

Light lepton-flavor-violating flavon: The messenger of neutrino mixing and lepton $g-2$

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In neutrino physics, a class of models with local or global family symmetries may be invoked, and then a flavon field is needed to realize full neutrino mixing. This flavon may leave hints in lepton $g-2$ and address their possible anomalies. In this work, we explore this idea in the $(B-L)_{ij}$ gauge extension to the standard model (SM), in which realistic neutrino mixing requires both a SM singlet flavon s and a vectorlike lepton (VLL) doublet. The dominant coupling between the flavon and leptons is in the manner of lepton-flavor-violation (LFV). Through an analytical analysis of the SM lepton-VLL mixing matrix, we find that the parameter space of the $s\bar{\mu}e$ -type flavon to explain $(g-2)_\mu$ has been completely excluded by the specific LFV process, muonium-antimuonium oscillation. But the $s\bar{\mu}\tau$ -type flavon still has the opportunity to account for the excess in $(g-2)_\mu$ or the deficit in $(g-2)_e$; however, it faces the strong constraint from $\tau \rightarrow \mu$ conservation and, in particular, the lepton flavor universality (LFU) test of Z boson decay, which arises due to our way to realize the LFV flavon. The surviving flavon is highly predictable, with mass in the narrow window $m_\tau \lesssim m_s \lesssim 2m_\tau$ and LFV coupling strength $\sim 10^{-2}$. Besides, it leaves a TeV scale VLL with a multilepton signature at the LHC.

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I. INTRODUCTION

Although the establishment of the Standard Model (SM) of particle physics achieves great success, it represents just a low-energy manifestation of the more fundamental theory and confronts many puzzles. We have been committed to solving these puzzles in order to unravel the mysterious mask of fundamental theories. The tiny but nonvanishing neutrino mass is a well-established puzzle, and therefore the neutrino extension furnishes a good basement on the way to searching for new physics.

The leptonic sector harbors several unresolved puzzles, among which the anomalous magnetic moment of the muon stands as a long-standing enigma. The first evidence for a deviation between the measured muon $g-2$ and the prediction of SM was reported by Brookhaven National Laboratory (BNL) [1], with subsequent high precision confirmation from Fermi National Accelerator Laboratory (FNAL) [2–6]. This discrepancy initially heightened expectations for new physics, though confidence has been

tempered by uncertainties in quantum chromodynamics (QCD), predominantly stemming from hadronic vacuum polarization (HVP). Divergent results between lattice QCD and dispersive-method approaches further complicate the interpretation. However, recent independent lattice calculations presented in WP25 [7] demonstrate convergence, aligning SM predictions with updated experimental analyses [8,9]. Resolving this puzzle requires advances in dispersive-method HVP calculations; current data favor a 5σ discrepancy [10],

$$\delta a_\mu = a_\mu^{\text{Exp}} - a_\mu^{\text{SM}} = (2.62 \pm 0.45) \times 10^{-9}. \quad (1.1)$$

Parallel developments in the electron sector reveal a similar tension. Improved precision in electron $g-2$ measurements [11] has uncovered a deviation between the SM predictions and the experimental results, with the current data seeming to favor a negative δa_e . Although claiming this as evidence for new physics remains premature, it unequivocally establishes $(g-2)_e$ as a promising probe for physics beyond the SM. Hence, the exploration of new physics mechanisms that could yield significant anomalous magnetic moment signals remains a theoretical topic of considerable importance, even as a prevailing view has emerged that these so-called puzzles are nearing resolution.

One may conjecture that the leptonic $g-2$ anomalies could be connected to the origin of the neutrino mass. Stated differently, the observed deviations in $(g-2)_e$ may

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reflect underlying mechanisms responsible for neutrino mass generation, particularly flavor mixing, which could also hint at the flavor-specific patterns seen in the data. In this work, we explore new physics scenarios following this intuition. Let us suppose that neutrino physics is attributed to the seesaw mechanism, which can be naturally embedded in a local $B - L$ extension of the SM [12–14]. Then, a natural candidate to account for the $(g - 2)_\mu$ discrepancy is a new massive gauge boson, Z' . However, since the resulting Z' couples to both quarks and electrons, it is strongly constrained and does not explain the anomaly. This limitation motivates the flavor-dependent local $(B - L)_{23}$ model [15], where only the second and third generations of fermions carry charges under the new gauge group. This construction yields a sub-GeV Z' that is largely blind to electrons and nucleons, thereby accommodating the δa_μ measurement. However, even in a flavor-specific construction, such as the $(B - L)_{13}$ model, where the Z' couple to electrons, it is unable to explain the negative δa_e value.

In this work, we consider another candidate, a light flavon with lepton flavor violation (LFV), to increase or decrease $(g - 2)_e$ in the gauged $(B - L)_{ij}$ model.¹ In a more general sense, this flavon may be present in any model with family symmetry G_X . This symmetry forbids certain family mixing, and generally a flavon field $\mathcal{F}$ spontaneously breaking G_X is indispensable; sometimes a vectorlike family of fermions $F = (F_L, F_R)$ is also introduced to mediate the G_X breaking effect on SM fermions f_i through coupling $\mathcal{F}\bar{f}_i F$, thus producing full mass mixing of fermions [27]. In general, those couplings give rise to FCNC, and, for example, for the Z' studied in [15], the $(g - 2)_\mu$ favored region may be threatened by the $s \rightarrow dZ'$ transition at the loop level [28]. Considering the local $(B - L)_{13}$, then from $\mathcal{F}\bar{\ell}_2 F$ we obtain a flavon with LFV coupling to $\bar{\mu}e$ or $\bar{\mu}\tau$, depending on the mixing between light and exotic charged leptons. Setting the flavon (CP -even in our work) as light as possible, the LFV flavon loop can lead to a sufficiently large positive δa_μ . Rather, if one considers the local $(B - L)_{23}$, the LFV flavon provides a way to understand the negative δa_e .

Of course, in a concrete model, whether the scenario of light LFV flavon can succeed requires a detailed analysis of the strong constraints of lepton flavor physics (such as charged lepton flavor decay, LFU violation of Z -boson decay, muonium and antimuonium oscillations, etc.) as well as the related constraints on the light boson. To arrive at the feasible parameter space in the $(B - L)_{13}$ model, we consider the minimal heavy-light mixing which facilitates approximate analytical analysis. We find that flavon of the type $s\bar{\mu}e$ has been ruled out, which confirms the conclusion

for the axionlike particle LFV in [29]. The $s\bar{\mu}\tau$ -type LFV flavon still has the opportunity to account for the $(g - 2)_e$ deviations, leaving a promising TeV scale VLL with multilepton signature at the LHC; besides, it predicts a flavon with mass close to m_τ with LFV coupling strength $\sim 10^{-2}$.

The work is organized as follows. We will first focus on the muon $g - 2$, and the discussions can be applied directly to the electron $g - 2$. Concretely, in Sec. II we present the local $(B - L)_{13}$ model for LFV flavon, and discuss the constraints from Z boson decay. In Sec. III, we investigate what kind of LFV flavon can successfully solve the muon $g - 2$ puzzle. In Sec. IV we carry out a similar investigation of the electron $g - 2$. The final section includes the conclusions and discussions.

II. LOCAL $(B - L)_{13}$ FOR MUON $g - 2$

As a concrete realization of the LFV flavon for the $(g - 2)_\mu$ puzzle, we work in the gauged $(B - L)_{13}$ extension to the SM, under which only the first and third generations of fermions are charged. From first glance, the leptonic sector of this model is similar to that of the well-known gauged $L_e - L_\tau$ model. However, they have a striking difference: here we have two RHNs for the sake of anomaly cancellation, and they are able to implement the minimal seesaw mechanism.

Although successfully explaining neutrino masses, the minimal $(B - L)_{13}$ model cannot fully account for neutrino mixing, that is, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [30,31], due to the flavorful $B - L$. Extra ingredients are still necessary to give realistic neutrino mixing. In this article, like the scheme in [15], we rely on the vectorlike lepton (VLL) $L_{L,R}$ and a flavon $\mathcal{F}_\ell$ to do this job by assigning a suitable $(B - L)_{13}$ charge. Then, the relevant field content of the model is listed in Table I. Given the above field content and symmetries, the most general Lagrangian of the leptonic sector reads

$$\begin{aligned} -\mathcal{L}_L = & Y_{22}^e \bar{\ell}_2 H e_{R2} + Y_{ij}^e \bar{\ell}_i H e_{Rj} + Y_{ij}^N \bar{\ell}_i \tilde{H} N_{Rj} \\ & + Y_i^e \bar{L}_L H e_{Ri} + Y_i^N \bar{L}_L \tilde{H} N_{Ri} \\ & + \lambda_2^e \bar{\ell}_2 L_R \mathcal{F}_\ell^* + M_i^e \bar{\ell}_i L_R + m_L \bar{L}_L L_R \\ & + \frac{\lambda_{ij}^N}{2} \Phi \bar{N}_{Ri}^c N_{Rj} + \text{H.c.}, \end{aligned} \quad (2.1)$$

where $\tilde{H} = i\sigma_2 H^*$, the Latin indices $i/j = 1, 3$. It is convenient to work on the basis that Y_{ij}^e and λ_{ij}^N are diagonal.

The Yukawa term $\lambda_2^e \bar{\ell}_2 L_R \mathcal{F}_\ell^*$ is essential, as it uniquely encodes the consequences of the flavorful $(B - L)_{13}$ symmetry breaking. This symmetry is spontaneously broken by the flavon vacuum expectation value (VEV), with $\mathcal{F}_\ell = \frac{v_\ell}{\sqrt{2}} + \frac{s+ia}{\sqrt{2}}$. Together with the Y_i^N mass terms, it

¹A light boson lies far below the weak scale and conserves the lepton flavor [15–20] widely used to understand the $(g - 2)_e$ anomaly. However, the light LFV flavon solution has been much less explored, despite previous works [21–26].

TABLE I. Field content and quantum numbers in the $(B - L)_{13}$ model.

	$\ell_e, \ell_\mu, \ell_\tau$	e_R, μ_R, τ_R	$N_{eR}, N_{\tau R}$	H	Φ	$L_{L/R}$	$\mathcal{F}_\ell$
$SU(2)_L \times U(1)_Y$	$(2, -1/2)$	$(1, -1)$	$(1, 0)$	$(2, 1/2)$	$(1, 0)$	$(2, -1/2)$	$(1, 0)$
$(B - L)_{13}$	$-1, 0, -1$	$-1, 0, -1$	$-1, -1$	0	2	-1	-1

subsequently communicates this symmetry breaking to the minimal seesaw sector via a light-heavy doublet mass mixing, thereby generating the missing Dirac mass terms,

$$(M_D)_{22} = \frac{M_2^\ell}{M_L} Y_1^N \frac{v_h}{\sqrt{2}}, \quad (M_D)_{23} = \frac{M_2^\ell}{M_L} Y_3^N \frac{v_h}{\sqrt{2}}, \quad (2.2)$$

with v_h the SM Higgs doublet VEV and $M_2^\ell \equiv \lambda_2^\ell v_f / \sqrt{2}$. In addition, this Yukawa coupling makes the flavon obtain LFV coupling to $\bar{\mu}e$ or $\bar{\mu}\tau$, due to the mixing between L_R and the corresponding light charged lepton species, which opens the possibility to solve the $(g - 2)_\mu$ puzzle through a LFV flavon at the GeV scale with $\lambda_2^\ell \sim 1$.

It is crucial to assess the compatibility of such a flavon with neutrino physics. A rough estimate of the resulting active neutrino mass matrix elements, derived from the aforementioned Dirac mass matrix, is presented below

$$m_\nu \sim M_D^2 / M_N \sim (\lambda_2^\ell)^2 \frac{(Y_i^N)^2}{\lambda^N} \frac{v_f^2 v_h^2}{M_L^2 \langle \Phi \rangle} \sim 0.01 \times \left(\frac{Y_i^N}{10^{-4}} \right) \left(\frac{v_f}{10 \text{ GeV}} \right)^2 \text{ eV}, \quad (2.3)$$

where we have set $\langle \Phi \rangle \sim \mathcal{O}(\text{TeV})$ and consider $M_L \sim \text{TeV}$, a required mass scale; all other Yukawa couplings are assumed to be in order one. The estimation shows that by adjusting Y_i^N , the flavon VEV scale v_f has a wide parameter space, so a light flavon mass $m_s \sim v_f$ (actually, it is not difficult to make $m_s \ll v_f$ in the scalar potential), remains well consistent with the neutrino data.

We have not specified the scalar potential of the model, $V(\Phi, \mathcal{F}_\ell)$, assuming that the two new scalars almost decouple from the SM Higgs doublet H . We will not delve into the details and will just point out its main features that are relevant to our subsequent discussions. This potential should be designed to generate both vacuum expectation values (VEV) for the two scalar fields, $\langle \mathcal{F}_\ell \rangle$ and $\langle \Phi \rangle$, with the latter related to RHN masses. Next, both the CP -odd and CP -even components of the flavon, a and s , can be the required LFV flavon, but in this work we take s as an example under the assumption that the effect of a is minor, with a comment soon later. It may be ascribed to the fact that a is the dominant Goldstone boson mode of $(B - L)_{13}$ spontaneously breaking, thus eaten by the gauge boson; otherwise, a is considerably heavier than

s as a result of the mixing effect between the CP -even components of $\mathcal{F}_\ell$ and Φ [32].

Comments are in order. First, the subsequent analysis proceeds in the limit of $g_{B-L} \rightarrow 0$, and thus the local $(B - L)_{13}$ effectively becomes a global one. This simplification frees us of any constraints associated with the new massive gauge boson. Second, as studied in Ref. [22], contrary to the flavor conservation pseudo scalar boson, the CP -odd component a having a nondiagonal coupling to $\bar{e}\mu$ moreover heavier than m_μ can also make a positive contribution to $(g - 2)_\mu$ at the one-loop level. Therefore, we can instead pick up a as the LFV flavon to enhance muon $g - 2$, but it is expected that such a switch will not give a substantial difference to the option in the current work. Last, $(B - L)_{13}$ may not be the unique option, and for instance, the alternative $(B - L)_{12}$ instead needs the Yukawa coupling $\lambda_3^\ell \bar{\ell}_3 L_R \mathcal{F}_\ell^*$ which also admits a LFV flavon coupling to $\bar{\tau}\mu$.

A. The charged lepton mass mixing with the Occam razor principle

The mass terms for the charged leptons, including the heavy vectorlike pair, are given by $(\bar{e}_L) M_e (e_R)$ with the most general mass mixing matrix restricted by the flavorful $B - L$ symmetry taking the following form

$$M_e \equiv \begin{pmatrix} \frac{v_h}{\sqrt{2}} Y_{11}^e & 0 & 0 & M_1^\ell \rightarrow 0 \\ 0 & \frac{v_h}{\sqrt{2}} Y_{22}^e & 0 & M_2^\ell \\ 0 & 0 & \frac{v_h}{\sqrt{2}} Y_{33}^e & M_3^\ell \rightarrow 0 \\ \frac{v_h}{\sqrt{2}} Y_1^e & 0 & \frac{v_h}{\sqrt{2}} Y_3^e \rightarrow 0 & m_L \end{pmatrix} \sim \begin{pmatrix} m_e & 0 & 0 & 0 \\ 0 & m_\mu & 0 & 0 \\ 0 & 0 & m_\tau & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}, \quad (2.4)$$

with m_L around the TeV scale, the largest mass scale in M_e . The Dirac mass terms between the light and heavy leptons are denoted as M_a^ℓ with $a = 1, 2, 3$, and especially the element $M_2^\ell = \lambda_2^\ell \frac{v_f}{\sqrt{2}} \neq 0$ is a result of the flavorful $(B - L)_{13}$ spontaneously breaking by the flavon field, which is indispensable to realize full neutrino mixing. Through biunitary transformation $[e_{L,R}]_a \rightarrow [\hat{e}_{L,R}]_a = [U_{L,R}]_{aa} [e_{L,R}]_a$, we move to the mass basis of charged leptons: $M_e = U_L \hat{M}_e U_R^\dagger$ with

$\hat{\mathcal{M}}_e = \text{diag}(m_e, m_\mu, m_\tau, m_4)$. Hereafter, we will drop the hat index for the states in the mass eigenstates.

Be careful that M_e contains several nondiagonal elements that have the potential to cause a large LFV. Let us use the Occam's razor principle to control the number of relevant parameters. From the side of neutrino physics, $M_{1,3}^e$ can be safely sent to zero, and $Y_{1,3}^e$ is also irrelevant. However, we will find that at least one of them is crucial to provide a feasible solution to the $(g-2)_\mu$ puzzle through the LFV coupling of the light flavon. For example, let us focus on the case with $Y_1^e \neq 0$. Then, the mixing effects of the charged lepton mass matrix M_e are encoded in the following 3×3 matrix with five elements,

$$M = \begin{pmatrix} Y_{11}^e \frac{v}{\sqrt{2}} & 0 & 0 \\ 0 & Y_{22}^e \frac{v}{\sqrt{2}} & M_2^e \\ Y_1^e \frac{v}{\sqrt{2}} & 0 & m_L \end{pmatrix} \equiv m_L \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & a \\ b & 0 & 1 \end{pmatrix}. \quad (2.5)$$

The mixing elements in U_L and U_R are obtained by diagonalizing the Hermitian matrices $MM^\dagger/|m_L|^2$ and $M^\dagger M/|m_L|^2$, respectively

$$\begin{aligned} \frac{MM^\dagger}{|m_L|^2} &= \begin{pmatrix} |y_e|^2 & 0 & y_e b^* \\ 0 & |y_\mu|^2 + |a|^2 & a \\ y_e^* b & a^* & |b|^2 + 1 \end{pmatrix}, \\ \frac{M^\dagger M}{|m_L|^2} &= \begin{pmatrix} |y_e|^2 + |b|^2 & 0 & b^* \\ 0 & |y_\mu|^2 & y_\mu^* a \\ b & y_\mu a^* & |a|^2 + 1 \end{pmatrix}. \end{aligned} \quad (2.6)$$

It is seen that a and b can generate 4-2 and 4-1 mixing in U_L , respectively, but the latter is suppressed by the chiral flip from the first generation, y_e . After $a \leftrightarrow b$ and $y_e \leftrightarrow y_\mu$, similar behaviors are obtained in U_R .

In our model, the performances of U_L and U_R in the LFV processes are quite different. For the moment, let us consider their imprints within the SM content, but we do not take into account the charged current, since it involves the unknown neutrino sector. Then, first, in the neutral current of the Z boson, U_L is gone after moving to the mass basis due to the family universe, including L_L . Next, in terms of the discussion in Appendix A, the LFV related to the SM Higgs boson can be written as

$$\begin{aligned} \mathcal{L}_h &= -\hat{Y}_{ij}^e h \bar{e}_i P_R e_j + \text{H.c.} \quad \text{with} \\ \hat{Y}_{i \neq j}^e &= -\left(U_L^\dagger \frac{M_e(v_h \rightarrow 0)}{v_h} U_R \right)_{i \neq j}. \end{aligned} \quad (2.7)$$

Thus, for the *Ansätze* like Eq. (2.5), $\hat{Y}_{i \neq j}^e = -\frac{m_L}{v_h} (U_L^\dagger)_{4i} (U_R)_{4j} - \frac{v_L}{v_h} Y_{i \neq j}^s$ with Y^s the Yukawa coupling

matrix of the flavon given in Eq. (3.1). These nondiagonal elements lead to radiative charged lepton flavor violating (cLFV) such as $\mu \rightarrow e\gamma$ and $\tau \rightarrow \mu\gamma$, thus being severely constrained. From [33,34], we get the upper limits $\sqrt{|\hat{Y}_{21}^e|^2 + |\hat{Y}_{21}^e|^2} < 3.6 \times 10^{-6}$ and $\sqrt{|\hat{Y}_{23}^e|^2 + |\hat{Y}_{32}^e|^2} < 0.016$. They are not of concern since the similar constraints from the light LFV flavon, which will be studied later, are much stronger. In the following subsection, we will show that within SM, the decay of the Z boson is able to impose tight bounds on the nondiagonal elements of U_R .

B. Constraints on U_R from Z boson decays

In terms of the previous analysis, a sizable Y_1^e or Y_3^e is dangerous, as it causes U_R to deviate significantly from the unit matrix and unlike U_L , this leaves strong imprints on the neutral current coupling to Z . The reason is attributed to L_R , which carries a hypercharge different from that of the SM right-handed leptons and thus gives rise to a large correction to the gauge coupling of the Z boson with the charged leptons,

$$\begin{aligned} \mathcal{L}_{NC} &\supset \frac{g}{c_W} Z_\mu (g_R^Z)_{ij} \bar{e}_R i \gamma^\mu e_{Rj} \quad \text{with} \\ (g_R^Z)_{ij} &= (s_W^2) \delta_{ij} + (\delta g_R^Z)_{ij}, \end{aligned} \quad (2.8)$$

where $c_W = \cos \theta_W$ and $s_W = \sin \theta_W$ with θ_W the Weinberg angle. In the presented couplings, the first term represents the SM prediction. The second term denotes the deviation originating from the mixing between the SM right-handed charged leptons (isospin singlets) and the VLL L_R . It carries an isospin charge of $T^3 = -1/2$ and contributes to the neutral current $\frac{g}{c_W} Z_\mu \bar{L}_R (-1/2) L_R$, which, upon transition to the mass basis, becomes [27],

$$\delta(g_R^Z)_{ij} = (U_R^\dagger)_{i4} (-1/2) (U_R)_{4j}. \quad (2.9)$$

The deviation is the second order of the mixing elements $(U_R)_{4i}$, of which the largest is $(U_R)_{41}$ in our setup. Hence, the most significant deviations that only involve the light leptons are $\delta(g_R^Z)_{22}$ and $\delta(g_R^Z)_{21}$, which are highly constrained by the precise data on Z boson decays.

In general, the branching ratios of Z boson decaying into a pair of charged leptons are written as $\text{Br}(Z \rightarrow \ell_i^\pm \ell_j^\mp) = \frac{1}{2} [\Gamma(Z \rightarrow \ell_i^+ \ell_j^-) + \Gamma(Z \rightarrow \ell_j^+ \ell_i^-)] / \Gamma_Z$, and the LEP electroweak working group gives the total width of the Z boson $\Gamma_Z = 2.4955 \pm 0.0023$ GeV [34,35]. The decay width at tree level is expressed as

$$\begin{aligned} \Gamma(Z \rightarrow \ell_i^+ \ell_j^-) &= \frac{G_F M_Z^3}{3\sqrt{2}\pi} [(g_L^Z)_{ij}^2 + (g_R^Z)_{ij}^2] \\ &\times \left(1 - \frac{m^2}{M_Z^2} \right)^2 \left(1 + \frac{m^2}{2M_Z^2} \right), \end{aligned} \quad (2.10)$$

with $m = \max\{m_i, m_j\}$, which can be safely neglected. In calculating, we take $G_F \approx 1.18 \times 10^{-5}$ GeV⁻² and

$M_Z \approx 91.19$ GeV. In our model, $(g_L^Z)_{ij}$ take their SM values and do not receive corrections, $(g_L^Z)_{ij} = (-1/2 + s_W^2)\delta_{ij}$, while g_R^Z receive modifications as Eq. (2.9).

1. Constraints on $\delta(g_R^Z)_{ii}$ from LFU tests

The i 'th flavor of the charged lepton mixes with L_R^- and thus modifies the flavor-conserving couplings $(g_R^Z)_{ii}$ by $|U_{4i}|^2$ [see Eq. (2.9)], which is not universal to various flavors and then leads to LFU violation of Z boson decay. Concretely, for the Z decay into some flavor ℓ_i , the deviation from the SM prediction is expressed as the ratio

$$\frac{\Gamma(Z \rightarrow \bar{\ell}_i \ell_i)}{\Gamma(Z \rightarrow \bar{\ell}_i \ell_i)_{\text{SM}}} = \frac{(-1/2 + s_W^2)^2 + (s_W^2 - \frac{1}{2}(U_R)_{4i}^*(U_R)_{4i})^2}{(-1/2 + s_W^2)^2 + (s_W^2)^2} \approx 1 - 1.8283|(U_R)_{4i}|^2. \quad (2.11)$$

The current measurements of LFU of Z boson decay are presented as the following [35]

$$\begin{aligned} \frac{\text{Br}(Z \rightarrow \mu^+ \mu^-)}{\text{Br}(Z \rightarrow e^+ e^-)} &= 1.0009 \pm 0.0028, \\ \frac{\text{Br}(Z \rightarrow \tau^+ \tau^-)}{\text{Br}(Z \rightarrow e^+ e^-)} &= 1.0019 \pm 0.0032, \end{aligned} \quad (2.12)$$

with a correlation of $+0.63$.

Consider the two mixing patterns that will be studied to solve the $(g-2)_\mu$ anomaly, corresponding to $Y_1^e \neq 0$ and $Y_3^e \neq 0$ in Eq. (2.4), respectively. For the former case, both the branch ratios of the $e^+ e^-$ and $\mu^+ \mu^-$ channels may be sizably decreased, and then we have the following approximation to LFU

$$\begin{aligned} \frac{\text{Br}(Z \rightarrow \mu^+ \mu^-)}{\text{Br}(Z \rightarrow e^+ e^-)} &\approx 1 - 1.8283(|(U_R)_{42}|^2 - |(U_R)_{41}|^2), \\ \frac{\text{Br}(Z \rightarrow \tau^+ \tau^-)}{\text{Br}(Z \rightarrow e^+ e^-)} &\approx 1 + 1.8283|(U_R)_{41}|^2. \end{aligned} \quad (2.13)$$

The scenarios $Y_1^e \neq 0$ and $Y_3^e \neq 0$, respectively, give a hierarchy between the two elements $|(U_R)_{41}|, |(U_R)_{43}| \gg |(U_R)_{42}|$. Therefore, applying the data in Eq. (2.12), we get the constraint $|(U_R)_{41}| \leq 0.053$. For the latter case, instead the branch ratios of the $\mu^+ \mu^-$ and $\tau^+ \tau^-$ may be modified and then one has

$$\begin{aligned} \frac{\text{Br}(Z \rightarrow \mu^+ \mu^-)}{\text{Br}(Z \rightarrow e^+ e^-)} &\approx 1 - 1.828|(U_R)_{42}|^2, \\ \frac{\text{Br}(Z \rightarrow \tau^+ \tau^-)}{\text{Br}(Z \rightarrow e^+ e^-)} &\approx 1 - 1.828|(U_R)_{43}|^2. \end{aligned} \quad (2.14)$$

Then, Eq. (2.12) imposes the bounds on the individual mixing elements, $|(U_R)_{42}| \leq 0.032$, and $|(U_R)_{43}| \leq 0.027$.

2. Constraints on $\delta(g_R^Z)_{i \neq j}$ from cLFV tests

The cLFV decay of the Z boson imposes a strong bound on the nondiagonal couplings in Eq. (2.9). The upper bounds of cLFV decays of the Z boson are listed as [36,37]

$$\begin{aligned} \text{Br}(Z \rightarrow e^\pm \mu^\mp) &\leq 7.5 \times 10^{-7}, \\ \text{Br}(Z \rightarrow e^\pm \tau^\mp) &\leq 5.0 \times 10^{-6}, \\ \text{Br}(Z \rightarrow \mu^\mp \tau^\mp) &\leq 6.5 \times 10^{-6}. \end{aligned} \quad (2.15)$$

The first and third bounds are relevant to our discussions. The first imposes a constraint to the $Y_1^e \neq 0$ scenario, where the $Z \rightarrow e^\pm \mu^\mp$ mode has the largest cLFV decay width,

$$\Gamma(Z \rightarrow \mu^+ e^-) \approx \frac{G_F M_Z^3}{3\sqrt{2}\pi} \frac{1}{4} |(U_R)_{41}^*(U_R)_{42}|^2, \quad (2.16)$$

obtained from Eq. (2.10). Similarly, the $Z \rightarrow \mu^\mp \tau^\pm$ mode restricts the $Y_3^e \neq 0$ scenario.

Note that these cLFV decays arise at the quartic order of mixing, and consequently their constraints on the individual mixing may be weaker than those from the LFU tests, which are sensitive to the quadratic order of mixing. Concretely, cLFV of Eq. (2.15) gives

$$\begin{aligned} |(U_R)_{41}^*(U_R)_{42}| &\leq 2.79 \times 10^{-6}, \\ |(U_R)_{43}^*(U_R)_{42}| &\leq 2.42 \times 10^{-5}. \end{aligned} \quad (2.17)$$

Due to the hierarchy between the mixing elements mentioned previously, the above bound on the larger element is indeed weaker than the corresponding LFU bound if the hierarchy is sufficiently strong.

III. CAN OUR LFV FLAVON RESOLVE THE MUON $g-2$ PUZZLE?

We have shown that LFV is not present in the SM loops and hence the $(g-2)_\mu$ cannot be enhanced by them. However, the flavon, introduced to accommodate full neutrino mixings via the Yukawa term $\lambda_2^e \bar{\ell}_2 L_R \mathcal{F}_\ell^* + H.c.$, retains LFV. This is because, unlike the SM Higgs boson, the flavon merely accounts for the mixing term in the lepton mass matrix. At the same time, this term is potential to lift the value of $(g-2)_\mu$, provided that the flavon s is sufficiently light and moreover obtains sizable LFV coupling to either μ and e or μ and τ . Note that the latter requirement explains why we turn on Y_1^e or Y_3^e in Eq. (2.4) rather than M_1^ℓ or M_3^ℓ : they generate sizable mixing between $\ell_{1,3R}$ and L_R , which is exactly what we need to produce the desired LFV flavon. In this section, we will try to find out the feasible scenarios of LFV flavon from our flavon-VLL system to resolve the $g-2$ puzzle, confronting many strong constraints related to the precise lepton flavor physics.

A. The profile of the light LFV flavon for $(g-2)_\mu$

Above all, let us derive the interactions between the flavon and the light charged leptons, from the single term given at the beginning of this section,

$$\mathcal{L}_{\text{flavon}} = \lambda_2^\ell \frac{s+ia}{\sqrt{2}} \bar{\mu}_L L_R^- + \text{H.c.} \rightarrow -Y_{ij}^s \bar{e}_i P_R e_j (s+ia) + \text{H.c.}, \quad (3.1)$$

with $Y_{ij}^s = \frac{\lambda_2^\ell}{\sqrt{2}} (U_L^*)_{2i} (U_R)_{4j}$, where the CP -odd flavon a is irrelevant here. From the analytical expression of Appendix B, one may further approximately express the flavon Yukawa couplings in terms of Lagrangian parameters. We list some of them, which are well approximated as follows

$$\begin{aligned} Y_{22}^s &= \frac{\lambda_2^\ell}{\sqrt{2}} (U_L^*)_{22} (U_R)_{42} \approx -\frac{\lambda_2^\ell}{\sqrt{2}} \frac{a^* |y_\mu|}{(1+|a|^2)^{3/2}}, \\ Y_{44}^s &= \frac{\lambda_2^\ell}{\sqrt{2}} (U_L^*)_{24} (U_R)_{44} \approx \frac{\lambda_2^\ell}{\sqrt{2}} \frac{a^*}{(1+|a|^2)^{1/2}}, \\ Y_{42}^s &= \frac{\lambda_2^\ell}{\sqrt{2}} (U_L^*)_{24} (U_R)_{42} \approx \frac{\lambda_2^\ell}{\sqrt{2}} \frac{a^* |ay_\mu|}{(1+|a|^2)^{3/2}}, \\ Y_{24}^s &= \frac{\lambda_2^\ell}{\sqrt{2}} (U_L^*)_{22} (U_R)_{44} \approx -\frac{\lambda_2^\ell}{\sqrt{2}} \frac{a^* / |a|}{(1+|a|^2)^{1/2}}. \end{aligned} \quad (3.2)$$

They are for the case where the $b = Y_{1,3}^e / m_L$ parameter is negligible. According to our *Ansätze*, $Y_{24}^s \simeq \frac{1}{\sqrt{2}} \lambda_2^\ell \sim \mathcal{O}(1)$, and thus it is of interest to study whether the $s - e_4 - \mu$ loop could fill the $(g-2)_\mu$ discrepancy, but later analysis negates this possibility. Moreover, the diagonal coupling $s\mu^+\mu^-$ is heavily suppressed.

Therefore, to enhance the $(g-2)_\mu$ value, we will rely on the sizable LFV coupling in two scenarios: (I) Y_{21}^s and/or Y_{12}^s ; (II) Y_{23}^s and/or Y_{32}^s . They correspond to $Y_1^e \neq 0$ and $Y_3^e \neq 0$ in Eq. (2.4), respectively. In some parameter space, which is reliable in our discussion, an approximate approach can be applied to diagonalize the 3×3 charged mass matrix (with the full details cast in Appendix B), to get the expression for the $Y_1^e \neq 0$ case,

$$\begin{aligned} Y_{21}^s &= \frac{\lambda_2^\ell}{\sqrt{2}} (U_L^*)_{22} (U_R)_{41} \approx \frac{\lambda_2^\ell}{\sqrt{2}} \frac{a^* / |a|}{\sqrt{1+|a|^2}} \frac{b |y_\mu|}{\sqrt{|y_\mu|^2 + |ab|^2}}, \\ Y_{12}^s &= \frac{\lambda_2^\ell}{\sqrt{2}} (U_L^*)_{21} (U_R)_{42} \\ &\approx \frac{\lambda_2^\ell}{\sqrt{2}} \frac{a^* y_e |b|}{(|y_\mu|^2 + |ab|^2)^{3/2}} \frac{|a| (|y_\mu|^2 - |b|^2)}{1+|a|^2}. \end{aligned} \quad (3.3)$$

The former is on the order $\lambda_2^\ell |b|$, while the latter is further suppressed by the Yukawa coupling of the electron

$y_e \equiv Y_{11}^e v_h / m_L \simeq m_e / m_L \sim \mathcal{O}(10^{-6})$, thus playing a minor role.

In the $s\bar{\mu}e$ scenario, we will see that a very light flavon can enhance $(g-2)_\mu$, but the enhancement is almost saturated at $m_s \sim m_\mu$. Therefore, m_s of interest is set to lie above m_μ . In fact, an even lighter flavon will open the decay channel $\mu \rightarrow e + s$ at tree level, which is strongly constrained by precise muon measurements such as its lifetime. Therefore, our discussion is restricted to the case $m_s > m_\mu + m_e$, and then the decay of the flavon is dominated by the $e\mu$ mode.² We have a similar conclusion for the flavon in the $s\bar{\mu}\tau$ scenario.

The LFV flavon also has a small diagonal coupling with the electron, and this induced coupling is estimated to be

$$Y_{11}^s = \frac{\lambda_2^\ell}{\sqrt{2}} (U_L^*)_{21} (U_R)_{41} \approx -\frac{\lambda_2^\ell}{\sqrt{2}} \frac{y_e a^* |y_\mu| |b|^2}{(|y_\mu|^2 + |ab|^2)^{3/2}}. \quad (3.4)$$

Then, we should concern ourselves whether that light flavon is excluded by electron beam dump experiments that produce the light flavon via a Bremsstrahlung-like process. However, note that the experimentally searched events are produced by long-lived scalars flying over macroscopic distances and decaying back to electron pairs, thus imposing no constraints on the flavon in our model which decays into $e + \mu$ instantly. By the way, the astrophysical constraints for the light bosons coupling to an electron are also irrelevant to the LFV flavon under consideration, since its mass is heavier than m_μ [38]. Anyway, in principle, Y_{11}^s can be made as small as will, as long as the full neutrino mixings, which require a properly large $|a|$, are achievable. However, according to a quite recent study published in arXiv [39], the supernova may impose a bound on the m_μ scale LFV flavon with properly coupling to $e\mu s$.

B. Muon $g-2$ from the LFV flavon loop

In this subsection, we will investigate the $(g-2)_\mu$ via the flavon loops, first giving the no-go via the $s - \mu - e_4$ loop in the minimal case with only $a = M_2^\ell / m_L \neq 0$. Then we analyze how the LFV loop of $s - e - \mu$, which benefits in free of any large mass scale suppression, may work at the price of one extra mixing parameter Y_1^e . Prior to those concrete analyses, we will have a general discussion on the formula of $g-2$ and further analyze the behavior of the loop function in two limits, the decoupling limit with one large mass scale (compared to the mass scales of the external legs) in the loop and the opposite case with only light loop mass scale. These discussions are also useful in

²A fine-tuned flavon mass within the very narrow window $m_\mu - m_e < m_s < m_\mu + m_e$ is considered in [25]; in this case, ϕ is long-lived given that it has no other decay channels.

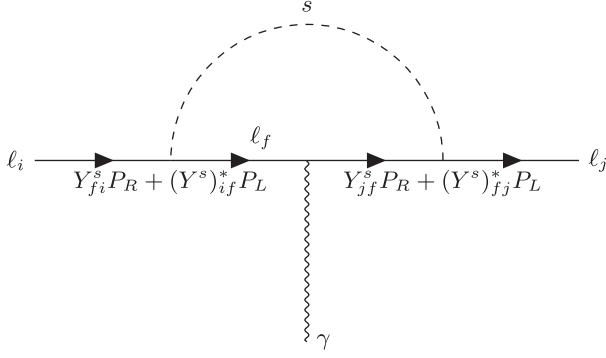


FIG. 1. The LFV flavon loop induce charged lepton $g-2$ for $\ell_i = \ell_j$ and cLFV decay $\ell_i \rightarrow \ell_j \gamma$ for $\ell_i \neq \ell_j$ with intermediate leptons ℓ_f .

the cLFV decay, which shares the same quantum process as the charged lepton $g-2$.

To keep the discussion most general, let us start from the effective Lagrangian of an LFV CP -even scalar boson coupling to different SM charged leptons ℓ_i , which has been given in Eq. (3.1). Then, following the calculation in Refs. [40,41], the one-loop contribution to the $g-2$ of $\ell_i = \{e, \mu, \tau\}$ via the LFV scalar boson loop $\ell_i - \ell_f - \ell_i$ (such as Fig. 1) is calculated to be

$$\delta a_{\ell_i}^s(\ell_f) = \frac{m_{\ell}^2}{16\pi^2 m_s^2} \left[(|Y_{if}^s|^2 + |Y_{fi}^s|^2) F_s\left(\frac{m_f^2}{m_s^2}, \frac{m_i^2}{m_s^2}\right) + \frac{m_f}{m_i} \text{Re}(Y_{if}^s Y_{fi}^s) G_s\left(\frac{m_f^2}{m_s^2}, \frac{m_i^2}{m_s^2}\right) \right]. \quad (3.5)$$

In the above expression, the first part does not require chirality flipping of the loop fermion, while the second part does, and their loop functions are respectively given by

$$F_s(x, y) = \int_0^1 dz \frac{z(1-z)^2}{z + (1-z)x - z(1-z)y},$$

$$G_s(x, y) = \int_0^1 dz \frac{2(1-z)^2}{z + (1-z)x - z(1-z)y}. \quad (3.6)$$

Their variables are $x \equiv m_f^2/m_s^2$, $y \equiv m_i^2/m_s^2$, the square of the ratio of the internal and external fermion mass to the flavon mass, respectively; in our work, we always have $y > 1$, namely the loop never develops an imaginary part.

In general, the loop functions $F_s(x, y)$ and $G_s(x, y)$ do not have simple analytical expressions. To demonstrate the behaviors of these loop functions, on the left of Fig. 2, we show them in a special limit (for instance, it is the case in the $s\bar{\mu}\tau$ scenario studied in Sec. III C 3) where the external fermion mass is negligible compared with the flavon mass m_s , giving $y \rightarrow 0$. In this limit, the denominator of the integral can be reduced to be $z + (1-z)x$ both for F_s and G_s , and then the integral can be implemented explicitly to get the widely used expression.

$$F_s(x) = -\frac{x^3 - 6x^2 + 3x + 6x \log x + 2}{6(1-x)^4},$$

$$G_s(x) = \frac{x^2 - 4x + 2 \log x + 3}{(1-x)^3}. \quad (3.7)$$

We always have $|F_s/G_s| < 1$, which is explicit in the first panel of Fig. 2. But if the scalar boson becomes as light as the external fermion, i.e., $y \rightarrow 1$, then we do not have a simple expression like the above, and one should use the complete numerical integral. We will encounter this limit in the $s\bar{\mu}e$ flavon scenario. For illustration, we give the numerical sample for a special case: $F_s(\frac{m_e^2}{m_\mu^2}, \frac{m_\mu^2}{m_\mu^2}) = 3.85022$, $G_s(\frac{m_e^2}{m_\mu^2}, \frac{m_\mu^2}{m_\mu^2}) = 629.236$.

It is tempting to consider whether the minimal case with $Y_{1,2,3}^e = 0$ and $M_{1,3}^\ell = 0$ in Eq. (2.4) can provide a viable solution to the $(g-2)_\mu$ puzzle. The contribution of flavon to $(g-2)_\mu$ is dominated by the $\mu - e - s$ loop. For $m_\mu \ll m_s < m_L$ one can use Eq. (3.7), and then $G_s(x) \approx -\frac{1}{x} \sim 10^{-3}$; moreover, $1/8 < F_s(x)/G_s(x) < 1/6$. The contribution can be well approximated to be

$$\delta a_\mu^s(e_4) \approx \frac{\kappa_m^2 |\lambda_2^\ell|^2}{32\pi^2} \frac{1/6 - |a|^2}{1 + |a|^2} \approx 3.16629 \times 10^{-9} \frac{1/6 - |a|^2}{1 + |a|^2}. \quad (3.8)$$

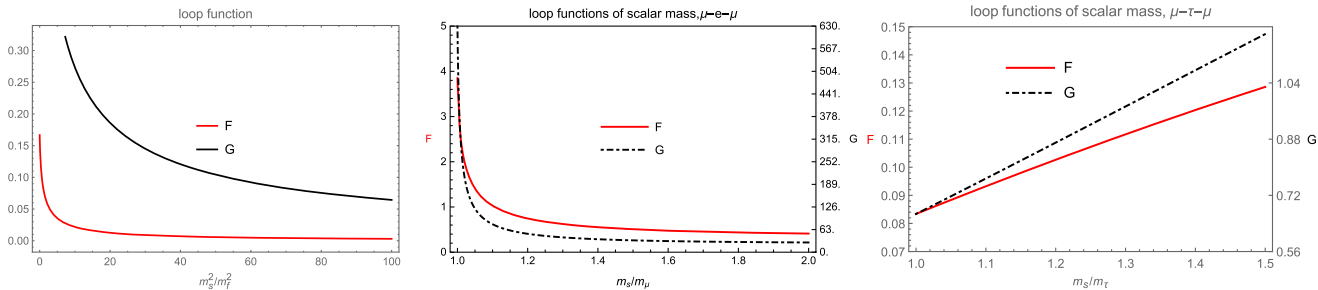


FIG. 2. Loop functions associated with the charged radiative process with the CP -even flavon running in the loop. The left panel is for $F_s(m_s^2/m_f^2)$ and $G_s(m_s^2/m_f^2)$ given in Eq. (3.7). For later consideration, we plot the loop functions for the $s\bar{\mu}e$ and $s\bar{\mu}\tau$ scenarios in the middle and right panels, respectively.

We have taken $|\lambda_2^e|=1$ and $\kappa_m = m_\mu/m_4 \approx |y_\mu|/(1+|a|^2) \simeq 10^{-3}$, namely a weak-scale heavy lepton, to make the final numerical estimation. Because λ_2^e is large, the dominant decay channel of e_4 is $\mu + s$ with $s \rightarrow \mu^+ \mu^-$, and as a result the light e_4 has definitely been ruled out by the 6μ signal at LHC [42].

C. The LFV flavon resolution against the cLFV constraints

Contrary to the chiral flip via the large mass of the heavy lepton e_4 , the contribution of the loop such as $\mu - e - s$, $\mu - \tau - s$ and $\mu - \mu - s$ can be enhanced through eliminating the large mass scale suppression in the loop, given that the flavon mass lies much below the weak scale. Here, the flavon predominantly couples to leptons violating the lepton flavor by virtue of the symmetry selection rule, and thus the last loop is negligible. The remaining scenarios face the strong constraints of lepton flavor violations. In the

following, we will examine cLFV through the example of $\mu - e$ conversion. The results are largely applicable to $\tau - \mu$ processes with minimal modifications.

1. Lepton flavor violation constraints

Key experimental probes for $\mu \rightarrow e$ transitions include radiative decay $\mu \rightarrow e\gamma$, three-body decay $\mu \rightarrow 3e$, and coherent $\mu - e$ conversion ($\mu N \rightarrow eN$). The conversion process entails a muon captured by a nucleus converting directly to an electron via an LFV interaction, with no neutrinos emitted. In our model, $\mu \rightarrow 3e$ presents two contributions as shown in Fig. 3: the tree-level channel mediated by the LFV flavon and the radiative channel mediated by the photon; the former can be neglected here due to the smallness of the flavon coupling to the electron Y_{11}^s , and the radiative channel is dominant by photon mediation. Then, all these $\mu \rightarrow e$ processes can be described by the following dimension-5 dipole operators,

$$\mathcal{L} = c_L \frac{e}{8\pi^2} m_\mu (\bar{e} \sigma^{\alpha\beta} P_L \mu) F_{\alpha\beta} + c_R \frac{e}{8\pi^2} m_\mu (\bar{e} \sigma^{\alpha\beta} P_R \mu) F_{\alpha\beta} + \text{H.c.}, \quad (3.9)$$

which are valid at the scale of muon mass. The Wilson coefficients $c_{L,R}$ are model dependent and are calculated by the matching at low energy. Specific to our model, they are calculated, such as $c_{L,R}(e\mu)$

$$c_L = \frac{1}{4m_\mu} \int_0^1 dx dy dz \delta(1-x-y-z) \left[\frac{xzm_e(Y_{22}^s)^* Y_{12}^s + yzm_\mu Y_{22}^s (Y_{21}^s)^* + (x+y)m_\mu (Y_{22}^s)^* (Y_{21}^s)^*}{zm_s^2 - xzm_e^2 - yzm_\mu^2 + (x+y)m_\mu^2 - xyq^2} \right. \\ \left. + \frac{xzm_e(Y_{21}^s)^* Y_{11}^s + yzm_\mu Y_{12}^s (Y_{11}^s)^* + (x+y)m_e (Y_{21}^s)^* (Y_{11}^s)^*}{zm_s^2 - xzm_e^2 - yzm_\mu^2 + (x+y)m_e^2 - xyq^2} \right], \quad (3.10)$$

where the two terms in the square bracket, respectively, denote that the leptons running in the loop are the muon and electron, and the latter is subdominant because $|Y_{11}^s| \ll |Y_{22}^s|$. c_R is obtained from c_L where the flavor indices are swapped in each Yukawa coupling, further

taking complex conjugation, for example, $Y_{12}^s \rightarrow (Y_{21}^s)^*$. Likewise for dimension-5 dipole operators related to muon and tau— $c_{L,R}$: $Y_{12}^s \rightarrow Y_{23}^s$, $Y_{21}^s \rightarrow Y_{32}^s$; and $m_\mu \rightarrow m_\tau$.

In terms of these operators, now we can calculate the observables. The results are well known, and we quote

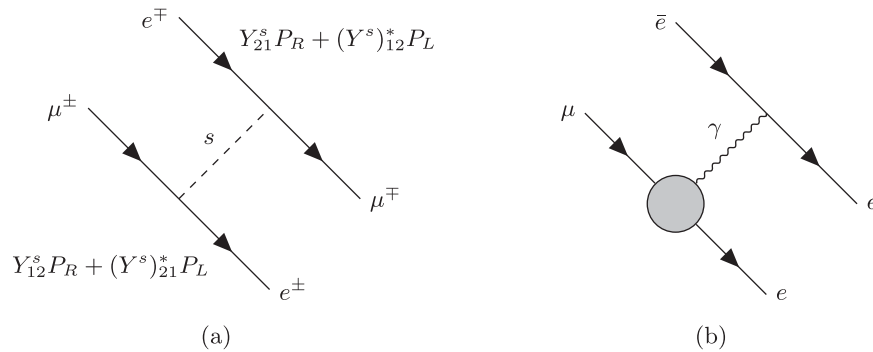


FIG. 3. Left: one schematic diagram of $\Delta L_{\mu,e} = \pm 2$ process of oscillation between muonium and antimuonium. Right: the $\Delta L_{\mu,e} = \pm 1$ process of $\mu \rightarrow 3e$, dominated by dipole operator; it also receives a tree level contribution such as the left figure with one final state μ replaced by e .

TABLE II. Some PDG data of muon and tau.

Decay models	Muon	Tau
Mean life	$\tau_\mu = 2.1969811 \times 10^{-6} \text{ s}$	$\tau_\tau = 2.903 \times 10^{-13} \text{ s}$
cLFV	$\text{Br}(\mu \rightarrow e\gamma) < 4.2 \times 10^{-13} 90\% \text{ C.L}$	$\text{Br}(\tau \rightarrow \mu\gamma) < 4.4 \times 10^{-8}$
Three-body	$\text{Br}(\mu \rightarrow 3e) < 1.0 \times 10^{-12} 90\% \text{ C.L}$	$\text{Br}(\tau \rightarrow 3\mu) < 2.1 \times 10^{-8}$

them in the following. First, the radiative two body decay width of $\mu \rightarrow e\gamma$ is

$$\Gamma(\mu \rightarrow e\gamma) = \frac{\alpha m_\mu^5}{64\pi^4} (|c_R|^2 + |c_L|^2). \quad (3.11)$$

Next, at leading in m_e/m_μ , the decay width of three-body $\mu \rightarrow 3e$ is dominated by the logarithmic term [43,44]

$$\Gamma(\mu \rightarrow 3e) \simeq \frac{\alpha^2 m_\mu^5}{192\pi^5} \left| \log \frac{m_\mu^2}{m_e^2} - 11/4 \right| (|c_L|^2 + |c_R|^2). \quad (3.12)$$

Their branch ratios are strongly constrained; we use the PDG data $\Gamma(\mu \rightarrow 3e)/\Gamma < 1.0 \times 10^{-12}$ with 90% CL. The current upper bounds on LFV for μ and τ are summarized in Table II [34]. Although the constraint from the $\mu - e$ conversion process is not competitive at present and

therefore is omitted from the current analysis, it is projected to become the most stringent in the future.

Specific to the muon, the flavon induces a $\Delta L_{\mu,e} = \pm 2$ process: muonium-antimuonium oscillation. Muonium (M) is the QED-bound state composed of a μ^- and an e^- . The search for this phenomenon is a key objective of the Muonium-to-Antimuonium Conversion Spectrometer (MACS) experiment PSI. The first data from the experiment have set an upper limit on the oscillation probability of $P_{M\bar{M}} \leq 8.2 \times 10^{-11}/S_B$ at 90% CL [45], where S_B is a suppression factor under external magnetic fields and is taken to be 1 here. In particular, this measurement yields a direct constraint on the LFV coupling $s\bar{\mu}e$ at the tree level (see the left panel of Fig. 3). Unlike the LFV decays discussed earlier, this constraint is obtained without invoking other flavon couplings. In our model, this probability is given by (with the calculation details provided in the Appendix C),

$$P_{M\bar{M}} = \frac{8/(\pi^2 a_B^6 \Gamma_\mu^2)}{(m_\mu^2 - m_s^2)^2 + \Gamma_s^2 m_s^2} \left(|c_{0,0}|^2 \left| \left(\frac{Y_{12}^s + (Y_{21}^s)^*}{2} \right)^2 - \left(1 + \frac{1}{\sqrt{1+X^2}} \right) \left(\frac{Y_{12}^s - (Y_{21}^s)^*}{2} \right)^2 \right|^2 \right. \\ \left. + |c_{1,0}|^2 \left| \left(\frac{Y_{12}^s + (Y_{21}^s)^*}{2} \right)^2 - \left(1 - \frac{1}{\sqrt{1+X^2}} \right) \left(\frac{Y_{12}^s - (Y_{21}^s)^*}{2} \right)^2 \right|^2 \right), \quad (3.13)$$

where, as in [46], we retain the complete propagator of the light flavon. In the above formula, $a_B \simeq 2.69 \times 10^5 \text{ GeV}^{-1}$ is the Bohr radius of muonium and $X = 6.31(B/1\text{T})$ with $B = 0.1 \text{ T}$ in the MEG, which results in the probability of the initial population state of muonium with (J, m_J) , $|c_{0,0}|^2 = 0.32$ and $|c_{1,0}|^2 = 0.18$ [47]. From the conceptual design of MACE [48], the sensitivity would go beyond $\mathcal{O}(10^{-14})$, setting even more stringent constraints, such as [49].

2. Rule out the $s\bar{\mu}e$ scenario with $Y_1^e \neq 0$

To obtain an LFV flavon for $(g-2)_\mu$, we have to turn on at least $Y_1^e \neq 0$ or $Y_3^e \neq 0$, and then $(U_R)_{41}$ or $(U_R)_{43}$ gains substantial value, as discussed in Appendix B. The $\mu - e - s$ loop is more promising than the $\mu - \tau - s$ loop since $m_\tau \gg m_\mu$, which may incur suppression again, and thus let us first focus on the former loop, whose contribution is given by

$$\delta a_\mu^s(e) = \frac{m_\mu^2}{16\pi^2 m_s^2} \left[(|Y_{21}^s|^2 + |Y_{12}^s|^2) F_s \left(\frac{m_e^2}{m_s^2}, \frac{m_\mu^2}{m_s^2} \right) \right. \\ \left. + \frac{m_e}{m_\mu} \text{Re}(Y_{21}^s Y_{12}^s) G_s \left(\frac{m_e^2}{m_s^2}, \frac{m_\mu^2}{m_s^2} \right) \right]. \quad (3.14)$$

For the sake of maximal enhancement, we consider the flavon mass around the order of the muon mass. Then, one should use the complete loop integral in Eq. (3.6) and the corresponding behaviors of F_s and G_s are shown in the center panel of Fig. 2.

Seemingly, there is no suppression in the loop functions from the heavy fermion scale, but it is hidden in the effective Yukawa couplings by means of heavy-light mixing. Therefore, we need to carefully analyze this suppression to see if it can be compensated by the light flavon's enhancement. To that end, we express $\delta a_\mu^s(e)$ in terms of the Lagrangian parameters. Considering the analytical approximation of Eq. (3.3) which shows the

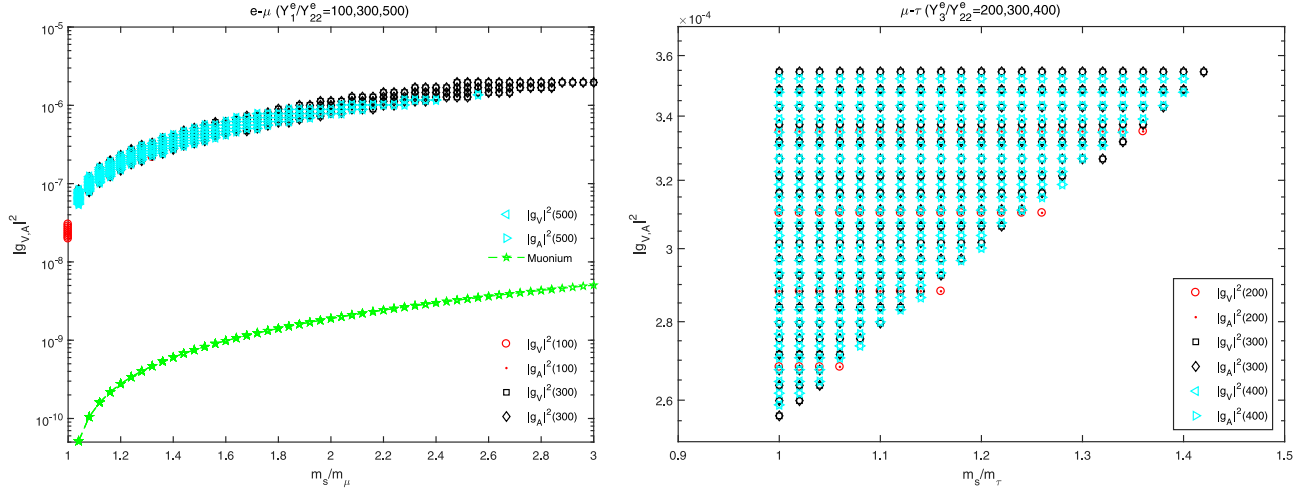


FIG. 4. Project the predicted LFV flavon in the mass-coupling plane in the $s\bar{\mu}e$ (left) and $s\bar{\mu}\tau$ (right) scenario. The LFV coupling is written as $g_V s\bar{\mu}e + ig_A s\bar{\mu}\gamma_5 e + H.c.$, and so on. In the left, the dashed green line represents the exclusion from the $M - \bar{M}$ conversion.

hierarchy $|Y_{21}^s| \gg |Y_{12}^s|$ and thus only the $|Y_{21}^s|^2$ term is left in Eq. (3.14), and then the final expression is simplified to be

$$\delta a_\mu^s(e) \approx \left| \frac{b}{y_\mu} \right|^2 \frac{|\lambda_2^\ell|^2}{32\pi^2} \frac{m_\mu^4}{m_L^2 m_s^2} F_s \left(\frac{m_e^2}{m_s^2}, \frac{m_\mu^2}{m_s^2} \right). \quad (3.15)$$

The suppression of heavy fermion scales as $m_\mu^2/m_L^2 \sim 10^{-8}$ for a TeV scale m_L ; taking $m_s = m_\mu$ for a numerical estimation which leads to $F_s \simeq 3.8$, then the required size can be easily obtained as long as b is a few times larger than y_μ .

According to the analysis in Eqs. (3.14) and (3.10), the dominant loop contributing to $(g-2)_\mu$ and the primary loop governing the Wilson coefficients $c_{L,R}$ exhibit similarities in both their coupling structures and loop functions. As a result, the parameter region favored by $(g-2)_\mu$ is constrained by bounds from $c_{L,R}$. It is noteworthy that $c_{L,R}$ receives additional suppression due to the hierarchy $|Y_{22}^s| \ll |Y_{21}^s|$, motivating us to explore the regime where $a = M_2^\ell/m_L \ll 1$. This is because the mixing parameter M_2^ℓ (or a) predominantly controls the diagonal flavon Yukawa coupling Y_{22}^s , whereas the LFV couplings such as $Y_{21,12}^s$ are proportional to Y_1^e (or b), as evident from Eqs. (3.2) and (3.3). Moreover, the contribution to $(g-2)_\mu$ is enhanced by the near-degeneracy between the muon and flavon masses, an enhancement that does not apply to $c_{L,R}$. Therefore, we perform a numerical scan over the full parameter space characterized by

$$Y_1^e/Y_{22}^e > 1, \quad M_2^\ell/m_L \ll 1, \quad m_s \gtrsim m_\mu, \quad (3.16)$$

and, moreover, we limit $m_L \gtrsim 1$ TeV considering the collider bound on VLL. We find an allowed corner at $m_L \lesssim 2.5$ TeV,

$m_s/m_\mu \gtrsim 1.3$, for $Y_1^e/Y_{22}^e = 40$, $M_2^\ell/m_L = 10^{-7}$ and $\lambda_2^\ell = 1$. Unfortunately, $M - \bar{M}$ conversion rules out the whole parameter space, which can be seen in the flavon profile obtained in this scenario, to the left of Fig. 4.

3. The last opportunity: The $s\bar{\mu}\tau$ scenario with $Y_3^e \neq 0$

The $s\bar{\mu}\tau$ scenario arising from $Y_3^e \neq 0$ is not constrained by the $M - \bar{M}$ conversion bound and thus remains viable. The analysis follows a reasoning similar to that in the previous cases. However, in the present scenario, the $\mu - \tau - s$ loop does not allow for a light mass scale. The integral denominator in Eq. (3.6) is dominated by the heavy fermion mass, that is, $zm_s^2 + (1-z)m_\tau^2 - z(1-z)m_\mu^2 \approx (1-z)m_\tau^2$. As a result, even a light flavon with a mass around m_τ produces only a modest enhancement. Adopting the approximations $F_s \simeq 1/6m_f^2$ and $G_s \simeq 1/m_f^2$, we estimate that

$$\delta a_\mu^s(\tau) \approx \frac{|\lambda_2^\ell|^2 m_\mu^2}{32\pi^2 m_\tau^2} \left[\frac{1}{6} \times (|(U_L)_{22}(U_R)_{43}|^2 + |(U_L)_{23}(U_R)_{42}|^2) + \frac{m_\tau}{m_\mu} \text{Re}((U_L)_{22}^*(U_R)_{43}(U_L)_{23}^*(U_R)_{42}) \right]. \quad (3.17)$$

We have to rely on the proper right-handed chiral rotation as large as possible to make $|(U_L)_{22}(U_R)_{43}|^2 + |(U_L)_{23}(U_R)_{42}|^2 \sim \mathcal{O}(10^{-4})$ for $\lambda_a^\ell \sim 1$; a complete parallel analysis to those made in Sec. III A, it basically requires $(U_R)_{43} \simeq \frac{b|y_\mu|}{\sqrt{|y_\mu|^2 + |ab|^2}} \gtrsim \mathcal{O}(0.01)$, with $b = Y_3^e v/(\sqrt{2}m_L)$ here while a remains the same. However, from Sec. II B it is known that the large element of U_R suffers a stringent constraint from LFU of Z decay, manifest in the $m_L - Y_3^e/Y_{22}^e$ plane to the left of Fig. 5, which gives the upper bound on Y_3^e/Y_{22}^e . Thus, a moderately large Yukawa

coupling λ_2^ℓ is needed to alleviate the burden on $(U_R)_{43}$. Of course, this scenario also gives rise to cLFV decay of τ , but the current experimental bounds are weaker than those of μ by about four orders of magnitude, see Table II. However, note that at the same time, compared to the $\mu - e - s$ loop solution to the $(g-2)_\mu$ puzzle, $\Gamma(\tau \rightarrow \mu\gamma)$ enjoys a significant enhancement due to the heavier τ mass [see Eq. (3.11)], so the cLFV τ decay yields an even stronger exclusion of the parameter space. Therefore, as in the previous case, we need to substantially suppress $a = M_2^\ell/m_L$ thus Y_{22}^s to reach the surviving parameter space.

In the numerical analysis, we have to carefully explore the viable parameter space, and we find that setting $\lambda_2^\ell \approx 2$

simultaneously with Y_3^e as large as possible works. In Fig. 6, we display the allowed region (the red-shaded triangle area) in the $m_L - m_s/m_\tau$ plane for three values of Y_3^e in units of the muon Yukawa coupling, $Y_3^e/Y_{22}^e = 200, 300, 400$; the larger Y_3^e shifts the viable region toward the heavier m_L region. As expected, compared to the $s\bar{\mu}e$ scenario, here we need $Y_3^e \gg Y_1^e$ to enhance $(U_R)_{43}$, thus compensating for the relative suppression from the heavier m_τ . We tried to find out the minimal λ_2^ℓ and obtained the limit solution with $\lambda_2^\ell = 1.7$, shown to the right of Fig. 5. It is the LFU bound that prevents the even smaller λ_2^ℓ .

It is of importance to depict the profile of the resulting flavon for future searches. In the left panel of Fig. 4, plotted

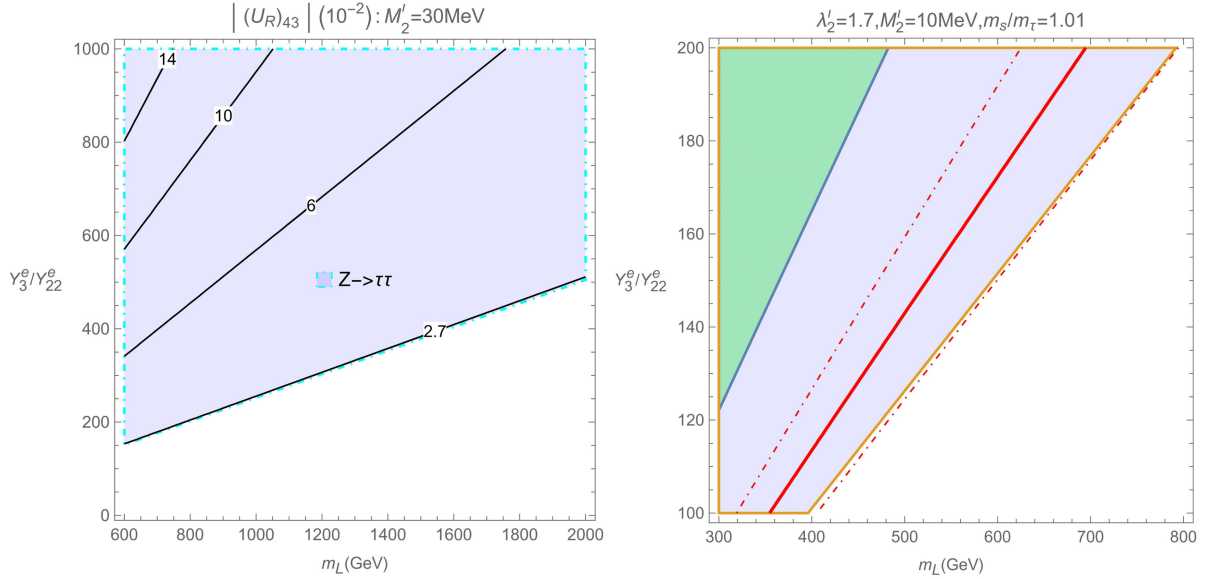


FIG. 5. Left: exclusion (blue shaded) from the LFU of Z boson decay in the $s\bar{\mu}\tau$ scenario, with contours for $|(U_R)_{43}|$ in the unit of 10^{-2} . Right: demonstration of the solution to the muon $g-2$ puzzle with a minimal λ_2^ℓ , where the band between the dashed lines lives in the edge of LFU exclusion and the green region is the exclusion of cLFV decay.

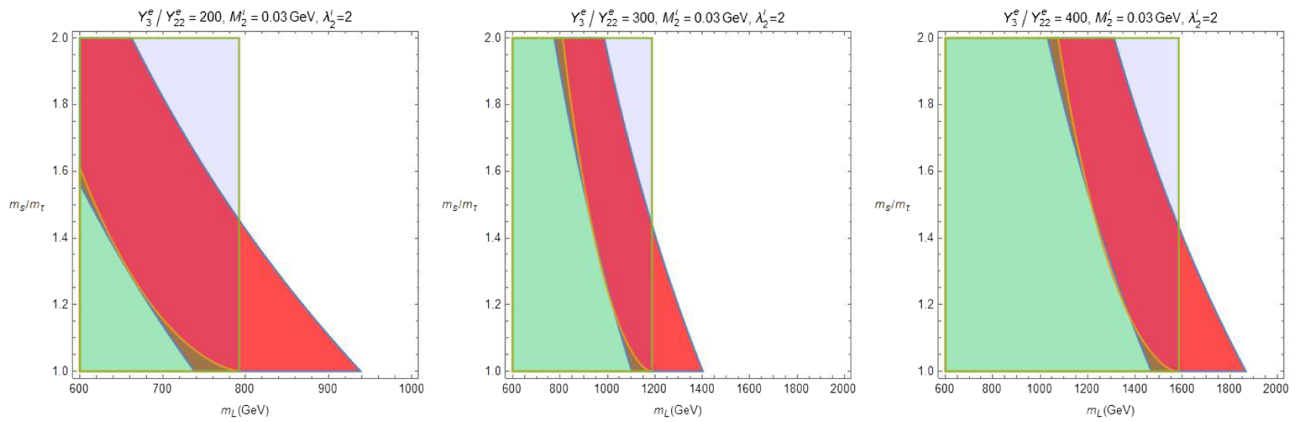


FIG. 6. The $\tau - \mu - s$ loop to explain the $(g-2)_\mu$ discrepancy (red-shaded region at the 2σ confidence level) confronting the LFU from Z decay (blue region) and cLFV of $\tau \rightarrow \mu\gamma$ (green region), taking $M_2^\ell = 0.03$ GeV and $\lambda_2^\ell = 2$ and $Y_3^e/Y_{22}^e = 200, 300, 400$ from left to right.

in the $m_s/m_\mu - |g_{V,A}|^2$ plane, we display the profile of $s\bar{\mu}e$ -type flavon, having a mass $\gtrsim \mathcal{O}(m_\mu)$ and LFV coupling $\sim \mathcal{O}(10^{-3})$, which is able to enhance the $(g-2)_\mu$ but is clearly excluded by $M - \bar{M}$ conversion. Again, we require the flavon mass to lie above m_τ to avoid a large decay branching ratio of $\tau \rightarrow \mu + s$. In the right panel of Fig. 4, the profile of the $s\bar{\mu}\tau$ -type flavon is shown, and we see that it yields a clear prediction, having mass $m_\tau \lesssim m_s \lesssim 1.5m_\tau$ and LFV coupling $\sim \mathcal{O}(10^{-2})$. We will comment on the prospect of detecting such a flavon in the part of the discussion.

IV. LOCAL $(B-L)_{23}$ FOR ELECTRON $g-2$

The persistence of a highly precise measurement and the theoretical prediction of $(g-2)_e$ is a great success of SM. The theoretical prediction is strongly affected by the precision of the fine structure constant α : $a_e^{\text{Theo}} = a_e^{\text{EM}}(\alpha) + a_e^{\text{EW}} + a_e^{\text{QCD}} + a_e^{\text{NP}}$, where the loop calculation for $a_e^{\text{EM}}(\alpha)$ has been carried out up to the fifth order for data [50,51]. Until now, the most precise determination on α has been attributed to two groups [52,53]. The recent $(g-2)_e$ result [11] improves the previous $(g-2)_e$ value three orders of magnitude, and this level of precision, for the first time, approaches the measurement uncertainty of α . Interestingly, for the prediction of the SM in $(g-2)_e$ with the two best measured α , it seems that they deviate from the measurement from two opposite sides

$$\begin{aligned} \delta a_e(\alpha_{\text{Cs}}) &= -(1.02 \pm 0.26) \times 10^{-12}, & 4.0\sigma; \\ \delta a_e(\alpha_{\text{Rb}}) &= (0.34 \pm 0.16) \times 10^{-12}, & 2.1\sigma. \end{aligned} \quad (4.1)$$

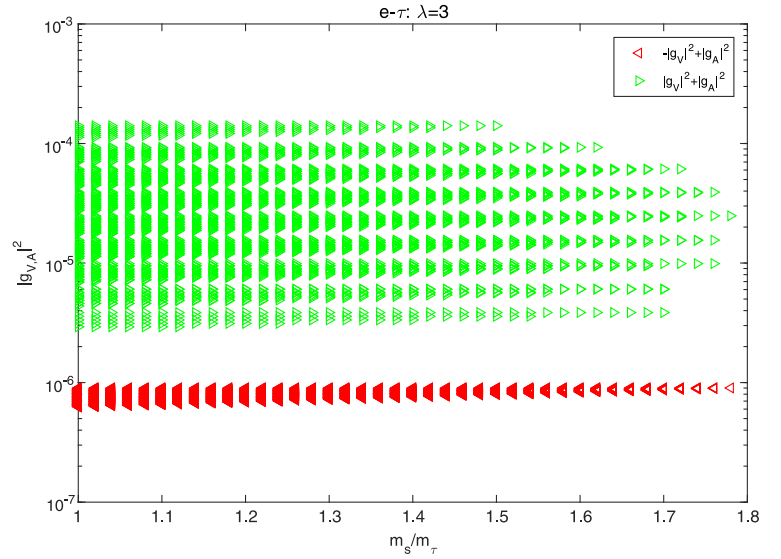
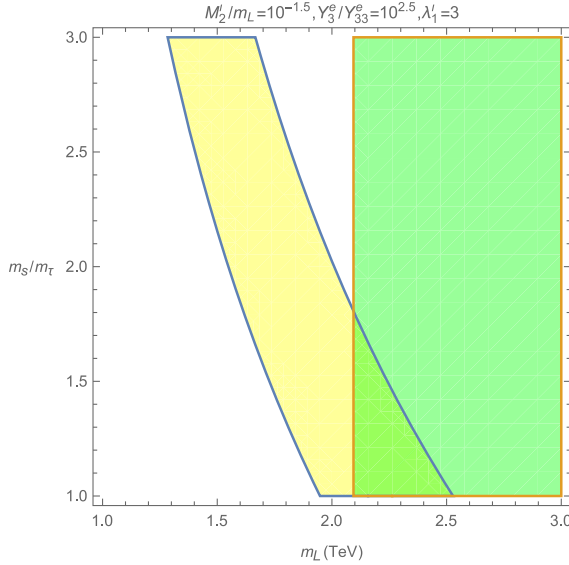


FIG. 7. Left: the yellow region in the $m_L - m_s/m_\tau$ plane can account for the required $\delta a_e < 0$, constrained by the LFU test of Z decay, which allows the green shaded region. Right: the resulting profile of flavon take $Y_{\tau e, e\tau}^s \sim 10^{-2,-3}$, where the core of $(g-2)_e$ is the difference between coupling square, $-|g_V|^2 + |g_A|^2 = -\Re(Y_{e\tau}^s Y_{\tau e}^s)$.

It is too early to claim a smoking gun of new physics, because both best α measurements suffer systematic errors that should be corrected. In this work, we choose the first deviation, with greater confidence, as a tentative anomaly, and investigate if it can be explained in the flavorful local $(B-L)_{ij}$ model.

To that end, this time we consider the local $(B-L)_{23}$ model, and then the charged lepton mass matrix takes the form of

$$M_e = \begin{pmatrix} \frac{v_h}{\sqrt{2}} Y_{11}^e & 0 & 0 & M_1^\ell \\ 0 & \frac{v_h}{\sqrt{2}} Y_{22}^e & 0 & M_2^\ell \rightarrow 0 \\ 0 & 0 & \frac{v_h}{\sqrt{2}} Y_{33}^e & M_3^\ell \rightarrow 0 \\ 0 & \frac{v_h}{\sqrt{2}} Y_2^e & \frac{v_h}{\sqrt{2}} Y_3^e & m_L \end{pmatrix}. \quad (4.2)$$

The discussions are similar to those of the muon case, and here the flavon gains an effective LFV coupling with $Y_{ij}^s = \frac{\lambda_i^e}{\sqrt{2}} (U_L^*)_{1i} (U_R)_{4j}$. We first try the $s\bar{\mu}e$ -type flavon with $Y_2^e \neq 0$, to find that it does not work. The reason is that the flavor-conserving coupling $s\bar{\mu}\mu$ is generated at the same time, but it enhances $(g-2)_\mu$ too much, contradicting the measurement. Note that this is different from the effective model analysis, where the viable parameter space survives when faced with the muonium-antimuonium oscillation [45].

Again, we turn to the $s\bar{\tau}e$ type flavon, which is free of the $(g-2)_\tau$ bound. However, contrary to $(g-2)_\mu$, the required $(g-2)_e$ discrepancy takes a minus sign. For the flavor-conserving case, at the one-loop level, this requires a CP -odd boson. But the negative sign can also be admitted

for a CP -even boson, provided that it has asymmetric couplings to different lepton flavors and leads to $\Re(Y_{e\tau}^s Y_{\tau e}^s) < 0$ for the chiral-flip contribution to $(g-2)_e$; the contribution without flipping the chirality of the loop fermion is always positive. The concrete expression can be obtained from Eq. (3.17) after some replacement,

$$\delta a_e^s(\tau) \approx \frac{m_e m_\tau |\lambda_1^\ell|^2}{32\pi^2 m_s^2} \Re((U_L^*)_{13}(U_R)_{41}(U_L^*)_{11}(U_R)_{43}) \times G_s\left(\frac{m_\tau^2}{m_s^2}, \frac{m_e^2}{m_s^2}\right). \quad (4.3)$$

It is seen that, without a very large λ_1^ℓ , we should rely on mixings $(U_R)_{43}$, $(U_R)_{41}$, and $(U_L)_{13}$ to be as large as possible. Likewise the cLFV test of Z boson imposes a bound, $|(U_R)_{41}| \leq 0.045$. As before, the first can be made large at the price of Y_3^e close to 1, and $(U_L)_{13} \propto (M_1^\ell/m_L)(v_h Y_3^e/m_L)$ can be sizable for a large flavor symmetry breaking parameter $M_1^\ell = \lambda_1^\ell v_f/\sqrt{2}$. As for $(U_R)_{41}$, it is nonvanishing but suppressed by the electron Yukawa coupling, $y_e M_1^\ell/m_L$.

Even though those large mixings are achieved, we should still worry about the LFU test of Z -boson decay, as in the situation in Sec. III C 3. We finally find the feasible corner shown in the left of Fig. 7. The minimal λ_1^ℓ is found to be not much less than 3. The resulting profile of LFV flavon is plotted in the right of Fig. 7, which shows that it typically has a smaller LFV coupling than the correspondence from the $(g-2)_\mu$ case.

V. CONCLUSION AND DISCUSSION

Lepton $g-2$ is a sensitive probe to new physics, in particular related to neutrino physics. In this work, we study a class of models with a lepton family symmetry, e.g., the local $(B-L)_{13}$, which is a candidate to realize the minimal seesaw. It presents a flavon-VLL system and our discussion expands mainly around these terms $\lambda_2^\ell \bar{\ell}_2 L_R \mathcal{F}_\ell + Y_i^e \bar{L}_L H e_{Ri} + m_L \bar{L}_L L_R$. This system admits a light LFV flavon to generate a large δa_μ . Two scenarios are considered in the spirit of Occam's razor principle to reduce the free parameters (which allows us to develop analytical approximations):

- (1) The flavon with LFV coupling to $e\mu$ by means of a sizable Y_1^e . If there was no muonium-antimuonium oscillation bound, we would have an attractive solution to the $(g-2)_\mu$ puzzle since it enjoys the maximal enhancement from a light flavon.
- (2) The flavon with LFV coupling to $\mu\tau$ by means of a sizable Y_3^e . In response to the strong constraints of the τ cLFV decay, in particular from the LFU of Z -boson decay which is specific to our model, this scenario leaves us with a flavon with mass

$m_\tau \lesssim m_s \lesssim 1.5m_\tau$, without invoking a λ_2^ℓ considerably larger than 2.

If we want to understand the tentative $\delta a_e < 0$ in our model framework, local $(B-L)_{23}$, which presents a flavon similar to the above second case, works.

At the LHC, the TeV scale VLL produces same-sign multilepton signals through $pp \rightarrow e_4^-(\rightarrow s\mu^-) + e_4^+(\rightarrow s\mu^+)$ followed by highly boosted $s \rightarrow e^\pm \mu^\mp$ and $s \rightarrow \tau^\pm \mu^\mp$ in the above two scenarios, respectively, and we notice that a similar feature (lepton jet) has been studied in Ref. [54]. Therefore, they have a very clean signal at the LHC, probably with little background, and can be easily hunted, and we will conduct a specific search in a future publication. Note that if the dominant decay channel of s is into a pair of invisible particles, such as neutrinos or dark matter, the signal will become much weaker and provides a potential way to rescue the low m_L region.

Detecting the predicted LFV flavon is also of importance. For the above two scenarios we studied, the $(g-2)_\mu$ inspired flavons have LFV coupling with strength typically $\sim 10^{-3}$ and $\sim 10^{-2}$, respectively. The former has already been excluded, but this kind of flavon, not related to the $(g-2)_\mu$ anomaly, is of interest in a wider sense of LFV at the muon collider, such as the projected μ TRISTAN [46]. The latter, as a viable solution, is of particular importance. In the Belle II experiment, light flavon can be produced via the process $e^+e^- \rightarrow \mu^\pm \tau^\mp s$. If $s \rightarrow \tau^\pm \mu^\mp$, then the same-sign lepton signal $\mu^\pm \mu^\pm \tau^\mp \tau^\mp$ is almost free from the SM background, and a large part of the parameter can be tested with the data of $\mathcal{O}(10)$ ab^{-1} [55].

Finally, let us mention the much less well-known $(g-2)_\tau$, which is measured as $0.0009^{+0.0032}_{-0.0031}$ [56]. In our model, associated with $\delta_{e,\mu}$ from the flavon, we find $a_\tau \lesssim 10^{-4}$, but future improvements to the measurement on $(g-2)_\tau$ may probe this associated signal.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: HIGGS LFV COUPLING

Within SM, the fermion masses come from a single Higgs VEV $\frac{v_h}{\sqrt{2}} \equiv \langle H^0 \rangle$ and therefore the mass matrix of the fermions is proportional to their Yukawa coupling matrix, which means that they can be diagonalized at the same time. Consequently, no FCNC is mediated by the SM Higgs boson. However, the presence of a heavy vectorlike lepton

pair spoils that proportionality, due to the mass terms independent of v_h , which are $m_L, M_{1,2,3}^\ell$ and also the mass term coming from the flavon VEV, shown in Eq. (2.4); the following discussions can be easily generalized to other parameter patterns. Then, they give rise to the SM Higgs boson induced LFV.

After EWSB and flavon developing VEV, the mass terms of the charged lepton and as well their Yukawa couplings with the SM Higgs boson h are given by

$$\mathcal{L}_Y = -(M_e)_{ab} \bar{e}_{La} e_{Rb} - Y_{ab}^e h \bar{e}_{La} e_{Rb} + \text{H.c.}, \quad (\text{A1})$$

with the charged lepton mass matrix M_e given by Eq. (2.4). Transforming to the mass basis, one has

$$\mathcal{L}_Y = -(\hat{M}_e)_{cc} \bar{\tilde{e}}_{Lc} \tilde{e}_{Rc} - (\hat{Y}^e)_{cd} h \bar{\tilde{e}}_{Lc} \tilde{e}_{Rd} + \text{H.c.}, \quad (\text{A2})$$

with

$$U_L^\dagger M_e U_R = \hat{M}_e, \quad U_L^\dagger (v_h Y^e) U_R = v_h \hat{Y}^e. \quad (\text{A3})$$

Now, the Yukawa coupling matrix $\hat{Y}^e$ is not diagonal. Subtract the above two equations, we can immediately write $\hat{Y}^e$ as the sum of two parts,

$$\hat{Y}^e = \text{diag}\left(\frac{m_e}{v_h}, \frac{m_\mu}{v_h}, \frac{m_\tau}{v_h}, \frac{m_4}{v_h}\right) - U_L^\dagger \left(\frac{M_e}{v_h} - Y^e\right) U_R, \quad (\text{A4})$$

where the first part, for the light species, gives the SM-like couplings, whereas the second part describes the deviations to the SM predictions, rewritten as $U_L^\dagger \frac{M_e(v_h \rightarrow 0)}{v_h} U_R$.

APPENDIX B: APPROXIMATELY DIAGONALIZE THE CHARGED LEPTON MASS MATRIX

In this appendix, we try to develop an approximate way to diagonalize the charged lepton mass matrix under the Occam's razor principle, Eq. (2.5). Its mass squared matrix $MM^\dagger$ and $M^\dagger M$ given in (2.6) are related to the left-handed and right-handed rotations, respectively; they both take a seesawlike hierarchical structure [58,59] with $\mathcal{O}(y_e) \ll$

$\mathcal{O}(a, b, y_\mu) \ll 1$ and thus may admit approximate diagonalization.

First, we can obtain U_L from the decomposition of the mass squared matrix $MM^\dagger/|m_L|^2$ with a good approximation. To that end, let us denote it as the block form

$$MM^\dagger/|m_L|^2 = \begin{pmatrix} |y_e|^2 & C \\ C^\dagger & \Lambda \end{pmatrix} \quad \text{with} \quad \Lambda = \begin{pmatrix} |y_\mu|^2 + |a|^2 & a \\ a^* & |b|^2 + 1 \end{pmatrix}, \quad (\text{B1})$$

and $C = (0 y_e b^*)$. The electron can be decomposed through the block chiral rotation

$$\tilde{U}_L = \begin{pmatrix} \sqrt{1 - BB^\dagger} & B \\ -B^\dagger & \sqrt{1 - B^\dagger B} \end{pmatrix} \quad \text{with} \quad B = C\Lambda^{-1} = \frac{C}{d} \begin{pmatrix} 1 + |b|^2 & -a \\ -a^* & |y_\mu|^2 + |a|^2 \end{pmatrix}, \quad (\text{B2})$$

with $d = \det \Lambda$. Then, we get electron mass and the mass-squared matrix for the 2-4 block

$$\tilde{m}_1^2 \approx |m_L|^2 \left(|y_e|^2 - \frac{|y_\mu|^2 + |a|^2}{(|y_\mu|^2 + |ab|^2)} |y_e|^2 |b|^2 \right), \quad (\text{B3})$$

$$\tilde{m}_2^2 \approx |m_L|^2 \begin{pmatrix} |y_\mu|^2 + |a|^2 & a \\ a^* & 1 + |y_e|^2 + \frac{|y_\mu|^2 + |a|^2}{(|y_\mu|^2 + |ab|^2)} |b|^2 |y_e|^2 \end{pmatrix}. \quad (\text{B4})$$

$\tilde{m}_2^2$ is also hierarchical and thus it can be approximately diagonalized similarly through

$$\tilde{U}'_L = \begin{pmatrix} \sqrt{1 - B'B'^\dagger} & B' \\ -B'^\dagger & \sqrt{1 - B'^\dagger B'} \end{pmatrix}. \quad (\text{B5})$$

Therefore, the final left-handed rotation $U_L = \tilde{U}_L \tilde{U}'_L$ has elements

$$\begin{aligned} (U_L)_{11} &\approx 1 - \frac{1}{2} \frac{|aby_e|^2(1 + |a|^2)}{(|y_\mu|^2 + |ab|^2)^2}, & (U_L)_{12} &\approx \frac{y_e b^* |a| \sqrt{1 + |a|^2}}{(|y_\mu|^2 + |ab|^2)}, & (U_L)_{14} &\approx 0; \\ (U_L)_{21} &\approx \frac{y_e^* b a}{|y_\mu|^2 + |ab|^2}, & (U_L)_{22} &\approx \left(1 - \frac{|y_e b a|^2 (1 + |a|^2)}{2(|y_\mu|^2 + |ab|^2)^2} \right) \frac{-a/|a|}{\sqrt{1 + |a|^2}}, & (U_L)_{24} &\approx \frac{a}{\sqrt{1 + |a|^2}}; \\ (U_L)_{41} &\approx \frac{-y_e^* b |a|^2}{|y_\mu|^2 + |ab|^2}, & (U_L)_{42} &\approx \left(1 - \frac{|y_e b a|^2 (1 + |a|^2)}{2(|y_\mu|^2 + |ab|^2)^2} \right) \frac{|a|}{\sqrt{1 + |a|^2}}, & (U_L)_{44} &\approx \frac{1}{\sqrt{1 + |a|^2}}. \end{aligned} \quad (\text{B6})$$

The elements related to muon and e_4 approximate the corresponding left-handed rotations in the scenario of two generations ($b = 0$). The final mass spectrum is

$$\begin{aligned}
m_1^2 &\approx |m_L|^2 |y_e|^2 \left(1 - \frac{|b|^2}{|y_\mu|^2 + |b|^2 |a|^2} (|y_\mu|^2 + |a|^2) \right), \\
m_2^2 &\approx |m_L|^2 \frac{|y_\mu|^2 + |a|^2 (|b|^2 + \frac{|y_\mu|^2 + |a|^2}{(|y_\mu|^2 + |ab|^2)} |y_e|^2 |b|^2)}{1 + |a|^2}, \\
m_4^2 &\approx |m_L|^2 (1 + |a|^2). \tag{B7}
\end{aligned}$$

Therefore, the corrections to the light charged lepton mass from the mixing effect are negligible, due to the smallness of a and b .

However, obtaining U_R from $M^\dagger M/m_L^2$ becomes more complicated. This is because after the first block rotation $\tilde{U}_R$, we are left with a remaining matrix $\tilde{\epsilon}$,

$$\begin{aligned}
\frac{M^\dagger M}{|m_L|^2} &= \left(\begin{array}{c|c} |y_e|^2 + |b|^2 & 0 \\ \hline 0 & |y_\mu|^2 \end{array} \middle| \begin{array}{c} b^* \\ y_\mu^* a \end{array} \right) \rightarrow \tilde{\epsilon} \\
&= |m_L|^2 \left(\begin{array}{c|c} |y_e|^2 + \frac{|ab|^2}{1+|a|^2} & -\frac{(ab)^* y_\mu}{1+|a|^2} \\ \hline -\frac{aby_\mu^*}{1+|a|^2} & \frac{|y_\mu|^2}{1+|a|^2} \end{array} \right), \tag{B8}
\end{aligned}$$

which in general lacks a hierarchical structure to be diagonalized with a simple expression. But we have two limits which may give rise to good analytical approximations. One is for $|ab| \ll |y_\mu|$, leading to $\tilde{\epsilon} \approx I$ and then

$$\begin{aligned}
(U_R)_{11} &\approx 1 - \frac{|b|^2}{2(1+|a|^2)^2}, & (U_R)_{12} &\approx \frac{a^* b^* y_\mu}{2(1+|a|^2)^2}, & (U_R)_{14} &\approx \frac{b^*}{1+|a|^2}; \\
(U_R)_{21} &\approx -\frac{aby_\mu^*}{2(1+|a|^2)^2}, & (U_R)_{22} &\approx -1 + \frac{|ay_\mu|^2}{2(1+|a|^2)^2}, & (U_R)_{24} &\approx \frac{ay_\mu^*}{1+|a|^2}; \\
(U_R)_{41} &\approx -\frac{b}{1+|a|^2}, & (U_R)_{42} &\approx \frac{a^* y_\mu}{1+|a|^2}, & (U_R)_{44} &\approx 1 - \frac{|b|^2 + |ay_\mu|^2}{2(1+|a|^2)^2}. \tag{B9}
\end{aligned}$$

As expected, the entries $(U_R)_{14,41,24,42}$ are controlled by the $(M^\dagger M)_{14,41,24,42}$ components. The other limit is for $|ab| \gg |y_e|$, and then the rotation matrix of Eq. (B8) has the analytical expression

$$\tilde{U}'_R = \left(\begin{array}{c|c|c} \frac{|y_\mu|}{\sqrt{|ab|^2 + |y_\mu|^2}} & \frac{|ab|}{\sqrt{|ab|^2 + |y_\mu|^2}} & 0 \\ \hline \frac{aby_\mu^*/|y_\mu|}{\sqrt{|ab|^2 + |y_\mu|^2}} & -\frac{aby_\mu^*/|ab|}{\sqrt{|ab|^2 + |y_\mu|^2}} & 0 \\ \hline 0 & 0 & 1 \end{array} \right). \tag{B10}$$

Our analytical analysis made in the text is based on the first limit.

The electron-tau mixing of $(B-L)_{23}$ model gives similar mass square matrix

$$\begin{aligned}
\frac{MM^\dagger}{|m_L|^2} &= \left(\begin{array}{c|c|c} |y_e|^2 + |a|^2 & 0 & a \\ \hline 0 & |y_\tau|^2 & y_e b^* \\ \hline a^* & y_e^* b & |b|^2 + 1 \end{array} \right), \\
\frac{M^\dagger M}{|m_L|^2} &= \left(\begin{array}{c|c|c} |y_e|^2 & 0 & ay_e^* \\ \hline 0 & |y_\tau|^2 + |b|^2 & b^* \\ \hline a^* y_e & b & |a|^2 + 1 \end{array} \right). \tag{B11}
\end{aligned}$$

The process of block diagonalization are

$$\begin{aligned}
M^\dagger M/|m_L|^2 &= \left(\begin{array}{c|c} |y_e|^2 & C \\ \hline C^\dagger & \Lambda \end{array} \right) \text{ with} \\
\Lambda &= \left(\begin{array}{c|c} |y_\tau|^2 + |b|^2 & b^* \\ \hline b & |a|^2 + 1 \end{array} \right), \tag{B12}
\end{aligned}$$

and $C = (0 \ ay_e^*)$.

We get analytical expression of related mass

$$m_1^2 \approx |m_L|^2 |y_e|^2 \frac{|y_\tau|^2}{(|y_\tau|^2 + |ab|^2 + |ay_\tau|^2)}; \tag{B13}$$

$$m_3^2 \approx |m_L|^2 \left(|y_\tau|^2 + \frac{|ab|^2}{1+|a|^2} \right); \tag{B14}$$

$$m_4^2 \approx |m_L|^2 (1 + |a|^2). \tag{B15}$$

APPENDIX C: MUONIUM-ANTIMUONIUM OSCILLATION

We follow [29,46,60] to carry out a detailed study of the muonium oscillation in our model, where the light flavon cannot be integrated out. As in the $K^0 - \bar{K}^0$ and neutrino system, the dynamics of the $M - \bar{M}$ mixing quantum system is described by their mass matrix, which consists of the elements of the effective Hamiltonian H_{eff} among the two states

$$\begin{aligned}
\mathcal{M} = \langle \alpha | H_{\text{eff}} | \beta \rangle &= \begin{pmatrix} m - i\Gamma_\mu/2 & \Delta m/2 \\ \Delta m/2 & m - i\Gamma_\mu/2 \end{pmatrix}, \text{ with} \\
|\alpha/\beta\rangle &= |M \text{ or } \bar{M}\rangle. \tag{C1}
\end{aligned}$$

where m and $\Gamma_\mu \simeq 3.00 \times 10^{-19}$ GeV are the averages of the masses and widths of M and $\bar{M}$, respectively. The off-diagonal element $\mathcal{M}_{12} = \langle M | H_{\text{eff}} | \bar{M} \rangle = \Delta m/2$ is crucial since it induces the $M - \bar{M}$ mass splitting and oscillation. It has been estimated without and with a magnetic field B

employed in the experimental apparatus [47,61]. Then, the time-integrated probability of $M - \bar{M}$ conversion is given by

$$P_{M\bar{M}} \approx \int_0^\infty dt \Gamma_\mu \exp(-\Gamma_\mu t) \sin^2(\mathcal{M}_{12}t) = \frac{2\mathcal{M}_{12}^2}{4\mathcal{M}_{12}^2 + \Gamma_\mu^2} \approx \frac{2\mathcal{M}_{12}^2}{\Gamma_\mu^2}. \quad (\text{C2})$$

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