

Electron-proton radiative recombination in the solar wind

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(Received 1 October 2025; accepted 17 November 2025; published 16 December 2025)

We investigate radiative recombination under conditions where both electrons and protons are in motion, thereby avoiding the stationary-ion approximation. The recombination cross section and rate coefficient are evaluated as functions of the kinematic parameters, with emphasis on regimes characteristic of the solar atmosphere, solar wind, and heliosphere. Particular attention is given to the impact of proton motion on the recombination rate. The total cross section is evaluated within the framework of nonrelativistic quantum electrodynamics, appropriate for particle energies typical of the solar atmosphere and solar wind. The results demonstrate that accounting for proton motion enhances the overall recombination rate.

DOI: [10.1103/gyhl-92fs](https://doi.org/10.1103/gyhl-92fs)

I. INTRODUCTION

Electron-proton radiative recombination is a fundamental process in quantum electrodynamics, with significant relevance across a wide range of physical phenomena—including astrophysical plasmas and controlled laboratory environments. In this process, a nonrelativistic free electron is captured by a proton, accompanied by the emission of a photon and the formation of a hydrogen atom in either its ground state or an excited bound state. The emitted photon carries away the excess energy of the system, thereby ensuring the conservation of both energy and momentum.

Most previous studies of radiative recombination have relied on two key simplifying assumptions: the stationary-ion approximation [1–8], in which the proton is treated as a fixed scattering center, and the assumption of thermal equilibrium [9–14], where both particles are assumed to follow Maxwell–Boltzmann distributions at the same temperature. A more comprehensive analysis, including approximate formulas valid in the high- and low-energy limits of incident electrons, is presented in Refs. [15,16].

By neglecting the motion of the proton, the fixed-proton approximation reduces the complexity of the theoretical treatment and facilitates analytical calculations. While the stationary-ion approximation is often justified by the large

proton-to-electron mass ratio, such simplifications may become inadequate in certain astrophysical environments. In certain context, proton, neutral-atom, and electron velocity distributions can decouple. For example, in the solar wind, the proton temperature is often correlated with wind speed, whereas the electron temperature is uncorrelated or even anticorrelated [17]. The velocity distributions or temperatures of different particle species are also known to decouple in other heliospheric and solar atmospheric context, including in the finite widths of shocks in the solar atmosphere [18] and solar wind [19], in the acceleration of nonthermal particles [20,21,22], in collisionless shocks or turbulence [23,24], and plasmas with strong temperature gradients or undergoing instabilities [25,26,27].

In such nonequilibrium environments, accurate modeling of radiative recombination necessitates accounting for the motion of both the electron and the proton. The resulting dynamics can substantially influence the recombination rate, the energy distribution of the resulting hydrogen atoms, and the spectral characteristics such as the line wing shapes and the line widths [28]. A detailed understanding of these effects is essential for the accurate interpretation of observational data and for reliable modeling of plasma behavior in the heliosphere and other astrophysical settings.

The solar wind is classified as fast or slow depending on the magnetic environments on the Sun from which it originates. The conditions in these regions determine how quickly and steadily plasma escapes into space. The fast solar wind, traveling at about 700–800 km/s, emerges principally from coronal holes—areas on the Sun’s surface with open magnetic field lines that allow plasma to stream

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freely outward. In contrast, the slow solar wind, moving at about 300–500 km/s, originates from the streamer belt, where closed magnetic loops trap plasma before it escapes. Although the slow solar wind originates from hotter coronal regions, it cools more as it expands into space, while the fast solar wind maintains higher speeds and relatively elevated temperatures due to its more direct outflow along open magnetic field lines. Near the Earth, at 1 AU, the temperatures of the particle species and other solar wind parameters vary significantly between different wind streams, as well as varying over the solar cycle. The fast solar wind typically shows hotter temperatures. Proton temperatures of $(0.5 \pm 0.6) \times 10^5$ K for the slow solar wind and $(1.0 \pm 0.8) \times 10^5$ K for the fast solar wind were reported based on data collected between 1998 and 2016 [29]. Other work provides values of $(0.4 \pm 0.2) \times 10^5$ K and $(2.4 \pm 0.6) \times 10^5$ K for proton temperatures in the slow and fast solar wind, along with $(1.3 \pm 0.5) \times 10^5$ K and $(1.0 \pm 0.2) \times 10^5$ K for the corresponding electron temperatures [30].

In this work, we investigate radiative recombination under conditions where both the electron and proton are in motion, thereby avoiding the commonly adopted stationary-ion approximation. Our objective is to calculate the recombination cross section and the corresponding recombination rate coefficient as functions of the kinematic parameters of the interacting particles, with particular attention to regimes characteristic of the solar atmosphere and solar wind. In addition, we aim to assess the significance of proton motion in determining the recombination rate coefficient. To this end, we calculate the total cross section for radiative recombination within the framework of nonrelativistic quantum electrodynamics, which is well suited for the particle energies typical of the solar atmosphere and solar wind. The initial state of the colliding particles is described by a Coulomb continuum wave function, while the final bound state is represented by a hydrogenic wave function. Since the Schrödinger equation with a Coulomb potential admits analytical solutions for both the discrete and continuous spectra, the radiative recombination cross sections can, in principle, be calculated exactly under arbitrary kinematical conditions.

The remainder of the paper is organized as follows. Section II introduces the basic formalism and governing equations, with a detailed treatment of the recombination cross sections in both the proton rest frame and the laboratory frame. Section III focuses on the analytical calculation of the transition matrix elements. In Sec. IV, the recombination cross section is calculated, and its dependence on physical parameters relevant to the solar atmosphere and solar wind is analyzed. Section V is devoted to the calculation of the recombination rate coefficient. Finally, Sec. VI summarizes the main findings and discusses their implications for the modeling of astrophysical plasmas.

II. GOVERNING EQUATIONS

We consider recombination of a nonrelativistic electron and proton into a hydrogen atom, accompanied by the emission of a photon:

$$e^- + p^+ \rightarrow H(n) + \gamma(\omega). \quad (1)$$

In Eq. (1), $H(n)$ denotes a hydrogen atom formed in either the ground or an excited state characterized by the principal quantum number n , while $\gamma(\omega)$ represents the emitted photon with frequency ω .

Let $\vec{v}_e$ and $\vec{v}_p$ denote the initial velocities of the electron and proton in the laboratory frame. The z axis of the laboratory frame is directed along the vector $\vec{v}_p$. To simplify the analysis, we introduce a reference frame in which the proton is at rest, specifically, a frame moving with velocity $\vec{V} = \vec{v}_p$ relative to the laboratory frame. In this proton rest frame, the electron approaches the proton with a relative velocity $\vec{v}'_e = \vec{v}_e - \vec{v}_p$. The radiative recombination cross section is derived in this frame, where the dynamics are more tractable, and the resulting expression is subsequently transformed back to the laboratory frame.

A. The proton rest frame

Let $\vec{k}'$ and ω' be the wave vector and frequency of the emitted photon in the proton rest frame. Within the first-order approximation of perturbation theory, the differential cross section for radiative recombination (1) in the proton rest frame is given by [2]

$$\left(\frac{d\sigma}{d\Omega_{\vec{k}'}} \right)_{\text{rest}} d\Omega_{\vec{k}'} = \frac{\omega'^3}{2\pi\hbar c^3 v'_e} \left| \langle \psi_f | e^{-i\vec{k}'\vec{r}} \left(\vec{\epsilon}'_j \hat{d}' \right) | \psi_i \rangle \right|^2 d\Omega_{\vec{k}'}. \quad (2)$$

In Eq. (2), ψ_i and ψ_f represent the wave functions of the electron in the initial (continuum) and final (discrete) states, respectively, $\hbar$ denotes the reduced Planck constant, and c is the speed of light. The unit vector $\vec{\epsilon}'_j$ represents the polarization of the emitted photon, $\hat{d}' = -e\vec{r}'$ is the dipole moment operator, $d\Omega_{\vec{k}'}$ denotes the differential solid angle in the direction of photon emission, and e is the charge of the electron.

In our analysis, we assume that the wavelength of the emitted photon is much larger than the characteristic size of the hydrogen atom formed after recombination. This condition justifies approximating the exponential term in equation (2) as unity. Consequently, under the long-wavelength approximation, Eq. (2) simplifies to

$$\left(\frac{d\sigma}{d\Omega_{\vec{k}'}} \right)_{\text{rest}} d\Omega_{\vec{k}'} = \frac{\omega'^3}{2\pi\hbar c^3 v'_e} \left| \vec{\epsilon}'_j \vec{d}'_{i,f} \right|^2 d\Omega_{\vec{k}'}, \quad (3)$$

where $\vec{d}'_{i,f} = \langle \psi_f | \vec{d}' | \psi_i \rangle$ denotes the dipole moment matrix element between the initial and final states.

A photon emitted during recombination may exhibit either linear or circular polarization. We consider photon emission with arbitrary polarization states, without imposing any specific restriction on the polarization. The polarization vector $\vec{\epsilon}'_j$ lies in the plane perpendicular to the direction of propagation $\vec{k}'$. Therefore, if the direction of polarization is resolved into two components, one, $\vec{\epsilon}'_1$, perpendicular to $\vec{d}'_{i,f}$ the other one, $\vec{\epsilon}'_2$, will lie in the plane determined by $\vec{k}'$ and $\vec{d}'_{i,f}$ at an angle of $\pi/2 - \theta'_k$ where θ'_k is the angle between $\vec{k}'$ and $\vec{d}'_{i,f}$ [2].

Under this configuration, the differential cross section is given by

$$\left(\frac{d\sigma}{d\Omega_{\vec{k}'}} \right)_{\text{rest}} d\Omega_{\vec{k}'} = \frac{\omega'^3}{2\pi\hbar c^3 v'_e} |\vec{d}'_{i,f}|^2 \sin^2 \theta'_k d\Omega_{\vec{k}'}, \quad (4)$$

where $\vec{d}'_{i,f} = \vec{i}(d'_x)_{i,f} + \vec{j}(d'_y)_{i,f} + \vec{k}(d'_z)_{i,f}$.

To obtain the total cross section, Eq. (4) must be integrated over all possible photon emission directions, i.e., over the entire solid angle $\Omega_{\vec{k}'}$. This yields the total cross section as [2,31]

$$\sigma_{\text{rest}}(v'_e) = \frac{4}{3} \frac{\omega'^3}{\hbar c^3 v'_e} |\vec{d}'_{i,f}|^2. \quad (5)$$

In the following, we consider the recombination of an electron with a bare nucleus of charge Z and derive all relevant equations for this system. In Eq. (5), the photon frequency ω' is determined by conservation of energy

$$\hbar\omega' = \frac{m_e v_e'^2}{2} + I_n, \quad (6)$$

where m_e is the electron mass, and $I_n = m_e e^4 Z^2 / 2\hbar^2 n^2$ is the binding energy of the electron in the hydrogenlike ion.

B. The laboratory frame

To relate the total cross section defined in the laboratory frame to that in the proton rest frame, we adopt the Galilean approximation to the Lorentz transformation. This approximation is valid in the limit $V \ll c$. The photon frequency ω in the laboratory frame is related to the frequency ω' in the proton rest frame by the nonrelativistic Doppler shift [32]

$$\omega = \omega' \left(1 + \frac{\vec{n}_k \cdot \vec{V}}{c} \right). \quad (7)$$

Here, $\vec{n}_k$ is the unit vector in the direction of photon propagation in the laboratory frame. This relation reflects the anisotropic frequency shift experienced due to the motion of the emitting system relative to the observer.

Averaging over all photon directions eliminates the term linear in $\vec{n}_k \cdot \vec{v}_p$ in Eq. (7). Therefore, the total cross section in the laboratory frame becomes

$$\sigma_{\text{lab}} = \sigma_{\text{rest}}(v'_e) + \mathcal{O}(v_e^2/c^2), \quad (8)$$

where $v'_e = \sqrt{v_e^2 - 2v_e v_p \cos \vartheta_e + v_p^2}$, and ϑ_e is the angle between the velocities $\vec{v}_e$ and $\vec{v}_p$ of the colliding electron and proton. The derived expressions show that the total cross section depends on three kinematical parameters: the electron velocity, the proton velocity, and the angle between their velocity vectors in the laboratory frame.

III. TRANSITION MATRIX ELEMENT

We calculate the transition matrix element appearing in Eq. (5) using parabolic coordinates: $\mu' = r'(1 + \cos \vartheta')$, $\nu' = r'(1 - \cos \vartheta')$, and $\varphi' = \arctg(y'/x')$. Here, r' denotes the radial variable, and ϑ' and φ' represent the polar and azimuthal angles, respectively. To facilitate the analytical calculations, we introduce new variables: $\xi^{(\pm)} = x' \pm iy'$ and $\xi^{(z)} = z'$, which in parabolic coordinates are expressed as $\xi^{(\pm)} = \sqrt{\mu'\nu'} e^{\pm i\varphi'}$ and $\xi^{(z)} = (\mu' - \nu')/2$. Accordingly, we obtain the following expression for the modulus of the transition matrix element:

$$|\vec{d}_{if}|^2 = e^2 \left[|(\xi^{(z)})_{i,f}|^2 + \frac{1}{2} (|(\xi^{(+)}_{i,f})|^2 + |(\xi^{(-)}_{i,f})|^2) \right]. \quad (9)$$

In parabolic coordinates, the normalized wave function of a bound electron in a hydrogenlike ion is given by [33]

$$\begin{aligned} \psi_f &= \frac{2^{\frac{1}{2}} (Z\alpha)^{\frac{3}{2}}}{n^2} f_{n_1,m} \left(\frac{Z\alpha\mu'}{n} \right) f_{n_2,m} \left(\frac{Z\alpha\nu'}{n} \right) \frac{e^{im\varphi'}}{\sqrt{2\pi}}, \\ f_{n_j,m}(\rho) &= \frac{1}{|m|!} \sqrt{\frac{(n_j + |m|)!}{n_j!}} F(-n_j, |m| + 1, \rho) e^{-\rho/2} \rho^{|m|/2}, \end{aligned} \quad (10)$$

where $\alpha = e^2 m_e / \hbar^2$ and $F(-n_j, |m| + 1, \rho)$ denotes a confluent hypergeometric function. The parabolic quantum numbers n_1 , n_2 , and m are related to the principal quantum number n by the condition $n = n_1 + n_2 + |m| + 1$.

For arbitrary direction of propagation of the incident electron, its wave function is expressed as [33]

$$\begin{aligned} \psi_i &= C_i e^{i\vec{k}'_e \cdot \vec{r}'} F(i\eta, 1, i(k'_e r - \vec{k}'_e \cdot \vec{r}')), \\ C_i &= e^{\pi\eta/2} \Gamma(1 - i\eta), \end{aligned} \quad (11)$$

where $\vec{k}'_e = m_e \vec{v}'_e / \hbar$ is the electron wave vector, $\eta = Ze^2 / \hbar v'_e$ is the Coulomb parameter, $F(i\eta, 1, i(k'_e r - \vec{k}'_e \cdot \vec{r}'))$ denotes a confluent hypergeometric function, C_i represents a normalizing factor, and $\Gamma(1 - i\eta)$ is a gamma function.

In parabolic coordinates, this wave function takes the form

$$\begin{aligned} \psi_i(\mu', \nu', \varphi') = & C_i e^{i\frac{k'_e}{2}(\mu' - \nu') \cos \vartheta'_e} e^{ik'_e \sqrt{\mu' \nu'} \cos \varphi' \sin \vartheta'_e} \\ & \cdot F\left(i\eta, 1, ik'_e \left(\frac{\mu' + \nu'}{2} - \frac{\mu' - \nu'}{2} \cos \vartheta'_e\right.\right. \\ & \left.\left. - \sqrt{\mu' \nu'} \cos \varphi' \sin \vartheta'_e\right)\right), \end{aligned} \quad (12)$$

where, ϑ'_e denotes the angle between $\vec{v}'_e$ and the axis $\vec{z}'$ of the proton rest frame. The azimuthal angles φ' and φ'_e are measured from plane $(\vec{z}'\vec{v}'_e)$; therefore $\varphi'_e = 0$. The trigonometric functions $\sin \vartheta'_e$ and $\cos \vartheta'_e$ are related to $\sin \vartheta_e$ and $\cos \vartheta_e$ by the following equations:

$$\sin \vartheta'_e = \frac{v_e \sin \vartheta_e}{(v_e^2 - 2v_e v_p \cos \vartheta_e + v_p^2)^{1/2}}, \quad (13)$$

$$\cos \vartheta'_e = \frac{v_e \cos \vartheta_e - v_p}{(v_e^2 - 2v_e v_p \cos \vartheta_e + v_p^2)^{1/2}}. \quad (14)$$

Substituting the derived expressions for the wave functions into the Eq. (9), we obtain for the transition matrix elements

$$(\xi^{(z)})_{i,f} = \frac{1}{8} \int_0^\infty \int_0^\infty \int_0^{2\pi} \psi_f^* \psi_i(\mu'^2 - \nu'^2) d\mu' d\nu' d\varphi', \quad (15)$$

and

$$\begin{aligned} (\xi^{(\pm)})_{i,f} = & \frac{1}{4} \int_0^\infty \int_0^\infty \int_0^{2\pi} \psi_f^* \psi_i \mu'^{1/2} \nu'^{1/2} (\mu' + \nu') \\ & \times e^{\pm i\varphi'} d\mu' d\nu' d\varphi'. \end{aligned} \quad (16)$$

To simplify the formulas, we omit the primes from the equations in what follows.

IV. RECOMBINATION CROSS SECTION

The flow of electrons and protons in the solar wind is approximately radial-directed away from the Sun—yet their propagation directions are not perfectly aligned. Observational studies [34] report that angular separations vary depending on solar wind conditions and the orientation of the magnetic field.

The equations derived in Secs. II and III enable, in principle, the calculation of radiative recombination cross sections for electron capture into arbitrary bound states of a hydrogenlike ion, applicable to any configuration of the colliding particles. In this work, we consider three representative cases: (i) the colliding particles move exactly parallel, (ii) they move nearly parallel, (iii) the colliding

particles move nonparallel and the angle between the electron and proton velocity vectors is not small. In all cases, the magnitudes of the particle velocities are treated as arbitrary.

A. Parallel motion of colliding particles

The calculations simplify significantly when the colliding particles propagate parallel to each other in the laboratory frame. From Eq. (13), it follows that when the electron propagates along the z axis—i.e., when $\vartheta_e = 0$ —we have $\sin \vartheta'_e = 0$ in the proton rest frame. As a result, the incident electron wave function becomes independent of the azimuthal angle φ and is separable, as indicated in Eq. (12). In this configuration, the first term in the matrix element (9) corresponds to electron capture into a state with $m = 0$, whereas the second and third terms contribute to the capture into states with $m = +1$ and $m = -1$, respectively.

The integration over the azimuthal angle φ yields the following expressions for the transition matrix elements:

$$\begin{aligned} (\xi^{(z)})_{i,n_1 n_2 m} = & \frac{\pi^{1/2} (Z\alpha)^{3/2} C_i}{4n^2} \left(J_{n_1, m}^{(2+|m|/2)} I_{n_2, m}^{(|m|/2)} \right. \\ & \left. - J_{n_1, m}^{(|m|/2)} I_{n_2, m}^{(2+|m|/2)} \right) \delta_{m,0}, \end{aligned} \quad (17)$$

and

$$\begin{aligned} (\xi^{(\pm)})_{i,n_1 n_2 m} = & \frac{\pi^{1/2} (Z\alpha)^{3/2} C_i}{2n^2} \left(J_{n_1, m}^{(3/2+|m|/2)} I_{n_2, m}^{(1/2+|m|/2)} \right. \\ & \left. + J_{n_1, m}^{(1/2+|m|/2)} I_{n_2, m}^{(3/2+|m|/2)} \right) \delta_{m,\pm 1}. \end{aligned} \quad (18)$$

The explicit expressions for $J_{n_1, m}^{(\tau+|m|/2)}$ and $I_{n_2, m}^{(\tau+|m|/2)}$ are presented in Appendix A.

Using the derived matrix elements, we obtain the electron capture cross section into the hydrogenlike ion state with $m = 0$ in the laboratory frame

$$\begin{aligned} \sigma_{n_1 n_2 0} = & \frac{2^8 \pi (Z\alpha)^3 e^2 (\omega/c)^3 |C_i|^2}{3 \hbar n^4 ((Z\alpha/n)^2 + k_e^2)^2 v_e} \left| \sum_{j=0}^{n_2} d_j \left(\frac{2Z\alpha}{n} \right)^j \right. \\ & \times \frac{1}{(Z\alpha/n + ik_e)^j} \left(\frac{F(-n_1, 3, 1, \beta) F(i\eta, 1 + j, 1, \beta_+)}{(Z\alpha/n - ik_e)^2} \right. \\ & \left. \left. - \frac{(j+1)(j+2)}{2} \frac{F(-n_1, 1, 1, \beta) F(i\eta, 3 + j, 1, \beta_+)}{(Z\alpha/n + ik_e)^2} \right) \right|^2. \end{aligned} \quad (19)$$

When $v_p = 0$ and $n_1 = n_2 = 0$, expression (19) reduces to the result previously obtained by Stobbe [1] for the radiative recombination cross section into the ground state of a hydrogenlike ion.

For the transition into the state with $m = \pm 1$, the recombination cross section is given by

$$\begin{aligned} \sigma_{n_1 n_2 \pm 1} &= \frac{2^{12} \pi (Z\alpha)^5 e^2 (\omega/c)^3 (n_1 + 1)! (n_2 + 1)! |C_i|^2}{3 \hbar n_1! n_2! n^6 ((Z\alpha/n)^2 + k_e^2)^4 v_e} \\ &\cdot \left| \sum_{j=0}^{n_2} d_j \left(\frac{2Z\alpha}{n} \right)^j \frac{(j+1)}{(Z\alpha/n + ik_e)^j} \right. \\ &\times \left(\frac{F(-n_1, 3, 2, \beta) F(i\eta, 2+j, 1, \beta_+)}{Z\alpha/n - ik_e} \right. \\ &\left. \left. + \frac{(j+2)}{2} \frac{F(-n_1, 2, 2, \beta) F(i\eta, 3+j, 1, \beta_+)}{Z\alpha/n + ik_e} \right) \right|^2. \quad (20) \end{aligned}$$

In Eqs. (19) and (20), $F(-n_1, b, c, \beta)$ and $F(i\eta, b, c, \beta_+)$ denote hypergeometric functions with variables $\beta = 2Z\alpha/(Z\alpha - ink_e)$ and $\beta_+ = 2ink_e/(Z\alpha + ink_e)$, respectively; $d_0 = 1$, $d_1 = -n_2/(1!(|m| + 1))$, $d_2 = n_2(n_2 - 1)/(2!(|m| + 1)(|m| + 2))$, ... are the coefficients of the confluent hypergeometric function $F(-n_2, |m| + 1, x)$ expressed in polynomial form. The hypergeometric functions $F(i\eta, b, c, \beta_+)$ are expressed in a polynomial form in Appendix A.

For our purpose, it is important to analyze the behavior of the cross sections (19) and (20) in the low-energy limit when the proton is held fixed. The calculation shows that, as $v_e \rightarrow 0$, the cross section diverges as $\sigma_{n_1 n_2 m} \sim 1/v_e^2$. The additional factor v_e in the denominator of Eqs. (19) and (20) originates from the normalization factor $|C_i|^2 = 2\pi\eta/(1 - \exp(-2\pi\eta)) \sim 1/v_e$.

Figure 1 presents the electron-proton recombination cross sections into the ground state of the hydrogen atom

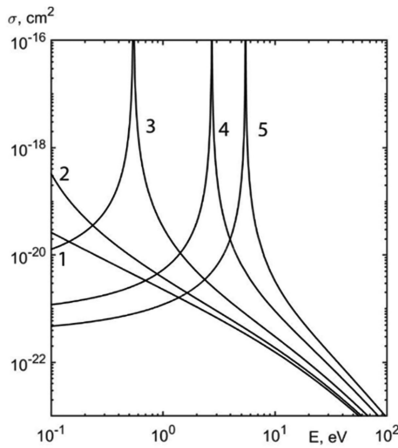


FIG. 1. The electron-proton recombination cross sections into the ground state of the hydrogen atom ($n_1 = n_2 = m = 0$). The proton energy is: 1 eV (curve 1), 100 eV (curve 2), 1 keV (curve 3), 5 keV (curve 4), and 10 keV (curve 5). The colliding particles propagate parallel to each other in the laboratory frame.

as functions of the incident electron energy, calculated using Eq. (19), for proton energies of 1 eV, 100 eV, 1 keV, 5 keV, and 10 keV. As shown in the figure, the cross sections for proton energies of 1 eV and 100 eV (curves 1 and 2) decrease monotonically with increasing electron energy over the range 0.1–100 eV. In contrast, the curves corresponding to higher proton energies (curves 3, 4, and 5) exhibit pronounced peaks at the electron energies where the electron velocity matches the proton velocity in the laboratory frame. This velocity resonance condition corresponds to the electron being nearly at rest in the proton rest frame, where the recombination cross section diverges, as predicted by Eq. (19). These divergences appear as sharp peaks in the cross-section curves. Similar resonance features are also present in curves 1 and 2, but they occur at lower electron energies (below 0.1 eV).

According to our calculations, the cross section is $\sigma_{000} = 2.1 \times 10^{-21} \text{ cm}^2$ for a fixed proton and an electron energy of 1 eV. This result is consistent with the radiative recombination cross section into the hydrogen ground state reported by Bethe and Salpeter [2].

Using Eqs. (19) and (20), we calculate the recombination cross sections into excited states of the hydrogen atom characterized by the total quantum number $n \leq 10$. The proton energies ranged from 1 eV to 10 keV. The cross sections for free-bound transitions into the states with $n_1 = 1$, $n_2 = m = 0$ and $n_1 = n_2 = 0$, $m = \pm 1$ are shown in Figs. 2 and 3, respectively. As in the previous case, the plotted curves exhibit sharp peaks when the electron velocity matches the proton velocity in the laboratory frame. This behavior is typical for transitions into other excited states as well. An important point for our purposes is that the recombination cross section decreases with increasing n .

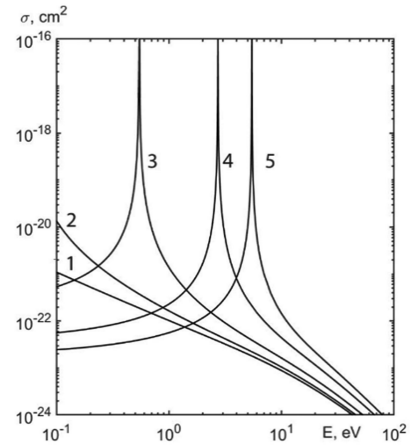


FIG. 2. As in Fig. 1 but for the recombination into the excited state of the hydrogen atom with $n_1 = 1$, $n_2 = 0$, and $m = 0$.

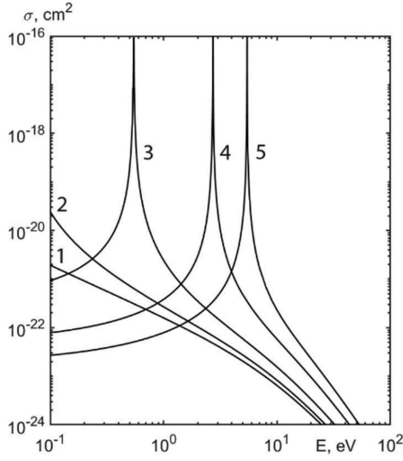


FIG. 3. As in Fig. 1 but for the recombination into the excited state of the hydrogen atom with $n_1 = n_2 = 0$ and $m = \pm 1$.

B. Nonparallel motion of colliding particles

We first consider the case where the colliding particles propagate nearly parallel to each other. For $\vartheta_e \ll 1$ and $v_e \neq v_p$, Eq. (13) yields $\sin \vartheta'_e \simeq v_e \vartheta_e / \sqrt{(v_e - v_p)^2 + v_e v_p \vartheta_e^2} \ll 1$. The right-hand side of Eq. (12) can be expanded in powers of the small parameter $ik_e \sqrt{\mu\nu} \cos \varphi \sin \vartheta'_e$, and the incident-electron wave function can then be expressed as

$$\psi_i = \psi_i^{(0)}(\mu, \nu) + \lambda(\vartheta_e) \sqrt{\mu\nu} \psi_i^{(1)}(\mu, \nu) \cos \varphi, \quad (21)$$

where $\psi_i^{(0)}$ is defined by the Eq. (12) with $\vartheta_e = 0$, $\lambda(\vartheta_e) = v_e \vartheta_e / \sqrt{(v_e - v_p)^2 + v_e v_p \vartheta_e^2}$ and

$$\psi_i^{(1)} = C_i e^{\frac{ik_e(\mu-\nu)}{2}} \left(ik_e F(i\eta, 1, ik_e \nu) - \frac{dF(i\eta, 1, ik_e \nu)}{d\nu} \right). \quad (22)$$

Using the wave functions given in Eqs. (10) and (22), we obtain the following expressions for the transition matrix elements:

$$\left(\xi^{(z)} \right)_{i,n_1 n_2 m} = \left(\xi_{\vartheta_e=0}^{(z)} \right)_{i,n_1 n_2 m} + \left(\xi_{\vartheta_e \neq 0}^{(z)} \right)_{i,n_1 n_2 m} \lambda(\vartheta_e), \quad (23)$$

and

$$\left(\xi^{(\pm)} \right)_{i,n_1 n_2 m} = \left(\xi_{\vartheta_e=0}^{(\pm)} \right)_{i,n_1 n_2 m} + \left(\xi_{\vartheta_e \neq 0}^{(\pm)} \right)_{i,n_1 n_2 m} \lambda(\vartheta_e), \quad (24)$$

where $(\xi_{\vartheta_e=0}^{(z)})_{i,n_1 n_2 m}$ and $(\xi_{\vartheta_e=0}^{(\pm)})_{i,n_1 n_2 m}$ are identical to those given in Eqs. (17) and (18), respectively, while

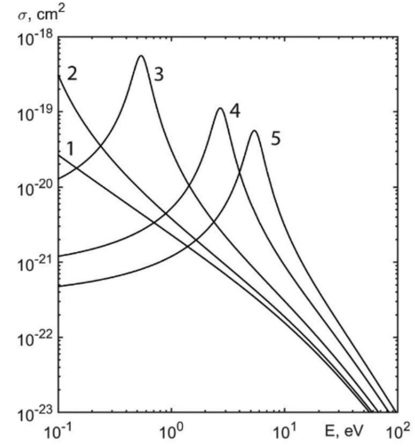


FIG. 4. As in Fig. 1 but for $\vartheta_e = 5^\circ$.

$$\begin{aligned} \left(\xi_{\vartheta_e \neq 0}^{(z)} \right)_{i,n_1 n_2 m} = & \frac{\pi^{1/2} (Z\alpha)^{3/2} C_i}{8n^2} \left[ik_e \left(J_{n_1, m}^{(5/2+|m|/2)} I_{n_2, m}^{(1/2+|m|/2)} \right. \right. \\ & - \left. J_{n_1, m}^{(1/2+|m|/2)} I_{n_2, m}^{(5/2+|m|/2)} \right) \\ & - \left(J_{n_1, m}^{(5/2+|m|/2)} \tilde{I}_{n_2, m}^{(1/2+|m|/2)} \right. \\ & \left. \left. - J_{n_1, m}^{(1/2+|m|/2)} \tilde{I}_{n_2, m}^{(5/2+|m|/2)} \right) \right] \delta_{m, \pm 1}, \quad (25) \end{aligned}$$

$$\begin{aligned} \left(\xi_{\vartheta_e \neq 0}^{(\pm)} \right)_{i,n_1 n_2 m} = & \frac{\pi^{1/2} (Z\alpha)^{3/2} C_i}{4n^2} \left[ik_e \left(J_{n_1, m}^{(2+|m|/2)} \times I_{n_2, m}^{(1+|m|/2)} \right. \right. \\ & + \left. J_{n_1, m}^{(1+|m|/2)} I_{n_2, m}^{(2+|m|/2)} \right) - \left(J_{n_1, m}^{(2+|m|/2)} \right. \\ & \times \tilde{I}_{n_2, m}^{(1+|m|/2)} + \left. J_{n_1, m}^{(1+|m|/2)} \tilde{I}_{n_2, m}^{(2+|m|/2)} \right) \\ & \left. \times (\delta_{m, 0} + \delta_{m, \pm 2}) \right]. \quad (26) \end{aligned}$$

The functions $\tilde{I}_{n_2, m}^{(\tau+|m|/2)}$ are defined by Eq. (A2), with $F(i\eta, 1, ik_e \nu)$ replaced by $dF(i\eta, 1, ik_e \nu)/d\nu$. All other notations remain consistent with those defined earlier.

Equations (23)–(26), together with the previously derived relations, determine the total cross sections in the regime where the angle between the electron and proton velocity vectors is small. For recombination into the ground state, the behavior of the cross section is illustrated in Fig. 4, where the angle between the particle velocities is fixed at five degrees.

As shown in Fig. 4, across the entire range of electron energies, the total cross section is larger when the recombining particles move nonparallel. This enhancement occurs because, unlike in the case of parallel propagation, both matrix elements (23) and (24) contribute to the cross section. Another distinction is that the cross sections no longer exhibit sharp maxima. For $\vartheta_e \neq 0$, the electron-

proton relative velocity remains finite even at $v_e = v_p$. As a result, the total cross section exhibits a broad maximum when the relative velocity reaches its minimum. These characteristics persist in recombination into both the ground and excited states of the hydrogen atom.

We proceed to analyze the configuration when the angle between the recombined particles is not small. To evaluate the matrix elements in Eq. (9) analytically, the incident-electron wave function is expressed as the following infinite series [35]:

$$\psi_i = C_i e^{i\frac{k_e}{2}(\mu-\nu)\cos\vartheta_e} \sum_{j=0}^{\infty} \sum_{p=0}^j \sum_{r=0}^p f_{j,p,r}^{(1)}(\mu, \nu) \cdot \sum_{l=0}^{\infty} \sum_{s=0}^l f_{l,s}^{(2)}(\mu, \nu) e^{i\zeta\varphi}, \quad (27)$$

where $\zeta = 2r - p + 2s - l$. Equation (27) is derived in Appendix B, where the explicit expressions for functions $f_{j,p,r}^{(1)}$ and $f_{l,s}^{(2)}$ are also provided.

By substituting the wave functions from Eqs. (10) and (27) into Eq. (9), the integration over the angular variable φ can be readily performed. This yields

$$(\xi^{(z)})_{i,f} = \frac{\pi^{1/2}(\mathcal{Z}\alpha)^{3/2}C_i}{4n^2} \sum_{j,p,r} \sum_{l,s} M_{n_1 n_2 m}^{(z)}(\vartheta_e) \delta_{m,\zeta}, \quad (28)$$

and

$$(\xi^{(\pm)})_{i,f} = \frac{\pi^{1/2}(\mathcal{Z}\alpha)^{3/2}C_i}{2n^2} \sum_{j,p,r} \sum_{l,s} M_{n_1 n_2 m}^{(\pm)}(\vartheta_e) \delta_{m,\zeta\pm 1}. \quad (29)$$

The functions $M_{n_1 n_2 m}^{(z)}$ and $M_{n_1 n_2 m}^{(\pm)}$ are defined in Appendix C. Equations (28) and (29), together with those presented in Appendix C, provide a complete algebraic expression for the recombination cross section for an arbitrary direction of propagation of the colliding particles. Due to its complexity, this expression is not presented here.

Equations (5), (8), and (9) define the recombination cross sections into the hydrogenlike ion state characterized by parabolic quantum numbers n_1 , n_2 , and m . However, we are interested in the recombination cross section summed over all parabolic quantum numbers n_1 and n_2 that satisfy the condition $n_1 + n_2 = n - |m| - 1$. For a given principal quantum number n and magnetic quantum number m , performing the summation yields

$$\sigma_{n,m} = \sum_{n_2=0}^{n-n_1-|m|-1} \sigma_{\text{rest}}(v_e). \quad (30)$$

Summing $\sigma_{n,m}$ over n and m then gives the total recombination cross section $\sigma(v_e, v_p, \vartheta_e)$ into all hydrogenlike ion states.

V. RECOMBINATION RATE COEFFICIENT

The obtained cross-section expressions enable evaluation of proton-motion effects on the solar wind recombination rate.

The electron-proton radiative recombination rate, R is expressed as the product of the electron and proton number densities, n_e and n_p , respectively, and the recombination rate coefficient α : $R = n_e n_p \alpha$. The recombination rate coefficient is defined as the thermal average of the product of the recombination cross section and the relative velocity of the electron-proton pair,

$$\alpha = \int d\vec{v}_e \int d\vec{v}_p f_e(\vec{v}_e) f_p(\vec{v}_p) v \sigma(v), \quad (31)$$

where $f_e(\vec{v}_e)$ and $f_p(\vec{v}_p)$ are the normalized velocity distribution (usually Maxwellian) of electrons and protons, σ is the recombination cross section, and $v = |\vec{v}_e - \vec{v}_p|$ is the relative speed of the particles.

Evaluating Eq. (31) is challenging because it involves a six-dimensional velocity integral, which is generally computed numerically and can be computationally expensive. Recently, however, by combining known results for Maxwellian (thermal), beam (drifting), and bi-Maxwellian (anisotropic) distributions, a compact one-dimensional integral representation of Eq. (31) has been derived [36]. In this work, we employ this simplified one-dimensional form to estimate the electron-proton recombination coefficient.

In solar-wind applications, electrons and protons typically share nearly the same bulk flow direction. Under this condition, both distributions can be approximated as drifting Maxwellians with bulk speeds $\vec{U}_e$ and $\vec{U}_p$ that are nearly parallel. The integral in Eq. (31) then reduces to an average over the relative speed distribution, characterized by the drift velocity $U_d = |\vec{U}_e - \vec{U}_p|$ (the difference of bulk speeds along the common direction), and the thermal width $\rho = \sqrt{\rho_e^2 + \rho_p^2}$ with $\rho_s = \sqrt{2k_B T_s / m_s}$ where k_B is the Boltzmann constant and T_s denotes the electron (or proton) temperature.

If the recombination cross section depends only on the magnitude of the relative speed, Eq. (31) simplifies to

$$\alpha(U_d, \rho) = \frac{1}{\sqrt{\pi}\rho U_d} \int_0^\infty \sigma(v) \left[e^{-\left(\frac{v-U_d}{\rho}\right)^2} - e^{-\left(\frac{v+U_d}{\rho}\right)^2} \right] v^2 dv. \quad (32)$$

Equation (32) remains valid for nearly parallel electron and proton flows and provides a practical tool for calculating the recombination rate coefficient directly from observed bulk speeds and thermal widths, expressed in terms of the electron and proton temperatures. In the limit $U_d \rightarrow 0$, Eq. (32) reduces to the standard expression for the recombination rate coefficient of an electron interacting

TABLE I. Recombination rate coefficients α in units of $10^{-14} \text{ cm}^3/\text{s}$. The electron and proton temperatures are taken as $T_e = 1.3 \times 10^5 \text{ K}$ and $T_p = 0.4 \times 10^5 \text{ K}$ for the slow solar wind, and $T_e = 1.0 \times 10^5 \text{ K}$ and $T_p = 2.4 \times 10^5 \text{ K}$ for the fast solar wind [30].

U_d (km/s)	α (slow wind)	α (fast wind)
1	3.194	3.873
10	3.194	3.873
50	3.193	3.871
100	3.190	3.867

with a stationary proton, i.e., the ordinary Maxwellian average without drift

$$\alpha(T_e) = 4\pi \left(\frac{m_e}{2\pi k_B T_e} \right)^{3/2} \int_0^\infty \sigma(v) e^{-\frac{m_e v^2}{2k_B T_e}} v^3 dv. \quad (33)$$

Table I presents the electron-proton recombination rate coefficients into the ground state of the hydrogen atom, calculated over a wide range of drift velocities for the slow and fast solar winds using Eq. (32), where the recombination cross section is defined by Eq. (19) with $Z = 1$.

For comparison, the corresponding rate coefficients for a stationary proton obtained from Eq. (33) for the electron temperatures $T_e = 1.3 \times 10^5 \text{ K}$ and $T_e = 1.0 \times 10^5 \text{ K}$ are $\alpha = 3.195 \times 10^{-14} \text{ cm}^3/\text{s}$ and $\alpha = 3.877 \times 10^{-14} \text{ cm}^3/\text{s}$, respectively. These results indicate that when electrons and protons move parallel to each other, the recombination rate coefficient exhibits only a weak dependence on the drift velocity and decreases slightly as U_d increases.

Using Eq. (32) where the recombination cross section is defined by Eqs. (5), (17), (18), (25), and (26), we calculate the recombination rate coefficients for the case when electrons and protons move almost parallel. The angle between the electron and proton velocity vectors has been taken as $\vartheta_e = 1^\circ, 5^\circ$, and 10° . For $\vartheta_e = 10^\circ$, the results of the calculations are presented in Table II.

The obtained results demonstrate that the recombination rate coefficient increases with the angle between the electron and proton velocity vectors. For $\vartheta_e = 10^\circ$, the deviation from the fixed-proton rate coefficient reaches about 2%. It is expected that a further increase in this angle will result in an even more pronounced enhancement of the recombination rate coefficient. To verify this expectation,

TABLE II. Same as in the Table I, but for $\vartheta_e = 10^\circ$.

U_d (km/s)	α (slow wind)	α (fast wind)
1	3.261	3.959
10	3.261	3.959
50	3.260	3.957
100	3.257	3.952

the rate coefficient should be calculated using the expression for the total cross section that remains valid when the angle between the velocity vectors of the recombining particles is not small [see Eqs. (28) and (29)].

VI. CONCLUSIONS

We calculate the electron-proton radiative recombination rate coefficient under conditions in which both particles are in motion. Such a situation arises in the solar wind where the proton velocity distribution can be broad and, in some cases, comparable to that of the electrons. Our calculations revealed that under conditions typical of the solar wind in the near-Earth region the effect of proton motion might be significant, leading to a noticeable increase in the overall recombination rate.

This result highlights the importance of considering the dynamics of both particle species when modeling recombination processes in space plasmas. By accounting for proton motion, we obtain a more accurate description of how electrons and protons interact, which can influence our understanding of how the solar wind evolves as it travels through the heliosphere.

Beyond the solar wind, these findings may also be relevant to other astrophysical environments where plasmas are hot and dynamic, such as stellar atmospheres, accretion flows, and interstellar clouds. In all such cases, the interplay between particle motions can alter key processes like energy loss, radiation emission, and the distribution of ion charge states.

This work provides a first step toward a more complete treatment. Future studies should evaluate thermal averages using Eq. (31) with Maxwellian or non-Maxwellian velocity distributions, include the angular dependence of the recombination cross section, and account for electron capture into excited hydrogen states for meaningful comparison with experimental data.

DATA AVAILABILITY

The numerical data underlying the results presented in this article, including the tables and plots generated in this work, are available from the corresponding author upon reasonable request.

APPENDIX A

The functions on the right-hand side of Eqs. (17) and (18), are defined as follows:

$$J_{n_1, m}^{(\tau + |m|/2)} = \frac{1}{|m|!} \left(\frac{Z\alpha}{n} \right)^{|m|/2} \sqrt{\frac{(n_1 + |m|)!}{n_1!}} \cdot \int_0^\infty e^{-\frac{Z\alpha/n - ik_e}{2}\mu} F\left(-n_1, |m| + 1, \frac{Z\alpha\mu}{n}\right) \mu^{\tau + \frac{|m|}{2}} d\mu, \quad (A1)$$

$$I_{n_2, m}^{(\tau+|m|/2)} = \frac{1}{|m|!} \left(\frac{Z\alpha}{n} \right)^{|m|/2} \sqrt{\frac{(n_2 + |m|)!}{n_2!}} \cdot \int_0^\infty e^{-\frac{Z\alpha/n + ik_e}{2}\nu} F\left(-n_2, |m| + 1, \frac{Z\alpha\nu}{n}\right) \cdot F(i\eta, 1, ik_e\nu) \nu^{\tau + \frac{|m|}{2}} d\nu. \quad (\text{A2})$$

These are standard tabulated integrals [37]:

$$\int_0^\infty e^{-\frac{Z\alpha/n - ik_e}{2}\mu} F(-n_1, |m| + 1, Z\alpha\mu/n) \mu^{\tau + |m|/2} d\mu = \frac{2^{\tau + |m|/2 + 1} (\tau + |m|/2)!}{(Z\alpha/n - ik_e)^{\tau + |m|/2 + 1}} F\left(-n_1, \tau + \frac{|m|}{2} + 1, |m| + 1, \beta\right) \quad (\text{A3})$$

and

$$\begin{aligned} & \int_0^\infty e^{-\frac{Z\alpha/n + ik_e}{2}\nu} F\left(-n_2, |m| + 1, \frac{Z\alpha\nu}{n}\right) \cdot F(i\eta, 1, ik_e\nu) \nu^{\tau + |m|/2} d\nu \\ &= \sum_{j=0}^{n_2} \frac{d_j}{j!} \left(\frac{Z\alpha}{n} \right)^j \int_0^\infty e^{-\frac{Z\alpha/n + ik_e}{2}\nu} F(i\eta, 1, ik_e\nu) \nu^{j + \tau + |m|/2} d\nu \\ &= \sum_{j=0}^{n_2} \frac{d_j}{j!} \left(\frac{Z\alpha}{n} \right)^j \frac{2^{j + \tau + |m|/2 + 1} (j + \tau + |m|/2)!}{(Z\alpha/n + ik_e)^{j + \tau + |m|/2 + 1}} \cdot F(i\eta, j + \tau + |m|/2 + 1, 1, \beta_+) \end{aligned} \quad (\text{A4})$$

In Eq. (A2) with $dF(i\eta, 1, ik_e\nu)/d\nu$, the method of integration by parts is employed.

Using the relation $F(i\eta, s, 1, \beta_+) = (1 - \beta_+)^{-i\eta} F(i\eta, 1 - s, 1, \beta_+ / (\beta_+ - 1))$ [33], the hypergeometric functions in Eqs. (A3) and (A4) can be expressed in the form

$$F(i\eta, s, 1, \beta_+) = e^{-2\eta \arctg(nk_e/Z)} F(i\eta, 1 - s, 1, \beta_-), \quad (\text{A5})$$

where $\beta_- = -2ink_e/(Z\alpha - ink_e)$. Since $s \geq 1$ is an integer, the hypergeometric function reduces to a polynomial.

APPENDIX B

Let us write the confluent hypergeometric function in (12) in the form $F(i\eta, 1, ik_e a(1 - 2b/a)/2)$, where $a = (\mu + \nu) - (\mu - \nu) \cos \vartheta_e$ and $b = \sqrt{\mu\nu} \cos \varphi \sin \vartheta_e$. The confluent hypergeometric function can be represented as an infinite series

$$\begin{aligned} & F(i\eta, 1, ik_e a(1 - 2b/a)/2) \\ &= \sum_{j=0}^{\infty} \frac{g_j}{(j!)^2} \left(\frac{ik_e a}{2} \right)^j (1 - 2b/a)^j, \end{aligned} \quad (\text{B1})$$

where $g_0 = 1$, $g_1 = i\eta$, $g_2 = i\eta(i\eta + 1)$, ...

The term $(1 - 2b/a)^j$ can be written as the following sum [38]:

$$\sum_{p=0}^j (-1)^p \binom{j}{p} \left(\frac{2\sqrt{\mu\nu} \cos \varphi \sin \vartheta_e}{\mu + \nu - (\mu - \nu) \cos \vartheta_e} \right)^p, \quad (\text{B2})$$

where $\binom{j}{p}$ are binomial coefficients. Analogously,

$$(\cos \varphi)^p = \frac{1}{2^p} \sum_{r=0}^p \binom{p}{r} e^{i(2r-p)\varphi}. \quad (\text{B3})$$

We thus obtain that

$$F(i\eta, 1, ik_e(a/2 - b)) = \sum_{j=0}^{\infty} \sum_{p=0}^j \sum_{r=0}^p f_{j,p,r}^{(1)} e^{i(2r-p)\varphi}, \quad (\text{B4})$$

where

$$\begin{aligned} f_{j,p,r}^{(1)} &= \frac{(-1)^p (ik_e)^j g_j}{2^j (j!)^2} \binom{j}{p} \binom{p}{r} (\sin \vartheta_e)^p \\ &\cdot (\mu + \nu - (\mu - \nu) \cos \vartheta_e)^{j-p} (\mu\nu)^{\frac{p}{2}}. \end{aligned} \quad (\text{B5})$$

In a similar manner, we obtain for the exponential term in (12)

$$e^{ik_e \sqrt{\mu\nu} \cos \varphi \sin \vartheta_e} = \sum_{l=0}^{\infty} \sum_{s=0}^l f_{l,s}^{(2)} e^{i(2s-l)\varphi}, \quad (\text{B6})$$

where

$$f_{l,s}^{(2)} = \frac{1}{l!} \left(\frac{ik_e}{2} \right)^l \binom{l}{s} (\mu\nu)^{\frac{l}{2}} (\sin \vartheta_e)^l. \quad (\text{B7})$$

APPENDIX C

The functions $M_{n_1 n_2 m}^{(\pm)}$ and $M_{n_1 n_2 m}^{(\pm)}$ are defined as follows:

$$\begin{aligned} M_{n_1 n_2 m}^{(z)} &= \int_0^\infty \int_0^\infty e^{\frac{ik_e}{2}(\mu - \nu) \cos \vartheta_e} f_{n_1, m} \left(\frac{Z\alpha\mu}{n} \right) f_{n_2, m} \left(\frac{Z\alpha\nu}{n} \right) \\ &\cdot f_{j,p,r}^{(1)}(\mu, \nu) f_{l,s}^{(2)}(\mu, \nu) (\mu^2 - \nu^2) d\mu d\nu, \end{aligned} \quad (\text{C1})$$

$$\begin{aligned} M_{n_1 n_2 m}^{(\pm)} &= \int_0^\infty \int_0^\infty e^{\frac{ik_e}{2}(\mu - \nu) \cos \vartheta_e} f_{n_1, m} \left(\frac{Z\alpha\mu}{n} \right) f_{n_2, m} \left(\frac{Z\alpha\nu}{n} \right) \\ &\cdot f_{j,p,r}^{(1)}(\mu, \nu) f_{l,s}^{(2)}(\mu, \nu) (\mu^{3/2} \nu^{1/2} + \mu^{1/2} \nu^{3/2}) d\mu d\nu. \end{aligned} \quad (\text{C2})$$

The integrals in Eqs. (C1) and (C2) can be expressed as a sum of products of integrals of the form:

$$J_{n_1 n_2 m}^{(\tau)} = \int_0^\infty e^{-\frac{Z\alpha \pm ink_e}{2n} x} F\left(-n_j, |m| + 1, \frac{Z\alpha}{n} x\right) x^\tau dx, \quad (\text{C3})$$

where τ is an integer. These integrals can be evaluated analytically [37].

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