

Lee–Yang edge singularities in a nonlocal Nambu–Jona-Lasinio model

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We investigate the QCD phase diagram and the associated Lee–Yang edge singularities using the two-flavor nonlocal Nambu–Jona-Lasinio model extended to complex chemical potential. We show explicitly there exists a strong correlation between the chiral phase transition and the patterns in the effective potential in the complex plane of order parameter, serving as a criterion to differentiate crossover from first-order transitions. Typically, the Lee–Yang edge singularities can be understood as a generalization of the critical end point (CEP) separating the crossover and the first-order transition, where the positive Nambu phase and the Wigner phase coalesce. We further analyze the scaling behavior near the CEP by extracting the critical exponent associated with the Lee–Yang singularities. Additionally, we confirm that the extrapolation of the Lee–Yang edge singularity trajectories provides an efficient method of determining the CEP location, even at a small real chemical potential. This provides a viable method for exploring regions of the QCD phase diagram that remain inaccessible to lattice QCD.

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I. INTRODUCTION

In recent years, considerable attention has been devoted to the study of strong interaction matter at finite temperature and density, due to its relevance to the early Universe matter generation and the ultrarelativistic heavy-ion collisions studies. The thermodynamic properties of strong interaction matter are embedded in the phase structure of QCD [1,2], which links to essential features of the interaction, such as the dynamical chiral symmetry breaking, quark/color confinement, and so on. Over the years, it has been realized that QCD phase structure can be very rich at high baryon chemical potential, typically with a potential critical end point (CEP) [3–9], and beyond that the moat regime or inhomogeneous phases [10,11], color-superconductivity [12], etc. Notably, the experimental search of CEP is a hot topic in recent studies, with numerous endeavors put in the studies of specific techniques and relevant observables [13–20]. It is then crucial to provide estimates not only on

the location of critical end point but also on the properties of critical scaling behavior through observables.

In turn, the phase structure at complex chemical potential can also provide valuable information on the properties of CEP mentioned above. In 1952, Lee and Yang investigated the distribution of zeros of the partition function in the complex chemical potential plane and their connection to phase transitions [21,22]. Inspired by their pioneering work, the criticality and analytic behavior of QCD has been further investigated by exploring the distribution of the Lee–Yang zeros and the respective singularities through various theoretical methods [23–36]. Nevertheless, systematic investigations of the thermodynamic potential and the (meta)stable phases in the complex order-parameter plane, together with their dynamical evolution as functions of temperature and complex chemical potential, are still scarce. Such studies are crucial for revealing the analytic structure of the QCD thermodynamic potential, since phase transitions are encoded in the motion and coalescence of its extrema, as well as in the associated branch points in the complex plane. In particular, the complex chemical potential region connects multiple aspects of QCD phase structure including the Roberge-Weiss transition (pure imaginary μ_B), Lee–Yang edge singularity (complex μ_B), and the presumed critical end point (real μ_B), the latter being still one of the fundamental questions that attracts numerous efforts from both theoretical and experimental sides (see, e.g., Ref. [36], etc.). Motivated by the scarcity of such systematic studies, we aim to perform a detailed

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analysis of their dynamical behavior at finite temperature and complex chemical potential and discuss whether there are specific patterns in the thermodynamic quantities presented at complex chemical potential relevant to the phase structure in this paper.

To date, however, theoretical calculations on QCD phase structure are still facing difficulties in the high chemical potential region. In particular, lattice QCD simulation as a first-principle approach is still not capable of tackling the case with a large, real chemical potential due to the notorious sign problem [37,38]. To circumvent the sign problem, lattice simulations often employ an imaginary chemical potential and use analytic continuation to extend the calculation to real chemical potential [39,40]. The analytical continuation from the imaginary chemical potential μ_I to the real chemical potential μ_R typically assumes a specific form, such as a Taylor expansion, which imposes a limiting convergence radius.

To go beyond these limitations, continuum QCD approaches have been developed, such as the Dyson–Schwinger equations (DSE) [41–43], the functional renormalization group (FRG) [44], holographic QCD [45–47], and so on, which enable the nonperturbative study on QCD at finite temperature and density including the dynamical properties of strong interaction.

These approaches involve typically rather sophisticated calculations. As a more tractable alternative, low-energy effective models of QCD, such as the Nambu–Jona-Lasinio (NJL) model [48–50], have been widely employed to investigate the dynamical chiral symmetry breaking (DCSB). However, the conventional NJL model with a contact interaction and mean-field approximation suffers from intrinsic limitations. Its non-renormalizable character necessitates the introduction of a regularization scheme; although Lorentz-invariant implementations are possible, the model remains fundamentally unable to encode the dynamics of QCD interactions, such as those shown in the momentum dependence of quark mass function arising from gluon exchange.

To address these issues, nonlocal extensions of the NJL model have been developed [51]. By introducing nonlocal interaction kernels, also referred to as the form factors in coordinate or momentum space, the nonlocal NJL model preserves the chiral symmetry structure while incorporating essential features of QCD’s low-energy dynamics. The nonlocal form factors can be motivated by the results from lattice QCD calculations [52], DSE, or the instanton liquid model [53]. By embedding QCD dynamics through an explicit momentum-dependent or “running”—coupling in the form of a nonlocal four-fermion interaction—the model offers a simple and realistic description of the DCSB. We then carry out our investigation within the framework of the nonlocal NJL model.

This paper is organized as follows. In Sec. II, we present the general formalism of the nonlocal NJL model. In

Sec. III, we investigate the phase diagram at complex chemical potentials and propose an improved method for identifying the critical end point (CEP) and the Lee–Yang edge singularities (LYEs). We further analyze the critical behavior by extracting the critical exponent $\beta\delta$ and assess the validity of the LYE-based extrapolation method for locating the CEP. Finally, Sec. IV provides a summary of our findings.

II. NONLOCAL NAMBU–JONA-LASINIO MODEL

A. Nonlocal NJL effective action

The basic idea of the nonlocal NJL model is to approximate QCD interactions with the nonlocal four-quark interactions. A scheme based on the features of the instanton liquid model (ILM) has been introduced in Ref. [51], in which the nonlocal form factor is associated to each quark field. At zero temperature and chemical potential, i.e., in the vacuum, the effective action takes the form:

$$\mathcal{S}_E = \int d^4x \left[\bar{\psi}(x)(-i\not{\partial} + m_q)\psi(x) - \frac{G}{2} J_\alpha(x) J_\alpha(x) \right]; \quad (1)$$

here m_q is the current quark mass, and the Euclidean operator $\not{\partial}$ is defined as

$$\not{\partial} = \gamma_4 \frac{\partial}{\partial \tau} + \vec{\gamma} \cdot \vec{\nabla}, \quad (2)$$

with $\gamma_4 = i\gamma_0$, $\tau = it$. The current $J_\alpha(x)$ involving nonlocal interactions is given by

$$J_\alpha(x) = \int d^4y \int d^4z r(y-x)r(x-z)\bar{\psi}(y)\Gamma_\alpha\psi(z); \quad (3)$$

here, $\Gamma_\alpha = (1, i\gamma_5 \vec{\tau})$ and $r(x-z)$ is a nonlocal regulator function.

To perform a standard bosonization of the theory for convenience, one introduces auxiliary fields $\varphi_\alpha(x) = (\sigma(x), \vec{\pi}(x))$, where σ and $\vec{\pi}$ are chiral partner boson fields representing the scalar-isoscalar and pseudoscalar-isovector mesonic degrees of freedom, respectively. Finally, one has the Euclidean generating functional of nonlocal NJL model in momentum space as

$$Z = \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp[-\mathcal{S}_E^{\text{bos}}], \quad (4)$$

with

$$\mathcal{S}_E^{\text{bos}} = -\ln \det \hat{A} + \frac{1}{2G} \int \frac{d^4p}{(2\pi)^4} \phi_\alpha^2(p); \quad (5)$$

here,

$$\begin{aligned}
A(p, p') &:= \langle p | \hat{A} | p' \rangle \\
&= (\not{p} + m_q)(2\pi)^4 \delta^{(4)}(p - p') \\
&\quad + r(p)r(p')\Gamma_\alpha \phi_\alpha(p - p'), \quad (6)
\end{aligned}$$

$$\phi_\alpha(p) = \int d^4x e^{-ip \cdot x} \varphi_\alpha(x), \quad (7)$$

$$r(p) = \int d^4z e^{-ip \cdot x} r(z). \quad (8)$$

Since $r(p)$ is Lorentz invariant, it should be a function of p^2 . We will then use the form $r(p^2)$ from now on.

B. Mean field approximation and gap equation

In a homogeneous and isotropic vacuum, the translationally invariant vacuum expectation values of the meson fields are given by $\bar{\sigma} = \langle \sigma \rangle$ and $\bar{\pi}_i = \langle \pi_i \rangle$. From symmetry considerations, the mean values of the pion fields vanish due to their nature as Goldstone bosons. The mesonic fields can thus be expanded around their mean values as

$$\sigma(x) = \bar{\sigma} + \delta\sigma(x), \quad \vec{\pi}(x) = \delta\vec{\pi}(x). \quad (9)$$

The bosonized Euclidean action $\mathcal{S}_E^{\text{bos}}$ can be expanded in powers of the mesonic fluctuations $\delta\sigma, \delta\vec{\pi}$:

$$\mathcal{S}_E^{\text{bos}} = \mathcal{S}_E^{\text{MF}} + \mathcal{S}_E^{(2)} + \dots \quad (10)$$

The grand canonical thermodynamic potential per four-volume $V^{(4)}$ in the mean-field approximation reads

$$\begin{aligned}
\Omega^{\text{MF}}(T, \mu) &= -\frac{T}{V} \ln \mathcal{Z}^{\text{MF}}(T, \mu) \\
&= -4N_c \int \frac{d^4p}{(2\pi)^4} \ln [p^2 + M^2(p^2)] + \frac{\bar{\sigma}^2}{2G}, \quad (11)
\end{aligned}$$

where the momentum-dependent mass function $M(p^2)$ is determined by the gap equation:

$$M(p^2) = m_q + r^2(p^2)\bar{\sigma} = m_q + \mathcal{C}(p^2)\bar{\sigma}. \quad (12)$$

Here, the form factor $\mathcal{C}(p^2) = r^2(p^2)$ is introduced to simplify the expression.

To extend the formulation to finite temperature T and chemical potential μ , the integration over the fourth momentum component is replaced by a Matsubara summation:

$$\int \frac{d^4p}{(2\pi)^4} F(p_4, \vec{p}) \rightarrow \oint_p F(p_4, \vec{p}), \quad (13)$$

$$\oint_p F(p_4, \vec{p}) := T \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} F(\omega_n - i\mu, \vec{p}), \quad (14)$$

where $\omega_n = (2n + 1)\pi T$ are the Matsubara frequencies for fermionic modes.

The equilibrium phase corresponds to the stationary point of the thermodynamic potential as a function of the condensate $\bar{\sigma}$:

$$\frac{\partial \Omega_{\text{MF}}}{\partial \bar{\sigma}} = 0, \quad (15)$$

leading to the gap equation:

$$\bar{\sigma} = 8N_c G \oint_p \frac{M(p^2)\mathcal{C}(p^2)}{p^2 + M^2(p^2)}. \quad (16)$$

Other relevant physical quantities, such as the chiral condensate $\langle \bar{q}q \rangle$ for each flavor and the corresponding thermal susceptibilities χ , can be derived from the thermodynamic potential as

$$\langle \bar{q}q \rangle = \frac{\partial \Omega}{\partial m_q} = -4N_c \oint_p \frac{M(p^2)}{p^2 + M^2(p^2)}, \quad (17)$$

$$\chi_T = -\frac{\partial \bar{\sigma}}{\partial T}, \quad \chi_\mu = -\frac{\partial \bar{\sigma}}{\partial \mu}. \quad (18)$$

Each solution $\bar{\sigma}_i(T, \mu)$ of Eq. (16) depends on the temperature and chemical potential, with i indexing possible multiple solutions. The thermal susceptibility $\chi(\bar{\sigma}_i)$ can be obtained by differentiating both sides of Eq. (15) with respect to T (or μ), yielding the following relations:

$$\begin{aligned}
\chi_T &= \left(\frac{\partial^2 \Omega_{\text{MF}}}{\partial T \partial \bar{\sigma}} \right) \bigg/ \left(\frac{\partial^2 \Omega_{\text{MF}}}{\partial \bar{\sigma}^2} \right), \\
\chi_\mu &= \left(\frac{\partial^2 \Omega_{\text{MF}}}{\partial \mu \partial \bar{\sigma}} \right) \bigg/ \left(\frac{\partial^2 \Omega_{\text{MF}}}{\partial \bar{\sigma}^2} \right). \quad (19)
\end{aligned}$$

C. The form factor $\mathcal{C}(p^2)$

Two commonly used schemes for the form factor $\mathcal{C}(p^2)$ are described here, denoted as FA and FB, respectively. It is well known that the presence of a nonperturbative QCD vacuum, characterized by a nonzero quark condensate $\langle \bar{q}q \rangle \neq 0$, dynamically generates a momentum-dependent quark mass. This results in a transition from current quarks to quasiparticle-like constituent quarks. The spontaneous breaking of chiral symmetry thus leads to a nontrivial dynamical quark mass $M(p^2)$, and the form factor $\mathcal{C}(p^2)$ can be obtained by fitting the lattice QCD (LQCD) results for $M(p^2)$ as shown in Eq. (12).



FIG. 1. Schematic view of the Dyson–Schwinger equation for the quark propagator. The gray blobs with straight lines stand for the full quark propagators, the gray blob with curly line is the gluon propagator, the white blob stands for the full quark-gluon interaction vertex, and the black dots stand for the classical vertex. With the nonlocal NJL interaction, the form factor $\mathcal{C}(p^2)$ matches the combination of gluon propagator and full quark-gluon vertex, as indicated in the gap equation in Eq. (16).

For the FA scheme [54], a simple ansatz is adopted:

$$\mathcal{C}(p^2) = \exp\left(-\frac{p^2}{\Lambda^2}\right), \quad (20)$$

where Λ serves as a soft cutoff and a nonperturbative scale, governing the momentum dependence of the interaction in the infrared regime. This exponential form provides a good description at low momenta, but it decays too rapidly at high momenta, thereby failing to reproduce the correct ultraviolet behavior.

A refined treatment is motivated by examining the Dyson–Schwinger equation (DSE) for the quark propagator, as shown in Fig. 1. The second term on the right-hand side includes the fully dressed quark and gluon propagators, as well as the quark-gluon vertex, which represents the quark self-energy $\Sigma(p)$ in the QCD vacuum. As a model for this term, the form factor $\mathcal{C}(p^2)$ should incorporate the chemical potential dependence, i.e., $\mathcal{C}((\omega_n - i\mu)^2 + \vec{p}^2)$.

At high momentum, a simplification is often taken by using the bare vertex γ_ν (rainbow truncation) and replacing the full gluon propagator by its free form, i.e.,

$$4\pi\alpha_s(p^2)D_{\mu\nu}(p) \rightarrow 4\pi\alpha_s(p^2)D_{\mu\nu}^{\text{free}}(p), \quad (21)$$

where $\alpha_s(p^2)$ is the running QCD coupling. In this approximation, the leading nontrivial contribution to the quark self-energy $\Sigma(p^2)$ takes the form:

$$\Sigma(p^2) = \pi \frac{N_c^2 - 1}{2N_c^2 N_f} \frac{\alpha_s(p^2)}{p^2} (3 + \xi) \langle \bar{\psi}\psi \rangle + \delta\Sigma, \quad (22)$$

where ξ is the gauge parameter, and $\delta\Sigma$ denotes the subleading corrections.

Based on this analysis, the FB scheme for the form factor is proposed as [55]:

$$\mathcal{C}(p^2) = \begin{cases} \exp\left(-\frac{p^2}{\Lambda^2}\right), & \text{for } p^2 < \Gamma^2, \\ \text{const.} \cdot \frac{\alpha_s(p^2)}{p^2}, & \text{for } p^2 \geq \Gamma^2. \end{cases} \quad (23)$$

Here, Γ denotes the matching scale. For momenta $p^2 > \Gamma^2$, the ultraviolet (UV) behavior of the interaction needs to be

modified. The optimal matching scale is found to be around 0.8 GeV; in this work, we adopt $\Gamma = 0.83$ GeV, following Ref. [55]. We further employ the standard one-loop expression for the running coupling,

$$\alpha_s(p^2) = \frac{4\pi}{\beta_0 \ln\left(\frac{p^2}{\Lambda_{\text{QCD}}^2}\right)}, \quad (24)$$

where $\beta_0 = 9$, corresponding to three active quark flavors, and $\Lambda_{\text{QCD}} = 0.25$ GeV. With these parameters, the running coupling reproduces $\alpha_s(m_Z^2) = 0.12$ at the Z -boson mass $m_Z = 91.2$ GeV.

In this work, we adopt the simpler FA form given in Eq. (20) to show our results. The results with the FB scheme are qualitatively the same as those of FA as seen in Fig. 6 in Sec. III. The model parameters to be determined are the coupling strength G in Eq. (1), the current quark mass m_q , and the cutoff parameter Λ in Eq. (20). These parameters can be fixed by fitting to physical observables such as the dynamical quark mass $M(p^2)$, the empirical value of the pion decay constant, and the pion mass m_π . Following Refs. [56,57], we take the parameter set:

$$G\Lambda^2 = 20.65, \quad m_q = 5.74 \text{ MeV}, \quad \Lambda = 752 \text{ MeV}. \quad (25)$$

III. CALCULATION AND NUMERICAL RESULTS

A. Real chemical potential

As discussed earlier, the quark condensate $\langle \bar{q}q \rangle$ is dynamically generated at low temperature T and low chemical potential μ . In this regime, quarks behave as quasiparticles with a momentum-dependent mass $M(p^2)$, as a consequence of dynamical chiral symmetry breaking. As T or μ increases, the condensate melts, and the dynamical mass vanishes. From Eq. (12), we observe that the solution $\bar{\sigma}$ of the gap equation differs from $M(p^2 = 0)$ only by the explicit chiral symmetry breaking term due to the current quark mass m_q . Therefore, $\bar{\sigma}$ can also be used as an order parameter for the chiral phase transition. Moreover, $\bar{\sigma}$ corresponds to the stationary point of the thermodynamic potential Ω , making it an instructive representation for how Ω evolves with temperature and chemical potential.

The obtained result is illustrated in Fig. 2, for the case of $T = 80$ MeV and at several values of the chemical potential. It shows that, at $\mu = 200$ MeV, the thermodynamic potential Ω exhibits a single global minimum at $\bar{\sigma} = 399.7$ MeV, referred to as the *positive Nambu solution*. As μ increases to 209.4 MeV, a new pair of extrema emerges: a local maximum and a local minimum. The newly formed minimum is identified as the *Wigner solution*. Upon further increase in μ , the *Wigner solution* becomes the global minimum, and the system undergoes a first-order phase transition at $\mu = 216.45$ MeV, where $\bar{\sigma}$ exhibits a

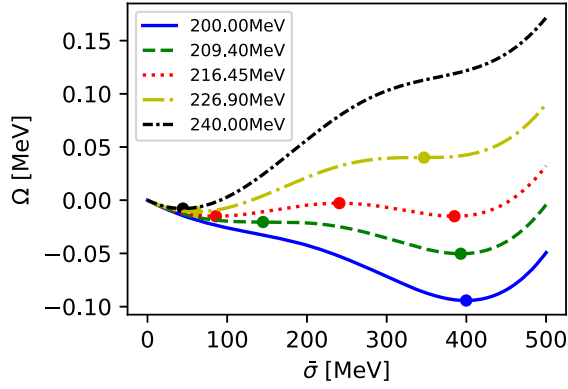


FIG. 2. Thermodynamic potential Ω as a function of $\bar{\sigma}$ at $T = 80$ MeV for several values of chemical potential μ . Solutions of the gap equation are indicated by dots.

discontinuous jump. Eventually, the positive Nambu solution vanishes at $\mu = 226.9$ MeV.

In contrast, at higher temperatures (e.g., $T = 100$ MeV), $\bar{\sigma}$ varies smoothly with μ , indicating a crossover rather than a genuine phase transition. Thus, a *critical end point* (CEP) exists at which the first-order phase transition line terminates. At the CEP, the three extrema of Ω —the positive Nambu solution, the Wigner solution, and the intermediate local maximum—coalesce into a single degenerate solution.

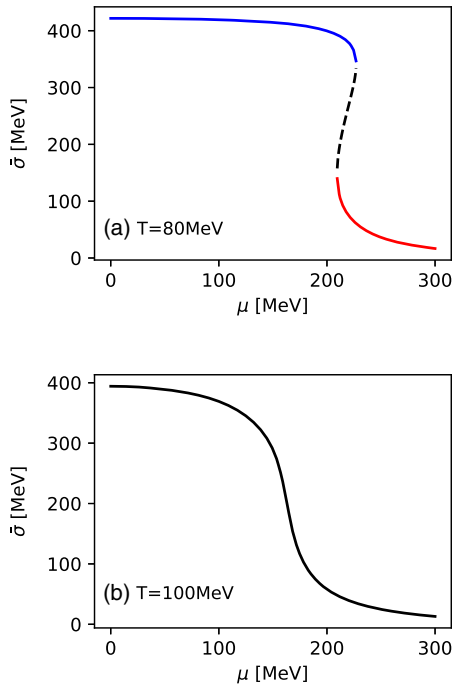


FIG. 3. Solutions of the gap equation $\bar{\sigma}(\mu)$ at two different temperatures: (a) At $T = 80$ MeV, where the Nambu solution and the Wigner solution are shown as the blue and red curves, respectively. We also show the metastable solution as the black-dashed curve. (b) At $T = 100$ MeV, where the phase transition is a crossover, with only one solution obtained.

Figure 3 shows the solutions of the gap equation as a function of μ in cases of $T = 80$ MeV and $T = 100$ MeV.

Solutions to the gap equation typically emerge or vanish in pairs, consisting of a local maximum ($\frac{\partial^2 \Omega}{\partial \bar{\sigma}^2} < 0$) and a local minimum ($\frac{\partial^2 \Omega}{\partial \bar{\sigma}^2} > 0$). When such a pair coalesces, the second derivative vanishes ($\frac{\partial^2 \Omega}{\partial \bar{\sigma}^2} = 0$), leading to a divergence in the susceptibility χ_T [see Eq. (19)]. As shown in Fig. 4, divergences at $\mu = 209.4$ MeV and $\mu = 226.9$ MeV signal the presence of *spinodal points* in the phase diagram. These points delineate the *spinodal region*, which enclose the *spinodal region*—a metastable domain associated with supercooling and superheating phenomena [58–60].

For physical quark masses and low chemical potentials ($\mu < \mu_{\text{CEP}}$), the chiral transition is a crossover, characterized by a smooth variation of the order parameter. The chiral crossover temperature $T_c(\mu_B)$ can be extracted from the temperature dependence of χ_T , where its peak signals the T_c at a given μ_B .

The curvature of the crossover line in the (T, μ_B) plane is a key observable, which can be a benchmark for the model:

$$\frac{T_c(\mu_B)}{T_c(0)} = 1 - \kappa \left(\frac{\mu_B}{T_c(0)} \right)^2 + \lambda \left(\frac{\mu_B}{T_c(0)} \right)^4 + \dots \quad (26)$$

Figure 5 shows the fitting result for $T_c(\mu_B)$ obtained from the maxima of χ_T in the range $\mu_B = 0$ –566.2 MeV. The fitted curvature is $\kappa = 0.01811(2)$, which is consistent with the predictions from previous studies in case of two flavors: $\kappa = 0.0175(7)$ [4], $\kappa = 0.0160(1)$ [61]. Additional lattice QCD results for κ can be found in Ref. [62].

The obtained phase diagram for the scheme FA and FB is shown in Fig. 6. It is clear that the critical end point is located at the intersection of spinodal lines at

$$T_{\text{CEP}}^{\text{FA}} = 90.10 \text{ MeV}, \quad \mu_{\text{q,CEP}}^{\text{FA}} = 188.73 \text{ MeV}, \quad (27)$$

$$T_{\text{CEP}}^{\text{FB}} = 101.40 \text{ MeV}, \quad \mu_{\text{q,CEP}}^{\text{FB}} = 207.31 \text{ MeV}. \quad (28)$$

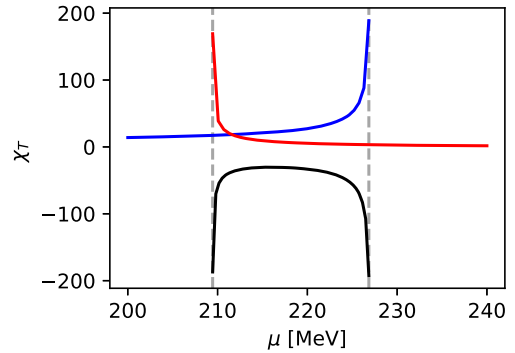


FIG. 4. Thermal susceptibility χ_T as a function of the chemical potential μ at $T = 80$ MeV. The blue, red, and black curves represent χ_T of the positive Nambu solution, the Wigner solution, and the local maximum solution, respectively.

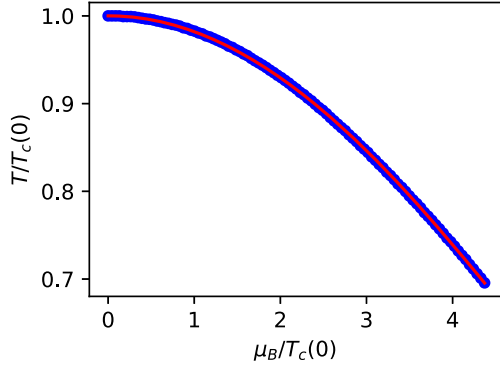


FIG. 5. Chiral crossover line extracted from the thermal susceptibility χ_T in the nonlocal NJL model, which is rescaled by the pseudocritical temperature $T_c(0) = 129.5$ MeV at vanishing μ .

B. Complex chemical potential

Before studying the properties of the system with a complex chemical potential via the nonlocal NJL model, let us briefly review the concept of Lee–Yang zeros. In a grand canonical ensemble, the partition function can be expressed

as a polynomial in the fugacity $\zeta = e^{\mu/T}$. Therefore, the distribution of the zeros of the partition function encodes all thermodynamic information of the system. The zeros coalesce into branch cuts emanating from the so-called Lee–Yang edge singularities (LYEs). A second-order (first-order) phase transition occurs when the Lee–Yang edge singularity (branch cut) appears at the real axis of μ or the axis of T .

When the chemical potential is analytically continued into the complex plane, the thermodynamic potential $\Omega(T, \mu, \bar{\sigma})$ also becomes complex. Consequently, the solutions of the gap equation are distributed in the complex plane, and the concept of a minimum becomes ill defined, since complex values cannot be ordered.

To locate the second-order phase transition, i.e., the LYE, several representative methods have been employed in recent studies [24,31,33,40]. One of them involves analyzing the radius of the convergence of the Taylor expansion of the thermodynamic potential Ω around $\mu = 0$ up to order N :

$$\Omega_N^E(T, \mu) = \sum_{n=1}^N \frac{1}{n!} \frac{\partial^n \Omega}{\partial \mu^n} \Big|_{\mu=0} \mu^n. \quad (29)$$

Alternatively, a resummation scheme for the thermodynamic potential, denoted by $\Omega_N^R(T, \mu)$, provides more direct insight into the location of LYE because, even at finite order, Ω_N^R includes infinite powers of μ [34,63].

In this article, we propose an improved method for locating the LYE. As mentioned earlier, the CEP lies at the intersection of spinodal lines, where three solutions of the gap equation coalesce. When $\mu_I = 0$, the spinodal lines can be identified by counting the number of real-valued solutions to the gap equation. However, when analytically continued into the complex $\bar{\sigma}$ plane, the total number of solutions remains constant—these solutions merely depart from or approach the real axis. The situation becomes more complicated when $\mu_I \neq 0$, as all solutions may become complex. Consequently, a modified approach is required to identify the LYE in the complex chemical potential plane.

A key observation is that the spinodal line separates regions in the phase diagram, which are characterized by different shapes of the thermodynamic potential surface.

When extended into the complex $\bar{\sigma}$ plane, the thermodynamic potential $\Omega(\bar{\sigma}; T, \mu)$ becomes a complex-valued function defined over a two-dimensional domain. It can be expressed as $\Omega = \Omega_R + i\Omega_I$, where both Ω_R and Ω_I are real functions of $\bar{\sigma}$, T , and μ . Assuming that Ω is analytic in $\bar{\sigma}$, the real part $\Omega_R(\bar{\sigma})|_{T, \mu}$ alone suffices to determine the full structure of the potential surface, due to the Cauchy–Riemann conditions.

Figure 7 displays the real part of the thermodynamic potential, $\Omega_R(\bar{\sigma})|_{T, \mu}$, in the complex $\bar{\sigma}$ plane at $T = 80$ MeV, $\mu_R = 200$ MeV, and $\mu_I/\pi T = 0$.

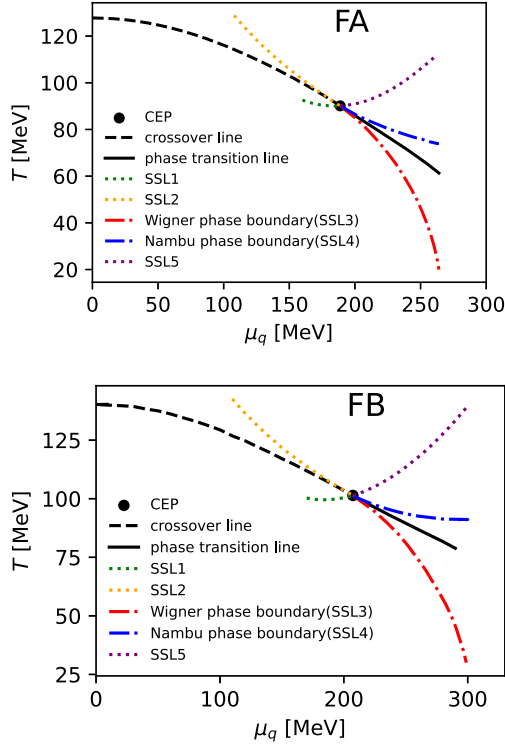


FIG. 6. The obtained phase diagram and the shape shifting lines (SSL) in the (T, μ) plane, computed from nonlocal NJL model within the FA and FB scheme. The red and blue dot-dashed lines (the spinodal lines) denote the boundaries of the Wigner phase and the positive Nambu phase, respectively, where each phase merges with the metastable phase, as shown in Fig. 2. The green, orange and purple dotted lines represent other shape shifting lines.

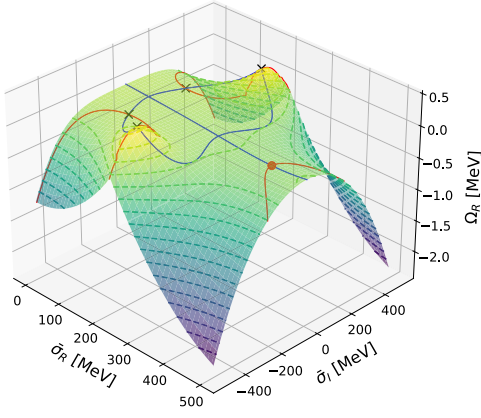


FIG. 7. Thermodynamic potential Ω in the complex plane of $\bar{\sigma}$ in the case of $T = 80$ MeV and $\mu = 180$ MeV. The red solid line corresponds to solutions of $\text{Re}(\frac{\partial \Omega}{\partial \bar{\sigma}}) = 0$, the blue line corresponds to solutions of $\text{Im}(\frac{\partial \Omega}{\partial \bar{\sigma}}) = 0$. The red point is the physical stationary point (the positive Nambu solution).

The condition of the stationary points, originally given by Eq. (15), becomes

$$\begin{aligned} \frac{\partial \Omega}{\partial \bar{\sigma}} &= \frac{\partial \Omega}{\partial \bar{\sigma}_R} = \frac{\partial \Omega_R}{\partial \bar{\sigma}_R} + i \frac{\partial \Omega_I}{\partial \bar{\sigma}_R} = \frac{\partial \Omega_R}{\partial \bar{\sigma}_R} - i \frac{\partial \Omega_R}{\partial \bar{\sigma}_I} = 0, \\ \Rightarrow \frac{\partial \Omega_R}{\partial \bar{\sigma}_R} &= 0, \frac{\partial \Omega_R}{\partial \bar{\sigma}_I} = 0. \end{aligned} \quad (30)$$

The red and blue lines in the figure represent the solutions to the real and imaginary parts of the gap equation, corresponding to the conditions $\partial \Omega_R / \partial \bar{\sigma}_R = 0$ and $\partial \Omega_R / \partial \bar{\sigma}_I = 0$, respectively. These curves jointly determine the stationary points and reflect the underlying geometry of the potential surface.

The red point marks the physical stationary point on the real axis, while the black crosses indicate other stationary solutions in the complex $\bar{\sigma}$ plane. For $\mu_I = 0$, the condition for a physical solution is that $\frac{\partial^2 \Omega_R}{\partial \bar{\sigma}_R^2} > 0$, which further requires $\frac{\partial^2 \Omega_R}{\partial \bar{\sigma}_I^2} < 0$. This implies that the physical solution now appears as a saddle point on the surface of Ω_R .

For $\mu_I / \pi T = 0$, the potential shapes characterized by the solutions of the real and imaginary parts of the gap equation are shown in Fig. 8. Figure 9 presents a similar classification of the potential shapes in the case $\mu_I / \pi T = 0.001$.

Looking over Fig. 8, one can recognize that, in the case of $\mu_R = 180$ MeV, Figs. 8(a) through 8(g) display the evolution of the potential shape across a crossover region as the temperature increases from 80 MeV to 100 MeV. In contrast, fixing $\mu_R = 200$ MeV, Figs. 8(h) through 8(n) demonstrate the shape evolution of a first-order phase transition from 80 MeV to 94 MeV.

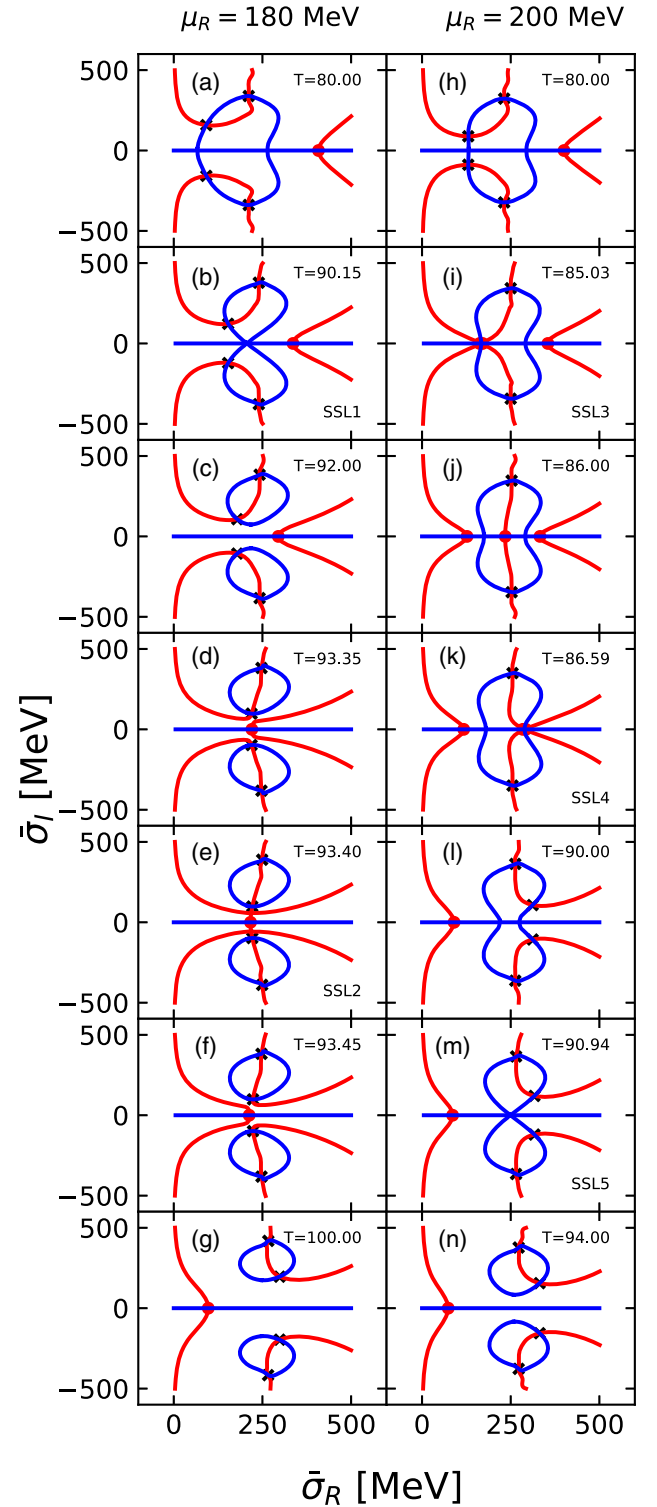


FIG. 8. Characteristic lines of Ω_R in the complex $\bar{\sigma}$ plane at $\mu_I / \pi T = 0$. Red (blue) lines denote solutions of the first (second) equation in Eq. (30). Different columns correspond to different values of μ_R , indicated at the top of each column, while the temperature (in MeV) is shown in the upper-right corner of each subplot. Labels for the shape-shifting lines in Fig. 6 are given in the lower-right corner.

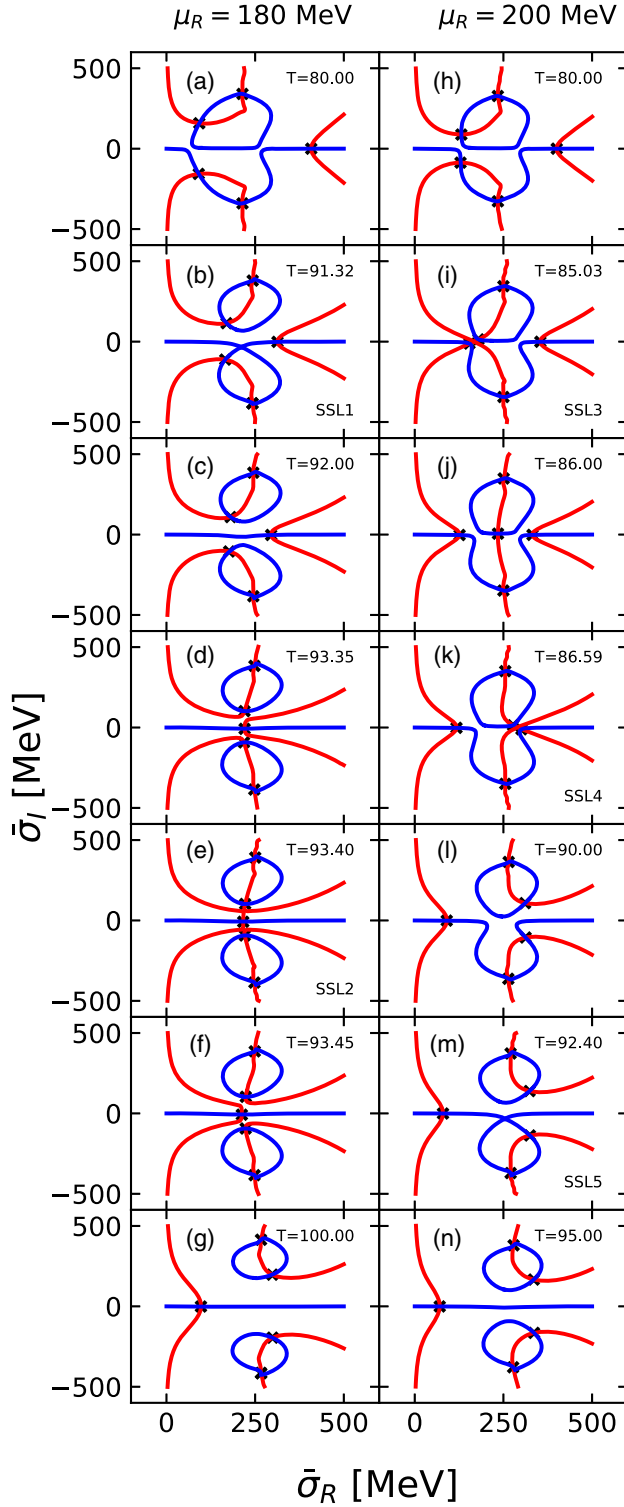


FIG. 9. Characteristic lines of Ω_R in the complex $\bar{\sigma}$ plane at $\mu_I/\pi T = 0.001$. Red (blue) lines denote solutions of the first (second) equation in Eq. (30). Different columns correspond to different values of μ_R , indicated at the top of each column, while the temperature (in MeV) is shown in the upper-right corner of each subplot. Labels for the shape-shifting lines are given in the lower-right corner.

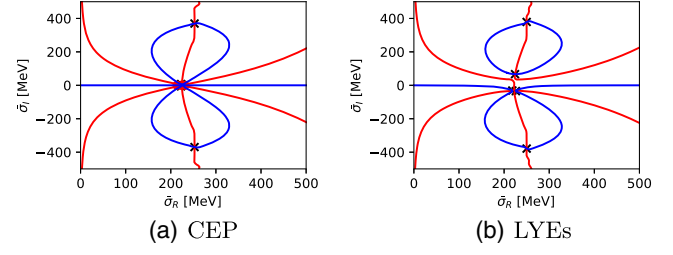


FIG. 10. Solutions of the gap equation in the complex $\bar{\sigma}$ plane at CEP and LYE. (a) CEP: $T = 90.10$ MeV, $\mu_R = 188.73$ MeV, $\mu_I/\pi T = 0$. (b) LYE: $T = 91.29$ MeV, $\mu_R = 185.57$ MeV, $\mu_I/\pi T = 0.001$.

Moreover, Figs. 8(b), 8(e), 8(i), 8(k), and 8(m) correspond to the scenarios where the potential shape qualitatively changes. Lines formed by connecting these scenarios in the phase diagram are referred to as *shape-shifting lines*. By saying *shape shifting*, we mean that the solution line of the real part or the imaginary part of gap equation self-intersects in the complex $\bar{\sigma}$ plane. For simplicity, we shall only focus on the lines at $\bar{\sigma}_R > 0$, since the explicit breaking from the physical current quark mass is relatively small. In particular, when μ_B is higher than that of CEP, the shape shifting lines also include the two spinodal lines. All of the shape shifting lines intersect at the critical end point (CEP) or the Lee–Yang edge singularities (LYEs), leading to a distinct potential structure at those points, as shown in Fig. 10.

This categorization allows one to determine the location of (T, μ) in the phase diagram based on the shape of the thermodynamic potential. For instance, the configurations shown in Figs. 8(j) and 8(l) indicate that (T, μ) lies near the first-order phase transition line with $\mu > \mu_{\text{CEP}}$, while Fig. 8(c) suggests that (T, μ) is in the crossover region with $\mu < \mu_{\text{CEP}}$.

When T or μ varies, the solutions of the gap equation trace out trajectories in the complex $\bar{\sigma}$ plane. As the shape of the thermodynamic potential Ω evolves distinctly across the first-order phase transition compared to the crossover, the trajectories of the solutions exhibit distinct patterns. Representative examples at $\mu_I/\pi T = 0.001$ are shown in Fig. 11.

In Fig. 11(a), the temperature varies from 80 to 100 MeV with a fixed $\mu_R = 180$ MeV, corresponding to the crossover region. In contrast, Fig. 11(b) depicts the case where $\mu_R = 200$ MeV, indicating a first-order phase transition. Keeping $\mu_I/\pi T = 0.001$ and $T \in [80, 100]$ MeV, the trajectories of the Nambu and Wigner solutions intersect at $\mu_R = 185.57$ MeV as shown in Fig. 11(c). When μ_R changes, the solution trajectories of the Nambu and Wigner solutions bifurcate.

Figures 10 and 11 imply that the positive Nambu solution $\bar{\sigma}_1$ coalesces with the Wigner solution $\bar{\sigma}_2$ at the

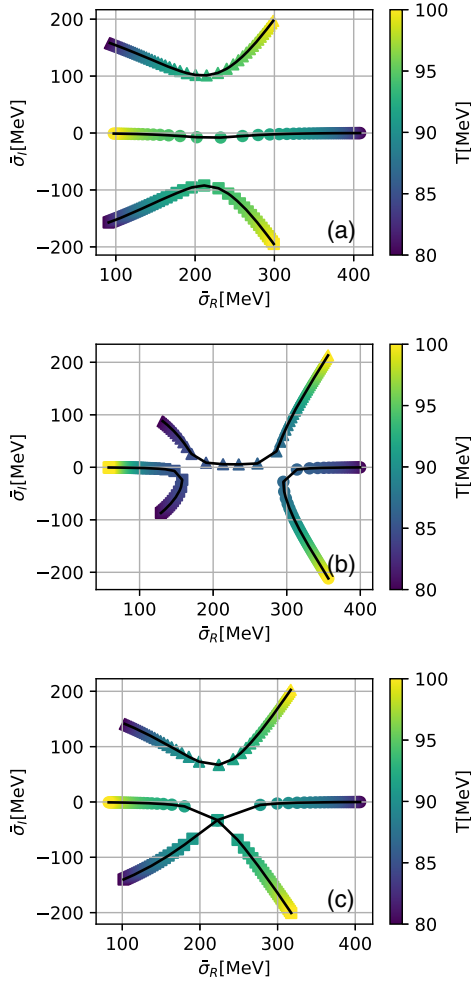


FIG. 11. Trajectories of the gap equation solutions (which can continuously evolve into physical states) at $\mu_l/\pi T = 0.001$. The temperature increases from 80 to 100 MeV, represented by a color gradient. (a) $\mu_R = 180$ MeV, corresponding to a crossover. (b) $\mu_R = 200$ MeV, where a first-order transition occurs. (c) $\mu_R = 185.57$ MeV, where the trajectory passes through the LYE. Each solution branch is depicted with a distinct marker shape.

CEP or more generally at the LYE. The coalescence of these solutions implies that

$$\left. \frac{\partial^2 \Omega}{\partial \bar{\sigma}^2} \right|_{(\bar{\sigma}_l, T_l, \mu_l)} = 0, \quad (31)$$

where $(T_l, \mu_l, \bar{\sigma}_l)$ are the temperature, chemical potential, and coalesced order parameter at the LYE (including CEP), respectively. This is guaranteed by the implicit function theorem. We present a general proof in the following.

We begin by introducing the notation:

$$F(\bar{\sigma}; T, \mu) := \frac{\partial \Omega(\bar{\sigma}; T, \mu)}{\partial \bar{\sigma}}. \quad (32)$$

From physical considerations, the thermodynamic potential Ω is expected to be continuously differentiable with respect to $(\bar{\sigma}, T, \mu)$. Upon complexification of the parameters, it is further assumed to be holomorphic. Consequently, $F(\bar{\sigma}; T, \mu)$ is also holomorphic in all its arguments. It is important to note that the thermodynamic potential evaluated at stationary points (including metastable ones), defined as $\Omega(T, \mu) := \Omega(\bar{\sigma}; T, \mu)|_{F(\bar{\sigma}; T, \mu)=0}$, may exhibit nonanalytic behavior at spinodal lines. However, this is not the function under consideration in the present analysis.

Since we are interested only in stationary points of the potential, we consider solutions of the equation

$$F(\bar{\sigma}; T, \mu) = 0, \quad (33)$$

as implied by Eq. (15).

To prove Eq. (31) at the Lee–Yang edge singularities (LYEs), we proceed by contradiction. Suppose, contrary to the claim, so that

$$\left. \frac{\partial F}{\partial \bar{\sigma}} \right|_{(\bar{\sigma}_l, T_l, \mu_l)} \neq 0. \quad (34)$$

We define an auxiliary function $H: \mathbb{C} \times \mathbb{C}^2 \rightarrow \mathbb{C} \times \mathbb{C}^2$ as

$$H(\bar{\sigma}; T, \mu) := (F(\bar{\sigma}; T, \mu), T, \mu). \quad (35)$$

The Jacobian matrix of H evaluated at the point $(\bar{\sigma}_l, T_l, \mu_l)$ is given by

$$J_H = \begin{pmatrix} \frac{\partial F}{\partial \bar{\sigma}} & \frac{\partial F}{\partial T} & \frac{\partial F}{\partial \mu} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (36)$$

$$\det(J_H)|_{(\bar{\sigma}_l, T_l, \mu_l)} = \left. \frac{\partial F}{\partial \bar{\sigma}} \right|_{(\bar{\sigma}_l, T_l, \mu_l)} \neq 0. \quad (37)$$

By the complex inverse function theorem, H is locally biholomorphic near $(\bar{\sigma}_l, T_l, \mu_l)$. Hence, in a neighborhood U , there exists a unique inverse function

$$H^{-1}(F(\bar{\sigma}; T, \mu), T, \mu) = (\bar{\sigma}, T, \mu). \quad (38)$$

Given that $F(\bar{\sigma}; T, \mu) = 0$, we obtain

$$H^{-1}(0, T, \mu) = (\bar{\sigma}, T, \mu), \quad (39)$$

which implies the existence of a unique holomorphic function $\bar{\sigma} = \bar{\sigma}(T, \mu)$ in U satisfying $F(\bar{\sigma}(T, \mu), T, \mu) = 0$.

This behavior is precisely observed in Fig. 11(a) and 11(b), where each stationary solution traces out a smooth, nonintersecting trajectory $\bar{\sigma}(T, \mu)$ as the temperature varies. The biholomorphic nature of the inverse mapping in

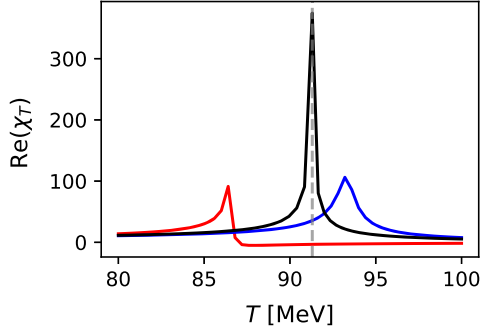


FIG. 12. Temperature dependence of the real part of the susceptibility $\text{Re}(\chi_T)$ for the positive Nambu solution, with T ranging from 80 to 100 MeV at $\mu_I/\pi T = 0.001$, shown for several values of μ_R . The red, blue, and black lines correspond to $\mu_R = 200.00$ MeV, $\mu_R = 180.00$ MeV, and $\mu_R = 185.57$ MeV, respectively. The black curve passes through the LYE, where a clear divergence in χ_T is observed.

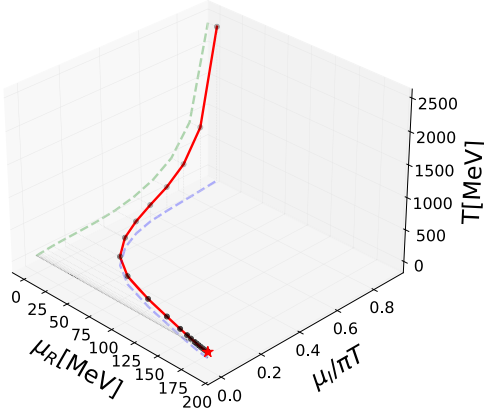


FIG. 13. Trajectory of Lee–Yang edge singularities (LYEs) in the complex chemical potential plane for $\mu_I/\pi T \in [0, 1]$.

Eq. (39) ensures local uniqueness, thereby prohibiting trajectory crossings.

However, this conclusion is contradicted by the behavior at the CEP (or more generally at the LYE), where two distinct solutions—the Nambu and Wigner branches—coexist at (T_I, μ_I) . As shown in Fig. 11(c), in any

neighborhood of $(T_I, \mu_I) = (91.29, 185.57)$ MeV, there exist two distinct solution branches $\bar{\sigma}_1(T, \mu)$ and $\bar{\sigma}_2(T, \mu)$ satisfying the gap equation. This multiplicity violates the local uniqueness guaranteed by the biholomorphic nature of the inverse mapping in Eq. (39). Therefore, the assumption Eq. (34) does not hold, and it follows that

$$\left. \frac{\partial F}{\partial \bar{\sigma}} \right|_{(\bar{\sigma}_I, T_I, \mu_I)} = 0. \quad (40)$$

As indicated by Eq. (19), the susceptibility χ diverges at the LYE (or CEP) due to the coalescence of distinct solutions. This behavior is consistent with the results shown in Fig. 4. Figure 12 illustrates an example of the temperature dependence of $\text{Re}(\chi_T)$ for the positive Nambu solution, with T varying from 80 to 100 MeV at fixed $\mu_R = 185.57$ MeV and $\mu_I/\pi T = 0.001$. A clear divergence is observed at $T = 91.29$ MeV, signaling the location of the LYE.

The resulting trajectories of Lee–Yang edge (LYE) singularities in the complex chemical potential plane are shown in Fig. 13.

Corresponding to Eq. (40), the correlation length of the system tends to diverge near the critical end point (CEP), and the associated critical behavior is governed solely by the symmetry and dimensionality of the system. Previous studies have shown that the trajectory of the Lee–Yang edge singularities (LYEs) near the CEP can be described by the scaling behavior of the Ising universality class [24,27,33]:

$$\mu_{\text{LY}}(T) - \mu_{\text{CEP}} = -c_1(T - T_{\text{CEP}}) + ic_2(T - T_{\text{CEP}})^{\beta\delta}, \quad (41)$$

where c_1 and c_2 are real-valued fitting coefficients.

Table I summarizes the fitting results for the distribution of LYE based on the scaling relation above. The coefficient of determination R^2 represents the square of the Pearson correlation coefficient and quantifies the quality of the fit.

By requiring a precision of three decimal places for $\beta\delta$ and $R^2 > 0.999999$, the critical region is found to be approximately $\Delta T \sim 1$ MeV and $\Delta\mu_R \sim 3$ MeV. In this

TABLE I. Fitting results for the critical scaling behavior of LYE in different intervals. Here, R^2 denotes the coefficient of determination, and $\beta\delta$ is the extracted critical exponent for each interval. The quantities $\Delta T = T - T_{\text{CEP}}$ and $\Delta\mu_R = \mu_R - \mu_{R,\text{CEP}}$ are given in MeV. Assuming $\beta\delta = 1.5$, the location of the CEP is determined by fitting the LYE trajectory within each interval.

	Interval 1	Interval 2	Interval 3	Interval 4
ΔT	[0.05, 1.18]	[1.18, 4.09]	[4.09, 9.94]	[9.94, 40.46]
$\Delta\mu_R$	[0.14, 3.17]	[3.17, 10.80]	[10.80, 25.88]	[25.88, 114.30]
$\Delta\frac{\mu_I}{\pi T}$	[0.00001, 0.001]	[0.001, 0.006]	[0.006, 0.02]	[0.02, 0.13]
$\beta\delta$	1.494(1)	1.470(2)	1.425(4)	1.50(5)
R^2	0.999998	0.999990	0.999973	0.996359
fitCEP	[90.10, 188.74]	[90.05, 188.82]	[89.73, 189.41]	[91.64, 188.50]

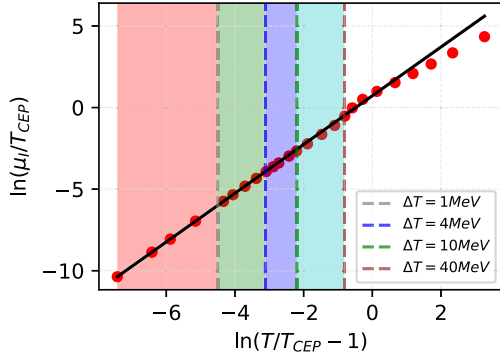


FIG. 14. Scaling behavior of the LYEs: μ_I versus $T - T_{\text{CEP}}$. The red, green, purple, and blue regions correspond to Intervals 1, 2, 3 and 4 in Table I, respectively. The black curve shows the scaling fit using the LYE data points within the Interval 1.

region, the extracted critical exponent in the nonlocal NJL model is $\beta\delta = 1.494(1)$, which is in close agreement with the mean-field values $\beta^{\text{MF}} = 0.5$ and $\delta^{\text{MF}} = 3.0$ [64,65]. Relaxing the constraint on the coefficient of determination to $R^2 > 0.99999$, the critical region broadens to approximately $\Delta T \sim 4$ MeV and $\Delta\mu_R \sim 10$ MeV. The corresponding fit is illustrated in Fig. 14.

It has been mentioned that lattice QCD faces significant challenges at nonzero quark chemical potential μ_R . Nevertheless, when μ_R/T is sufficiently small, the positions of Lee–Yang edge singularities (LYEs) can still be identified. Consequently, the critical end point (CEP) can be determined by extrapolating the trajectory of LYEs using Eq. (41):

$$T = \left(\frac{\text{Im}(\mu_{\text{LY}}(T))}{c_2} \right)^{\frac{2}{\beta\delta}} + T_{\text{CEP}}, \quad (42)$$

$$\text{Re}(\mu_{\text{LY}}(T)) = -c_1(T - T_{\text{CEP}}) + \mu_{\text{CEP}}. \quad (43)$$

By fitting Eq. (42), the CEP temperature T_{CEP} is extracted. Subsequently, inserting this value into Eq. (43) allows one to determine the corresponding chemical potential μ_{CEP} .

To assess the validity of this method, we perform fits in several intervals, as shown in Fig. 15, assuming a critical exponent of $\beta\delta = 1.5$. When the fitting region satisfies $\Delta T < 4$ MeV, $\Delta\mu_R < 10$ MeV, and $\Delta(\mu_I/\pi T) < 0.001$, the extrapolated CEP position closely matches the true value $T_{\text{CEP}}, \mu_{\text{CEP}} = (90.10, 188.73)$ MeV, with an error less than 0.1 MeV.

However, as μ_R/T becomes smaller, the extrapolation based on Eq. (41) becomes less reliable. For example, in the case of interval 3, the extracted critical exponent $\beta\delta$ is lower than expected, while in interval 4, the coefficient of determination is significantly reduced. Despite these deviations, the estimated location of the CEP remains reasonably accurate.

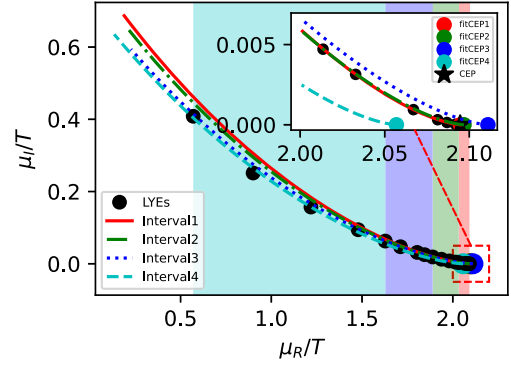


FIG. 15. Determining the position of CEP via extrapolation of LYEs through Eqs. (42) and (43), within the intervals given by Table I. The red, green, purple, and blue regions correspond to Intervals 1, 2, 3 and 4 in Table I, respectively.

In short, the extrapolation method based on analyzing LYEs proves to be efficient for locating the CEP, provided that the positions of the LYEs can be determined with sufficient high precision.

IV. SUMMARY

In this work, we have investigated the QCD phase diagram in the two-flavor nonlocal Nambu–Jona-Lasinio (NJL) model. The model introduces a momentum-dependent form factor that characterizes the dynamical properties of strong interaction for the combined effects from gluon propagator and quark-gluon vertex. Using model parameters fixed in the vacuum as given in Ref. [56], the chiral phase transition line is studied, with its curvature found to be $\kappa = 0.01811(2)$, which is in good agreement with previous estimation. In addition, the critical end point (CEP) is found at $T = 90.10$ MeV and $\mu = 188.73$ MeV.

This further allows for the investigation of QCD phase transition in the presence of complex chemical potential, where the chiral order parameter $\bar{\sigma}$ is also extended to be a complex one. We have proposed an improved method for identifying the Lee–Yang edge singularities (LYEs), based on the classification of the potential Ω according to its shape. The phase diagram is divided into several regions characterized by distinct potential shapes, and the boundaries between them are referred to as “shape-shifting lines.” These lines intersect at the CEP or more generally at LYEs.

At LYEs, the positive Nambu solution and the Wigner solution always coalesce, implying that $\partial^2\Omega/\partial\bar{\sigma}^2 = 0$ and a divergence in the susceptibility. This behavior is further demonstrated in the trajectories of the gap equation solutions, where the LYEs correspond to the intersection point of these distinct solution branches.

By employing this method, the trajectory of the LYEs has been determined. The extracted critical exponent $\beta\delta$ is $1.494(1)$ in the region $\Delta T \sim 1$ MeV and $\Delta\mu_R \sim 3$ MeV,

which is consistent with the mean-field theoretical prediction $\beta\delta = 1.5$.

We further highlight the practical significance of the LYE-based extrapolation method for locating the CEP. While lattice QCD is known to suffer from severe limitations at nonzero real chemical potential, our analysis demonstrates that the trajectory of the LYE, which can be accessed when μ_R/T is small, provides a viable path forward. By performing critical scaling fits in appropriate parameter intervals and assuming a mean-field critical exponent $\beta\delta = 1.5$, we show that the extrapolated position of the CEP can be determined with high precision—within 0.1 MeV of the true value—when the fitting region satisfies $\Delta T < 4$ MeV, $\Delta\mu_R < 10$ MeV, and $\Delta(\mu_I/\pi T) < 0.001$. Although the reliability of the extrapolation decreases as the fitting region moves further from the CEP (i.e., as μ_R/T becomes smaller), the deviations remain moderate. This suggests that, given sufficiently accurate determination of LYE, the extrapolation method offers a robust and complementary approach to identifying the CEP in the QCD phase diagram.

Finally, we comment on possible extensions of the present work. Here, we have restricted ourselves to the two-flavor nonlocal NJL model. The explicit u - d flavor symmetry breaking, induced by their small current-mass

difference, is known to have only a negligible impact on the chiral phase structure in the absence of external magnetic field. In contrast, the inclusion of the strange quark may qualitatively change the patterns in the chiral phase transition that we found at complex μ_B , as suggested by the Columbia plot already at vanishing μ_B . A consistent $(2+1)$ -flavor generalization of the nonlocal NJL model, which requires a dedicated modeling of the strange-quark sector and its form factor, is therefore highly desirable. A systematic investigation of combining strange quarks on the analytic structure of the thermodynamic potential and on the distribution of Lee–Yang edge singularities is left for future work.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [66].

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