

Holographic timelike c -function

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The integration of high-energy degrees of freedom along the renormalization group (RG) flow in Poincaré-invariant theories is captured by a monotonic c -function. For such theories, holographic c -functions have been constructed using entanglement entropy. However, in theories with broken Lorentz invariance, these constructions generally fail, reflecting both the violation of entanglement RG monotonicity and its limitations in capturing essential features of nonrelativistic RG flows. Since many quantum many-body systems lack Lorentz invariance, it is important to identify a quantity that captures the decrease of effective degrees of freedom along such flows. We show that the recently introduced holographic timelike entanglement entropy naturally gives rise to a new c -function applicable to all such theories, which moreover obeys a geometric upper bound on its rate of change. We further demonstrate the existence of this c -function in theories with Lifshitz and hyperscaling-violating fixed points, showing that, provided the null energy condition and thermodynamic stability are satisfied, it obeys the expected RG monotonicity.

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I. INTRODUCTION

The renormalization group (RG) flow in quantum field theories reveals a rich structure and serves as a powerful tool for their analysis. One of its most well-known features is the existence of a so-called c -function, which decreases along RG flows and coincides with the central charge at the conformal fixed points of the theory. This property was first proven in two-dimensional quantum field theories [1] and was later reformulated in terms of entanglement entropy, a quantity that measures how entangled a given spatial subregion is with its surroundings.

A generalization to even spacetime dimensions was conjectured in [2] and proven in [3] for RG flows between conformal fixed points. For certain conformal field theories (CFTs) in arbitrary dimensions, generalized c -theorems have been established within the holographic framework [4], prompting extensive studies and reformulations using entanglement entropy [5], particularly in the context of holography [6–12].

An intuitive understanding of these results comes from interpreting the central charge as a measure of the number of degrees of freedom in the theory. Since RG flow involves integrating out high-energy degrees of freedom [13],

a decrease in this measure along the flow is expected. A generalized version of the holographic c -theorem is anticipated to hold in theories with nonrelativistic fixed points, which are common in quantum matter and many-body condensed matter systems. Such generalizations have been explored holographically [12,14], though a universally appropriate c -function for these systems remains elusive. This absence stems from a deeper reason: it is known that RG flow can violate the monotonicity of entanglement entropy under RG transformations [15].

This is not unexpected, since entanglement entropy is defined on a fixed time slice and thus may not capture the full dynamics of nonrelativistic RG flows. As a result, it is not surprising that it fails to support the existence of a proper c -function. This observation raises a fundamental question: does there exist a quantity that respects RG flow monotonicity in these broader contexts, and if so, how can it be applied to the physics of quantum matter?

In this work, we introduce a novel c -function based on the recently proposed timelike entanglement entropy in holography [16,17]. This quantity is defined holographically as the area of extremal surfaces—spacelike and timelike—that are anchored to a time subinterval on the boundary [16,18]. Remarkably, we show that the timelike c -function constructed from this pseudoentropy is monotonic along holographic RG flows. This monotonicity arises nontrivially from the geometric properties of the extremal surfaces involved.

Our proposed c -function maintains the correct monotonic behavior even in physical theories with broken

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Lorentz or scale invariance. Specifically, we show that the derivative of the timelike c -function along the RG flow can be expressed in a compact way in terms of the bulk geometry of the holographic dual. As a general test, we verify the most general homogeneous spaces with monomial metric elements, namely with Poincaré invariance, anisotropic Lifshitz scalings, and scale covariant hyperscaling violation all admit a monotonic c -function under RG flow, provided that the null energy conditions (NEC), thermodynamic stability conditions, and an effective dimension condition are satisfied. These theories are the most interesting and physically relevant due to the symmetries they possess. Moreover, we emphasize that in the most general fully arbitrary Lorentz violating backgrounds, NEC and thermodynamic stability do not always map one to one onto the conditions required for timelike entanglement entropy monotonicity. This is not unexpected; such fully arbitrary backgrounds typically cannot arise as solutions from physically motivated bulk actions. In fact the satisfaction of the timelike c -theorem may suggest that only specific subclasses of these anisotropic geometries, like those we analyzed, may arise as well-behaved consistent holographic theories.

II. HOLOGRAPHIC SETUP

We consider the homogeneous holographic spacetime in the following coordinate system

$$ds_{d+1}^2 = -e^{2B(r)} dt^2 + e^{2A(r)} dx^2 + dr^2, \quad (1)$$

where $d-1$ is the number of dimensions of the spatial plane spanned by $\vec{x}$ and the boundary of the spacetime is taken to be at $r \rightarrow \infty$, without loss of generality. Static homogeneous spacetimes can always be diagonalized and brought to the form of (1); therefore, this represents the most generic case.

For a boundary time interval $-\frac{T}{2} \leq t \leq \frac{T}{2}$, with $x_1 = 0$ and the remaining coordinates $x_i = \sigma_i$, the bulk surface is parametrized by $t(r)$. The action corresponding to the timelike entanglement entropy is given by

$$S = 2 \int dr e^{(d-2)A(r)} \sqrt{1 - e^{2B(r)} t'^2}, \quad (2)$$

with the associated equations of motion

$$t_s = 2 \int dr \frac{e^{\Lambda_m - B}}{\sqrt{s e^{2\Lambda(r)} + e^{2\Lambda_m}}}, \quad s^2 := 1. \quad (3)$$

The positive value $s = +1$ corresponds to the spacelike branch of the surface, while $s = -1$ corresponds to the timelike branch. Here, $\Lambda(r) := B(r) + (d-2)A(r)$ and

$\Lambda_m := \Lambda(r_m)$, where r_m is the turning point of the symmetric timelike surface at which $t'_-(r)$ diverges. The turning point enters the equation of motion via the integration constant. Substituting (3) into (2), we obtain the area in terms of the geometry,

$$S_s = 2 \int dr \frac{e^{2\Lambda-B}}{\sqrt{e^{2\Lambda} + s e^{2\Lambda_m}}}, \quad (4)$$

where S_s denotes the total timelike entanglement entropy. As before, the subscript $s = +1$ corresponds to the spacelike part of the bulk surface described by the solution $t_+(r)$, contributing the real part of the timelike entanglement entropy S_+ . The integration bounds in this case lie between r_∂ and r_b , where r_∂ denotes the boundary of the theory and r_b is the deepest point in the bulk reached by the extremal surface. For example, in Anti-de Sitter (AdS) space in Poincaré coordinates, with the boundary at $r_\partial \rightarrow \infty$, we have $r_b = 0$.

The subscript $s = -1$ corresponds to the timelike portion of the surface, described by the solution $t_-(r)$, and contributes the imaginary part of the timelike entanglement entropy S_- . The timelike surface extends from the turning point r_m to r_b . The imaginary and real contributions of the timelike surface are denoted by $S_{\text{Im}} := iS_-$ and $S_{\text{Re}} := S_+$, respectively. Similarly, the time integrals describing these surfaces are denoted by $t_{\text{Re}} := t_+$ and $t_{\text{Im}} := t_-$, providing a unified notation.

To describe the common class of solutions studied in [16,18], the spacelike surface t_{Re} originates from the boundary at $t = T/2$ (with $r = r_\partial \rightarrow \infty$), and in theories without horizons or IR walls, it asymptotes to the deep infrared as $r = r_b \rightarrow 0$, reaching $t = T_{\text{Re}}/2 \rightarrow \infty$. The right, symmetric branch of the timelike surface t_{Im} begins at the mirror-symmetric extremal point of the surface at $t = 0$ (with $r = r_m$), and similarly, in the absence of horizons or IR walls, it asymptotes to the deep IR as $r = r_b \rightarrow 0$, with $t = T_{\text{Im}}/2 \rightarrow \infty$.

III. HOLOGRAPHIC TIMELIKE c -FUNCTION

The timelike entropy supports a holographic, monotonic c -function for any type of homogeneous theory. To construct a suitable function, we consider the derivatives of t_s and S_s , corresponding to the timelike and spacelike surfaces, with respect to the turning point r_m of the timelike surface. We will demonstrate that a unique and naturally motivated c -function exists.

We begin by analyzing the timelike surface. Applying the Leibniz integral rule yields the derivative of the area,

$$\frac{1}{2} \frac{\partial S_{\text{Im}}}{\partial r_m} = \frac{e^{2\Lambda-B}}{\sqrt{e^{2\Lambda_m} - e^{2\Lambda}}} \Big|_{r_m} + \Lambda'_m e^{2\Lambda_m} \int_{r_m}^{r_b} dr \frac{e^{2\Lambda-B}}{(e^{2\Lambda_m} - e^{2\Lambda})^{\frac{3}{2}}}. \quad (5)$$

Similarly for the time interval we obtain

$$\frac{1}{2} \frac{\partial t_{\text{Im}}}{\partial r_m} = \frac{e^{\Lambda-B}}{\sqrt{e^{2\Lambda_m} - e^{2\Lambda}}} \Big|_{r_m} + \Lambda'_m e^{\Lambda_m} \int_{r_m}^{r_b} dr \frac{e^{2\Lambda-B}}{(e^{2\Lambda_m} - e^{2\Lambda})^{\frac{3}{2}}}. \quad (6)$$

By combining Eqs. (5) and (6), we obtain the derivative of the timelike part of the area with respect to its time interval,

$$\frac{\partial S_{\text{Im}}}{\partial r_m} = e^{\Lambda_m} \frac{\partial t_{\text{Im}}}{\partial r_m} \Rightarrow \frac{\partial S_{\text{Im}}}{\partial t_{\text{Im}}} = e^{\Lambda_m}. \quad (7)$$

For the spacelike surface, we follow a similar strategy to compute the desired derivative,

$$\frac{\partial S_{\text{Re}}}{\partial t_{\text{Re}}} = -e^{\Lambda_m}. \quad (8)$$

Here, the minus sign arises because the boundary of these integrals is independent of r_m .

We observe that the derivatives of the spacelike and timelike areas with respect to their corresponding time intervals are equal in absolute value. Motivated by this observation and the holographic definition of the c -function via entanglement entropy [6,11,12,27,28], we propose a candidate monotonic timelike c -function based on the timelike entanglement entropy. The appropriate function is,

$$c = c_{\text{Re}} + c_{\text{Im}}, \quad (9)$$

with

$$c_{(\text{Im,Re})} := t_{(\text{Im,Re})}^{d_x} \frac{\partial S_{(\text{Im,Re})}}{\partial t_{(\text{Im,Re})}} \quad (10)$$

where d_x is a constant with physical significance, which we will later relate to an effective dimensionality of the observables in the theory.

We can reexpress the c -function (9) in terms of the boundary interval T , by absorbing the usual positive normalization constants into the definition,

$$c = T^{d_x} \frac{\partial S}{\partial T}, \quad (11)$$

with $S = S_{\text{Re}} + S_{\text{Im}}$ and $T = t_{\text{Im}} - t_{\text{Re}}$. The expressions of the c -function of (10) and of (11) have the same monotonicity along the RG flow and are equivalent.

To investigate the monotonicity of the c -function along the RG flow in generic holographic theories, we analyze the derivatives of its components. It is crucial to note that the time interval can be rewritten as,

$$t_s = -2 \int dr \frac{e^{\frac{\Lambda}{d_x} - B}}{\Lambda'} \cdot F_s, \quad F_s := \frac{\Lambda' e^{-\frac{\Lambda}{d_x} + \Lambda_m}}{\sqrt{e^{2\Lambda_m} + s e^{2\Lambda}}}. \quad (12)$$

We treat the timelike and spacelike parts separately and study each component individually and fix direction of the integral's boundaries appropriately.

A. The timelike contribution to the c -function

By performing integration by parts on the time interval expression (12), we obtain,

$$-\frac{t_{\text{Im}}}{2} = \frac{e^{\frac{\Lambda}{d_x} - B}}{\Lambda'} \mathcal{F}_- \Big|_{r_m}^{r_b} - \int_{r_m}^{r_b} \frac{e^{\frac{\Lambda}{d_x} - B}}{\Lambda'} \mathcal{F}_- \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right). \quad (13)$$

The monotonicity of c_{Im} is computed from (10) as,

$$\frac{\partial c_{\text{Im}}}{\partial r_m} = t_{\text{Im}}^{d_x-1} e^{\Lambda_m} \left(d_x \frac{\partial t_{\text{Im}}}{\partial r_m} + \Lambda'_m t_{\text{Im}} \right). \quad (14)$$

We then express the derivative of t_{Im} with respect to itself, making use of the fact that the derivative of the $\mathcal{F}$ function can conveniently be rewritten in terms of $\mathcal{F}$. After a lengthy algebraic manipulation, we arrive at,

$$\begin{aligned} \frac{\partial c_{\text{Im}}}{\partial r_m} = & 2e^{\Lambda_m} t_{\text{Im}}^{d_x-1} d_x \Lambda'_m \left[-\frac{t'_{\text{Im } b}}{\Lambda'_b} \right. \\ & \left. + \int_0^{t_{\text{Im}}} dt \frac{1}{\Lambda'} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right) \right], \end{aligned} \quad (15)$$

where we have traded the r integral using the differential of the time-interval length from (3), and we emphasize that the derivative of the surface, $t'_{\text{Im } b}$, evaluated at the bulk point r_b , appears explicitly in the expression.

B. The spacelike contribution to the c -function

We now consider the spacelike contribution. Performing integration by parts on the time interval (12), the resulting expression involves $\mathcal{F}_+$. We proceed in a similar way as in the previous subsection. Using the properties of the derivatives of $\mathcal{F}_+$, we relate the derivative of t_{Re} to the function itself. After some algebra, the derivative of the spacelike component of the c -function from (10) eventually takes the following form:

$$\begin{aligned} \frac{\partial c_{\text{Re}}}{\partial r_m} = & 2e^{\Lambda_m} t_{\text{Re}}^{d_x-1} d_x \Lambda'_m \left[\left(\frac{t'_{\text{Re } b}}{\Lambda'_b} - \frac{t'_{\text{Re } \partial}}{\Lambda'_\partial} \right) \right. \\ & \left. - \int_{\frac{T}{2}}^{t_{\text{Re}}} dt \frac{1}{\Lambda'} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right) \right]. \end{aligned} \quad (16)$$

Here, T is the time interval at the point where the surface originates from the boundary of the theory, and $T_{\text{Re}} \rightarrow \infty$ denotes the time interval as $r \rightarrow r_b$, where the surfaces

asymptote to the IR. Notice the crucial detail that the expression in the first line depends on the derivatives t'_{Re} evaluated at $r = r_b$ and $r = r_\partial$.

C. The c-theorem

The monotonicity of the c -function (9) is determined by the sum of the derivatives of the spacelike and timelike contributions. The fact that these two contributions are intertwined stems from the asymptotic convergence of the spacelike and timelike surfaces. This becomes evident from the expression for the gradient of the normal vector along both surfaces [19], which reads,

$$|\mathcal{T}|^2 = -\frac{e^{2A(d-1)}}{e^{2\Lambda} + s e^{2\Lambda_m}}. \quad (17)$$

The smooth merging of the surfaces occurs at a radial position r_b . In this limit, $e^{2\Lambda_b} \rightarrow 0$, leading to $|\mathcal{T}|^2 \sim -s^{-1}$, and the transition from timelike to spacelike surfaces happens smoothly, preserving the correct signature of the normal vector field. This, in turn, imposes the following condition on the derivatives of t at the merging point in the deep IR:

$$t'^2_{\text{Re } b} \simeq t'^2_{\text{Im } b}. \quad (18)$$

Moreover, in the absence of horizons or IR walls, we have $T_\infty := T_{\text{Re}} \simeq T_{\text{Im}} \simeq \infty$, reflecting the fact that both surfaces asymptotically approach the IR. Using these properties, and inserting the expressions for the timelike (15) and spacelike (16) contributions, the derivative of the total c -function (9) becomes,

$$\begin{aligned} \frac{\partial c}{\partial r_m} &= 2e^{\Lambda_m} T_\infty^{d_x-1} d_x \Lambda'_m \left[\frac{1}{\Lambda'_b} (t'_{\text{Re } b} - t'_{\text{Im } b}) - \frac{1}{\Lambda'_\partial} t'_{\text{Re } \partial} \right. \\ &\quad \left. + \int_0^{\frac{T}{2}} dt \frac{1}{\Lambda'} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right) \right]. \end{aligned} \quad (19)$$

The first two terms cancel due to the condition (18), while the remaining term on the first line vanishes because the surface intersects the boundary orthogonally. This can also be seen from (3), since $e^{A_\partial} \rightarrow \infty$. Furthermore, to obtain (19), we combined the integrals as follows: $\int_0^{\frac{T_\infty}{2}} dt(\dots) - \int_{\frac{T}{2}}^{\frac{T_\infty}{2}} dt(\dots) = \int_0^{\frac{T}{2}} dt(\dots)$. Eventually, the derivative of the c -function can be compactly written as,

$$\frac{\partial c}{\partial r_m} = 2e^{\Lambda_m} T_\infty^{d_x-1} d_x \Lambda'_m \int_0^{\frac{T}{2}} dt \frac{1}{\Lambda'} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right). \quad (20)$$

This expression is subject to the natural holographic condition at the boundary,

$$\frac{t'_{\text{Re } \partial}}{\Lambda'_\partial} = 0. \quad (21)$$

Equation (20) is one of the main results of this work. It yields the c -theorem,

$$\frac{\partial c}{\partial r_m} \geq 0, \quad (22)$$

for a UV boundary located at infinity in the holographic spacetime. This inequality translates into a constraint on the geometric components of the metric, as encoded in (20). The condition takes an averaged form: the integral in (20), multiplied by $d_x \Lambda'_m$, determines the nature of the monotonic behavior of the c -function.

However, a stronger version of this condition can also be imposed: namely, that the integrand in (20), when multiplied by $d_x \Lambda'_m$, maintains a definite sign pointwise along the radial flow. This stricter requirement leads to a fully general formulation of a holographic c -theorem. Moreover, the c -function in (20) exhibits an automatic upper bound on its rate of change along the RG flow,

$$0 \leq \partial_{r_m} c \leq e^{\Lambda_m} T_\infty^{d_x-1} T \Lambda'_m, \quad (23)$$

as discussed in Appendix A.2, thereby constraining how rapidly effective degrees of freedom can be coarse-grained in consistent holographic theories.

D. NEC and thermodynamic stability conditions

Holographic theories, especially those with broken Lorentz invariance, may suffer from instabilities. To avoid such pathologies, we impose two minimal physical requirements: the null energy condition and thermodynamic stability.

The NEC for the metric fields along the holographic RG flow serves as a minimal consistency condition ensuring causal propagation, nonrepulsive gravity, avoiding instabilities and superluminal modes [29–32]. We obtain two independent conditions which can be written as [12,14]

$$B'' - A'' + (B' - A')(B' + (d-1)A') \geq 0, \quad (24)$$

$$-A(r)'^2 + B'(r)A'(r) - A''(r) \geq 0. \quad (25)$$

Thermodynamic stability is analyzed by considering a black hole solution of the metric (1) with the usual blackening factor $f(r)$ satisfying $f(r_h) = 0$ at the black hole horizon $r = r_h$. To avoid a conical singularity, the Euclidean compactified time circle must be identified with a period set by the Hawking temperature,

$$T = \frac{f'(r_h)}{4\pi} e^{B(r_h)}. \quad (26)$$

The thermal entropy density is proportional to the area of the black hole horizon. By absorbing constants and positive-definite prefactors, the normalized specific heat, derived from the thermal entropy S_{th} , is proportional to

$$\frac{\partial S_{th}}{\partial T} \sim \frac{(d-1)A'(r_h)}{\partial_{r_h} f'(r_h) + f'(r_h)B'(r_h)} \geq 0. \quad (27)$$

The three conditions (24), (25), and (27), taken together, guarantee the physical stability and naturalness of the theory.

E. Effective dimensionality

In this subsection, we consider the theory at a fixed radial scale r_f , and therefore at a certain energy scale. The parameter d_x can be thought as an effective spatial dimension of the c -function in the theory we are looking at. The integrand of the derivative of the c -function (20) vanishes at a fixed radial distance r_f when

$$d_{x_r} = \frac{\Lambda'(r_f)^2}{\Lambda''(r_f) + B'(r_f)\Lambda'(r_f)}, \quad (28)$$

and therefore $\partial_{r_m} c$ vanishes. For theories with $\Lambda''(r_f) \geq 0$, the NEC (24) and (25) constrain d_{x_r} to remain positive along the entire RG flow.

The physical meaning of d_x can be understood independently by analyzing the scaling symmetry of the metric. Introducing a new radial coordinate $\tilde{r}(r)$ at a particular scale r_f , the following scaling transformation $t \rightarrow \lambda^{\tilde{\beta}} t$, $x \rightarrow \lambda^{\tilde{\alpha}} x$, $\tilde{r} \rightarrow \lambda^{-\tilde{\delta}} \tilde{r}$, leaves the transformed metric (1)

$$ds^2 = -\tilde{r}^{2\tilde{\beta}} dt^2 + \tilde{r}^{2\tilde{\alpha}} dx_i^2 + \tilde{r}^{2\tilde{\delta}} d\tilde{r}^2, \quad (29)$$

scale covariant, provided $\tilde{\alpha} = \tilde{\delta}(\alpha - 1 - \delta)$ and $\tilde{\beta} = \tilde{\delta}(\beta - 1 - \delta)$. The timelike entanglement entropy then scales as

$$S \sim \tilde{r}^{\alpha(d-2)+\delta+1} \sim t^{-\frac{\alpha(d-2)+\delta+1}{\beta-\delta-1}}. \quad (30)$$

To ensure that the c -function is dimensionless we obtain from (10) and (30) the expression

$$d_x = \frac{\beta + \alpha(d-2)}{\beta - \delta - 1}. \quad (31)$$

For the scale covariant theories, or at fixed points with scale covariance the result (31) matches with a direct use of the Eq. (28) making evident the role of d_{x_r} , or d_x . This allows us to recast the c -function derivative (20) in an even more compact form as

$$\frac{\partial c}{\partial r_m} = 2e^{\Lambda_m} T_\infty^{d_x-1} \Lambda'_m \int_0^{\frac{T}{2}} dt \left(1 - \frac{d_x}{d_{x_r}}\right), \quad (32)$$

with $d_{x_r}(r)$ given by (28).

IV. HOLOGRAPHIC TIMELIKE c -THEOREM

A. Timelike c -theorem in Poincaré invariant theories

Let us consider a conformal theory with $A(r) = B(r)$, with a bulk geometry described by the metric (1). The evolution of the geometry along the RG flow is encoded in the conformal factor $A(r)$. Such geometries are typical solutions of Einstein gravity coupled to a scalar field with an appropriate nontrivial profile along the RG flow, for instance, [4,33]. The scalar dilaton is dual to a relevant operator $\mathcal{O}$, and the RG flow can be thought of as being triggered by perturbing the theory at its UV critical point by this operator. In our analysis, the details of the dilaton's profile and potential are not essential.

The effective dimension can be obtained using (28),

$$d_{x_r} = \frac{(d-1)A'^2}{A'' + A'^2} \quad (33)$$

while the NEC (25) gives: $A'' \leq 0$. The integrand of (20) becomes up to positive factors,

$$\frac{\partial c}{\partial r_m} \sim A'_m \int_0^{\frac{T}{2}} dt \left(1 - \frac{d_x}{d_{x_r}}\right), \quad (34)$$

which implies that, as long as the norm of the effective dimension d_{x_r} decreases along the RG flow—or alternatively, as long as a constant d_x is equal to or smaller than the minimum effective dimension, the c -theorem automatically holds.

To prove this statement, note that the NEC implies A' is a decreasing function along the RG flow. Since we have placed the boundary of the theory at $r \rightarrow \infty$, we have $A_{UV} \geq 0$ and $A'_{\min} = A'_{UV}$. Therefore, it suffices to show that the minimum value of A' at the boundary is positive to ensure the monotonicity of the c -function (34). Indeed, this is the case: since $A_{UV} \sim A(r) \rightarrow \infty$ at the boundary, we have $A'_{\min} \geq 0$, which then implies $A' \geq 0$ along the entire RG flow.

As an explicit example consider the case where at UV, $d_x = d - 1$, which is equal to the lower bound of a positive d_{x_r} , by the saturation of the NEC. The monotonicity of the timelike holographic c -function is then always

$$\frac{\partial c}{\partial r_m} \sim A'_m \int_0^{T/2} dt \frac{(-A'')}{A'^2} \geq 0, \quad (35)$$

which is ensured by the NEC.

Hence, the timelike holographic c -function for a Poincaré invariant theory is monotonic as long as the

NEC are satisfied and the effective dimension decreases. This case also serves as a consistency check, confirming that the timelike c -function is an appropriate choice for Poincaré invariant theories.

B. Timelike c -theorem in nonrelativistic theories

1. Lifshitz theories

Consider a Lifshitz theory where $A = r$ and $B = zr$, where z is the Lifshitz exponent, a parameter characterizing Lorentz violation. Applying the analysis of Sec. III E we find that the metric in Poincaré coordinates has exponents $\alpha = 1$, $\beta = z$ and $\delta = -1$. The effective dimension here is given by (28), or equivalently by (31) and becomes

$$d_x = 1 + \frac{d-2}{z}, \quad (36)$$

while the NEC reduces to the simple inequality $z \geq 1$. When $z = 1$ we recover the usual AdS spacetime and dual to the conformal theory. The NEC constrains $d_{x_r} > 0$ and $\Lambda' = z + d - 2 > 0$. For the trivial, scale-invariant Lifshitz RG flow, the holographic timelike c -function (20), with $d_x = d_{x_r}$ given by (36), yields a vanishing derivative: $\partial_{r_m} c = 0$, as expected. For a nontrivial Lifshitz RG flow, the validity of the c -theorem (32) is guaranteed as long as $d_x \leq d_{x_r}$, since already $\Lambda' > 0$ by the NEC.

Let us elaborate further by considering a theory where the Lifshitz exponent $z_r(r)$ varies very slowly along the RG flow with fixed points characterized by z_{UV} and z_{IR} . d_x is constant and is determined at a fixed point, while d_{x_r} is the effective dimension at a given radial distance. Then (32) approximately gives

$$\frac{\partial c}{\partial r_m} \sim \Lambda'_m \int_0^{\frac{r}{z}} dt \frac{1}{z\Lambda'} (z - z_r) \geq 0. \quad (37)$$

The NEC enforces $\Lambda' \geq 0$ and $z_r(r)$ denotes the slowly varying Lifshitz exponent at r . Therefore, the c -theorem reduces to the simple condition: $z \geq z_r$.

For example, the validity of the c -theorem can be demonstrated in theories with conformal UV fixed points, $z_{UV} = 1$, that satisfy the NEC, so that $z_{\min} = z_{UV}$, and have a slowly decreasing $z_r(r)$. Taking $d_x := d_{x_{IR}}$ and thus $z := z_{IR} = z_{\max}$ ensures $z \geq z_r$ and the c -theorem (37) holds.

In summary, the NEC together with the universal condition $d_x \leq d_{x_r}$ automatically ensures the validity of the c -theorem for Lifshitz RG flows.

2. Hyperscaling violation theories

Next, let us consider a more intricate case: a hyperscaling violation theory characterized by spatial homogeneity and scale covariance, as a stringent test for the c -theorem. Here we have

$$\beta = z - \frac{\theta}{d-1}, \quad \alpha = 1 - \frac{\theta}{d-1}, \quad \delta = \alpha - 2, \quad (38)$$

where z and θ are the dynamical Lifshitz exponent and hyperscaling violation exponent, respectively.

The effective dimension (31) of the theory is

$$d_x = 1 + \frac{d - \theta - 2}{z}, \quad (39)$$

which, when compared with (36), clearly reflects the effect of the hyperscaling violation exponent: the modes propagate effectively in $d - \theta$ spatial dimensions rather than in d , leading to consequences analogous to those seen in the thermodynamics and thermal entropy of the theory.

The c -function evolves along the RG flow as

$$\frac{\partial c}{\partial r_m} \sim \Lambda'_m \int_0^{\frac{r}{z}} \left(1 - \frac{d_x}{d_{x_r}}\right) dt, \quad (40)$$

where $\Lambda'_m = (d-1)(\theta - d - z + 2)/(\theta r_m)$. The NEC (24), (25) and stability conditions (27) impose $d_{x_r} \geq 0$ for any physical hyperscaling violation theory, but now they do not uniquely fix the sign of Λ' . We will show that as long as the natural conditions are satisfied, the c -theorem holds for $d_x \leq d_{x_r}$, as in our previous RG flows.

To establish the existence of the c -theorem, we carefully study each parameter subregime in our coordinate system. There are three consistent subregimes to consider: A) For $\alpha > 1$, we have the range $r \in [0, \infty)$ in the coordinate system (1), with a boundary ∂ at ∞ . Once the NEC and the thermodynamic stability conditions are satisfied, the function Λ'_m is automatically positive and therefore, from (40), the derivative $\partial_{r_m} c \geq 0$. The c -theorem holds for the range of parameters $\alpha > 1$.

For the range of parameters described by B) $\alpha < 0$ the boundary in the coordinate system (1) is again at infinity. The NEC and the thermodynamic stability then impose $\Lambda'_m \geq 0$ and therefore $\partial_{r_m} c \geq 0$. The c -theorem is satisfied for this case as well.

For subregime C), where $0 < \alpha < 1$ the boundary is at the minimum radial value, $r = 0$. The NEC and the thermodynamic stability now conveniently impose the opposite conditions compared to previous cases: $\Lambda'_m \leq 0$, and hence, from (40): $\partial_{r_m} c \leq 0$. The c -theorem is again satisfied.

This is a particularly elegant derivation: the derivative of the c -function is naturally constrained by the NEC and thermodynamic stability conditions to change sign depending on the location of the UV fixed point along the radial direction. It takes negative values, as expected, when the UV fixed point lies at the minimum of the radial coordinate—precisely the condition under which the c -theorem holds. Everything fits together seamlessly to ensure the validity of the c -theorem.

3. Anisotropic hyperscaling-violating theories

In [34], theories in which rotational symmetry in the spatial plane is further broken were considered. This breaking of spatial rotational invariance leads to direction-dependent effective dimensionalities, generalizing (28), and consequently to a family of directional c -functions. It was shown in [34] that these anisotropic c -functions remain monotonic along the RG flow even in such highly nontrivial backgrounds, with their evolution governed by a general expression analogous to (20). This provides one of the most stringent consistency tests of monotonicity.

V. DISCUSSION

We have constructed a c -function based on holographic timelike entanglement entropy that is applicable to non-relativistic theories. We demonstrate that renormalization group flows without Lorentz invariance obey a timelike entanglement monotonicity, thereby establishing a direct connection between RG irreversibility and timelike entanglement. This constitutes a systematic improvement over previous attempts to construct c -functions from spacelike entanglement entropy, whose monotonicity is known to fail once Lorentz symmetry is broken.

Many quantum many-body systems are inherently non-Lorentz-invariant, motivating the study of their holographic RG flows as a powerful framework for probing strongly coupled dynamics and phase transitions. Our results open new foundational directions for analyzing such RG flows holographically and strongly suggest that, in the absence of Lorentz symmetry, a field-theoretic c -function should be constructed from timelike entanglement measures rather than from spacelike entanglement entropy. Our work also suggests a more general lesson: when space and time scale differently, RG irreversibility must be probed through observables sensitive to the temporal direction.

Moreover, we identify an upper bound on the rate of change of the timelike c -function along the RG flow, constraining how rapidly effective degrees of freedom can be coarse grained in consistent holographic theories. This bound reflects the locality and internal consistency of the RG process, and provides a quantitative measure of how fast RG irreversibility can proceed under controlled physical conditions.

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DATA AVAILABILITY

The data supporting this study's findings are available within the article.

APPENDIX A: THE DERIVATIVE OF THE c -FUNCTION

In this section, we provide additional technical details on the derivation of the derivative of the c -function (20). The timelike interval length, written as in (12), is

$$t_{\text{Im}} = -2 \int_{r_m}^{r_b} dr \frac{e^{\frac{\Lambda}{d_x} - B}}{\Lambda'} F_-, \quad (\text{A1})$$

and can be rewritten in the form (13). After some algebra, its derivative takes the form

$$\begin{aligned} -\frac{1}{2} \frac{\partial t_{\text{Im}}}{\partial r_m} &= \frac{\partial_{r_m} \mathcal{F}_{b-}}{\Lambda'_b} e^{\frac{\Lambda_b}{d_x} - B_b} - \frac{\partial_{r_m} \mathcal{F}_{m-}}{\Lambda'_m} e^{\frac{\Lambda_m}{d_x} - B_m} \\ &\quad - \int_{r_m}^{r_b} dr \frac{e^{\frac{\Lambda}{d_x} - B}}{\Lambda'} \partial_{r_m} \mathcal{F}_- \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right). \end{aligned} \quad (\text{A2})$$

To further process the expression above, we note that the derivatives of the $\mathcal{F}$ function relate to the function itself as follows:

$$\frac{\partial_{r_m} \mathcal{F}_{m-}}{\mathcal{F}_{m-}} = -\frac{\Lambda'_m}{d_x}, \quad \partial_{r_m} \mathcal{F}_- + \frac{\Lambda'_m \mathcal{F}_-}{d_x} = -\frac{\Lambda'_m e^{\frac{\Lambda_m}{d_x} - B_m}}{\sqrt{e^{2\Lambda_m} - e^{2\Lambda}}}. \quad (\text{A3})$$

In the first expression $\mathcal{F}_{m-}$ is evaluated before taking the derivative. Using these relations and referring back to (13), we observe that the derivative of t_{Im} depends on t_{Im} itself. Consequently (A2) can be rewritten as,

$$\begin{aligned} &-\frac{1}{2} \left(\frac{\partial t_{\text{Im}}}{\partial r_m} + \frac{\Lambda'_m}{d_x} t_{\text{Im}} \right) \\ &= \frac{e^{\frac{\Lambda_b}{d_x} - B_b}}{\Lambda'_b} \left(\partial_{r_m} \mathcal{F}_{b-} + \frac{\Lambda'_m \mathcal{F}_{b-}}{d_x} \right) \\ &\quad + \Lambda'_m e^{\Lambda_m} \int_{r_m}^{r_b} dr \frac{e^{-B}}{\Lambda' \sqrt{e^{2\Lambda_m} - e^{2\Lambda}}} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right). \end{aligned} \quad (\text{A4})$$

The first line simplifies with the help of the second identity in (A3). The above expression is essentially proportional to the derivative of the timelike component c_{Im} as can be seen from (14). From this, we obtain a compact expression,

$$\frac{\partial c_{\text{Im}}}{\partial r_m} = -2t_{\text{Im}}^{d_x-1} \Lambda'_m e^{2\Lambda_m} d_x \left[-\frac{e^{-B_b}}{\Lambda'_b \sqrt{e^{2\Lambda_m} - e^{2\Lambda_b}}} + \int_{r_m}^{r_b} dr \frac{e^{-B}}{\Lambda' \sqrt{e^{2\Lambda_m} - e^{2\Lambda}}} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right) \right]. \quad (\text{A5})$$

Note that in the first term the derivative $t'_{\text{Im},b}$ (3) of the timelike surface at the deepest point in the bulk r_b appears, and in the integral term we encounter the differential of the time interval

$$dt_{\text{Im}} = -\frac{e^{\Lambda_m-B}}{\sqrt{e^{2\Lambda_m} - e^{2\Lambda}}} dr, \quad (\text{A6})$$

which, when substituted into (A5), yields the final expression (15).

Next, we analyze the spacelike part of the c -function, where some differences in the computation arise. The time interval is given by

$$t_{\text{Re}} = -2 \int_{r_\partial}^{r_b} dr \frac{e^{\frac{\Lambda}{d_x}-B}}{\Lambda'} F_+, \quad (\text{A7})$$

and can be rewritten as

$$-\frac{t_{\text{Re}}}{2} = \frac{e^{\frac{\Lambda}{d_x}-B}}{\Lambda'} \mathcal{F}_+ \Big|_{r_\partial}^{r_b} - \int_{r_\partial}^{r_b} dr \frac{e^{\frac{\Lambda}{d_x}-B}}{\Lambda'} \mathcal{F}_+ \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right).$$

Taking the derivative of t_{Re} yields

$$-\frac{1}{2} \frac{\partial t_{\text{Re}}}{\partial r_m} = \frac{e^{\frac{\Lambda_b}{d_x}-B_b}}{\Lambda'_b} \partial_{r_m} \mathcal{F}_{b+} - \frac{e^{\frac{\Lambda_\partial}{d_x}-B_\partial}}{\Lambda'_\partial} \partial_{r_m} \mathcal{F}_{\partial+} - \int_{r_\partial}^{r_b} dr \frac{e^{\frac{\Lambda}{d_x}-B}}{\Lambda'} \partial_{r_m} \mathcal{F}_+ \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right).$$

Using

$$\partial_{r_m} \mathcal{F}_+ = -\frac{\Lambda'_m}{\sqrt{e^{2\Lambda} + e^{2\Lambda_m}}} e^{-\frac{\Lambda}{d_x} + \Lambda_m} - \frac{\mathcal{F}_+}{d_x} \Lambda'_m, \quad (\text{A8})$$

we observe that the derivative of t_{Re} can be expressed in terms of the time-interval itself, and the above expressions simplify conveniently as

$$-\frac{1}{2} \left(\frac{\partial t_{\text{Re}}}{\partial r_m} + \frac{\Lambda'_m}{d_x} t_{\text{Re}} \right) = -\Lambda'_m e^{\Lambda_m} \left[\frac{e^{-B_b}}{\Lambda'_b \sqrt{e^{2\Lambda_b} + e^{2\Lambda_m}}} - \frac{e^{-B_\partial}}{\Lambda'_\partial \sqrt{e^{2\Lambda_\partial} + e^{2\Lambda_m}}} \right] + \Lambda'_m \int_{r_\partial}^{r_b} dr \frac{e^{\Lambda_m-B}}{\Lambda' \sqrt{e^{2\Lambda} + e^{2\Lambda_m}}} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right). \quad (\text{A9})$$

Nicely, the first two terms are proportional to $t'_{\text{Re},b}$ and $t'_{\text{Re},\partial}$, the derivatives of the spacelike surface at the deepest point in the bulk and at the boundary, respectively. Furthermore, the integrand's differential is naturally expressed in terms of the time differential,

$$dt_{\text{Re}} = -\frac{e^{\Lambda_m-B}}{\sqrt{e^{2\Lambda} + e^{2\Lambda_m}}} dr. \quad (\text{A10})$$

Equation (A9) essentially gives $\partial_{r_m} c_{\text{Re}}$, since from (10), we then obtain

$$\frac{\partial c_{\text{Re}}}{\partial r_m} = -t_{\text{Re}}^{d_x-1} e^{\Lambda_m} d_x \left(\frac{\partial t_{\text{Re}}}{\partial r_m} + \frac{\Lambda'_m t_{\text{Re}}}{d_x} \right). \quad (\text{A11})$$

A straightforward substitution of (A9) completes the derivation of (16). We highlight how elegantly all the geometric properties of the surface combine to yield a compact and easily accessible final expression.

APPENDIX B: BOUNDS ON THE RATE OF THE C-FUNCTION ALONG THE RG FLOW

The boundary time interval is given by $T = t_{\text{Im}} - t_{\text{Re}}$. We can compute the derivative

$$\frac{\partial c}{\partial T} = \frac{\partial c}{\partial r_m} \frac{\partial r_m}{\partial T}, \quad (\text{B1})$$

where the first factor on the right-hand side is given by the expression (20), which encodes the monotonicity conditions we have analyzed so far. To compute the second factor, we use Eqs. (A4), (A6) for the t_{Im} contribution, and (A9) and (A10) for the t_{Re} contribution. This leads us to the compact expression,

$$\frac{\partial c}{\partial T} = \frac{d_x \partial_{r_m} c}{(-T \Lambda'_m + e^{-\Lambda_m} T_\infty^{1-d_x} \partial_{r_m} c)}. \quad (\text{B2})$$

The c -theorem gives $\partial_{r_m} c \geq 0$, and $\partial_T c \leq 0$ since probing larger intervals corresponds to lower energies, along which the c -function decreases. Then depending on the sign of d_x , $\partial_{r_m} c$ is bounded from above or below with respect to $T \Lambda'_m$. For example, for positive d_x the c -theorem implies

$$0 \leq \partial_{r_m} c \leq e^{\Lambda_m} T_\infty^{d_x-1} T \Lambda'_m. \quad (\text{B3})$$

The c -theorem sets a lower bound and an upper bound, not necessarily independent of each other, on the derivative of the c -function for RG flows.

To be more precise we rewrite the expression in terms of the metric elements as

$$\frac{\partial c}{\partial T} = T_\infty^{d_x-1} e^{\Lambda_m} d_x \left[1 - \frac{T}{2d_x \int_0^T dt \frac{1}{\Lambda'} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right)} \right]^{-1} \quad (\text{B4})$$

and therefore, for instance, for positive d_x the c -theorem imposes

$$\frac{T}{2d_x \int_0^{\frac{T}{2}} dt \frac{1}{\Lambda'} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right)} \geq 1, \quad (\text{B5})$$

where the value of the integral is constrained with respect to the rescaled time interval T/d_x .

One may wonder whether this is a sensible bound and whether it is consistent with the c -theorem as we have formulated it so far. In fact, it does. To see this, let us rewrite (B5) for positive d_x , in terms of d_{x_r} (28),

$$\frac{T}{2 \int_0^{\frac{T}{2}} dt \left(1 - \frac{d_x}{d_{x_r}} \right)} \geq 1. \quad (\text{B6})$$

Once the c -theorem condition $d_x \leq d_{x_r}$ is imposed, the first inequality in (B3) is satisfied. Moreover, the integral appearing in the above expression is positive and (B6) can be rewritten as $\int_0^{\frac{T}{2}} dt f(t) \leq \frac{T}{2}$. From the properties of the Riemann integral, it is easy to see that as long as the integrand $f(t) \leq 1$, the inequality (B6) is automatically satisfied. This directly implies that the upper bound is trivially respected for positive values of d_x considered here, and that no additional constraints arise. In other words, this serves as an additional nontrivial consistency check: the conditions we imposed for the c -theorem by studying $\partial_{r_m} c$

are sufficient to ensure the correct monotonicity along the RG flow when examined through $\partial_T c$.

APPENDIX C: EQUIVALENCE BETWEEN THE c -FUNCTION EXPRESSIONS

In this section we show that the two expressions of the c -function (10) and (11) are equivalent. Let us study the monotonicity of (11) by taking the derivative with respect to r_m to obtain

$$\frac{\partial c}{\partial r_m} = 2T^{d_x-1} e^{\Lambda_m} \left[d_x \frac{\partial T}{\partial r_m} + T \Lambda'_m \right]. \quad (\text{C1})$$

We can compute, using (A4) and (A9), the derivative

$$\frac{\partial T}{\partial r_m} = 2\Lambda'_m \left[-\frac{T}{2d_x} + \int_0^{\frac{T}{2}} dt \frac{1}{\Lambda'} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right) \right]. \quad (\text{C2})$$

Substituting (C2) into (C1), we obtain

$$\frac{\partial c}{\partial r_m} = 4e^{\Lambda_m} T^{d_x-1} d_x \Lambda'_m \int_0^{\frac{T}{2}} dt \frac{1}{\Lambda'} \left(\frac{\Lambda'}{d_x} - \frac{\Lambda''}{\Lambda'} - B' \right). \quad (\text{C3})$$

The right-hand side is equal, up to a trivial positive constant, to (20), thus establishing the equivalence of the monotonicity conditions for the two expressions (10) and (11).

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