

Macdonald index from a refined Kontsevich-Soibelman operator

George Andrews^{*}

Department of Mathematics, Penn State University, University Park, Pennsylvania 16802, USA

Anindya Banerjee[†]

Department of Physics, University of Cincinnati, Cincinnati, Ohio 45211, USA

Ranveer Kumar Singh[‡] and Runkai Tao[§]

*NHETC and Department of Physics and Astronomy, Rutgers University,
126 Frelinghuysen Road, Piscataway, New Jersey 08855, USA*



(Received 20 November 2025; accepted 10 April 2026; published 18 May 2026)

We propose a refinement of the Kontsevich-Soibelman operator for a class of four-dimensional $\mathcal{N} = 2$ superconformal field theories characterized by the following conditions: (1) Their Coulomb branch admits a source/sink chamber, i.e., a chamber in which the Bogomol'nyi-Prasad-Sommerfield (BPS) quiver consists of only source and sink nodes. (2) The nodes with valency greater than 2 of the BPS quiver in a source/sink chamber are either all sources or all sinks. We present strong evidence that the trace of this refined operator is related to the Macdonald index of the theory. In particular, we conjecture new closed form expressions for the Macdonald indices of the $(A_1, \mathfrak{g})$ Argyres-Douglas theories for any simply laced Lie algebra $\mathfrak{g}$.

DOI: [10.1103/gv2h-8hs8](https://doi.org/10.1103/gv2h-8hs8)

I. INTRODUCTION

All known examples of interacting four-dimensional (4D) $\mathcal{N} = 2$ quantum field theories have a continuous moduli space of vacua known as the Coulomb branch. This is a singular manifold, and powerful techniques for analyzing the [protected or Bogomol'nyi-Prasad-Sommerfield (BPS)] spectrum and dynamics of these theories may be derived from the operation of moving on this moduli space [1,2]. In particular, many observables are related to representations of the monodromy group associated with the singularities.

In this work, we will make use of such an object, the q -deformed version [3] of the Kontsevich-Soibelman monodromy operator $M(q)$ [6,7], to understand the spectrum of BPS states in these theories [5]. In particular, we will focus on superconformal field theories (SCFTs), where the full $SU(2) \times U(1)$ symmetry is nonanomalous and by virtue of the state-operator correspondence, one may study

BPS local operators. Furthermore, in such SCFTs, the monodromy operator satisfies $M^r = I$ for some finite positive integer r [8,9]. We will make use of a counting function that tracks the multiplicities of various superconformal representations that appear in the spectrum of local operators in the theory (note that the superconformal index has ambiguities associated with degeneracies and mutually canceling contributions). The superconformal index becomes more unambiguous when enumerating the operator content for more protected local operators (like half-BPS operators). We will focus here on quarter-BPS operators where there are some ambiguities, so that these lie at the present threshold of our ability to reconstruct from the index.

The SCFT lives at the origin of moduli space, and moving out on the moduli space is accomplished by turning on vacuum expectation values for certain universal scalar operators that exist in these theories. The underlying argument in our work is that the low energy effective theory obtained in the infrared by moving away from the origin of the moduli space contains enough information to partially reconstruct the BPS spectrum of the SCFT living at the origin. Such a use for the monodromy operator was first advocated in [10] where the Schur indices of several classes of 4D $\mathcal{N} = 2$ SCFTs was computed. In this paper, we propose a prescription to generalize the monodromy operator for a “special” class of 4D $\mathcal{N} = 2$ SCFTs. We will use this generalized operator to study the same Schur

^{*}Contact author: geal@psu.edu

[†]Contact author: ab1702@scarletmail.rutgers.edu

[‡]Contact author: rks158@scarletmail.rutgers.edu

[§]Contact author: runkai.tao@physics.rutgers.edu

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

subsector of the local operator spectrum of these theories [11], but compute a refined counting function associated with this sector known as the Macdonald index [12]. This may also be understood as a special limit of the full superconformal index. See Refs. [13–17] for other existing ways of computing the Macdonald index of 4D $\mathcal{N} = 2$ theories. We expect a similar generalization to work for a much larger class of SCFTs. In the present work, we use our prescription to conjecture a new closed form expression for the Macdonald indices of $(A_1, \mathfrak{g})$ Argyres-Douglas theories.

II. KONTSEVICH-SOIBELMAN OPERATOR

Any 4D $\mathcal{N} = 2$ theory at a generic point on the Coulomb branch is IR free and is characterized by a $U(1)^r$ gauge theory. The massive spectrum of the theory consists of particles carrying electric and magnetic charges valued in the charge lattice Γ equipped with a Dirac pairing $\langle \cdot, \cdot \rangle$. The spectrum of BPS states is particularly interesting because they behave nicely under certain supersymmetry-preserving RG flows. The spectrum of BPS states with a given charge $\gamma \in \Gamma$ is captured by the protected spin character (PSC) defined as follows [18]: let $\mathcal{H}'_{\text{int}}$ be the internal Hilbert space of the BPS particle with charge γ . Then the PSC of γ is defined by

$$\Omega(\gamma, y) := \text{Tr}_{\mathcal{H}'_{\text{int}}} (-1)^R y^{J+R} = \sum_{n \in \mathbb{Z}} \Omega_n(\gamma) y^n, \quad (1)$$

where J is the generator of the Cartan subalgebra of the little group $SU(2)$ of the Lorentz group $SO(1,3)$ and R is the Cartan generator of the $SU(2)$ R-symmetry group. As one moves in the moduli space, the BPS spectrum is piecewise constant but may jump discontinuously across certain codimension one subloci, called walls of marginal stability. To understand this jump, one needs to construct wall-crossing invariants. Define the quantum torus algebra to be the formal span of generators

$$\mathcal{A} := \text{Span}_{\mathbb{C}} \{X_\gamma : \gamma \in \Gamma\}, \quad (2)$$

with the multiplication on $\mathcal{A}$ defined as

$$X_\gamma X_{\gamma'} = q^{\langle \gamma, \gamma' \rangle / 2} X_{\gamma+\gamma'} = q^{\langle \gamma, \gamma' \rangle} X_{\gamma'} X_\gamma, \quad (3)$$

where q is a formal variable. For each $\gamma \in \Gamma$, we construct the following element of $\mathcal{A}$:

$$U_\gamma = \prod_{n \in \mathbb{Z}} E_q((-1)^n q^{n/2} X_\gamma)^{(-1)^n \Omega_n(\gamma)}, \quad (4)$$

where $E_q(X)$ is the quantum dilogarithm

$$E_q(z) := \prod_{i=0}^{\infty} (1 + q^{i+\frac{1}{2}} z)^{-1} = \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}} z)^n}{(q)_n}. \quad (5)$$

The Kontsevich-Soibelman (KS) operator is defined as

$$\mathcal{O}(q) := \prod_{\gamma \in \Gamma}^{\curvearrowright} U_\gamma, \quad (6)$$

where the product is taken in the order of increasing $\text{Arg}(Z(\gamma))$ with respect to an arbitrarily chosen phase θ , where $Z(\gamma)$ is the central charge corresponding to the charge γ . This operator is related to the KS operator by $\mathcal{O}(q) = M(q)^{-1}$. The celebrated result of Kontsevich and Soibelman [4] is that the operator $\mathcal{O}(q)$ is a wall-crossing invariant. Thus, if one can relate physically relevant quantities to this operator, then one can compute it at any point in the moduli space and glean information about the UV theory at the origin of the moduli space. This was the motivation behind the conjecture of Cordova-Shao [10] relating the Schur index of a 4D $\mathcal{N} = 2$ SCFT to the trace of the KS operator.

A. Trace of KS operator and Schur index

Let us begin by quoting the conjecture from [10]: it states that the Schur index of a 4D $\mathcal{N} = 2$ SCFT is given by an appropriately defined trace,

$$\mathcal{I}_S(q, z_1, \dots, z_{N_f}) := (q)_\infty^{2r} \text{Tr} \mathcal{O}(q), \quad (7)$$

where r is the rank of the theory, $z_1, \dots, z_{N_f}$ are fugacities for flavor symmetries, and

$$(a)_\infty := (a; q)_\infty := \prod_{i=0}^{\infty} (1 - aq^i). \quad (8)$$

The factor $(q)_\infty^{2r}$ is simply the Schur index of r free $U(1)$ vector multiplets which is the contribution of the free $U(1)^r$ vector multiplets at a generic point on the Coulomb branch. Recall that a charge $\gamma \in \Gamma$ is called a flavor charge if $\langle \gamma, \Gamma \rangle = 0$. Let $\{\gamma_{f_i}\}_{i=1}^{N_f}$ be an integral basis of the sublattice of flavor charges. For any charge γ , let

$$\gamma_f = \sum_{i=1}^{N_f} f_i(\gamma) \gamma_{f_i} \quad (9)$$

be the projection on to the subspace of flavor charges. We define

$$\text{Tr}[X_\gamma] = \begin{cases} \prod_i \text{Tr}[X_{\gamma_{f_i}}]^{f_i(\gamma)} & \langle \gamma, \Gamma \rangle = 0, \\ 0 & \text{else.} \end{cases} \quad (10)$$

Imposing the cyclicity of the trace, the trace of any element of $\mathcal{A}$ can then be written in terms of q and N_f variables

$\text{Tr}[X_{\gamma_{f_i}}]$. These variables are functions of the flavor fugacities $z_1, \dots, z_n$ and the explicit functional relation can be determined as explained in [10]. In [10], the trace of the KS operator was computed for a large class of examples and the conjecture (7) was verified.

III. THE REFINED KS OPERATOR

In this section, we define a refined KS operator for a special class of 4D $\mathcal{N} = 2$ SCFTs.

A. BPS quiver and source/sink chamber

The BPS quiver of a 4D $\mathcal{N} = 2$ theory [19] captures the spectrum of BPS states of the theory at a given point the Coulomb branch. We will not go into relevance of quivers in 4D $\mathcal{N} = 2$ theories in this work and instead refer readers to [19–22] for more details. The rules to draw the BPS quiver are as follows [8,23]:

- (1) Draw a node for each basis element of the charge lattice.
- (2) If for two nodes γ_i, γ_j , $\langle \gamma_i, \gamma_j \rangle > 0$, draw $\langle \gamma_i, \gamma_j \rangle$ arrows from node i to node j and if $\langle \gamma_i, \gamma_j \rangle < 0$ then draw $|\langle \gamma_i, \gamma_j \rangle|$ arrows from j to i .

Note that the quiver depends on a choice of basis and choosing a different basis gives a different quiver. Some 4D $\mathcal{N} = 2$ theories admit a source/sink chamber, meaning that the charge lattice at a given point in this chamber admits a basis in which the BPS quiver has only source and sink nodes [24]. The (conjectured) crucial property [8,23] of the source/sink chamber is that the central charges corresponding to the nodes satisfy

$$\text{Arg}(Z(\gamma)) > \text{Arg}(Z(\gamma')), \text{ for all } \gamma \in \text{sink}, \gamma' \in \text{source}. \quad (11)$$

Moreover, it is clear that

$$[X_\gamma, X_{\gamma'}] = 0, \quad \gamma, \gamma' \in \text{sources or } \gamma, \gamma' \in \text{sinks}. \quad (12)$$

Thus, one can compute the KS operator in the source/sink chamber,

$$\mathcal{O}(q) = \prod_{\gamma \in \text{source}} U_{-\gamma} \prod_{\gamma \in \text{sink}} U_{-\gamma} \prod_{\gamma \in \text{source}} U_{\gamma} \prod_{\gamma \in \text{sink}} U_{\gamma}. \quad (13)$$

B. Refined KS operator

We consider 4D $\mathcal{N} = 2$ SCFTs that satisfy the following two properties:

- (1) The theory admits a source/sink chamber.
- (2) The nodes in the BPS quiver at a point in the source/sink chamber with valency greater than 2 are either all sources or all sinks.

The $(A_1, \mathfrak{g})$ Argyres-Douglas theories introduced in [8] are examples of this special class of theories. Their BPS quivers are simply laced tree graphs that have a finite

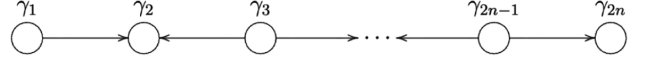


FIG. 1. BPS quiver of the (A_1, A_{2n}) theory in the source/sink chamber. Figure adapted from [10].

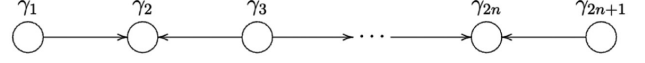


FIG. 2. BPS quiver of the (A_1, A_{2n+1}) theory in the source/sink chamber. Figure adapted from [10].

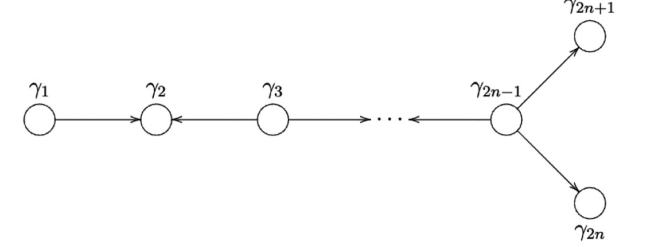


FIG. 3. BPS quiver of the (A_1, D_{2n+1}) theory in the source/sink chamber. Figure adapted from [10].

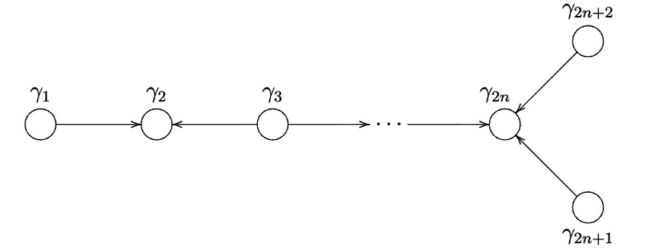


FIG. 4. BPS quiver of the (A_1, D_{2n+2}) theory in the source/sink chamber. Figure adapted from [10].

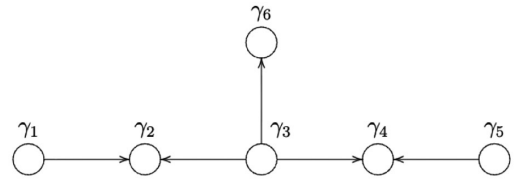
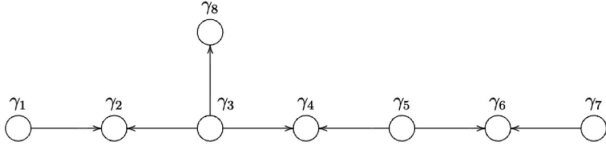


FIG. 5. BPS quiver of the (A_1, E_6) theory in the source/sink chamber. Figure adapted from [10].

number of mutation-equivalent graphs in their equivalence class under quiver mutations. See footnote 10 in [23] for more details. Moreover, their moduli space admits a source/sink chamber. Each node in the BPS quiver in the source/sink chamber represents a single hypermultiplet with trivial internal degrees of freedom. Moreover, the quivers for these theories shown in Figs. 1–7 clearly satisfy the second condition. Define the functions

FIG. 6. BPS quiver of the (A_1, E_8) theory in the source/sink chamber. Figure adapted from [10].

$$E_{q,T}(X_\gamma) = \sum_{n=0}^{\infty} \frac{(-qT)^{\frac{1}{2}} X_\gamma^n}{(q)_n},$$

$$\tilde{E}_{q,T}(X_\gamma) = \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}} X_\gamma)^n}{(qT)_n}. \quad (14)$$

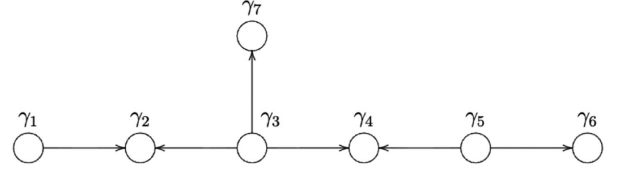
Define the refined KS operator in the source/sink chamber by

$$\begin{aligned} \mathcal{O}(q, T) := & \prod_{\gamma \in \text{source}} \prod_{n \in \mathbb{Z}} E_{q,T}((-1)^n q^{n/2} X_{-\gamma})^{(-1)^n \Omega_n(-\gamma)} \\ & \times \prod_{\gamma \in \text{sink}} \prod_{n \in \mathbb{Z}} \tilde{E}_{q,T}((-1)^n q^{n/2} X_{-\gamma})^{(-1)^n \Omega_n(-\gamma)} \\ & \times \prod_{\gamma \in \text{source}} \prod_{n \in \mathbb{Z}} E_{q,T}((-1)^n q^{n/2} X_\gamma)^{(-1)^n \Omega_n(\gamma)} \\ & \times \prod_{\gamma \in \text{sink}} \prod_{n \in \mathbb{Z}} E_q((-1)^n q^{n/2} X_\gamma)^{(-1)^n \Omega_n(\gamma)}. \end{aligned} \quad (15)$$

If the BPS quiver has a source node with valency greater than 2, then we define the refined KS operator as

$$\begin{aligned} \mathcal{O}(q, T) := & \prod_{\gamma \in \text{source}} \prod_{n \in \mathbb{Z}} \tilde{E}_{q,T}((-1)^n q^{n/2} X_{-\gamma})^{(-1)^n \Omega_n(-\gamma)} \\ & \times \prod_{\gamma \in \text{sink}} \prod_{n \in \mathbb{Z}} E_{q,T}((-1)^n q^{n/2} X_{-\gamma})^{(-1)^n \Omega_n(-\gamma)} \\ & \times \prod_{\gamma \in \text{source}} \prod_{n \in \mathbb{Z}} E_q((-1)^n q^{n/2} X_\gamma)^{(-1)^n \Omega_n(\gamma)} \\ & \times \prod_{\gamma \in \text{sink}} \prod_{n \in \mathbb{Z}} E_{q,T}((-1)^n q^{n/2} X_\gamma)^{(-1)^n \Omega_n(\gamma)}. \end{aligned} \quad (16)$$

Note that, at the moment, we can define the refined KS operator only in the source/sink chamber.

FIG. 7. BPS quiver of the (A_1, E_7) theory in the source/sink chamber. Figure adapted from [10].

IV. MACDONALD INDEX FROM MODIFIED KS OPERATOR: A CONJECTURE

Recall that the Macdonald index of the free vector multiplet is given by

$$\mathcal{I}_M^{\text{vec}}(q, T) = (q)_\infty (qT)_\infty. \quad (17)$$

We conjecture that the Macdonald index of the class of 4D $\mathcal{N} = 2$ SCFTs for which the modified KS operator is defined is given by

$$\mathcal{I}_M(q, T, z_1, \dots, z_{N_f}) = (q)_\infty^r (qT)_\infty^r \text{Tr} \mathcal{O}(q, T), \quad (18)$$

where r is the rank of the theory. Assuming the conjecture in [10], this formula trivially satisfies the correct Schur limit $T \rightarrow 1$.

A. Evidence for the conjecture

We now use our conjecture to calculate the Macdonald index of $(A_1, \mathfrak{g})$ Argyres-Douglas theories. In this section, we present the detailed computation for the (A_1, A_k) theory and summarize our conjectures for other $(A_1, \mathfrak{g})$ theories.

1. (A_1, A_k) theory

Let us start with the $k = 2n$ case. The BPS quiver of the (A_1, A_{2n}) theory in the source/sink chamber is shown in Fig. 1.

Each node in the quiver represents a single hypermultiplet. The refined KS operator is thus given by

$$\begin{aligned} \mathcal{O}(q, T) = & \prod_{i=1}^n E_{q,T}(X_{-\gamma_{2i-1}}) \prod_{i=1}^n \tilde{E}_{q,T}(X_{-\gamma_{2i}}) \\ & \times \prod_{i=1}^n E_{q,T}(X_{\gamma_{2i-1}}) \prod_{i=1}^n E_q(X_{\gamma_{2i}}). \end{aligned} \quad (19)$$

Using the formula for the functions $E_{q,T}$ and $\tilde{E}_{q,T}$, we get

$$\mathcal{O}(q, T) = \sum_{\substack{\ell_1, \dots, \ell_{2n} \geq 0 \\ k_1, \dots, k_{2n} \geq 0}} \frac{(-1)^{\sum_{i=1}^{2n} (k_i + \ell_i)} q^{\frac{1}{2} \sum_{i=1}^{2n} (k_i + \ell_i)} T^{\frac{1}{2} \sum_{i=1}^n (\ell_{2i-1} + k_{2i-1})}}{\prod_{i=1}^n (q)_{k_{2i-1}} (q)_{\ell_{2i-1}} \prod_{i=1}^n (q)_{k_{2i}} (qT)_{\ell_{2i}}} \prod_{i=1}^n X_{-\gamma_{2i-1}}^{\ell_{2i-1}} \prod_{i=1}^n X_{-\gamma_{2i}}^{\ell_{2i}} \prod_{i=1}^n X_{\gamma_{2i-1}}^{k_{2i-1}} \prod_{i=1}^n X_{\gamma_{2i}}^{k_{2i}}. \quad (20)$$

To compute the trace of the operator, we use the quantum torus algebra to commute the $X_{-\gamma_{2i-1}}^{\ell_{2i-1}}$ past $X_{-\gamma_{2i}}^{\ell_{2i}}$ and then use the trace formula (10) noting that the theory has no flavor charges. This gives

$$\mathcal{I}_M^{A_{2n}}(q, T) = (q)_\infty^n (Tq)_\infty^n \times \sum_{k_1, \dots, k_{2n} \geq 0} \frac{T^{\sum_{i=1}^n k_{2i-1}} q^{\sum_{i=1}^n k_{2i} + \sum_{i=1}^{2n-1} k_i k_{i+1}}}{\prod_{i=1}^n (q)_{k_{2i-1}}^2 \prod_{i=1}^n (q)_{k_{2i}} (Tq)_{k_{2i}}}. \quad (21)$$

For the first few values of n , one can compute the series and find a perfect match with the known form of the Macdonald index [14–16,25]. In fact, we can go further and prove that the above series is identical to the known form of the Macdonald index,

$$\mathcal{I}_M^{A_{2n}}(q, T) = \sum_{\ell_1, \dots, \ell_n \geq 0} \frac{T^{\ell_1 + \dots + \ell_n} q^{\ell_1^2 + \dots + \ell_n^2 + \ell_1 + \dots + \ell_n}}{(q)_{\ell_1 - \ell_2} (q)_{\ell_2 - \ell_3} \dots (q)_{\ell_{n-1} - \ell_n} (q)_{\ell_n}}. \quad (22)$$

The proof of this nontrivial identity is presented in the Appendix.

The BPS quiver of the (A_1, A_{2n+1}) theory in the source/sink chamber is shown in Fig. 2. The computation of the trace for (A_1, A_{2n+1}) is similar to the (A_1, A_{2n}) case, albeit with the complication of a flavor generator. There is $U(1)$ flavor symmetry for $n > 1$ and $SU(2)$ flavor symmetry for $n = 1$. Following the calculation in [10], we obtain

$$\begin{aligned} \mathcal{I}_M^{A_{2n+1}}(q, T, z) &= (q)_\infty^n (Tq)_\infty^n \sum_{\substack{\ell_1, \dots, \ell_{2n+1} \geq 0 \\ k_1, \dots, k_{2n+1} \geq 0}} \frac{(-1)^{\sum_{i=1}^{2n+1} (k_i + \ell_i)} T^{\frac{1}{2} \sum_{i=1}^{2n+1} (k_{2i-1} + \ell_{2i-1})} q^{\frac{1}{2} \sum_{i=1}^{2n+1} (k_i + \ell_i) + \sum_{j=1}^n \ell_{2j} (\ell_{2j-1} + \ell_{2j+1})}}{\prod_{i=1}^{n+1} (q)_{k_{2i-1}} (q)_{\ell_{2i-1}} \prod_{i=1}^n (Tq)_{k_{2i}} (q)_{\ell_{2i}}} \\ &\times \text{Tr}[X_{\gamma_f}]^{\ell_1 - k_1} \prod_{i=1}^n \delta_{k_{2i}, \ell_{2i}} \prod_{j=1}^n \delta_{(-1)^{j+1} k_1 + k_{2j+1}, (-1)^{j+1} \ell_1 + \ell_{2j+1}}, \end{aligned} \quad (23)$$

where we take $\text{Tr}[X_{\gamma_f}] = z^2$ for $n = 1$ and $\text{Tr}[X_{\gamma_f}] = (-1)^{(n+1)} z$ for $n > 1$ [10]. Let $\chi_n(z)$ denote the character of the n -dimensional representation of $SU(2)$. The first few terms in the expansion are shown as follows:

$$\begin{aligned} \mathcal{I}_M^{A_3}(q, T, z) &= 1 + qT\chi_3 + q^2T(\chi_1 + \chi_3) + q^2T^2\chi_5 + q^3T(\chi_1 + \chi_3) + q^3T^2(\chi_3 + \chi_5) + q^3T^3\chi_7 \\ &\quad + q^4T(\chi_1 + \chi_3) + q^4T^2(\chi_1 + 2\chi_3 + 2\chi_5) + q^4T^3(\chi_5 + \chi_7) + q^4T^4\chi_9 + \dots \\ \mathcal{I}_M^{A_5}(q, T, z) &= 1 + qT + q^3T^{\frac{3}{2}}(z + z^{-1}) + q^2(2T + T^2) + q^{\frac{5}{2}}(T^{\frac{3}{2}} + T^{\frac{5}{2}})(z + z^{-1}) + 2q^3(T + T^2) \\ &\quad + q^3T^3(1 + z^{-2} + z^2) + q^{\frac{7}{2}}(T^{\frac{3}{2}} + 2T^{\frac{5}{2}} + T^{\frac{7}{2}})(z + z^{-1}) + q^4(2T + 5T^2 + T^3 + (1 + z^2 + z^{-2})(T^3 + T^4)) + \dots \end{aligned} \quad (24)$$

This matches with the results of [13,15].

2. (A_1, D_k) theory

The BPS quiver of the (A_1, D_k) theory in the source/sink chamber is shown in Figs. 3 and 4. Let us start with the $k = 2n + 1$ case. Note that the only node with valency greater than 2 is a source, so the refined KS operator is given by

$$\mathcal{O}(q, T) = \prod_{i=1}^n \tilde{E}_{q,T}(X_{-\gamma_{2i-1}}) \prod_{i=1}^n E_{q,T}(X_{-\gamma_{2i}}) E_{q,T}(X_{-\gamma_{2n+1}}) \prod_{i=1}^n E_q(X_{\gamma_{2i-1}}) \prod_{i=1}^n E_{q,T}(X_{\gamma_{2i}}) E_{q,T}(X_{\gamma_{2n+1}}). \quad (25)$$

Following the calculation of trace in [10], we obtain

$$\mathcal{I}_M^{D_{2n+1}}(q, T, z) = (q)_\infty^n (Tq)_\infty^n \sum_{\substack{\ell_1, \dots, \ell_{2n+1} \geq 0 \\ k_{2n}, k_{2n+1} \geq 0}} \frac{q^{\sum_{i=1}^{2n+1} \ell_i + \frac{1}{2} \sum_{i,j=1}^{2n+1} b_{ij}^{D_{2n+1}} \ell_i \ell_j} T^{\ell_{2n} + \ell_{2n+1} + \sum_{i=1}^{n-1} \ell_{2i} 2(\ell_{2n+1} - k_{2n+1})}}{(q)_{k_{2n}} (q)_{k_{2n+1}} (q)_{\ell_{2n}} (q)_{\ell_{2n+1}} \prod_{i=1}^{n-1} (q)_{\ell_{2i}}^2 \prod_{i=1}^n (q)_{\ell_{2i-1}} (Tq)_{\ell_{2i-1}}} \delta_{k_{2n} + k_{2n+1}, \ell_{2n} + \ell_{2n+1}}, \quad (26)$$

where z is the flavor fugacity, $b_{ij}^{D_{2n+1}} = -C_{ij}^{D_{2n+1}} + 2\delta_{ij}$, and $C_{ij}^{D_{2n+1}}$ is the Cartan matrix of the D_{2n+1} Lie algebra. The first few terms for (A_1, D_5) are given as follows:

$$\begin{aligned} \mathcal{I}_M^{D_5}(q, T, z) = & 1 + qT\chi_3 + q^2T(\chi_1 + \chi_3) + q^2T^2\chi_5 + q^3T(\chi_1 + \chi_3) + q^3T^2(2\chi_3 + \chi_5) + q^3T^3\chi_7 \\ & + q^4T(\chi_1 + \chi_3) + q^4T^2(5\chi_3 + 2\chi_5) + q^4T^3(\chi_7 + 2\chi_5) + q^4T^4\chi_9 + \dots \end{aligned} \quad (27)$$

The computation for (A_1, D_{2n+2}) is similar. We obtain

$$\begin{aligned} \mathcal{I}_M^{D_{2n+2}}(q, T, x, y) = & (q)_\infty^n (Tq)_\infty^n \sum_{\substack{\ell_1, \dots, \ell_{2n+2} \geq 0 \\ k_1, \dots, k_{2n+2} \geq 0}} \frac{(-1)^{\sum_{i=1}^{2n+2} (k_i + \ell_i)} q^{\frac{1}{2} \sum_{i=1}^{2n+2} (k_i + \ell_i) + \frac{1}{2} \sum_{i,j=1}^{2n+2} b_{ij}^{D_{2n+2}} \ell_i \ell_j}}{(q)_{k_{2n+1}} (q)_{\ell_{2n+1}} (q)_{k_{2n+2}} (q)_{\ell_{2n+2}} \prod_{i=1}^n (Tq)_{k_{2i}} (q)_{\ell_{2i}} \prod_{i=1}^n (q)_{k_{2i-1}} (q)_{\ell_{2i-1}}} \\ & \times T^{\frac{1}{2}(k_{2n+1} + k_{2n+2} + \ell_{2n+1} + \ell_{2n+2}) + \frac{1}{2} \sum_{i=1}^n (k_{2i-1} + \ell_{2i-1})} \left(\text{Tr}[X_{\gamma_{f_1}}] \right)^{\ell_{2n+2} - k_{2n+2}} \left(\text{Tr}[X_{\gamma_{f_2}}] \right)^{\ell_1 - k_1} \\ & \times \left(\prod_{i=1}^n \delta_{k_{2i}, \ell_{2i}} \right) \left(\prod_{i=1}^{n-1} \delta_{(-1)^{i+1} k_1 + k_{2i+1}, (-1)^{i+1} \ell_1 + \ell_{2i+1}} \right) \delta_{(-1)^{n+1} k_1 + k_{2n+1} + k_{2n+2}, (-1)^{n+1} \ell_1 + \ell_{2n+1} + \ell_{2n+2}}, \end{aligned} \quad (28)$$

where $b_{ij}^{D_{2n+2}} = -C_{ij}^{D_{2n+2}} + 2\delta_{ij}$ and $C_{ij}^{D_{2n+2}}$ is the Cartan matrix of the D_{2n+2} Lie algebra, and we follow the normalization for $\text{SU}(2) \times \text{U}(1)$ flavor fugacities given in [10],

$$\text{Tr}[X_{\gamma_{f_1}}] = y^2, \quad \text{Tr}[X_{\gamma_{f_2}}] = \begin{cases} xy & \text{if } n \text{ is odd,} \\ -\frac{x}{y} & \text{if } n \text{ is even.} \end{cases} \quad (29)$$

For (A_1, D_4) , the flavor symmetry is enhanced to $\text{SU}(3)$ and the fugacities x, y are related to the $\text{SU}(3)$ flavor fugacities as $x = z_2^{-3/2}, y = z_1 z_2^{-1/2}$ [10]. Let χ_n and $\tilde{\chi}_n$ denote the character of the n -dimensional representation of $\text{SU}(2)$ and $\text{SU}(3)$, respectively. Then the first terms in the expansion of the Macdonald index are given by

$$\begin{aligned} \mathcal{I}_M^{D_4}(q, T, z_1, z_2) = & 1 + qT\tilde{\chi}_8 + q^2T(\tilde{\chi}_1 + \tilde{\chi}_8) + q^2T^2\tilde{\chi}_{27} + q^3T(\tilde{\chi}_1 + \tilde{\chi}_8) + q^3T^2(\tilde{\chi}_8 + \tilde{\chi}_{10} + \tilde{\chi}_{\overline{10}} + \tilde{\chi}_{27}) + q^3T^3\tilde{\chi}_{64} + \dots, \\ \mathcal{I}_M^{D_6}(q, T, x, y) = & 1 + qT(\chi_1 + \chi_3) + q^2T^2(x + x^{-1})\chi_2 + q^2T(2\chi_1 + \chi_3) + q^2T^2(\chi_1 + \chi_3 + \chi_5) + q^2T^3(x + x^{-1})\chi_2 \\ & + q^2T^2(x + x^{-1})(\chi_2 + \chi_4) + q^3T(2\chi_1 + \chi_3) + q^3T^2(2\chi_1 + 4\chi_3 + \chi_5) \\ & + q^3T^3((\chi_1 + (x^{-2} + x^2 + 1)\chi_3 + \chi_5 + \chi_7)) + \dots \end{aligned} \quad (30)$$

The Macdonald index for (A_1, D_4) matches with Table 2 in [13] and (A_1, D_6) matches with the general formula of [13].

3. (A_1, E_n) theory

Let us start with (A_1, E_6) theory. The BPS quiver for the (A_1, E_6) theory is shown in Fig. 5. The refined KS operator is given by

$$\mathcal{O}(q, T) = \prod_{i=1}^3 \tilde{E}_{q,T}(X_{-\gamma_{2i-1}}) \prod_{i=1}^3 E_{q,T}(X_{-\gamma_{2i}}) \prod_{i=1}^3 E_q(X_{\gamma_{2i-1}}) \prod_{i=1}^3 E_{q,T}(X_{\gamma_{2i}}). \quad (31)$$

Computing the trace, the Macdonald index is given by

$$\begin{aligned} \mathcal{I}_M^{E_6}(q, T) = & (q)_\infty^3 (qT)_\infty^3 \sum_{\ell_1, \dots, \ell_6=0}^{\infty} \frac{T^{\ell_2 + \ell_4 + \ell_6} q^{\sum_{i=1}^6 \ell_i + \frac{1}{2} \sum_{i,j=1}^6 b_{ij}^{E_6} \ell_i \ell_j}}{\prod_{i=1}^3 (qT)_{\ell_{2i-1}} (q)_{\ell_{2i-1}} \prod_{i=1}^3 (q)_{\ell_{2i}}^2} \\ = & 1 + q^2T + q^3(T + T^2) + q^4(T + 2T^2) + q^5(T + 2T^2) + q^6(T + 3T^2 + 2T^3) + \dots, \end{aligned} \quad (32)$$

where $b_{ij}^{E_6} = -C_{ij}^{E_6} + 2\delta_{ij}$ and $C_{ij}^{E_6}$ is the Cartan matrix of the E_6 Lie algebra. The BPS quiver of (A_1, E_8) theory is shown in Fig. 6. We now describe the computation of the Macdonald index for the (A_1, E_7) theory. Unlike the other (A_1, E_n) theories,

(A_1, E_7) has a $U(1)$ flavor symmetry. Note also that there is a source node with valency greater than 2. Thus, we must use (16) to write the refined KS operator. Performing the computation parallel to [10] with the refined KS operator, we get

$$\begin{aligned} \mathcal{I}_M^{E_7}(q, T, z) &= (q)_\infty^3 (qT)_\infty^3 \sum_{\substack{\ell_1, \dots, \ell_7 \geq 0 \\ k_1, \dots, k_7 \geq 0}} \frac{(-1)^{\sum_{i=1}^7 (k_i + \ell_i)} T^{\frac{1}{2}(\sum_{i=1}^7 (k_i + \ell_i) + \frac{1}{2} \sum_{i=1}^3 (k_{2i} + \ell_{2i}))} q^{\frac{1}{2} \sum_{i=1}^7 (k_i + \ell_i)} q^{k_2(\ell_1 + \ell_3) + k_4(\ell_3 + \ell_5) + k_6\ell_5 + k_7\ell_3}}{(q)_{k_7}(q)_{\ell_7} \prod_{i=1}^3 (qT)_{k_{2i-1}} (q)_{\ell_{2i-1}} \prod_{i=1}^3 (q)_{k_{2i}} (q)_{\ell_{2i}}} \\ &\quad \times (-z)^{-\ell_7 + k_7} \delta_{k_4 + k_7, \ell_4 + \ell_7} \delta_{k_6 - k_7, \ell_6 - \ell_7} \prod_{i \in \{1, 2, 3, 5\}} \delta_{k_i, \ell_i} \\ &= 1 + qT + q^{\frac{3}{2}} T^{\frac{3}{2}} (z + z^{-1}) + q^2 (2T + T^2) + q^{\frac{5}{2}} (T^{\frac{3}{2}} + T^{\frac{5}{2}}) (z + z^{-1}) + q^3 (2T + 3T^2 + T^3 (z^{-2} + 1 + z^2)) \\ &\quad + q^{\frac{7}{2}} (T^{\frac{3}{2}} + 3T^{\frac{5}{2}} + T^{\frac{7}{2}}) (z + z^{-1}) + \dots, \end{aligned} \quad (33)$$

where z is the flavor fugacity. The computation of the Macdonald index of (A_1, E_8) theory identical to that of (A_1, E_6) theory and is given by

$$\begin{aligned} \mathcal{I}_M^{E_8}(q, T) &= (q)_\infty^4 (qT)_\infty^4 \sum_{\ell_1, \dots, \ell_8=0}^{\infty} \frac{T^{\sum_{i=1}^4 \ell_{2i}} q^{\sum_{i=1}^8 \ell_i + \frac{1}{2} \sum_{i,j=1}^8 b_{ij}^{E_8} \ell_i \ell_j}}{\prod_{i=1}^4 (qT)_{\ell_{2i-1}} (q)_{\ell_{2i-1}} \prod_{i=1}^4 (q)_{\ell_{2i}}^2} \\ &= 1 + q^2 T + q^3 (T + T^2) + q^4 (T + 2T^2) + q^5 (T + 2T^2 + T^3) + q^6 (T + 3T^2 + 3T^3) + \dots, \end{aligned} \quad (34)$$

where $b_{ij}^{E_8} = -C_{ij}^{E_8} + 2\delta_{ij}$ and $C_{ij}^{E_8}$ is the Cartan matrix of the E_8 Lie algebra. It is expected that $(A_1, E_{6,8}) \cong (A_2, A_{3,4})$ [26]. We confirmed that the Macdonald indices of $(A_1, E_{6,8})$ theory match with the Macdonald indices of $(A_2, A_{3,4})$ theory computed in [27,28].

V. DISCUSSION

The intriguing relation we have found between the Macdonald index of a 4D $\mathcal{N} = 2$ SCFT and the trace of the modified KS operator raises a lot of interesting questions. The modified function $E_{q,T}(z)$ can be recognized as the Macdonald index of a half-hypermultiplet. This means that our prescription assigns a full hypermultiplet to the source nodes in case the BPS quiver does not have a source node with valency greater than 2 and to the sink nodes in case the BPS quiver has a source node with valency greater than 2. However, the function $\tilde{E}_{q,T}$ does not seem to be related to any superconformal multiplet. Perhaps a simplified model of BPS bound states in terms of “halos” [5,18,20] could be useful in understanding the physical meaning of the function $\tilde{E}_{q,T}$. Using bound state halos in terms of noninteracting bosons and fermions, Ref. [20] could prove some special cases of the KS wall-crossing formula. Hence, it will be useful to understand (and possibly prove) the Cordova-Shao conjecture and our prescription in terms of halos.

We list here some possible future directions. (1) The first order problem is to perform more checks of our conjecture. (2) Second, we would like to understand how the refined

KS operator transforms under quiver mutations [8,23]. More generally, we would like to give a chamber independent prescription for the refined KS operator. Once we have a chamber independent prescription, our formulas for Argyres-Douglas theories give strong evidence that the modified KS operator is a wall-crossing invariant. (3) Finally, we would like to interpret our prescription (and, in particular, the Cordova-Shao) conjecture in terms of BPS halos.

Note added. Recently, the paper [29] appeared which also conjectured expressions for the Macdonald indices of $(A_1, \mathfrak{g})$ theories using the KS monodromy operator.

ACKNOWLEDGMENTS

The authors would like to thank Gregory Moore, Leonardo Rastelli, and Anirudh Deb for comments on the draft and Ashoke Sen for some useful discussions on related topics. R. K. S. would like to thank LITP, Stanford University for hospitality where this work began. The work of A. B. is supported by the U.S. Department of Energy under Grant No. DE-SC1019775. The work of R. K. S. is supported by the U.S. Department of Energy under Grant No. DE-SC0010008.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: PROOF OF A q -SERIES IDENTITY

In this appendix, we prove that the conjecture (21) for the Macdonald index of (A_1, A_{2n}) theory agrees with the known form (22) of the Macdonald index. More precisely, we prove the following:

Theorem A.1. We have the following identity

$$(q)_\infty^n (Tq)_\infty^n \sum_{k_1, \dots, k_{2n} \geq 0} T^{k_1+k_3+\dots+k_{2n-1}} q^{k_1+\dots+k_{2n}} \frac{q^{k_1 k_2 + k_2 k_3 + \dots + k_{2n-1} k_{2n}}}{\prod_{i=1}^n (q)_{k_{2i-1}}^2 \prod_{i=1}^n (q)_{k_{2i}} (Tq)_{k_{2i}}} = \sum_{\ell_1, \dots, \ell_n \geq 0} \frac{T^{\ell_1+\dots+\ell_n} q^{\ell_1^2+\dots+\ell_n^2+\ell_1+\dots+\ell_n}}{(q)_{\ell_1-\ell_2} (q)_{\ell_2-\ell_3} \dots (q)_{\ell_{n-1}-\ell_n} (q)_{\ell_n}}. \quad (\text{A1})$$

We will prove this in three steps. Let us denote the lhs by L and rhs by R . For $1 \leq i \leq n-1$, define

$$S_i = (q)^{n-i} (Tq)^{n-i} \sum_{\substack{k_1, \dots, k_{2n-2i} \geq 0 \\ n_1, \dots, n_i \geq 0}} T^{k_1+\dots+k_{2n-2i}} q^{k_1+\dots+k_{2n-2i}} \frac{q^{k_1 k_2 + \dots + k_{2n-2i-1} k_{2n-2i}}}{\prod_{r=1}^{n-i} (q)_{k_{2r-1}}^2 \prod_{r=1}^{n-i} (q)_{k_{2r}} (Tq)_{k_{2r}}} \\ \times \frac{q^{n_1^2+\dots+n_i^2+n_1+\dots+n_i} (Tq^{n_i+1})_{2n-2i}}{(q)_{n_1-n_2} (q)_{n_2-n_3} \dots (q)_{n_{i-1}-n_i} (q)_{n_i}}. \quad (\text{A2})$$

We prove the result in three steps.

- (1) $L = S_1$.
- (2) $R = S_{n-1}$.
- (3) $S_i = S_{i+1}$ for $1 \leq i \leq n-2$.

We will need the following identities:

Lemma A1. We have

$$\sum_{N \geq 0} \frac{T^N}{(q)_N (cq)_N} = \frac{1}{(t)_\infty} \sum_{N \geq 0} \frac{q^{N^2} T^{N^2} c^N}{(q)_N (cq)_N}. \quad (\text{A3})$$

Proof. We use Eq. (3.3.13), p. 39 in [30],

$$\sum_{N=0}^{\infty} \frac{(a)_N (b)_N t^N}{(q)_N (c)_N} = \frac{(abt/c)_\infty}{(t)_\infty} \sum_{N=0}^{\infty} \frac{(c/a)_N (c/b)_N (abt/c)^N}{(q)_N (c)_N}. \quad (\text{A4})$$

We replace c by cq and take $a, b \rightarrow \infty$. Note that

$$\lim_{a \rightarrow b} (c/a)_N a^N = \lim_{a \rightarrow 0} (1-c/a)(1-c/aq) \dots (1-cq^{N-1}/a) a^N \\ = \lim_{a \rightarrow 0} (-c)^N \cdot q^{\frac{N(N-1)}{2}}. \quad (\text{A5})$$

Thus, we get

$$\sum_{N \geq 0} \frac{T^N}{(q)_N (c)_N} = \frac{1}{(t)_\infty} \sum_{N=0}^{\infty} \frac{c^N q^{N^2-N}}{(q)_N (c)_N} T^N. \quad (\text{A6})$$

Replace $c \rightarrow cq$ to get the result. ■

We also have

$$\sum_{N \geq 0} \frac{T^N}{(q)_N} = \frac{1}{(T, q)_\infty} = \frac{1}{(T)_\infty}. \quad (\text{A7})$$

Lemma A2. We have

$$\sum_{j \geq 0} \frac{(cq^{N+1})_j t^j}{(q)_j (cq)_j} = \frac{1}{(t)_\infty} \sum_{j=0}^N \binom{N}{j}_q \frac{q^j c^j t^j}{(cq)_j}. \quad (\text{A8})$$

Proof. We get this by taking $a \rightarrow 0$ and $b = cq^{N+1}$ in (A4). We have

$$\sum_{j=0}^{\infty} \frac{(cq^{N+1})_j t^j (a)_j}{(q)_j (cq)_j} \\ = \frac{1}{(t)_\infty} \sum_{j=0}^{\infty} \frac{(cq/a)_j a^j (cq/cq^{N+1})_j (cq^{N+1} t/cq)^j}{(q)_j (cq)_j} \\ = \frac{1}{(t)_\infty} \sum_{j=0}^{\infty} \frac{(cq/a)_j a^j (q^{-N})_j (q^N t)^j}{(q)_j (cq)_j}. \quad (\text{A9})$$

Now,

$$\lim_{a \rightarrow 0} (cq/a)_j a^j = \lim_{a \rightarrow 0} (1-cq/a)(1-cq^2/a) \\ \times \dots (1-cq^{j+1}/a) a^j \\ = (-c)^j q^{1+2+\dots+j+1} = (-c)^j q^{j(j+1)/2}. \quad (\text{A10})$$

Next,

$$(q^{-N})_j = (q^{-N}; q)_j = \frac{(q)_N}{(q)_{N-j}} (-1)^j q^{\binom{j}{2}-Nj}. \quad (\text{A11})$$

We get

$$\begin{aligned}
& \sum_{j=0}^{\infty} \frac{(cq^{N+1})_j t^j}{(q)_j (cq)_j} \\
&= \frac{1}{(t)_{\infty}} \sum_{j=0}^{\infty} \frac{c^j q^{(j^2+j)/2} q^{j(j-1)/2-N_j} q^{N_j} T^j}{(q)_j (cq)_j} \frac{(q)_N}{(q)_{N-j}} \\
&= \frac{1}{(T)_{\infty}} \sum_{j=0}^N \frac{c^j q^{j^2} T^j}{(cq)_j} \binom{N}{j}_q. \tag{A12}
\end{aligned}$$

$$H_j = \frac{T^{k_{2j-1}} q^{k_{2j}+k_{2j-1}+k_{2j}k_{2j-1}+k_{2j-1}k_{2j-2}}}{(q)_{k_{2j-1}}^2 (q)_{k_{2j}} (tq)_{k_{2j}}} H_{j-1} \tag{A15}$$

and

$$N_j = \frac{t^{n_j} q^{n_j^2+n_j} (q)_{n_{j-1}}}{(q)_{n_j} (q)_{n_{j-1}-n_j}} N_{j-1} = t^{n_j} q^{n_j^2+n_j} \binom{n_{j-1}}{n_j}_q N_{j-1}. \tag{A16}$$

■

Introduce the notation

$$H_j := \frac{T^{k_1+k_3+\dots+k_{2j-1}} q^{k_1+k_2+\dots+k_{2j}+k_1k_2+\dots+k_{2j-1}k_{2j}}}{\prod_{r=1}^j (q)_{k_{2r-1}}^2 \prod_{r=1}^j (q)_{k_{2r}} (Tq)_{k_{2r}}}, \tag{A13}$$

$$N_j := \frac{T^{n_1+\dots+n_j} q^{n_1^2+\dots+n_j^2+n_1+\dots+n_j}}{(q)_{n_1-n_2} (q)_{n_2-n_3} \dots (q)_{n_{j-1}-n_j} (q)_{n_j}}. \tag{A14}$$

We have

$$S_i = (q)_{\infty}^{n-i} (tq)_{\infty}^{n-i} \sum_{\substack{k_1, \dots, k_{2n-2i} \geq 0 \\ n_1, \dots, n_i \geq 0}} H_{n-i} N_i (Tq^{n_i+1})_{k_{2n-2i}}. \tag{A17}$$

Let us first prove that $L = S_1$.*Proposition A.1.* Let L, S_i be as above. Then we have

$$L = S_1. \tag{A18}$$

Then we can write

Proof. We have

$$\begin{aligned}
L &= (q)_{\infty}^n (tq)_{\infty}^n \sum_{k_1, \dots, k_{2n} \geq 0} H_n = (q)_{\infty}^n (tq)_{\infty}^n \sum_{k_1, \dots, k_{2n} \geq 0} \frac{t^{k_{2n-1}} q^{k_{2n}+k_{2n-1}+k_{2n}k_{2n-1}+k_{2n-1}k_{2n-2}}}{(q)_{k_{2n-1}}^2 (q)_{k_{2n}} (tq)_{k_{2n}}} H_{n-1} \\
&= (q)_{\infty}^n (tq)_{\infty}^n \sum_{k_{2n} \geq 0} \frac{q^{k_{2n}+k_{2n}k_{2n-1}}}{(q)_{k_{2n}} (tq)_{k_{2n}}} \sum_{k_1, \dots, k_{2n-1} \geq 0} \frac{t^{k_{2n-1}} q^{k_{2n-1}+k_{2n-1}k_{2n-2}}}{(q)_{k_{2n-1}}^2} H_{n-1}. \tag{A19}
\end{aligned}$$

Using Lemma A1 with $T = q^{1+k_{2n-1}}, c = T$, we get

$$\sum_{k_{2n} \geq 0} \frac{q^{(1+k_{2n-1})k_{2n}}}{(q)_{k_{2n}} (Tq)_{k_{2n}}} = \frac{1}{(q^{1+k_{2n-1}})_{\infty}} \sum_{n_1 \geq 0} \frac{q^{n_1^2} q^{n_1(1+k_{2n-1})} T^{n_1}}{(q)_{n_1} (Tq)_{n_1}}. \tag{A20}$$

Substituting this, we get

$$\begin{aligned}
L &= (q)_{\infty}^n (tq)_{\infty}^n \sum_{k_1, \dots, k_{2n-1} \geq 0} \frac{t^{k_{2n-1}} q^{k_{2n-1}+k_{2n-1}k_{2n-2}}}{(q)_{k_{2n-1}}^2} H_{n-1} \frac{q^{n_1^2} q^{n_1(1+k_{2n-1})} t^{n_1}}{(q^{1+k_{2n-1}})_{\infty} (q)_{n_1} (tq)_{n_1}} \\
&= (q)_{\infty}^n (tq)_{\infty}^n \sum_{\substack{k_1, \dots, k_{2n-2} \geq 0 \\ n_1 \geq 0}} H_{n-1} \frac{q^{n_1^2} t^{n_1}}{(q)_{n_1} (tq)_{n_1}} \sum_{k_{2n-1} \geq 0} \frac{t^{k_{2n-1}} q^{k_{2n-1}+k_{2n-1}k_{2n-2}}}{(q)_{k_{2n-1}}^2} \frac{q^{n_1(1+k_{2n-1})}}{(q^{1+k_{2n-1}})_{\infty}} \\
&= (q)_{\infty}^n (tq)_{\infty}^n \sum_{\substack{k_1, \dots, k_{2n-2} \geq 0 \\ n_1 \geq 0}} H_{n-1} \frac{q^{n_1^2+n_1} t^{n_1}}{(q)_{n_1} (tq)_{n_1}} \sum_{k_{2n-1} \geq 0} \frac{t^{k_{2n-1}} q^{k_{2n-1}(1+k_{2n-2}+n_1)} (q^{1+k_{2n-1}})_{\infty}}{(q)_{k_{2n-1}} (q^{1+k_{2n-1}})_{\infty} (q)_{\infty}} \\
&= (q)_{\infty}^{n-1} (tq)_{\infty}^n \sum_{\substack{k_1, \dots, k_{2n-2} \geq 0 \\ n_1 \geq 0}} H_{n-1} \frac{q^{n_1^2+n_1} t^{n_1}}{(q)_{n_1} (tq)_{n_1}} \sum_{k_{2n-1} \geq 0} \frac{t^{k_{2n-1}} q^{k_{2n-1}(1+k_{2n-2}+n_1)}}{(q)_{k_{2n-1}}} \\
&= (q)_{\infty}^{n-1} (tq)_{\infty}^n \sum_{k_1, \dots, k_{2n-2} \geq 0} H_{n-1} \frac{q^{n_1^2+n_1} t^{n_1}}{(q)_{n_1} (tq)_{n_1}} \frac{1}{(tq^{n+k_{2n-2}+1})_{\infty}}, \tag{A21}
\end{aligned}$$

where we used

$$\begin{aligned} & \sum_{k_{2n-1} \geq 0} \frac{t^{k_{2n-1}} q^{k_{2n-1}(1+k_{2n-2}+n_1)}}{(q)_{k_{2n-1}}} \\ &= \sum_{k_{2n-1} \geq 0} \frac{(tq^{1+n_1+k_{2n-2}})_{k_{2n-1}}}{(q)_{k_{2n-1}}} = \frac{1}{(tq^{1+n_1+k_{2n-2}})_\infty}. \end{aligned} \quad (\text{A22})$$

We now prove that

$$S_{n-1} = R. \quad (\text{A24})$$

Thus, we have

$$\begin{aligned} L &= (q)_\infty^{n-1} (tq)_\infty^n \sum_{\substack{k_1, \dots, k_{2n-2} \geq 0 \\ n_1 \geq 0}} H_{n-1} N_1 \cdot \frac{(tq^{n_1+1})_\infty}{(tq)_\infty (tq^{1+n_1+k_{2n-2}})_\infty} \\ &= (q)_\infty^{n-1} (tq)_\infty^{n-1} \sum_{\substack{k_1, \dots, k_{2n-2} \geq 0 \\ n_1 \geq 0}} H_{n-1} N_1 (tq^{n_1+1})_{k_{2n-2}} \\ &= S_1. \end{aligned} \quad (\text{A23})$$

Proposition A.2. Let R, S_i be as above. Then we have

$$R = S_{n-1}. \quad (\text{A25})$$

Proof. We have

$$\begin{aligned} S_{n-1} &= (q)_\infty (tq)_\infty \sum_{n_1, \dots, n_{n-1} \geq 0} N_{n-1} \frac{T^{k_1} q^{k_1+k_2+k_1 k_2}}{(q)_{k_1}^2 (q)_{k_2} (tq)_{k_2}} (tq^{n_{n-1}+1})_{k_2} \\ &= (q)_\infty (tq)_\infty \sum_{n_1, \dots, n_{n-1} \geq 0} N_{n-1} \frac{T^{k_1} q^{k_1}}{(q)_{k_1}^2} \sum_{k_2 \geq 0} \frac{q^{k_2+k_1 k_2}}{(q)_{k_2} (tq)_{k_2}} (tq^{n_{n-1}+1})_{k_2}. \end{aligned} \quad (\text{A26})$$

Using Lemma A2 for the sum over k_2 with $c = T$, $T = q^{k_1+1}$, we get

$$\begin{aligned} \sum_{k_2 \geq 0} \frac{q^{k_2+k_1 k_2}}{(q)_{k_2} (tq)_{k_2}} (tq^{n_{n-1}+1})_{k_2} &= \sum_{k_2 \geq 0} \frac{q^{(1+k_1)k_2} (tq^{n_{n-1}+1})_{k_2}}{(q)_{k_2} (tq)_{k_2}} \\ &= \frac{1}{(q^{1+k_2})_\infty} \sum_{n_1, \dots, n_{n-1} \geq 0} \binom{n_{n-1}}{n_n}_q \frac{q^{n_n^2} T^{n_n} q^{n_n(1+k_1)}}{(Tq)_{n_n}}. \end{aligned} \quad (\text{A27})$$

Substituting we get

$$\begin{aligned} S_{n-1} &= (q)_\infty (tq)_\infty \sum_{\substack{n_1, \dots, n_{n-1} \geq 0 \\ k_1 \geq 0}} N_{n-1} \frac{T^{k_1} q^{k_1}}{(q)_{k_1}^2 (q^{1+k_2})_\infty} \sum_{n_1, \dots, n_{n-1} \geq 0} \binom{n_{n-1}}{n_n}_q \frac{q^{n_n^2+n_n} q^{n_n k_1} T^{n_n}}{(q)_{n_n} (Tq)_{n_n}} \\ &= (q)_\infty (tq)_\infty \sum_{n_1, \dots, n_{n-1} \geq 0} N_{n-1} \binom{n_{n-1}}{n_n} \frac{q^{n_n^2+n_n} T^{n_n}}{(q)_{n_n} (Tq)_{n_n}} \sum_{k_1 \geq 0} \frac{T^{k_1} q^{k_1+n_n k_1}}{(q^{k_1+1})_\infty (q)_{k_1}^2}. \end{aligned} \quad (\text{A28})$$

We have

$$\sum_{k_1 \geq 0} \frac{T^{k_1} q^{k_1+n_n k_1}}{(q^{k_1+1})_\infty (q)_{k_1}^2} = \sum_{k_2 \geq 0} \frac{T^{k_1} q^{(1+n_n)k_1}}{(q)_{k_1} (q)_\infty} = \frac{1}{(tq^{1+n_n})_\infty} \frac{1}{(q)_\infty}. \quad (\text{A29})$$

Thus, we get

$$\begin{aligned}
S_{n-1} &= (tq)_\infty \sum_{n_1, \dots, n_n \geq 0} N_{n-1} \binom{n_{n-1}}{n_n} \frac{q^{n_1^2 + n_n} T^{n_n}}{(q)_{n_n} (Tq)_{n_n} (tq^{n_n+1})_\infty} \\
&= \sum_{n_1, \dots, n_n \geq 0} N_{n-1} \binom{n_{n-1}}{n_n}_q q^{n_1^2 + n_n} T^{n_n} \\
&= \sum_{n_1, \dots, n_n \geq 0} N_n \\
&= R.
\end{aligned} \tag{A30}$$

We finally prove that $S_i = S_{i+1}$.

Proposition A.3. Let S_i be as above. Then we have

$$S_i = S_{i+1} \quad \text{for } 1 \leq i \leq n-2. \tag{A31}$$

Proof. For ease of notation, we will write

$$j_2 = k_{2n-2i}, \quad j_1 = k_{2n-2i-1}. \tag{A32}$$

■ Then we have

$$\begin{aligned}
S_i &= (q)_\infty^{n-i} (tq)_\infty^{n-i} \sum_{\substack{k_1, \dots, k_{2n-2i-2} \geq 0 \\ j_1, j_2 \geq 0 \\ n_1, \dots, n_i \geq 0}} H_{n-i-1} \frac{T^{j_1} q^{j_2 + j_1 + j_2 j_1 + j_1 k_{2n-2i-2}}}{(q)_{j_1}^2 (q)_{j_2} (tq)_{j_2}} \cdot N_i (tq^{n_i+1})_{j_2} \\
&= (q)_\infty^{n-i} (tq)_\infty^{n-i} \sum_{\substack{k_1, \dots, k_{2n-2i-2} \geq 0 \\ j_1 \geq 0 \\ n_1, \dots, n_i \geq 0}} H_{n-i-1} N_i \frac{T^{j_1} q^{j_1 + j_1 k_{2n-2i-2}}}{(q)_{j_1}^2} \sum_{j_2 \geq 0} \frac{q^{j_2 + j_1 j_2}}{(q)_{j_2} (tq)_{j_2}} (tq^{n_i+1})_{j_2} \\
&= (q)_\infty^{n-i} (tq)_\infty^{n-i} \sum_{\substack{k_1, \dots, k_{2n-2i-2} \geq 0 \\ j_1 \geq 0 \\ n_1, \dots, n_i, n_{i+1} \geq 0}} H_{n-i-1} N_i \frac{T^{j_1} q^{j_1 + j_1 k_{2n-2i-2}}}{(q)_{j_1}^2 (q^{1+j_1})_\infty} \binom{n_i}{n_{i+1}}_q \frac{q^{n_{i+1}^2} T^{n_{i+1}} q^{n_{i+1}(t+j_1)}}{(tq)_{n_{i+1}}} \\
&= (q)_\infty^{n-i-1} (tq)_\infty^{n-i} \sum_{\substack{k_1, \dots, k_{2n-2i-2} \geq 0 \\ n_1, \dots, n_i, n_{i+1} \geq 0}} H_{n-i-1} N_{i+1} \frac{1}{(tq^{k_{2n-2i-2} + n_{i+1} + 1})_\infty (tq)_{n_{i+1}}} \\
&= (q)_\infty^{n-i-1} (tq)_\infty^{n-i-1} \sum_{\substack{k_1, \dots, k_{2n-2i-2} \geq 0 \\ n_1, \dots, n_i, n_{i+1} \geq 0}} H_{n-i-1} N_{i+1} (tq^{n_{i+1}+1})_{k_{2n-2i-2}} \\
&= S_{i+1}.
\end{aligned} \tag{A33}$$

■
■

Proof of Theorem A1. This proof follows from the above propositions and induction on i .

-
- [1] N. Seiberg and E. Witten, *Nucl. Phys.* **B426**, 19 (1994).
 - [2] N. Seiberg and E. Witten, *Nucl. Phys.* **B431**, 484 (1994).
 - [3] The original Kontsevich-Soibelman operator was introduced in context of Donaldson-Thomas invariants in [4] and used in context of 4D $\mathcal{N} = 2$ theories in [5] to study wall crossing.
 - [4] M. Kontsevich and Y. Soibelman, [arXiv:0811.2435](#).
 - [5] D. Gaiotto, G. W. Moore, and A. Neitzke, *Commun. Math. Phys.* **299**, 163 (2010).
 - [6] T. Dimofte and S. Gukov, *Lett. Math. Phys.* **91**, 1 (2010).
 - [7] T. Dimofte, S. Gukov, and Y. Soibelman, *Lett. Math. Phys.* **95**, 1 (2011).
 - [8] S. Cecotti, A. Neitzke, and C. Vafa, [arXiv:1006.3435](#).
 - [9] S. Cecotti, J. Song, C. Vafa, and W. Yan, *J. High Energy Phys.* **11** (2017) 013.
 - [10] C. Cordova and S.-H. Shao, *J. High Energy Phys.* **01** (2016) 040.
 - [11] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. van Rees, *Commun. Math. Phys.* **336**, 1359 (2015).
 - [12] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *Commun. Math. Phys.* **319**, 147 (2013).
 - [13] M. Buican and T. Nishinaka, *J. High Energy Phys.* **02** (2016) 159.
 - [14] C. Bhargava, M. Buican, and H. Jiang, *Phys. Rev. Lett.* **132**, 031602 (2024).
 - [15] J. Song, *J. High Energy Phys.* **08** (2017) 044.
 - [16] G. Andrews, A. Banerjee, C. Bhargava, R. K. Singh, and R. Tao, [arXiv:2507.06294](#).
 - [17] M. J. Kang, C. Lawrie, and J. Song, [arXiv:2507.12510](#).
 - [18] D. Gaiotto, G. W. Moore, and A. Neitzke, *Adv. Theor. Math. Phys.* **17**, 241 (2013).
 - [19] F. Denef, *J. High Energy Phys.* **10** (2002) 023.

- [20] F. Denef and G. W. Moore, [J. High Energy Phys.](#) **11** (2011) 129.
- [21] J. Manschot, B. Pioline, and A. Sen, [J. High Energy Phys.](#) **07** (2011) 059.
- [22] J. Manschot, B. Pioline, and A. Sen, [J. High Energy Phys.](#) **11** (2012) 023.
- [23] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, and C. Vafa, [Commun. Math. Phys.](#) **323**, 1185 (2013).
- [24] A node with all arrows emanating from it is called a source node and a node with all arrows ending on it is called a sink node.
- [25] J. Song, [J. High Energy Phys.](#) **02** (2016) 045.
- [26] Y. Wang and D. Xie, [Phys. Rev. D](#) **94**, 065012 (2016).
- [27] A. Watanabe and R.-D. Zhu, [J. High Energy Phys.](#) **02** (2020) 004.
- [28] P. Agarwal, S. Lee, and J. Song, [J. High Energy Phys.](#) **06** (2019) 102.
- [29] H. Kim, H. Kim, and J. Song, [J. High Energy Phys.](#) **03** (2026) 213.
- [30] G. Andrews, *The Theory of Partitions*, Cambridge Mathematical Library (Cambridge University Press, Cambridge, England, 1998).