}\sigma_{\mu\nu}. \quad (3)$$

Some of the coefficients in $\mathcal{L}$ are more interesting than others. For example, m' is not Lorentz violating, but rather represents a Majorana mass term that may be absorbed into m in the noninteracting Lorentz-invariant theory by a field redefinition.

The specific form of interaction the fermions experience is not especially important, as regards the discrete symmetry structure we shall be discussing. For the purpose of definite calculations, we shall use a Yukawa interaction, with scalar and interaction Lagrange density

$$\mathcal{L}_\phi = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \bar{\psi}G\psi\phi. \quad (4)$$

However, nothing about our analysis should depend qualitatively on having this particular interaction. Generally, calculations of Lorentz-violating fermion self-energies in a Yukawa theory and an Abelian gauge theory have similar levels of complexity. Each theory can have certain vertex insertions not present in the other. In a gauge theory, the gauge symmetry requires that Γ modify not just free fermion propagation but also the boson-fermion vertex. In a Yukawa theory, the vertex can be modified in a different way; G , like M , may include all possible Dirac matrix structures,

$$G = g + ig'\gamma_5 + G_1 = g + ig'\gamma_5 + I^\mu\gamma_\mu + J^\mu\gamma_5\gamma_\mu + \frac{1}{2}L^{\mu\nu}\sigma_{\mu\nu}. \quad (5)$$

The full one-loop renormalization of the Yukawa theory including all these terms (as well as additional Lorentz violation in the pure scalar propagation sector) was dealt with in Ref. [11]; however, that level of generality is not necessary here, so we shall restrict attention to a theory with a standard Yukawa vertex $G = g$.

Without the condition of Lorentz symmetry—which amounts to a condition that the operators in the Lagrange density of the theory should have no free Lorentz indices—we can see that there are many more dimension-three and dimension-four operators than in a

conventional QFT. However, there is actually some redundancy in the SME description, and there exists an exact equivalence between two different types of Lorentz-violating fermion theories. This redundancy is at the root of all the observations we shall make in this paper. There is a transformation of the SME Lagrange density with solely a f^μ term into one with just a $c^{\nu\mu}$ term instead [13]. The theory with f^μ , defined by

$$\mathcal{L}_f = \bar{\psi}[i(\gamma^\mu + if^\mu\gamma_5)\partial_\mu - m]\psi, \quad (6)$$

is converted into

$$\mathcal{L}_c = \bar{\psi}'[i(\gamma^\mu + c_{\text{eff}}^{\nu\mu}\gamma_\nu)\partial_\mu - m]\psi' \quad (7)$$

by a local linear transformation acting on the four-component fermion field,

$$\psi' = e^{\frac{i}{2}f^\nu\gamma_\nu\gamma_5 G(-f^2)}\psi, \quad (8)$$

where $G(\xi) = \frac{1}{\sqrt{\xi}}\tan^{-1}\sqrt{\xi}$. The $c^{\nu\mu}$ tensor in the transformed Lagrange density is

$$c_{\text{eff}}^{\nu\mu} = \frac{f^\mu f^\nu}{f^2} \left(\sqrt{1 - f^2} - 1 \right) \approx -\frac{1}{2}f^\nu f^\mu, \quad (9)$$

with the approximate form applying when $|f^2|$ is small. We shall work in this approximate regime, since physical Lorentz violation, if it exists, is a small effect.

The equivalence between f^μ and $c_{\text{eff}}^{\nu\mu}$ has a number of interesting consequences—some relatively straightforward, others rather more subtle. While calculations may be performed in either version of the theory, in some cases it may be significantly easier to use one than the other. It seems that the $c^{\nu\mu}$ -type theory is typically simpler to work with, one reason for this being that effects that appear at first order in $c_{\text{eff}}^{\nu\mu}$ are necessarily second order in f^μ . In fact, from the form of $c_{\text{eff}}^{\nu\mu}$, it is clear that observable phenomena can only depend on even powers of the f^μ coefficients, and this leads to another curious observation—that the individual components of the f^μ operator do not have physically realizable CPT characteristics. While the operators appear, according to usual properties of Dirac bilinears, to be odd under CPT , there is no physical CPT violation, because odd powers of f^μ never appear in observables.

Moreover, because the theories with f^μ and $c_{\text{eff}}^{\nu\mu}$ are entirely equivalent, it seems that by understanding the quantum corrections in one version of the theory, we should be able to draw conclusions about the $\mathcal{O}(\hbar)$ behavior of the other. Taking the $c^{\nu\mu}$ and f^μ terms to be physically equivalent, it would appear that this equivalence should manifest itself in the renormalization group (RG) scalings of the two operator types. We shall express the RG β -function for a quantity x using $\beta_x = x\Psi(x)$. With this notation, it seems that, at leading order, we ought to have

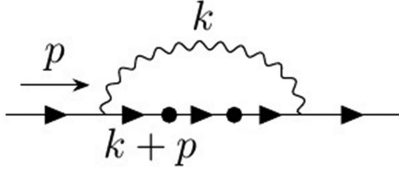


FIG. 1. Divergent one-loop diagram with two Lorentz-violating f^μ vertex insertions on the internal fermion line.

$\Psi(f^\mu) = \frac{1}{2}\Psi(c^{\nu\mu})$, because this would make the RG evolution of $c^{\nu\mu}$ and $f^\nu f^\mu$ the same. However, explicit calculations of β -functions did not initially bear this out. Instead, it was found that (regardless of what type of interactions between fermions were introduced, gauge or Yukawa) β_f seemed to vanish at leading order, while β_c did not.

The reason for this discrepancy was initially somewhat tricky to track down [14]. It seemed clear that at least part of the difficulty had to be related to the fact that the explicit calculations of β_f were only performed to first order in the Lorentz violation coefficient, while the equivalence to the $c^{\nu\mu}$ theory appeared at second order. However, it is not necessarily obvious how this could make a difference in the concrete determination of a β -function. The ultimate reason this issue arises turns out to be that there are different ways that the counterterms may be chosen to cancel the divergences in the theory with nonzero f^μ coefficients. A one-loop fermion self-energy diagram (Fig. 1) with two f^μ insertions on the internal fermion line is, as naively expected, divergent. The divergence can be canceled in either of two different ways: by a single $\delta c^{\nu\mu}$ counterterm, of the type shown in Fig. 2, or by the pair of diagrams shown in Fig. 3, each with two insertions—one of f^ν and one of δf^μ . Since the prefactors of the divergent logarithms appearing in the counterterms determine the one-loop β -functions for the associated operators, this means that there are two different ways for the SME coefficients to evolve under changes of the renormalization scale.

In fact, there are more than just two choices for how to renormalize the theory. Linear combinations of the $\delta c^{\nu\mu}$ diagram and the paired δf^μ diagrams may be used to eliminate the divergences. This means that there is actually

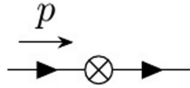


FIG. 2. Diagram with a single $\delta c^{\nu\mu}$ fermion insertion to cancel the $\mathcal{O}(f^2)$ divergence in the diagram from Fig. 1.



FIG. 3. The two counterterm diagrams that together cancel the divergence with alternate f^ν and δf^μ insertions.

a continuous family of one-loop renormalization schemes, with corresponding continuous families of β -function pairs. In a theory in which the force carriers exchanged between fermions are Yukawa bosons, the divergence in Fig. 1 may be canceled by any combination of counterterms

$$\delta c^{\nu\mu} = -\frac{Xg^2}{6}\eta f^\nu f^\mu, \quad (10)$$

$$\delta f^\mu = -\frac{(1+2X)g^2}{6}\eta f^\mu, \quad (11)$$

where g is the Yukawa coupling, and

$$\eta = \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}\mathcal{M}^\epsilon} \quad (12)$$

is the logarithmic divergence. Any value of the parameter X will work. $X = 0$ corresponds to the choice discussed previously, motivated by the observation that the f^μ is physically equivalent to a $c^{\nu\mu}$.

Note that, if $\delta c^{\nu\mu}$ is nonzero, the theory must actually contain a tree-level $c^{\nu\mu}$ coefficient as well as an f^μ , and there must be an additional contribution to (10) corresponding to the self-renormalization of the $c^{\nu\mu}$ operator. Including the contributions from both types of SME terms, the RG β -functions depend on the scheme parameter X according to

$$\beta_{c^{\nu\mu}} = \frac{2g^2}{3(4\pi)^2}c^{\nu\mu} - \frac{Xg^2}{3(4\pi)^2}f^\nu f^\mu, \quad (13)$$

$$\beta_{f^\mu} = \frac{(1-X)g^2}{3(4\pi)^2}f^\mu. \quad (14)$$

Although equivalence between the f^μ and $c^{\nu\mu}$ coefficients is exact, according to the first form stated in (9), the specific relations between the scheme-dependent β -functions are not themselves exact, but are valid only in our small $|f^2|$ approximation. The scheme dependences of the RG functions would be more complicated in the exact theory.

In the theory with both $c^{\nu\mu}$ and f^μ appearing in the Lagrangian, the coefficients cannot actually be measured independently. Up to the order we have considered, physical quantities such as the fermion energy-momentum relation will only depend on the particular linear combination $c_{\text{eff}}^{\nu\mu} = c^{\nu\mu} - f^\nu f^\mu/2$. According to (13) and (14), the RG flow for this physically observable combination is given by

$$\frac{\partial}{\partial(\log p/\mathcal{M})}\left(c^{\nu\mu} - \frac{1}{2}f^\nu f^\mu\right) = \frac{2g^2}{3(4\pi)^2}\left(c^{\nu\mu} - \frac{1}{2}f^\nu f^\mu\right), \quad (15)$$

independent of X . The physical quantity $c_{\text{eff}}^{\nu\mu}$ has the same RG evolution as $c^{\nu\mu}$ itself would have in a version of the theory with no f^μ present at all—a fact which, with a bit of careful consideration, should come as no surprise.

So the RG functions for certain operators in the SME are not unique and instead depend on the renormalization scheme, although not in a way that seems to lead to any direct inconsistencies. In fact, it is not unknown for RG β -functions to depend on the renormalization scheme even in conventional Lorentz-invariant QFTs. What is striking here, however, is that the scheme dependence arises already at one-loop order. Even at the classical level, the description of the fermion sector contain a redundancy, since it is possible to exchange a f^μ coefficient for the equivalent $c^{\nu\mu}$ in tree-level calculations. It is not so surprising then that the redundancy has a manifestation at the quantum level as well.

III. DISCRETE OPERATORS IN THE SME

Moreover, there is another manifestation of the ambiguity and the peculiar radiative corrections associated with it. The Lagrange density $\mathcal{L}_c$ is manifestly even under the discrete C and CPT symmetries. This follows from the usual transformation properties of the vector bilinear in the Dirac theory; normally, objects with even numbers of Lorentz indices are expected to be CPT even. However, the equivalent theory described by $\mathcal{L}_f$ appears superficially to be CPT odd, because it involves an operator with a single free index. The necessary resolution is that the physical discrete C , P , and T operators need to be transformed in accordance with (8). The operators no longer take their usual forms, but in the theory with f^μ , they depend explicitly on the f^μ coefficients, and this creates another interesting situation to explore.

For detailed calculations, we shall focus solely on the parity operator. The reasons for this are purely pragmatic. The actions of charge conjugation and time reversal involve complex conjugations, and so the matrix forms of the corresponding operators depend on the representation of the Dirac matrices. In contrast, the parity operator in the standard theory is always γ^0 , independent of how the Dirac matrices are chosen.

Another comment about the representation of the Dirac matrices is in order at this point. The field redefinitions that transfer Lorentz violation from the f^μ to $c^{\nu\mu}$ operators may be reinterpreted as transformations of the Dirac matrices. In essence, a theory in which the only source of Lorentz violation is a f^μ term is really a theory with a $c^{\nu\mu}$ term and a different choice of representation for the Clifford algebra of Dirac matrices.

The explicit matrix form of the parity operator in the $\mathcal{L}_f$ theory is [15]

$$P\psi P = S_P\psi(t, -\vec{x}) = \left[\gamma^0 + i \frac{f^0}{\sqrt{1-f^2}} \gamma_5 + \frac{(1-\sqrt{1-f^2})f^0 f^\nu}{f^2 \sqrt{1-f^2}} \gamma_\nu \right] \psi(t, -\vec{x}). \quad (16)$$

If the f^μ is transformed away according to (8), S_P transforms into the usual form $S'_P = \gamma^0$. Note also that $S_P = \gamma^0 S_P^\dagger \gamma^0 \equiv \bar{S}_P$, so the parity transformation matrix is the same whether it is S_P acting to the left of ψ , as in (16), or $\bar{S}_P$ acting to the right of $\bar{\psi}$.

Although the form (16) is exact, it may be convenient to take the leading order (in f^μ) approximation, in light of the observed smallness of physical Lorentz violation,

$$S_P \approx \gamma^0 + i f^0 \gamma_5. \quad (17)$$

Another simplification occurs if the axial vector f^μ is lightlike. In that case, when the square roots in (16) are expanded as power series in f^2 , the series terminate with their leading terms, giving

$$S_P = \gamma^0 + i f^0 \gamma_5 + \frac{1}{2} f^0 f^\nu \gamma_\nu. \quad (18)$$

We can now make more concrete the previous observation that the operators parametrized by the f^μ coefficients do not typically have well-defined behaviors under discrete spacetime symmetries. With the modified P operation (16) a spacelike f^j term in the Lagrange density is neither even nor odd under parity.

What is potentially most consequential about the modified P (and C and T) is that it can have a nontrivial RG evolution. If the renormalization scheme is chosen with $X \neq 1$, so that $\beta_f \neq 0$, S_P will have the $\mathcal{O}(f)$ RG scale dependence

$$\frac{\partial}{\partial(\log p/\mathcal{M})} S_P \approx i \gamma_5 \frac{\partial f^0}{\partial(\log p/\mathcal{M})} = i \gamma_5 \frac{(1-X)g^2 f^0}{3(4\pi)^2}. \quad (19)$$

So the matrix operator implementing parity is manifestly dependent on the scale unless $X = 1$.

For solving the RG equation, the key quantity is the ratio between $\Psi(f^\mu)$ (governing the scale evolution of the Lorentz violation coefficient) and $\Psi(g)$ for the evolution of the Yukawa coupling, in the absence of the SME terms. The conventional RG behavior of g is

$$g(p) = \frac{g}{\left[1 - \frac{10g^2}{(4\pi)^2} (\log p/\mathcal{M}) \right]^{1/2}}. \quad (20)$$

The one-loop RG evolution of a SME operator x will be set by the denominator in (20), raised to a power proportional to $\Psi(x)/\Psi(g)$. In particular, with the β -function (14),

$$f^\mu(p) = \frac{f^\mu}{\left[1 - \frac{10g^2}{(4\pi)^2} (\log p/\mathcal{M})\right]^{(1-X)/30}}. \quad (21)$$

This is sufficient for the solution of the approximate RG equation (19), but it is actually possible to do quite a bit better.

At this one-loop order, the scaling is the same for all the components of f^μ , which means that the preferred space-time direction is unchanged by the action of the RG. Moreover, the fact that all the components of $f^\mu(p)$ have the same scale dependence means that the evolution of the f^μ -exact expression (16) for S_P is straightforward. The γ_5 term inside the brackets in (16) has f^0 times a power series in f^2 and thus contains all the odd powers of f^μ components. The γ_ν term contains all the even powers, with $f^0 f^\nu$ times a different power series in f^2 . The RG dependence of each term is set simply by the integer degree at which the f^μ components appear in it. For the lightlike case, the evolution of the terminating expression (18) is particularly simple.

IV. IMPLICATIONS

Whether this observation—that C , P , and T can exhibit nontrivial RG behavior—has any physical significance may initially seem unclear. Indeed, we may be inclined to think not, since the RG evolution is explicitly scheme dependent. However, it is certainly a peculiar observation meriting more detailed analysis. There are many features of the SME, particularly at the quantum level, that are not what we might have expected, based solely on experience with conventional Lorentz-invariant theories, and this is another such a feature. There is, moreover, at least one somewhat analogous situation that has definite physical importance—related to the tricky question of the proton spin and how relatively little of the total nucleon spin (or axial charge) may appear to derive from the spins of the underlying quarks. A fermion spin operator may naively appear to be purely a matrix operator Σ_0 in spinor space, therefore having no scale dependence. However, because of the chiral anomaly, the matrix elements of the total quark axial charge are modified to [16,17]

$$\Sigma = \Sigma_0 - \frac{N_f \alpha_s}{2\pi} \Delta G, \quad (22)$$

and both the strong coupling α_s and the gluon polarization ΔG are scale-dependent quantities. While the anomalous scale dependence of the axial charge turned out to be a relatively minor part of the solution to the “proton spin crisis,” it does provide a striking example where the scale dependence of a seemingly fixed matrix operator has direct physical consequences.

It also turns out that something similar (although not so unequivocal, because of the renormalization scheme

dependence) may be said about the RG flow of the C , P , and T operators in the SME with a fermion f^μ term. While the RG flow of these operators differs from that of Σ , in that the scale dependence can always be eliminated by a choice of X , the observation that there are schemes in which these operators have nontrivial RG behavior is nonetheless important—because there are situations in which considering a theory with f^μ as the only source of Lorentz violation may actually be quite useful. Any fermion theory with a $c^{\nu\mu}$ tensor that is a simple tensor product of two four-vectors may be reformulated in terms of a f^μ instead, reducing the number of SME parameters in the fermion sector from nine to only four.

This reformulation also simplifies the forms of some important observables in the theory. The impact of $c^{\nu\mu}$ -type Lorentz violation on the behavior of ultrarelativistic fermions is controlled primarily by the maximum achievable velocity (MAV) of the particles’ species. At these high energies, decay and collision processes are essentially always nearly collinear, because of relativistic beaming. The kinematics and dynamics look very similar to those of standard $(1+1)$ -dimensional special relativity, only with the uniform speed of light replaced by a direction-dependent MAV, which is (up to first order in $c^{\nu\mu}$) [18]

$$v_{\text{MAV}} = 1 - c_{00} - c_{(0j)} \hat{v}_j - c_{jk} \hat{v}_j \hat{v}_k. \quad (23)$$

$\hat{v}$ is a unit three-vector in the direction of the motion, and $c_{(0j)}$ is the symmetrized combination $c_{(0j)} = c_{0j} + c_{j0}$. If instead of using $c^{\nu\mu}$, the theory is formulated with the corresponding f^μ (satisfying $c^{\nu\mu} \approx -\frac{1}{2} f^\nu f^\mu$), the expression for the MAV is more compact and seems to have a clearer physical meaning,

$$v_{\text{MAV}} = 1 + \frac{1}{2} (f^\mu \hat{v}_\mu)^2, \quad (24)$$

where $\hat{v}^\mu$ is now a lightlike four-vector $(1, \hat{v})$.

Owing to the simplicity of this expression, using the equivalent theory with f^μ as the only source of fermionic Lorentz violation can be extremely advantageous when making measurements of v_{MAV} —provided the $c^{\nu\mu}$ matrix may be assumed to have the form of a tensor product. And it actually turns out that v_{MAV} is frequently the natural object of study when looking at how high-energy observations can be used to test Lorentz symmetry. Because of gauge invariance, any charged particle minimally coupled to the electromagnetic field interacts with photons through its velocity [19], and this makes v_{MAV} a natural observable in high-energy electron-photon interaction processes. Measurements of the synchrotron and inverse Compton spectra from high-energy electrons (whether in terrestrial laboratories of energetic astrophysical environments) may be used to place constraints on v_{MAV} for the electron field [20]. Similarly, the observed absences of processes that are

forbidden in conventional relativistic physics, such as vacuum Cerenkov radiation, $e^- \rightarrow e^- + \gamma$, or photon decay via $\gamma \rightarrow e^- + e^+$ also give bounds on the electron v_{MAV} [21,22].

So there can certainly be situations when it may be notationally and computationally advantageous to choose a theory with only f^μ -type (as opposed to $c^{\nu\mu}$ -type) Lorentz violation. In such cases, it is also clearly the most convenient to choose a renormalization prescription with $X = 0$, so that if there are no $c^{\nu\mu}$ coefficients at tree level, none will be generated by one-loop quantum corrections either. In that case, the RG-improved fermion propagator, to the order represented by the $X = 0$ solution (21) is

$$S_F^{(1,f)}(q) = \frac{i}{\not{p} + i[f(\sqrt{q^2}) \cdot q]\gamma_5 - m}, \quad (25)$$

and if we would like to understand the discrete symmetry characteristics of intermediate states with virtual (off-shell) fermion excitations, it is necessary to use the RG-evolved C , P , and T operators.

The coefficient of the Lorentz-violating term in the propagator $S_F^{(1,f)}$ depends on q^2 , and this helps with the understanding of what the scale dependence of S_P means. The quantity q^2 is a measure of how off shell a virtual

particle is. The C , P , and T operations on real fermions, which all have $q^2 = m^2$, are not affected by the RG flow—which is reassuring from a physical viewpoint. However, the virtual quanta exchanged along internal lines in a Feynman diagram behave differently under the discrete transformations when the actions of C , P , and T on the entire diagram are considered. For these virtual intermediates, this is an authentic physical effect if $X \neq 1$, not a mathematical artifact. We may be used to having the action of these operations on real and virtual Dirac spinors be the same, but there is evidently nothing that requires this to be the case when Lorentz symmetry is broken.

The standard Feynman propagator has very natural behavior under P . The spatial components of the four-momentum q are inverted, while the time component is left unchanged,

$$S_P S_F^{(0)} S_P = S_P \left(\frac{i}{\not{q} - m} \right) S_P = \frac{i}{q^0 \gamma^0 + \vec{q} \cdot \vec{\gamma} - m}. \quad (26)$$

With the inclusion of $c^{\nu\mu}$ coefficients, the generalization is straightforward. The terms in the denominator of the propagator with c^{00} and c^{jk} are unchanged, but the c^{0j} term changes sign,

$$S_P S_F^{(0,c)} S_P = S_P \left(\frac{i}{\not{q} + c^{00} q^0 \gamma^0 - c^{0j} q^j \gamma^0 - c^{j0} q^0 \gamma^j + c^{jk} q^k \gamma^j - m} \right) S_P \quad (27)$$

$$= \frac{i}{\not{q} + c^{00} q^0 \gamma^0 + c^{0j} q^j \gamma^0 + c^{j0} q^0 \gamma^j + c^{jk} q^k \gamma^j - m}. \quad (28)$$

(The behavior under reflection across a single plane, inverting just one spatial coordinate—in contrast to P , which inverts all three—is slightly more complicated.)

To get equivalent parity behavior in a f^μ theory that is itself equivalent to a tensor product $c^{\nu\mu}$ theory, it is necessary to use the modified parity operator of (16) to act on

$$S_F^{(0,f)} = \frac{i}{\not{q} + i(f \cdot q) \gamma_5 - m}. \quad (29)$$

With the unmodified parity operation, all four components of the $i(f \cdot q) \gamma_5$ in the denominator would change sign, because $\{\gamma_5, \gamma^0\} = 0$. It is necessary to use the (correct at leading order) $S_P = \gamma^0 + i f^0 \gamma_5$ instead to get the proper result,

$$S_P S_F^{(0,f)} S_P = \frac{i}{\not{q} + i f^0 p^0 \gamma_5 - i(\vec{f} \cdot \vec{q}) \gamma_5 - m}, \quad (30)$$

instead. The unchanged $i f^0 q^0 \gamma_5$ component only appears thanks to the additional term in S_P and the way it anticommutes with $\not{q}$; the invariance of this term depends on the $\mathcal{O}(f)$ addition to S_P having the precise coefficient it does.

We can now see that if we instead use the RG-improved propagator $S_F^{(1,f)}$ from (25) for a virtual fermion intermediate, it will not have the correct parity behavior (inverting $\vec{q}$ but leaving q^0 unaffected) unless the parity matrix S_P takes the form $S_P = \gamma^0 + i f^0 (\sqrt{q^2}) \gamma_5$ at leading order, with the scale-dependent $f^0(p)$ from (21). In other words, in a formulation of the theory with f^μ coefficients as the only source of fermion-sector Lorentz violation, it is absolutely necessary to have a parity operator P with nontrivial RG scale dependence in order to have the correct behavior under discrete symmetries.

Physically, the modifications to S_P must be small. With a coupling constant $g \sim 10^{-2}$ and $|f^\mu| \lesssim 10^{-7}$ —as inferred

from the electron v_{MAV} data—the fractional change in S_P with a change of scale is $\lesssim 10^{-14}(\log q^2/m^2)$. However, for heavier fermions—especially strongly interacting heavy quarks—the experimentally established suppression of the RG flow for S_P is much less severe, at a $\lesssim 10^{-4}(\log q^2/m^2)$ level potentially accessible in precision experiments. Moreover, regardless of the experimental status, the potential scale dependence of S_P demonstrates the theoretical possibility of an interesting new feature in QFTs.

The same essential arguments applied to P hold, rather straightforwardly, for C and T ; only keeping track of additional complex conjugation operators is necessary. We can therefore conclude that there are definite situations in which the nontrivial RG scalings of C , P , and T are—if not unavoidable, since we can always choose a formulation of the fermion sector without a RG-evolving f^μ (or without any f^μ at all)—potentially quite useful and relevant for

physically interesting calculations. This shows again that theories with broken Lorentz symmetry may be interesting theoretical laboratories for understanding just what kinds of behavior are permitted in QFT. Especially at the quantum level, these theories have shown interesting types of behavior that are not normally seen in conventional Lorentz-invariant QFTs.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the author upon reasonable request.

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