

Determination of $\alpha_s(M_Z)$ via a high-precision effective coupling $\alpha_s^{g_1}(Q)$

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We propose a novel method to determine the strong coupling of quantum chromodynamics (QCD) and fix its running behavior at all scales by using the Bjorken sum rules (BSR). The BSR defines an effective coupling $\alpha_s^{g_1}(Q)$ which includes the nonperturbative higher-twist corrections and perturbative QCD (pQCD) corrections to the leading-twist part. For the leading-twist part of $\alpha_s^{g_1}(Q)$, we adopt the infinite-order scale-setting procedure of the principle of maximum conformality (PMC_∞) to deal with its pQCD corrections, which reveals the intrinsic conformality of the series and eliminates conventional renormalization scheme-and-scale ambiguities. Using the PMC_∞ approach, we not only eliminate *the first kind of residual scale dependence* due to uncalculated higher-order terms, but also resolve the previous “self-consistency problem.” The light-front holographic QCD model is used for $\alpha_s^{g_1}(Q)$ in the infrared region, which also reveals a conformal behavior at $Q \rightarrow 0$. As a combination, we obtain a precise $\alpha_s^{g_1}(Q)$ at all scales, which matches well with the known experimental data with p -value $\sim 98\%$, and determines the strong coupling constant at the scale M_Z , $\alpha_s(M_Z) = 0.1220^{+0.0011}_{-0.0012}(\Delta\kappa)^{-0.0046}_{+0.0007}(\text{the.})$, where the first error comes from $\Delta\kappa$ of LFHQCD model and the second error is from *the second kind of residual scale dependence*.

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I. INTRODUCTION

The quantum chromodynamics (QCD) strong coupling constant α_s is one of the fundamental parameters in QCD, indicating the strength of gluon-quark interactions. Knowing its precise magnitude is crucial for achieving highly precise QCD predictions and probing the intricate internal structure of hadrons. The α_s is scale-dependent. Owing to asymptotic freedom in QCD theory [1,2], its magnitude decreases progressively with increasing energy scale Q and becomes negligibly small at scales far exceeding the QCD scale Λ_{QCD} . On this basis, the perturbative QCD (pQCD) corrections to physical observables can be routinely expanded as a perturbative series with increasing α_s -powers. It is then important to know the magnitude α_s at

any scales. Practically, one usually extracts the magnitude of α_s at some typical energy scale such as the Z-boson mass M_Z via comparison with experimental data, and then determines its value at any scale by using the following renormalization group equation (RGE) [3,4],

$$\mu^2 \frac{\partial}{\partial \mu^2} \left(\frac{\alpha_s(\mu)}{\pi} \right) = \beta(\alpha_s) = - \sum_{i=0}^{\infty} \beta_i \left(\frac{\alpha_s(\mu)}{\pi} \right)^{i+2}, \quad (1)$$

where μ stands for the energy scale, α_s and the perturbatively calculable $\{\beta_i\}$ -coefficients are generally scheme-dependent, whose first several terms under the $\overline{\text{MS}}$ -scheme and V-scheme are put in the Appendix. And to achieve a reliable and precise pQCD prediction, it is also imperative to find an appropriate method for determining the correct effective value of α_s (and hence the corresponding effective scale) of the considered process.

The Bjorken sum rule (BSR) defines an quantity $\Gamma_1^{p-n}(Q)$ [5,6] to describe the difference between the spin structure functions of the proton and neutron, $g_1^{p,n}(x)$, with the Bjorken scaling variable x and the energy scale Q . The measurable physical observable $\Gamma_1^{p-n}(Q)$ provides a platform for extracting the strong coupling constant α_s at various energy scales. Experimentally, the HERMES

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Collaboration at DESY [7], the COMPASS Collaboration at CERN [8], the E142 and E143 Collaborations at SLAC [9] and Deur *et al.* at Jefferson lab (JLab) [10–13] issued the data for $\Gamma_1^{p-n}(Q)$ within the Q^2 -range of 0.05 GeV² to 10 GeV². Recently, Deur *et al.* incorporate new data within the range of $Q^2 \in [0.02, 4.75]$ GeV² [14], thereby facilitating the extraction of α_s with higher precision.

Theoretically, one defines an effective coupling $\alpha_s^{g_1}(Q)$ for the BSR $\Gamma_1^{p-n}(Q)$,

$$\begin{aligned} \Gamma_1^{p-n}(Q) &= \int_0^1 dx [g_1^p(x) - g_1^n(x)] \\ &= \left| \frac{g_A}{g_V} \right| \frac{1}{6} [1 - \alpha_s^{g_1}(Q)], \end{aligned} \quad (2)$$

where $\alpha_s^{g_1}(Q) = \alpha_s^{g_1}(Q)/\pi$, and the ratio of nucleon axial charge $|g_A/g_V| = 1.2754 \pm 0.0013$ [15]. The effective coupling $\alpha_s^{g_1}(Q)$ contains not only perturbatively calculable leading-twist terms but also the nonperturbative higher-twist terms. The nonperturbative higher-twist terms are in the forms of $1/Q^{2n}$ under the operator product expansion form [5,6]. On the one hand, when Q is large enough, the effective coupling $\alpha_s^{g_1}(Q)$ will be dominated by perturbative contributions and can be expanded as a power series over α_s . Till now, the pQCD corrections to $\alpha_s^{g_1}(Q)$ in the massless limit have been calculated up to $\mathcal{O}(\alpha_s^4)$ level under the conventional $\overline{\text{MS}}$ -scheme [16–18], and the singlet contributions to BSR have been given up to $\mathcal{O}(\alpha_s^4)$ -level in Refs. [19,20]. Beyond the massless limit, the heavy quark mass (m_c and m_b) corrections to $\alpha_s^{g_1}(Q)$ have been calculated up to $\mathcal{O}(\alpha_s^2)$ -level [21]. On the other hand, it has been observed that the higher-twist terms will have sizable or even dominant contributions in middle and low Q -ranges. References [12,13,22–24] extracted the higher-twist corrections by comparing the JLab data with the leading-twist term of BSR.

In low-energy region, one of the successful nonperturbative models is the light-front holographic QCD (LFHQCD) [25], which is achieved by using an approximate conformal theory describing QCD. The LFHQCD model is a constructed by using both the AdS/CFT duality [26–28] and the light-front quantization [29,30]. It incorporates a Gaussian $e^{+\kappa^2 z^2}$ -term to modify the action of the 5-dimensional AdS, where the mass scale κ serves as a confinement scale as well. The LFHQCD suggests the analytic form of the strong coupling as $\alpha_s^{\text{LFHQCD}}(Q) \propto e^{-Q^2/4\kappa^2}$ [25], which parametrizes all the nonperturbative corrections, including the higher-twist contributions. In another case, the method of Dyson-Schwinger equation (DSE) [31] defines a process-independent (PI) effective charge $\hat{\alpha}_{\text{PI}}(Q)$ [32,33], which is derived from a gluon two-point function $D_{\mu\nu}^{\text{PB}}(Q, \zeta)$ under the pinch technique [34–37] and background field method [37,38]. After taking the solutions of DSE and lattice results, the PI effective

charge α_{PI} has a freezing value as $\hat{\alpha}_{\text{PI}}(0) = (0.97 \pm 0.04)\pi$. Recent reviews offer a comparative analysis of the two distinct effective couplings, $\alpha_s^{g_1, \text{LFHQCD}}(Q)$ and $\hat{\alpha}_{\text{PI}}(Q)$, within the low-energy region [39,40]. The finite predictions for effective coupling at a low Q -range under two different theories indicate that the interaction strengths may not significantly increase at certain energy scales under specific conditions, nor completely vanish but instead remain at a stable, nonzero level.

In high-energy region, based on the pQCD theory, the effective coupling $\alpha_s^{g_1}(Q)$ at leading-twist can be written as an expansion of $\alpha_s(\mu_r) = \alpha_s(\mu_r)/\pi$ as follows,

$$\alpha_s^{g_1}(Q) = \sum_{i=1}^n r_i^\delta(Q, \mu_r) \alpha_s^{\delta, i}(\mu_r), \quad (3)$$

where the perturbative coefficients r_i^δ are calculated under a chosen renormalization scheme δ with the renormalization scale μ_r . During the renormalization procedure, one chooses a typical renormalization scheme, e.g., $\overline{\text{MS}}$ -scheme, and introduces the renormalization scale μ_r to cancel the divergent large logarithms in the loop momentum integrations. As a guiding principle, a valid perturbative prediction for any physical observable must be independent of the initial choices of renormalization scheme and scale; this is the central property of renormalization group invariance (RGI) [41–44]. After applying the standard regularization and renormalization procedures, one will obtain a finite pQCD prediction free from UV divergences. Nevertheless, high-loop calculations are intricate, and one always obtains a fixed-order series. In this case, the renormalization scale μ_r is usually taken as the typical momentum Q of the process and varying it within $[Q/2, 2Q]$ to estimate its uncertainty. This simple treatment results in an unphysical uncertainty, i.e., the conventional renormalization scale uncertainty. It is however unnecessary and reduces the precision of the pQCD prediction, and directly violates the RGI. In high-energy region, the relatively small α_s effectively mitigates the influence of this additional uncertainty, thereby preserving the predictive accuracy of the series. However, the substantial difference in α_s values between the intermediate and low-energy domains inevitably reduces the matching precision of the smooth transition of $\alpha_s^{g_1}(Q)$ from small to large scales, making the matched parameters show sizable scale uncertainty [23,45].

The occurrence of scale uncertainty for the initial fixed-order series arises from the mismatching between the perturbative coefficients and the values of α_s . Based on the RGI, the principle of maximum conformality approach (PMC) [46–50] has been introduced to correctly determine the α_s -value and eliminate the μ_r -dependence with the help of RGE. That is, the PMC determines the magnitude of α_s by systematically using the nonconformal $\{\beta_i\}$ -terms of the pQCD series, which then turns into a scheme-invariant conformal series and eliminates the

μ_r -dependence simultaneously. It has been noted that the perturbative expansion of asymptotic series in powers of the strong coupling α_s is linked to the factorially divergent renormalon terms ($n!\beta_0^n\alpha_s^n$ [51–53] using the approximation of $\beta_{i(\neq 0)} \sim \beta_0^{i+1}$). At high enough orders (e.g., $n \sim 1/a_s$ [53]), the large magnitude of the renormalon terms may exceed the α_s -power suppression, leading to divergent behavior at higher-orders. This results in a small radius of convergence for the weak-coupling pQCD series and is called the asymptotic series problem [54]. The PMC series eliminates the RGE-involved renormalon terms,¹ achieve a larger radius of convergence and a more convergent series, and then can soften the asymptotic series problem to a certain degree. It has been proved that the PMC predictions are independent of the renormalization scheme and scale [55–57]. The PMC series can be applied to give a reasonable estimation of the magnitude of one-order higher uncalculated higher-order terms [58].

The standard PMC procedures suggested in Refs. [46–50] is a multiscale setting approach, simply called as the PMCM approach, which uses the RGE and fixes the effective couplings for each order by absorbing different types of $\{\beta_i\}$ -terms into the corresponding α_s via an order-by-order method. Since the same type of $\{\beta_i\}$ -term appears in different orders, the corresponding PMCM scales at each order exhibit a perturbative nature. So even though the PMCM removes conventional scheme-and-scale ambiguities, apparently, the uncalculated/unknown higher-order terms will cause two kinds of residual scale dependence [59], e.g., the last perturbative terms of all the PMCM scales are unknown, which is called *the first kind of residual scale dependence*; and since there is no $\{\beta_i\}$ -terms to fix the PMCM scale for the highest-order terms of the pQCD series, its magnitude cannot be strictly fixed, which is called *the second kind of residual scale dependence*. In large Q -range, such residual scale dependence would be suppressed by small magnitude of α_s and the perturbative nature of the PMC scales. But those two residual scale dependences could be sizable in low and intermediate Q -range.

To achieve a smooth transition for the effective coupling $\alpha_s^{q_1}(Q)$ within whole energy region, one needs to fix a transition scale Q_0 [60–62]. It shows the transition between perturbative and nonperturbative domain, whose magnitude is fixed by requiring $\alpha_s(Q_0)$ from either side to be exactly the same and their derivatives from either side also to be exactly the same. Reference [63] has applied the PMCM approach to compute the effective coupling and Q_0 . It has been found that the PMC series converges much faster than

the conventional $\overline{\text{MS}}$ pQCD series, which results in a significantly smaller uncertainty for the PMC prediction. Thus the PMC provides a determination of $\alpha_s^{q_1}(Q)$ compatible with the data and the conventional pQCD calculation, but without scheme-dependence and with significantly improved precision. However a “self-consistency problem” has also been observed, i.e., the domain of applicability of the nonperturbative LFHQCD and the domain of applicability of the perturbative PMC predictions do not overlap if the $\overline{\text{MS}}$ -scheme is used as the auxiliary scheme, which is caused by the fact that the PMC scales at certain orders in the $\overline{\text{MS}}$ -scheme are in some cases smaller than the transition scale Q_0 .

The PMC single-scale approach (PMCs) [64,65] and the PMC infinite-order scale-setting approach (PMC_∞) [66] have been suggested to suppress the residual scale dependence. The PMCs method uses a single PMC scale Q_* to absorb all $\{\beta_i\}$ -terms of the whole series. By using the PMCs, *the second kind of residual scale dependence* is removed, the single effective PMC scale displays stability and convergence with increasing order in pQCD. Though the PMC prediction is scheme independent, a proper choice of scheme could have some subtle differences. Using the PMCs to deal with the BSR, Ref. [45] shows that transforming the series from the $\overline{\text{MS}}$ -scheme into the physical V -scheme [67–71], avoids the above-mentioned self-consistency problem. However due to perturbative nature of the PMCs scale and its perturbative series not being convergent enough, *the first kind of residual scale dependence* arising from the unknown higher-order terms still exerts a significant influence on the theoretical forecasting of PMCs.

The PMC_∞ approach determines the coupling constant $\alpha_s(\mu_i)$ at each order by using the intrinsic conformality (iCF) of the series, which sets the scales by requiring that all the scale-dependent $\{\beta_i\}$ -terms at each order vanish separately. This provides the required conformal coefficients via an order-by-order method. The PMC_∞ is also a multiscale-setting approach, but different from the PMCs, the $\alpha_s(\mu_i)$ -values at low orders are determined after identifying all β_0 -terms at each order, then the newly fixed PMC scales at each order are no-longer in perturbative form, thus it will not have the *first kind of residual scale dependence*. While, like the early PMCM, the last term of the resultant PMC_∞ series lacks enough information to determine its exact α_s value, i.e., it still has *the second kind of residual scale dependence*. For a good convergent series of the pQCD approximant, *the second kind of residual scale dependence* could be negligible when enough higher-order terms are known, thus improving the precision of the pQCD prediction.

A detailed comparison of various PMC scale-setting approaches can be found in Ref. [72], which shows that different scale-setting approaches may have its own advantages and one may choose a proper one to achieve a even more precise fixed-order prediction after removing

¹In general cases, there may be IR renormalon terms in the series that need to be resummed or be reabsorbed into the nonperturbative parameters, and the remaining conformal coefficients may still be large for some processes, so the use of PMC to solve the asymptotic series problem need further studies.

conventional scheme-and-scale ambiguities. In this paper, we will adopt the PMC_∞ method to eliminate such residual scale dependence with the attempt of further improving the precision of theoretical prediction.

The remaining parts of the paper are organized as follows. In Sec. II, we present the main prescriptions for calculating $a_s^{g_1}(Q)$ via its perturbative relations to the V -scheme strong coupling by using both the PMC_∞ and the PMCs procedures. In Sec. III, we show the extracted values of $\alpha_s(M_z)$ through the effective coupling $a_s^{g_1}(Q)$, i.e., by meticulously comparing the pQCD predictions with either LFHQCD results or the experimental data. In Sec. IV, we summarize the results.

II. CALCULATION METHOD

Taking the physical V -scheme strong coupling [defined or calculated in Refs. [69–71]] as the expansion basis, the effective coupling defined in Eq. (3) up to $n = 4$ can be expressed as

$$a_s^{g_1}(Q) = \sum_{i=1}^4 r_i^V(Q, \mu_r) a_s^{V,i}(\mu_r), \quad (4)$$

where $a_s^V(\mu_r) = \alpha_s^V(\mu_r)/\pi$, and the perturbative coefficients r_i^V are expressed in terms of the $\{\beta_i^V\}$ -series as

$$r_1^V(Q, \mu_r) = r_{1,0}^V, \quad (5)$$

$$r_2^V(Q, \mu_r) = r_{2,0}^V + \beta_0^V r_{2,1}^V, \quad (6)$$

$$r_3^V(Q, \mu_r) = r_{3,0}^V + \beta_1^V r_{3,1}^V + 2\beta_0^V r_{3,1}^V + \beta_0^{V,2} r_{3,2}^V, \quad (7)$$

$$\begin{aligned} r_4^V(Q, \mu_r) = & r_{4,0}^V + \beta_2^V r_{2,1}^V + 2\beta_1^V r_{3,1}^V + \frac{5}{2}\beta_0^V \beta_1^V r_{3,2}^V \\ & + 3\beta_0^V r_{4,1}^V + 3\beta_0^{V,2} r_{4,2}^V + \beta_0^{V,3} r_{4,3}^V, \end{aligned} \quad (8)$$

where the conformal coefficients $r_{i,0}^V$ are μ_r -independent, and the general formulation of coefficients $r_{i,j \neq 0}^V$ is

$$r_{i,j}^V = \sum_{k=0}^j C_j^k \ln^k(\mu_r^2/Q^2) \hat{r}_{i-k,j-k}^V, \quad (9)$$

with $C_j^k = j!/k!(j-k)!$ and $\hat{r}_{i,j}^V = r_{i,j}^V|_{\mu_r=Q}$. Direct calculations of the coefficients $\hat{r}_{i,j}^V$ up to four-loop level are currently unavailable. However, these coefficients $\hat{r}_{i,j}^V$ can be derived by using the calculated $\overline{\text{MS}}$ -scheme ones [16–21] with the help of the relation between the $\overline{\text{MS}}$ -scheme strong coupling and the V -scheme strong coupling, which has also been calculated up to four-loop level [73,74]. For convenience, we put the transition procedures between those two schemes and the V -scheme coefficients $r_{i,j}^V$ in the Appendix.

Based on the iCF assumption of standard PMC_∞ method, the perturbative expansion of effective coupling $a_s^{g_1}(Q)$ exhibits a particular structure, which collects all the terms with the same conformal coefficients into the same conformal subset [66]. For the $a_s^{g_1}(Q)$ expansion known up to $\mathcal{O}(a_s^{V,4})$, there are four different subsets,

$$a_s^{g_1}(Q) = \sum_{i=1}^4 a_{s,i}^{g_1}(Q). \quad (10)$$

Each subset $a_{s,i}^{g_1}(Q)$ with $i = (1, \dots, 4)$ collects together the same category of nonconformal terms and ensures the scheme independence of each subset via the commensurate scale relations among different orders [75]. In the iCF forms, the effective coupling $a_s^{g_1}(Q)$ is not only renormalization scale invariant for the entire series but also exhibits the ordered scale invariance, where each subset is renormalization scale invariant under a global intrinsic scale μ_i^V . To be specific, the required four subsets have the following expression

$$\begin{aligned} a_{s,1}^{g_1}(Q) = & r_{1,\text{IC}}^V \left\{ a_s^{V,1}(\mu_r) + \beta_0^V \ln \frac{\mu_r^2}{\mu_1^{V,2}} a_s^{V,2}(\mu_r) + \left(\beta_1^V + \beta_0^{V,2} \ln \frac{\mu_r^2}{\mu_1^{V,2}} \right) \ln \frac{\mu_r^2}{\mu_1^{V,2}} a_s^{V,3}(\mu_r) \right. \\ & \left. + \left[\beta_2^V + \frac{5}{2}\beta_0^V \beta_1^V \ln \frac{\mu_r^2}{\mu_1^{V,2}} + \beta_0^{V,3} \left(\ln \frac{\mu_r^2}{\mu_1^{V,2}} \right)^2 \right] \ln \frac{\mu_r^2}{\mu_1^{V,2}} a_s^{V,4}(\mu_r) \right\}, \end{aligned} \quad (11)$$

$$a_{s,2}^{g_1}(Q) = r_{2,\text{IC}}^V \left[a_s^{V,2}(\mu_r) + 2\beta_0^V \ln \frac{\mu_r^2}{\mu_2^{V,2}} a_s^{V,3}(\mu_r) + \left(2\beta_1^V + 3\beta_0^{V,2} \ln \frac{\mu_r^2}{\mu_2^{V,2}} \right) \ln \frac{\mu_r^2}{\mu_2^{V,2}} a_s^{V,4}(\mu_r) \right], \quad (12)$$

$$a_{s,3}^{g_1}(Q) = r_{3,\text{IC}}^V \left[a_s^{V,3}(\mu_r) + 3\beta_0^V \ln \frac{\mu_r^2}{\mu_3^{V,2}} a_s^{V,4}(\mu_r) \right], \quad (13)$$

$$a_{s,4}^{g_1}(Q) = r_{4,\text{IC}}^V a_s^{V,4}(\mu_r), \quad (14)$$

where $r_{i,\text{IC}}^V$ are the renormalization scale-independent iCF coefficients and μ_i^V are the PMC scales. The coefficients $r_{i,\text{IC}}^V$ can be identified from the known $r_i^V(Q, \mu_r)$ by using the standard PMC_∞ procedure. For examples, at the leading order (LO) which is already conformal, $r_{1,\text{IC}}^V(Q) = r_1^V(Q, \mu_r)$; and at the next-to leading order (NLO), $r_{2,\text{IC}}^V(Q) = r_2^V(Q, \mu_r)|_{n_f=33/2}$, where fixing the quark flavor number $n_f = 33/2$ makes the β_0^V -term vanish. For higher-order terms,

$$r_{3,\text{IC}}^V(Q) = r_3^V(Q, \mu_r)|_{n_f=33/2} - r_{1,\text{IC}}^V \bar{\beta}_1^V \ln \frac{\mu_r^2}{\mu_1^{V,2}} \Big|_{n_f=33/2}, \quad (15)$$

$$r_{4,\text{IC}}^V(Q) = r_4^V(Q, \mu_r)|_{n_f=33/2} - r_{1,\text{IC}}^V \bar{\beta}_2^V \ln \frac{\mu_r^2}{\mu_1^{V,2}} \Big|_{n_f=33/2} - 2r_{2,\text{IC}}^V \bar{\beta}_1^V \ln \frac{\mu_r^2}{\mu_2^{V,2}} \Big|_{n_f=33/2}, \quad (16)$$

where $\bar{\beta}_1^V = \beta_1^V(n_f = 33/2)$ and $\bar{\beta}_2^V = \beta_2^V(n_f = 33/2)$. It has been proved that the conformal coefficients $r_{i,\text{IC}}^V \equiv r_{i,0}^V$ [72]. The PMC scales μ_i^V are determined by requiring each subsets satisfying the condition of RGI, and its first three ones are

$$\ln \frac{\mu_r^2}{\mu_1^{V,2}} = \frac{r_2^V(Q, \mu_r) - r_{2,\text{IC}}^V}{r_{1,\text{IC}}^V \beta_0^V}, \quad (17)$$

$$\ln \frac{\mu_r^2}{\mu_2^{V,2}} = \frac{r_3^V(Q, \mu_r) - r_{3,\text{IC}}^V - r_{1,\text{IC}}^V [\beta_1^V + \beta_0^{V,2} \ln(\mu_r^2/\mu_1^{V,2})] \ln(\mu_r^2/\mu_1^{V,2})}{2r_{2,\text{IC}}^V \beta_0^V}, \quad (18)$$

$$\begin{aligned} \ln \frac{\mu_r^2}{\mu_3^{V,2}} = & \frac{r_4^V(Q, \mu_r) - r_{4,\text{IC}}^V}{3r_{3,\text{IC}}^V \beta_0^V} - \frac{r_{2,\text{IC}}^V [2\beta_1^V + 3\beta_0^{V,2} \ln(\mu_r^2/\mu_2^{V,2})] \ln(\mu_r^2/\mu_1^{V,2})}{3r_{3,\text{IC}}^V \beta_0^V} \\ & - \frac{r_{1,\text{IC}}^V [\beta_2^V + \frac{5}{2}\beta_0^V \beta_1^V \ln(\mu_r^2/\mu_1^{V,2}) + \beta_0^{V,3} \ln^2(\mu_r^2/\mu_1^{V,2})] \ln(\mu_r^2/\mu_1^{V,2})}{3r_{3,\text{IC}}^V \beta_0^V}. \end{aligned} \quad (19)$$

By incorporating the relations between $r_i^V(Q, \mu_r)$ and $\hat{r}_{i,j}^V$ into above expressions for the PMC scales μ_i^V , the derived first three ones are

$$\ln \frac{Q^2}{\mu_1^{V,2}} = \frac{\hat{r}_{2,1}^V}{\hat{r}_{1,0}^V}, \quad (20)$$

$$\ln \frac{Q^2}{\mu_2^{V,2}} = \frac{\hat{r}_{3,1}^V}{\hat{r}_{2,0}^V} + \beta_0^V \left(\frac{\hat{r}_{3,2}^V}{2\hat{r}_{2,0}^V} - \frac{\hat{r}_{2,1}^V}{2\hat{r}_{1,0}^V \hat{r}_{2,0}^V} \right), \quad (21)$$

$$\begin{aligned} \ln \frac{Q^2}{\mu_3^{V,2}} = & \frac{\hat{r}_{4,1}^V}{\hat{r}_{3,0}^V} + \beta_0^V \left(\frac{\hat{r}_{4,2}^V}{\hat{r}_{3,0}^V} - \frac{\hat{r}_{3,1}^V}{\hat{r}_{2,0}^V \hat{r}_{3,0}^V} \right) \\ & + \beta_1^V \left(\frac{\hat{r}_{3,2}^V}{2\hat{r}_{3,0}^V} - \frac{\hat{r}_{2,1}^V}{2\hat{r}_{1,0}^V \hat{r}_{3,0}^V} \right) \\ & + \frac{\beta_0^{V,2}}{\hat{r}_{3,0}^V} \left(\frac{\hat{r}_{4,3}^V}{3} + \frac{\hat{r}_{3,1}^V \hat{r}_{2,1}^V}{\hat{r}_{1,0}^V \hat{r}_{2,0}^V} - \frac{\hat{r}_{3,1}^V \hat{r}_{3,2}^V}{\hat{r}_{2,0}^V} - \frac{\hat{r}_{2,1}^V}{3\hat{r}_{1,0}^V} \right) \\ & + \frac{\beta_0^{V,3}}{\hat{r}_{2,0}^V \hat{r}_{3,0}^V} \left(\frac{\hat{r}_{3,2}^V \hat{r}_{2,1}^V}{2\hat{r}_{1,0}^V} - \frac{\hat{r}_{3,2}^V}{4} - \frac{\hat{r}_{2,1}^V}{4\hat{r}_{1,0}^V} \right). \end{aligned} \quad (22)$$

After fixing the PMC scales μ_i^V for each conformal subsets, then we can transform the initial series as the following series

$$a_s^{g_1}(Q)|_{\text{PMC}_\infty} = \sum_{i=1}^4 r_{i,\text{IC}}^V(Q) a_s^{V,i}(\mu_i^V). \quad (23)$$

It is a conformal series, which incorporates accurate coupling values $a_s^{V,i}(\mu_i^V)$ and exhibits scale invariance under the condition that all the RGE-involved $\{\beta_i\}$ -terms vanish. This ensures that the series exhibits renormalization scale invariance and is free from the *first kind of residual scale dependence*. Moreover, the PMC_∞ approach reveals the intrinsic conformality of the QCD perturbation series and achieves the ordered scale invariance, i.e., retaining scale invariance at any order. However, given the absence of higher-order information, the PMC_∞ last scale μ_4^V remains uncertain, it still has the *second kind of residual scale dependence*. Fortunately, this uncertainty associated with the N^4LO -term is suppressed by a high power of $\alpha_s^{V,i}$ and is further mitigated by the excellent convergence properties of the series (23). If not specially stated, we will implicitly take $\mu_4^V = \mu_3^V$ for the subsequent discussions.

To compare the effects of the PMCs and the PMC_∞ methods in eliminating the residual scale dependence due to unknown higher-order terms, we also apply the PMCs method to deal with the pQCD corrections to $a_s^{g_1}(Q)$. Using the standard PMCs procedures, the resultant PMC series is

$$a_s^{g_1}(Q)|_{\text{PMCs}} = \sum_{i=1}^4 r_{i,0}^V(Q) a_s^{V,i}(Q_*^V), \quad (24)$$

where by using the given $N^3\text{LO}$ -level pQCD series, the single-scale Q_*^V can be fixed up to $\mathcal{O}(\alpha_s^3)$ -level, i.e., the next-to-next-to-leading log ($N^2\text{LL}$) accuracy,

$$\ln \frac{Q_*^{V,2}}{Q^2} = S_0 + S_1 a_s^{V,1}(Q_*^V) + S_2 a_s^{V,2}(Q_*^V), \quad (25)$$

whose expansion coefficients S_i are

$$S_0 = -\frac{\hat{r}_{2,1}^V}{\hat{r}_{1,0}^V}, \quad (26)$$

$$S_1 = \frac{2(\hat{r}_{2,0}^V \hat{r}_{2,1}^V - \hat{r}_{1,0}^V \hat{r}_{3,1}^V)}{\hat{r}_{1,0}^{V,2}} + \frac{(\hat{r}_{2,1}^{V,2} - \hat{r}_{1,0}^V \hat{r}_{3,2}^V)}{\hat{r}_{1,0}^{V,2}} \beta_0^V, \quad (27)$$

$$S_2 = -\frac{(2\hat{r}_{2,1}^{V,3} - 3\hat{r}_{1,0}^V \hat{r}_{3,2}^V \hat{r}_{2,1}^V + \hat{r}_{1,0}^{V,2} \hat{r}_{4,3}^V)}{\hat{r}_{1,0}^{V,3}} \beta_0^{V,2} - \frac{\hat{r}_{2,0}^V (5\hat{r}_{2,1}^{V,2} - 2\hat{r}_{1,0}^V \hat{r}_{3,2}^V) + 3\hat{r}_{1,0}^V (\hat{r}_{1,0}^V \hat{r}_{4,2}^V - 2\hat{r}_{2,1}^V \hat{r}_{3,1}^V)}{\hat{r}_{1,0}^{V,3}} \beta_0^V \\ + \frac{3(\hat{r}_{2,1}^{V,2} - \hat{r}_{1,0}^V \hat{r}_{3,2}^V)}{2\hat{r}_{1,0}^{V,2}} \beta_1^V + \frac{4\hat{r}_{2,0}^V (\hat{r}_{1,0}^V \hat{r}_{3,1}^V - \hat{r}_{2,0}^V \hat{r}_{2,1}^V)}{\hat{r}_{1,0}^{V,3}} + \frac{3\hat{r}_{1,0}^V (\hat{r}_{2,1}^V \hat{r}_{3,0}^V - \hat{r}_{1,0}^V \hat{r}_{4,1}^V)}{\hat{r}_{1,0}^{V,3}}. \quad (28)$$

The PMCs series (24) do not have the *second kind of residual scale dependence*, and because the single-scale Q_*^V is of perturbative nature, it still has the *first kind of residual scale dependence*. We will use the Padé approximation approach (PAA) [76] to estimate the magnitude of the $N^3\text{LL}$ -term, i.e., the $\pm S_3 a_s^{V,3}(Q_*^V)$ -term, which leads to ΔQ_*^V . Generally, such kind of residual scale dependence will be reduced by double suppression of the exponential-suppression and α_s -suppression. However, if the series of the right-hand-side of Eq. (25) does not converge fast enough, because the error of Q_*^V works for the LO-terms of the series, the *second kind of residual scale dependence* could be sizable.

III. NUMERICAL RESULTS AND DISCUSSIONS

Using the RGE, the solution for the running behavior of $\alpha_s(\mu_r)$ can be expressed as an expansion over inverse powers of the logarithm $l_\Lambda = \ln \mu_r^2 / \Lambda^2$. The universal α_s running behavior up to four-loop level is [46,77]

$$a_s(\mu_r) = \frac{1}{\beta_0 l_\Lambda} \left\{ 1 - \frac{\beta_1}{\beta_0^2} \frac{\ln l_\Lambda}{l_\Lambda} + \frac{1}{\beta_0^2 l_\Lambda^2} \right. \\ \times \left[\frac{\beta_1^2}{\beta_0^2} (\ln^2 l_\Lambda - \ln l_\Lambda - 1) + \frac{\beta_2}{\beta_0} \right] \\ + \frac{1}{\beta_0^3 l_\Lambda^3} \left[\frac{\beta_1^3}{\beta_0^3} \left(-\ln^3 l_\Lambda + \frac{5}{2} \ln^2 l_\Lambda + 2 \ln l_\Lambda - \frac{1}{2} \right) \right. \\ \left. \left. - 3 \frac{\beta_1 \beta_2}{\beta_0^2} \ln l_\Lambda + \frac{\beta_3}{2\beta_0} \right] \right\}, \quad (29)$$

where the asymptotic scale Λ and $\{\beta_{i(\geq 2)}\}$ -functions are scheme-dependent. The Λ plays a pivotal role in defining the boundary for the truncation of infinite integrations encountered in pQCD calculations. It can be associated

with the typical hadron size and its value is not predicted by the QCD theory but must be extracted from a measurement of α_s at a given reference scale. Typically, the value of Λ is determined by fixing a α_s -value at a standard scale, such as the mass of the Z boson (M_Z), within a specified renormalization scheme.

To do numerical calculation, we adopt the PDG averaged value $\alpha_s(M_Z) = 0.1180 \pm 0.0009$ [15] to determine α_s running behavior, which is shown in Fig. 1. At the mass scale $M_Z = 91.1876$ GeV and $n_f = 5$ quark flavors, we obtain $\Lambda_{\overline{\text{MS}}}^{n_f=5} = 208 \pm 10$ MeV under the $\overline{\text{MS}}$ -scheme, and $\Lambda_V^{n_f=5} = 285 \pm 14$ MeV under the V-scheme.

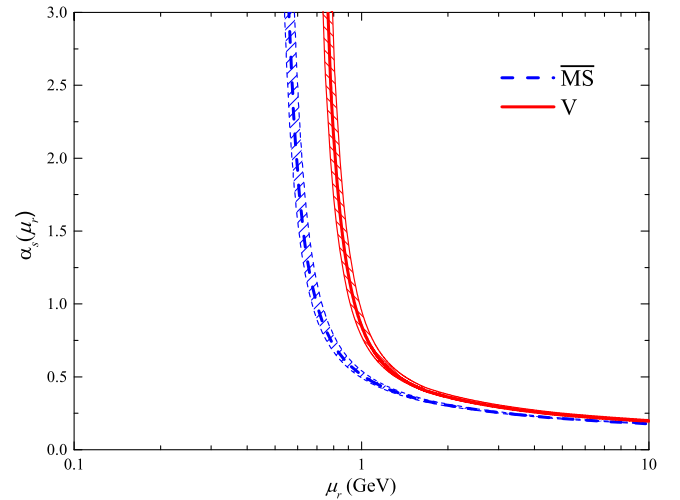


FIG. 1. The four-loop $\alpha_s(\mu_r)$ running behavior under $\overline{\text{MS}}$ -scheme and V-scheme. The blue band shows the running behavior with $\Lambda_{\overline{\text{MS}}}^{n_f=5} = 208 \pm 10$ MeV under the $\overline{\text{MS}}$ -scheme and the red band shows the running behavior of α_s with $\Lambda_V^{n_f=5} = 285 \pm 14$ MeV under the V-scheme.

TABLE I. Results for effective coupling $a_s^{g_1}(m_c)$ at each order (LO, NLO, N²LO and N³LO) under the conventional (Conv.), the PMC_∞ and PMCs methods, respectively. The conventional predictions are highly scale dependent, whose central values are for $a_s^{g_1}(m_c)|_{\text{Conv}}$ with $\mu_r = m_c$ and the uncertainties are for $\mu_r \in [1, 4]$ GeV.

	LO	NLO	N ² LO	N ³ LO	Total
Conv.	$0.129_{-0.046}^{+0.141}$	$0.037_{+0.003}^{-0.042}$	$-0.009_{+0.029}^{-0.258}$	$0.007_{+0.006}^{+0.261}$	$0.164_{-0.008}^{+0.102}$
PMC _∞	0.108	0.026	0.024	0.006	0.164
PMCs	0.108	0.041	0.028	0.007	0.184

A. PQCD corrections of $\alpha_s^{g_1}$ up to N³LO-level

Under $\overline{\text{MS}}$ -scheme, Refs. [16–20] have provided the analytic perturbation corrections of the effective coupling $a_s^i(Q)$ up to four-loop. In our present calculations, we have completed the transformation of Eq. (4) into the V -scheme and the heavy quark mass effects are also considered in the perturbation coefficient r_2 [21], with $m_c = 1.67 \pm 0.07$ GeV and $m_b = 4.78 \pm 0.06$ GeV [15] are taken. At the typical momentum $Q = m_c$, the effective coupling $a_s^{g_1}(m_c)$ takes the form

$$a_s^{g_1}(m_c)|_{\text{Conv}} = a_s^{V,1}(m_c) + 2.238a_s^{V,2}(m_c) - 4.417 \\ \times a_s^{V,3}(m_c) + 25.651a_s^{V,4}(m_c), \quad (30)$$

in which μ_r is chosen to be m_c . The results for different choices of μ_r can be derived with the help of RGE. Considering the error by taking $\mu_r \in [1, 4]$ GeV, contributions of the LO-terms, the NLO-terms, the N²LO-terms and the N³LO-terms for the effective coupling $a_s^{g_1}(m_c)$ are shown in Table. I. Table. I gives the results for conventional (Conv.), the PMCs and PMC_∞ methods, respectively. The value for $a_s^{g_1}(m_c)$ under the conventional scale setting method is $a_s^{g_1}(m_c)|_{\text{Conv}} = 0.164_{-0.008}^{+0.102}$, which shows the scale error $\sim (+62\%_{-5\%})$.

After applying the PMC_∞ method, the corresponding iCF perturbative series is

$$a_s^{g_1}(m_c)|_{\text{PMC}_\infty} = a_s^{V,1}(\mu_1^V) + 3.550a_s^{V,2}(\mu_2^V) + 22.073 \\ \times a_s^{V,3}(\mu_3^V) + 55.702a_s^{V,4}(\mu_4^V), \quad (31)$$

where the PMC scales μ_i^V are functions of the chosen momentum m_c and independent of the renormalization scale μ_r . From the Eqs. (20)–(22), the three PMC_∞ scales of Eq. (31) are $\mu_1^V(m_c) = 2.236$ GeV, $\mu_2^V(m_c) = 3.754$ GeV, and $\mu_3^V(m_c) = 2.425$ GeV, respectively. Additionally, we also give the PMCs prediction with the single-scale Q_*^V for comparison.

$$a_s^{g_1}(m_c)|_{\text{PMCs}} = a_s^{V,1}(Q_*^V) + 3.550a_s^{V,2}(Q_*^V) + 22.073 \\ \times a_s^{V,3}(Q_*^V) + 55.702a_s^{V,4}(Q_*^V), \quad (32)$$

where $Q_*^V(m_c) = 2.250$ GeV.

Table. I shows the relative magnitudes between the LO-term, the NLO-term, the N²LO-term and the N³LO-term for $a_s^{g_1}(m_c)|_{\text{PMC}_\infty}$ in Eq. (31) and $a_s^{g_1}(m_c)|_{\text{PMCs}}$ in Eq. (32). It indicates that the PMC predictions are independent of the choice of μ_r . The ratios of the magnitudes of NⁱLO-terms over that of the LO-terms are introduced to show the convergence of pQCD series under various scale-setting approaches, e.g., $N^i\text{LO}(\mu_r)/\text{LO}(\mu_r)$ for conventional series, $N^i\text{LO}(\mu_i^V)/\text{LO}(\mu_1^V)$ for PMC_∞ series, and $N^i\text{LO}(Q_*)/\text{LO}(Q_*)$ for PMCs series. More explicitly, for the conventional approach, the ratios are

$$1 : 0.287_{+0.195}^{-0.305} : -0.070_{+0.310}^{-0.919} : 0.054_{+0.102}^{+0.938}$$

with $\mu_r = m_c$ and the uncertainties are for $\mu_r \in [1, 4]$ GeV. For the PMC_∞ and PMCs approaches, we have

$$1 : 0.241 : 0.222 : 0.056, \\ 1 : 0.380 : 0.259 : 0.065,$$

respectively. The conventional series is highly scale dependent. For the conventional series with $\mu_r = 1$ GeV, the magnitude of the N²LO-terms is even larger than that of the NLO-terms. It is noted that since the magnitude of α_s in the charm energy region is ~ 0.4 , the improvement of perturbative convergence due to the elimination of the divergent nonconformal $\{\beta_i\}$ -terms by applying the PMC approaches is not as obvious as in the previous cases, e.g. the convergent behavior of PMC_∞-series starts at the N³LO-terms. However, unlike the highly scale-dependent conventional series, its scale-invariant perturbative behavior could be regarded as the intrinsic property of the pQCD series.

Figure 2 illustrates the variation of PMC scales for the $\alpha_s^{g_1}$ expansion over V -scheme strong coupling versus the momentum transfer Q . Figure 2 shows that all the PMC scales will increase with the increment of Q . And such rapid escalation of μ_i^V also explains why our presently derived PMC scales under the V -scheme can solve the previously encountered “self-consistency problem” [63] within the region $Q \gtrsim 1$ GeV.

For the PMCs procedure, the single-scale Q_*^V is determined by absorbing all types of $\{\beta_i\}$ -terms into an overall effective α_s , and the unknown higher-order $\{\beta_i\}$ -terms directly leads to the first kind of residual scale dependence.

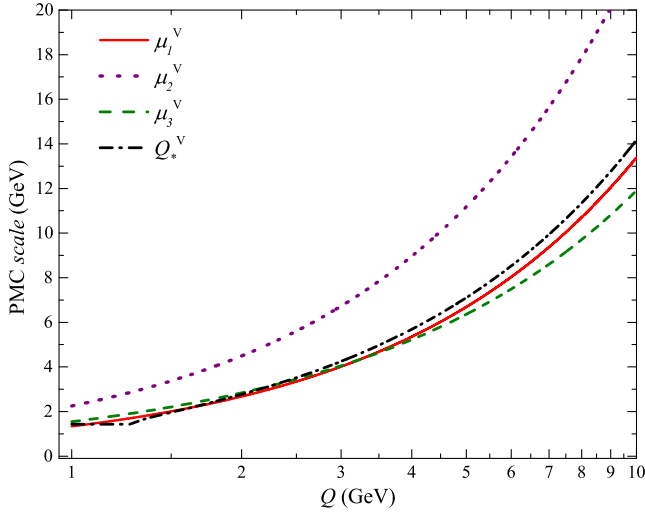


FIG. 2. The PMC scales for the $\alpha_s^{g_1}$ expansion over V -scheme strong coupling versus the momentum transfer Q , respectively. The red solid, purple dotted, and green dashed lines represent the first three PMC_∞ scales μ_i^V , respectively. The PMCs scale Q_*^V is represented by a black dot-dashed line.

Using Eq. (25), we obtain $Q_*^V(Q = m_c) = 2.250$ GeV with the coefficients $S_0 = 0.583$, $S_1 = 7.360$, and $S_2 = -67.254$. We then adopt the PAA approach to estimate the $N^3\text{LL}$ -level perturbative coefficient S_3 for $\ln(Q_*^{V,2}/Q^2)$ at $Q = m_c$, and by using the [0/1]-type generating function [78] we obtain

$$S_3^{\text{PAA}} = \frac{S_2^2}{S_1}. \quad (33)$$

By taking $\pm S_3^{\text{PAA}} \alpha_s^{V,3}(Q_*^V)$ as an estimation of the *first kind of residual scale dependence*, we obtain $Q_*^V(Q = m_c) = 2.250_{-0.169}^{+0.765}$ GeV, which leads to $a_s^{g_1}(m_c)|_{\text{PMCs}} = 0.184_{-0.038}^{+0.014}(\Delta Q_*^V)$, indicating that the residual scale dependence for PMCs is about $(-21\%_{+8\%})$. Based on the known PMCs series in Eq. (32), the [0/3]-type generating function [78] yields the perturbative coefficient $\hat{r}_{5,0}$ as

$$\hat{r}_{5,0}^{\text{PAA}} = \frac{\hat{r}_{2,0}^{V,4} - 3\hat{r}_{1,0}^V \hat{r}_{2,0}^{V,2} \hat{r}_{3,0}^V + \hat{r}_{1,0}^{V,2} \hat{r}_{3,0}^{V,2} + 2\hat{r}_{1,0}^{V,2} \hat{r}_{2,0}^V \hat{r}_{4,0}^V}{\hat{r}_{1,0}^{V,3}}. \quad (34)$$

Numerically, the [0/3]-type PAA prediction gives the conformal coefficient $\hat{r}_{5,0}^V(Q = m_c) = 207.0$, and the conventional coefficient $\hat{r}_5^V(Q = m_c) = 225.7_{-49.7}^{+2092.9}$ with uncertainties from $\mu_r \in [1, 4]$ GeV. The ECH method provides $r_5^{\overline{\text{MS}}} = 1557.4 \pm 32.8$ with errors from the preferred renormalization scheme [79].

Different from the PMCs scale Q_*^V , the PMC_∞ effective scales are fixed order-by-order and free of perturbative

nature. Its uncertainty comes from the unknown last PMC scale μ_4^V in Eq. (31), which needs a higher-order r_5^V -term to determine. Thus the PMC_∞ does not have the *first kind of residual scale dependence*, but has the *second kind of residual scale dependence*. As an estimation of the *second kind of residual scale dependence* for $Q = m_c$, we take $\mu_4^V(Q) \equiv \mu_3^V(Q)$ and vary it from Q to $2Q$ to estimate its uncertainty, which gives $a_s^{g_1}(Q = m_c)|_{\text{PMC}_\infty} = 0.164_{-0.003}^{+0.009}(\Delta \mu_4^V)$. As the decoupling mass corrections have been calculated up to $\mathcal{O}(\alpha_s^2)$ -level [21], we make a simple discussion on the charm and bottom quark mass effects on the effective coupling $a_s^{g_1}(Q)$. For that purpose, we set the momentum transfer $Q = m_{c,0}$ with $m_{c,0} = 1.67$ GeV. The results for other momentum transfers can be done via a similar way. And the uncertainties for $a_s^{g_1}(Q = m_{c,0})$ are estimated by taking $\Delta m_c = \pm 0.07$ GeV and $\Delta m_b = \pm 0.06$ GeV. We then obtain $\Delta a_s^{g_1}(m_{c,0})|_{\text{Conv}} = (+0.00082_{-0.00898})(\Delta m_c) \mp 0.00002(\Delta m_b)$, which shows that $\Delta m_{c,b}$ -errors $\sim (+0.50\%_{-5.48\%})$. Here the errors for $a_s^{g_1}(m_{c,0})|_{\text{Conv}}$ is a combination of the mass corrections and renormalization scale dependence, since μ_r is fixed to m_c for the scale-dependent conventional series. On the other hand, the PMC_∞ prediction is, $\Delta a_s^{g_1}(m_{c,0})|_{\text{PMC}_\infty} = \mp 0.00005(\Delta m_c) \mp 0.00002(\Delta m_b)$ with the $\Delta m_{c,b}$ -errors $\sim \mp 0.03\%$. And the PMCs result is $\Delta a_s^{g_1}(m_{c,0})|_{\text{PMCs}} = \mp 0.00008(\Delta m_c) \pm 0.00003(\Delta m_b)$ with $\Delta m_{c,b}$ -errors $\sim \mp 0.05\%$. Both indicate that the heavy quark mass effects are negligible for the PMC series.

Above numerical results show that the PMCs and PMC_∞ series are consistent with each other with reasonable errors, which also show that due to the convergent behavior occurs at the $N^3\text{LO}$ -level, the residual scale errors of the PMCs and PMC_∞ series are smaller than the conventional scale error of the initial pQCD series. This shows the importance of a proper scale-setting approach. Moreover, the convergent behavior of the PMC_∞ series for $a_s^{g_1}(Q)$ leads to a much smaller residual scale dependence, e.g., the *second kind of residual scale dependence* is much smaller than the *first kind of residual scale dependence* of PMCs series. In fact, even by varying $\mu_4^V(Q = m_c)$ up to $8Q$, the magnitude of $\Delta a_s^{g_1}(Q = m_c)$ will only be changed to 0.006. Thus, a precise prediction for the properties of $a_s^{g_1}(Q)$ may be achieved by using PMC_∞ series. So in the following, we will utilize the PMC_∞ approach to do the discussions.

B. The running behavior of $\alpha_s^{g_1}(Q)$ at all scales

Many low-energy models have been suggested to explain the infrared behavior of α_s , cf., the reviews [80,81]. In this subsection, we utilize the LFHQCD method for $a_s^{g_1}$ low-energy behavior and determine its running behavior at all scales in combination with the above perturbative behavior. The LFHQCD transforms the modified AdS_5 coupling into momentum space and gives

$$a_s^{g_1, \text{LFHQCD}}(Q) = e^{-Q^2/4\kappa^2}, \quad (35)$$

where the free parameter κ can be derived from the light meson and baryon spectroscopy, which gives $\kappa = 0.523 \pm 0.024$ GeV [82]. In infrared region, the LFHQCD model gives $a_s^{g_1, \text{LFHQCD}}(0) = 1$. To achieve a smooth scale-running behavior of $a_s^{g_1}(Q)$ across the whole Q -range, one can match the low-energy LFHQCD model behavior with the perturbative behavior by fixing proper values for the scales Q_0 and κ . Their values can be fixed by two criterions, e.g., (1) the value of $a_s^{g_1}(Q_0)$ calculated from both sides is exactly the same, and (2) the derivatives of $a_s^{g_1}(Q)$ at the $Q = Q_0$ point calculated from both sides is exactly the same. Here Q_0 serves as a threshold of perturbative and nonperturbative separation, for the case of $Q > Q_0$, $a_s^{g_1}(Q)$ is calculated by using Eqs. (4), (23), and (24), respectively; and for the case of $Q < Q_0$, $a_s^{g_1}(Q)$ is calculated by using Eq. (35).

Figures 3 and 4 show the matching of the low-energy LFHQCD model of $a_s^{g_1}(Q)$ with its pQCD series up to $N^3\text{LO}$ -level QCD corrections under conventional and PMC_∞ approaches, respectively, which can be adopted to fix the parameters Q_0 and κ . The available experiment data points are also presented [7–14, 83–89] in Figs. 3 and 4. In small Q -region, the pQCD series of conventional scale-setting approach is highly scale dependent. In Fig. 3, the blue solid line is for the case of $\mu_r = Q$, which leads to $\kappa = 0.517$ GeV and $Q_0 = 1.145$ GeV. The lower limit $\mu_r = Q_0/2$ reaches the nonperturbative region, so we vary μ_r within a conservative range of $[Q, 2Q]$. For each μ_r

within this range, one can achieve a smooth transition from the low to large energy region, with $\kappa \in [0.396, 0.517]$ GeV and $Q_0 \in [0.786, 1.145]$ GeV, which shows the μ_r -uncertainty $\sim [-23.4\%, 0\%]$ in κ and $\sim [-31.4\%, 0\%]$ in Q_0 . After applying the PMC_∞ approach to eliminate such scale dependence, Fig. 4 shows a more precise matching of the LFHQCD model with the PMC pQCD series can be achieved. In Fig. 4, the central red solid line represents the matching with the PMC_∞ prediction $a_s^{g_1}|_{\text{PMC}_\infty}(Q)$, and the red band shows the combined errors caused by the estimated residual scale dependence for $\Delta\mu_4^Y \in [Q, 2Q]$ and the error from $\Delta\Lambda_V^{n_f=5} = 14$ MeV. As for the two input parameters, we have $\kappa = 0.442^{+0.115}_{-0.030}$ GeV with errors $\sim (+26.0\%, -6.8\%)$, and $Q_0 = 0.903^{+0.344}_{-0.067}$ GeV with errors $\sim (+38.1\%, -7.4\%)$. Because of a faster increase of the PMC scales versus Q for the V -scheme coupling, all the PMC scales are larger than Q_0 , i.e., $\mu_1^Y(Q_0) = 1.209^{+0.460}_{-0.089}$ GeV, $\mu_2^Y(Q_0) = 2.034^{+0.771}_{-0.150}$ GeV, $\mu_3^Y(Q_0) = 1.412^{+0.464}_{-0.093}$ GeV at $Q_0 = 0.903^{+0.344}_{-0.067}$ GeV. This confirms our previous observation that by applying the PMC, the “self-consistency problem” can be avoided. A comparison of Figs. 3 and 4 indicates that the PMC approach does improve the precision of theoretical predictions for $a_s^{g_1}(Q)$. It is found that the conventional line of $a_s^{g_1}(Q)$ under the choice of $\mu_r = Q$ is within the error band of PMC prediction.

The quality parameter $\chi^2/\text{d.o.f}$ (where the symbol “d.o.f” is the short notation of the degrees of freedom) has been introduced in the literature to show the fitness of the theoretical predictions with the data. The $\chi^2/\text{d.o.f}$

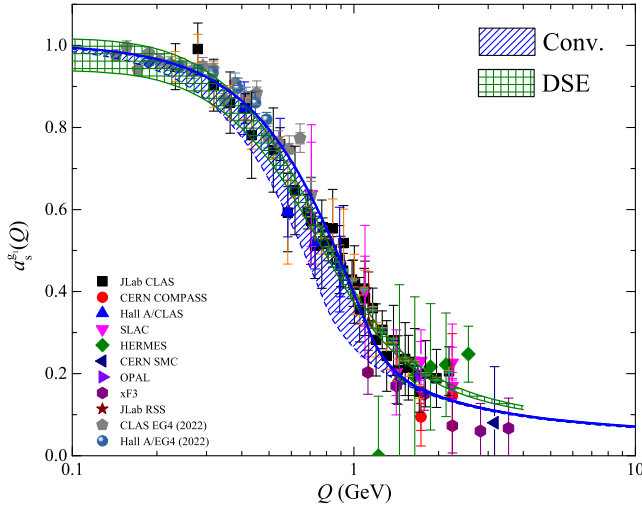


FIG. 3. The matching of the LFHQCD model of $a_s^{g_1}(Q)$ with its pQCD series up to $N^3\text{LO}$ -level QCD corrections, under conventional scale-setting method. The central blue line represents the conventional pQCD series when $\mu_r = Q$, and the blue band is obtained after considering the renormalization scale uncertainty with $\mu_r \in [Q, 2Q]$. As a comparison, the DSE prediction [32,33] is also given by the green band.

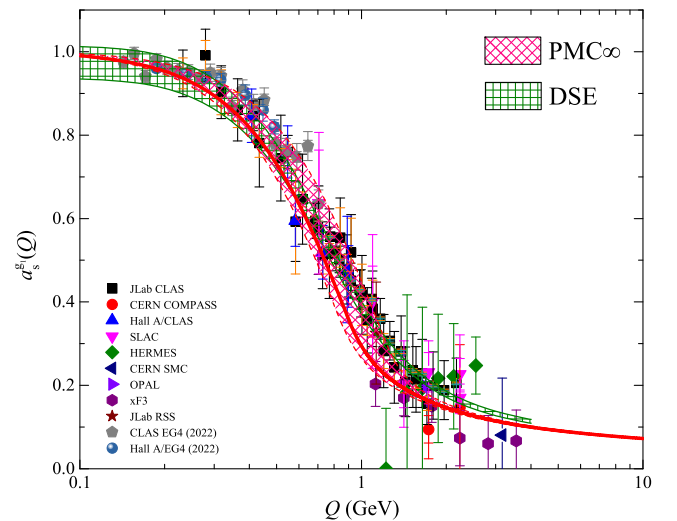


FIG. 4. The matching of the LFHQCD model of $a_s^{g_1}(Q)$ with its pQCD series up to $N^3\text{LO}$ -level QCD corrections, under PMC_∞ approach. The red band represents a combination of the residual scale dependence estimated by using $\Delta\mu_4^Y \in [Q, 2Q]$ and the error from $\Delta\Lambda_V^{n_f=5} = 14$ MeV. As a comparison, the DSE prediction [32,33] is also given by the green band.

indicates the quality of fit χ^2 over the sum of the number of experiment data point n minus the number of input parameters (for the present case, κ , Q_0 and k), which gives

$$\chi^2/\text{d.o.f} = \frac{1}{n-3} \sum_{m=1}^n \frac{[a_s^{g_1, \text{exp.}}(Q_m) - a_s^{g_1, \text{the.}}(Q_m)]^2}{\sigma_{m, \text{exp.}}^2 + \sigma_{m, \text{the.}}^2}, \quad (36)$$

where “exp.” stands for the experimental value, and each data point Q_m with the experimental errors $\sigma_{m, \text{exp.}}^2 = \sigma_{m, \text{stat}}^2 + k\sigma_{m, \text{sys}}^2$, which includes both statistical error and systematic error. The statistical errors are uncorrelated, and the correlations between the systematic error are estimated by a factor k [24,90,91], where we use $k = 0.5$ in the subsequent calculations. The symbol “the.” stands for the theoretical prediction, and $\sigma_{m, \text{the.}}^2$ is its uncertainty. The conventional theoretical uncertainty comes from $\Delta\mu_r$, and the theoretical errors for the PMC_∞ method arise from the combination of $\Delta\mu_4^V$ and $\Delta\Lambda_V^{n_f=5}$. When doing the calculations of $\chi^2/\text{d.o.f}$, we take $n = 79$ which includes the latest JLab experiment data [10–14]. The conventional prediction is highly scale dependent, and for definiteness, we take the results $a_s^{g_1}(Q)|_{\text{Conv}}$ at $\mu_r = Q$ to calculate its $\chi^2/\text{d.o.f}$, which is $\simeq 0.4$. As for the PMC_∞ prediction, we obtain $\chi^2/\text{d.o.f} \simeq 0.7$. The two small $\chi^2/\text{d.o.f}$ -values indicate good agreement with experimental data. The smaller $\chi^2/\text{d.o.f}$ -value obtained by the conventional pQCD series is mainly due to the choice of μ_r , e.g., $\chi^2/\text{d.o.f} \simeq 1.0^2$ when $\mu_r = 1.5Q$. Different from the conventional results, the PMC predictions are μ_r -independent, and the good fit of the PMC to the experimental data further indicates the two parameters (κ and Q_0) in PMC are reasonable and reliable, which corresponds to a p -value [15] $\sim 98\%$.

The defined coupling $a_s^{g_1}(Q)$ incorporates both the perturbative part and nonperturbative higher-twist contributions. Based on the OPE, the higher-twist contributions to the BSR exhibit a form of $1/Q^{2n}$. Up to $n = 4$, the higher-twist corrections to $a_s^{g_1}(Q)$ are

$$a_s^{g_1, \text{n.p.}} = -6 \left| \frac{g_V}{g_A} \right| \left(\frac{\mu_4^{p-n}}{Q^2} + \frac{\mu_6^{p-n}}{Q^4} \right).$$

The coefficient of the $\mathcal{O}(Q^{-2})$ -term satisfies [92–94]

$$\mu_4^{p-n} = \frac{M^2}{9} (a_2^{p-n} + 4d_2^{p-n} + 4f_2^{p-n}),$$

where the nucleon mass $M \approx 0.94$ GeV. The leading-twist target mass correction a_2^{p-n} and the twist-3 matrix

²When taking $\mu_r = 1.5Q$, the conventional method yields $\chi^2/\text{d.o.f} \simeq 1.0$ with $\kappa = 0.446$ GeV and $Q_0 = 0.925$ GeV, where the theoretical uncertainty arises from $\mu_r \in [Q, 2Q]$. This $\chi^2/\text{d.o.f} \simeq 1.0$ corresponds to a p -value of $\sim 43\%$. The conventional prediction for $\alpha_s(M_Z) = 0.1222 \pm 0.0010_{-0.0022}^{+0.0040}$ (under $\mu_r = 1.5Q$) includes uncertainties from $\Delta\kappa$ and $\mu_r \in [Q, 2Q]$.

element d_2^{p-n} are fixed at $Q^2 = 1\text{GeV}^2$, with their combined contributions satisfying $a_2^{p-n} + 4d_2^{p-n} \approx 0.063$ [13]. Then, the remaining component in μ_4^{p-n} is f_2^{p-n} , which follows the evolution equation

$$f_2^{p-n}(Q^2) = f_2^{p-n}(1) \left(\frac{a(Q)}{a(1)} \right)^{\gamma_0/8\beta_0},$$

where $\gamma_0/8\beta_0 = 32/81$ for $n_f = 3$ [92]. The higher-twist contributions to the BSR are typically extracted by comparing the highly precise perturbative predictions with experimental data in the low and intermediate energy regions. To obtain accurate higher-twist contributions, it is essential to incorporate a general low- α_s model into the new extraction procedure. A detailed discussion on this point can be found in Ref. [23]. In the following, the LFHQCD model is employed to determine the IR behavior of $a_s^{g_1}(Q)$. The LFHQCD parametrizes the nonperturbative corrections through the asymptotic scale κ , which inherently includes higher-twist effects. Given the dominance of higher-twist contributions in the low and intermediate energy regions, we incorporate both the twist-4 and twist-6 corrections into the coupling $a_s^{g_1}(Q)$ within the $Q \geq Q_0$ region. By minimizing the $\chi^2/\text{d.o.f}$ value while satisfying the two matching conditions, we obtain the PMC results

$$Q_0 = 0.966_{-0.031}^{+0.179}(\text{the.})_{-0.023}^{-0.028}(\text{exp.}) \text{ GeV},$$

$$\kappa = 0.522_{-0.002}^{+0.003}(\text{the.})_{-0.020}^{-0.021}(\text{exp.}) \text{ GeV},$$

$$f_2^{p-n}(1 \text{ GeV}^2) = -0.220_{-0.021}^{+0.088}(\text{the.}) \pm 0.013(\text{exp.}),$$

$$\mu_6 = 0.056_{+0.001}^{+0.100}(\text{the.}) \pm 0.005(\text{exp.}) \text{ GeV}^4,$$

where the first error, labeled as “the.,” is the combined error caused by $\Delta\mu_4$ and $\Delta\Lambda_V^{n_f=5}$, and the second error, labeled as “exp.,” arises from experimental errors in the 79 input measurement data points. In the energy range of $Q_0 \leq Q \leq 10$ GeV, the contribution of higher-twist terms to the effective coupling $a_s^{g_1}(Q)$ decreases from $\sim 27.3\%$ to 2.7% . After comparing the LFHQCD prediction with those of other low- α_s models within the framework of standard OPE expressions [23], the LFHQCD prediction yields a significantly smaller $\chi^2/\text{d.o.f}$ value of $\simeq 0.7$. Here, the theoretical uncertainty $\sigma_{\text{the.}}^2$ has been neglected in calculating the $\chi^2/\text{d.o.f}$ value.

As a final remark, we also add the DSE model for a process-independent effective charge $\hat{a}_{\text{PI}}(Q)$ in Figs. 3 and 4. Inspired by the results of the strong coupling from the ghost-gluon vertex by lattice QCD [95], the DSE model connects the pinch technique gluon self-energy and the ghost-gluon vertex, and produces a modified strong coupling through a unique gluon dressing function

$$\hat{a}_{\text{PI}}(Q) = \frac{\alpha_0}{\pi} \frac{D(Q)}{D(Q)} \left[\frac{F(Q; \mu_r)/F(0; \mu_r)}{1 - L(Q; \mu_r)F(Q; \mu_r)} \right]^2, \quad (37)$$

where the ratio of $D(Q)/\mathcal{D}(Q)$ is a RGI function, which is defined in Refs. [32,33]. The function $F(Q;\mu_r)$ is a redefined dressing function for the ghost propagator and $L(Q;\mu_r)$ is the longitudinal part of the gluon-ghost vacuum polarization. The two functions depend on the renormalization scale μ_r and are evaluated at $\mu_r = 3.6$ GeV [33]. The parameter $\alpha_0 = 0.97(4)\pi$, indicating the freezing value of the effective charge, is determined by the lattice calculation. Fig. 4 shows the $\hat{\alpha}_{\text{PI}}(Q)$ and PMC prediction $\alpha_s^{g_1}(Q)$ are compatible in the small Q -region below the transition scale Q_0 , whereas above the threshold, the PMC prediction show better agreement with the data.

C. Determination of $\alpha_s(M_Z)$ -value through the precise coupling $\alpha_s^{g_1}(Q)$

The strong coupling α_s is not an observable. So, the determination of the α_s -value is usually through the comparison of some observables that can be precisely calculated and be precisely measured, e.g., the e^+e^- hadronic cross section and the τ , W and Z branching fractions to hadrons, and the hadronic decay width of heavy quarkonium [96–98].

In this subsection, we endeavor to harness the precision of PMC predictions for $\alpha_s^{g_1}$ to accurately determine the $\alpha_s(M_Z)$ -value. And in the preceding alignment of LFHQCD and PMC predictions for $\alpha_s^{g_1}(Q)$, it is noteworthy that Λ_{QCD} , being another parameter in the realm of nonperturbative physics, can likewise be determined through this matching procedure. For the purpose, by inversely taking the mass parameter $\kappa = 0.523 \pm 0.024$ GeV derived from the calculation of hadron spectrum [82] as an input, the other two matching parameters (Q_0 and Λ_{QCD}) are shown in Table. II. For the case of $\mu_r = Q$, the conventional scale-setting method yields $Q_0 = 1.1580 \pm 0.0541(\Delta\kappa)_{-0.1632}^{+0.1988}(\text{the.})$ GeV and $\Lambda_V^{n_f=3} = 433 \pm 20(\Delta\kappa)_{+118}^{-173}(\text{the.})$ MeV, where the first error “ $\Delta\kappa$ ” stems from $\Delta\kappa = \pm 0.024$ GeV and the second error “the.” is from $\mu_r \in [Q/2, 2Q]$. For the rapidly growth of PMC scales, there are two $\Lambda_V^{n_f}$ -scales: $\Lambda_V^{n_f=3}$ for $\alpha_s(\mu_{1,3}^V)$ and $\Lambda_V^{n_f=4}$ for $\alpha_s(\mu_2^V)$, which are satisfying the quark

threshold continuity condition of the coupling α_s , i.e., $\alpha_s(\mu_r = m_q, \Lambda_V^{n_f+1}, n_f + 1) = \alpha_s(\mu_r = m_q, \Lambda_V^{n_f}, n_f)$ at $\mu = m_q$ with quark mass m_q of the $(n_f + 1)^{\text{th}}$ flavor. The PMC_∞ method obtains $Q_0 = 1.0712 \pm 0.0498(\Delta\kappa)_{-0.0066}^{+0.1048}(\text{the.})$ GeV and $\Lambda_V^{n_f=3} = 503 \pm 22(\Delta\kappa)_{+14}^{-85}(\text{the.})$ MeV, where the second error “the.” is from $\Delta\mu_4^V \in [Q, 2Q]$. After the matching procedure under the $\overline{\text{MS}}$ scheme, the conventional results obtained by choosing $\mu_r = Q$ yield $\Lambda_{\overline{\text{MS}}}^{n_f=3} = 339 \pm 19$ MeV [99]. Recently, Ref. [100] provides a new expression of QCD effective coupling, which results in $\kappa = 0.58 \pm 0.03$ GeV and $\Lambda_{g1}^{n_f=3} = 0.92 \pm 0.05$ GeV.

We then calculate the α_s -value by considering the 4-loop running equation for the strong coupling $a_s(Q)$ given in Eq. (29). And the input QCD scale $\Lambda_{\overline{\text{MS}}}^{n_f=5}$ can be obtained through the asymptotic scale transformation relation between V and $\overline{\text{MS}}$ schemes in the Appendix. The conventional prediction is

$$\alpha_s(M_Z) = 0.1183 \pm 0.0011(\Delta\kappa)_{+0.0062}^{-0.0106}(\text{the.}), \quad (38)$$

which shows the μ_r -uncertainty $\sim (-9.0\%_{+5.2\%})$. The PMC_∞ prediction is

$$\alpha_s(M_Z) = 0.1220_{-0.0012}^{+0.0011}(\Delta\kappa)_{+0.0007}^{-0.0046}(\text{the.}). \quad (39)$$

whose net error is small $\sim (-3.8\%_{+0.6\%})$. The conventional result for $\alpha_s(M_Z)$ agrees with the PMC prediction within the respective uncertainties of both. The PMC_∞ prediction is consistent with the PDG average value $\alpha_s(M_Z) = 0.1180 \pm 0.0009$ [15] and other similar predictions within errors, e.g., a global fit to electro electroweak data gives $\alpha_s(M_Z) = 0.1193 \pm 0.0028$ [101], and a new determination gives $\alpha_s(M_Z) = 0.1227 \pm 0.0016(\text{the.})_{-0.0132}^{+0.0117}(\text{exp.})$ using the R_{uds} data during the tau-charm energy [98].

IV. SUMMARY

In the present paper, we have proposed a new method to determine the magnitude of α_s at all scales with the help of

TABLE II. The fitting parameters Q_0 , $\Lambda_V^{n_f=3}$, and $\alpha_s(M_Z)$ -value under the choice of $\kappa = 0.523 \pm 0.024$ GeV. The first and second errors for the Q_0 and $\alpha_s(M_Z)$ under conventional method are from the uncertainties of $\Delta\kappa = \pm 0.024$ GeV and $\mu_r \in [Q/2, 2Q]$, respectively. The first and second errors of the Q_0 and $\alpha_s(M_Z)$ under PMC_∞ method are from the uncertainties of $\Delta\kappa = \pm 0.024$ GeV and $\mu_4 \in [Q, 2Q]$, respectively.

	Q_0 (GeV)	$\Lambda_V^{n_f=3}$ (MeV)	$\alpha_s(M_Z)$
Conv. ($\mu_r = Q$)	$1.1580 \pm 0.0541_{-0.1632}^{+0.1988}$	$433 \pm 20_{+118}^{-173}$	$0.1183 \pm 0.0011_{+0.0062}^{-0.0106}$
PMC_∞	$1.0712 \pm 0.0498_{-0.0066}^{+0.1048}$	$503 \pm 22_{+14}^{-85}$	$0.1220_{-0.0012}^{+0.0011}_{-0.0046}^{+0.0117}$

BSR. The renormalization scheme-and-scale errors are always treated as an important systematic error of pQCD prediction. We adopt the PMC_∞ approach to deal with the pQCD corrections to $\alpha_s^{g_1}(Q)$, which reveals the intrinsic conformality of series and eliminates the conventional renormalization scheme-and-scale ambiguities. Thus the precision of pQCD prediction is greatly improved. Furthermore, to remove the previous “self-consistency problem” that occurred with $\overline{\text{MS}}$ -scheme series, we have adopted the V -scheme series to do our analysis. The resulting PMC_∞ series has only *the second kind of residual scale dependence*, which is negligible for the present case with up to four-loop QCD corrections. The LHFQCD model is used for the IR-behavior of $\alpha_s^{g_1}(Q)$. As a combination, we obtain a precise $\alpha_s^{g_1}(Q)$ at all scales, which agree well with the data with its p -value $\sim 98\%$. We then determine the strong coupling constant at the scale M_Z , i.e., $\alpha_s(M_Z) = 0.1220^{+0.0011}_{-0.0012}(\Delta\kappa)^{-0.0046}_{+0.0007}(\text{the.})$, where the first error comes from $\Delta\kappa = \pm 0.024$ GeV of LHFQCD model and the second error is from the unknown higher-order terms in pQCD series. It is consistent with the PDG average value within errors, thus being another example of showing the importance of PMC. The higher-twist contributions are dominant in the low and intermediate energy regions. In the $\chi^2/\text{d.o.f}$ calculation, we incorporate twist-4 and twist-6 corrections into the coupling $\alpha_s^{g_1}(Q)$ for $Q \geq Q_0$, and obtain the higher-twist coefficients as: $f_2^{p-n}(1 \text{ GeV}^2) = -0.220^{+0.088}_{-0.021}(\text{the.}) \pm 0.013(\text{exp})$ and $\mu_6 = 0.056^{+0.100}_{-0.001}(\text{the.}) \pm 0.005(\text{exp}) \text{ GeV}^4$.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: TRANSITIONS OF THE COEFFICIENTS FROM $\overline{\text{MS}}$ -SCHEME TO V -SCHEME

The transition between the perturbative coefficients under V - and $\overline{\text{MS}}$ -scheme is utilizing the established

relationships between the two coupling constants: α_s^V and $\alpha_s^{\overline{\text{MS}}}$. The two couplings under different schemes are satisfying the renormalization equation, i.e.,

$$\beta^V(\alpha_s^V) = \frac{\partial \alpha_s^V}{\partial \alpha_s^{\overline{\text{MS}}}} \beta^{\overline{\text{MS}}}(\alpha_s^{\overline{\text{MS}}}). \quad (\text{A1})$$

The coefficients in the two β -functions are the same at the first two orders,³ i.e., $\beta_0^V = \beta_0^{\overline{\text{MS}}} = \frac{1}{4}(11 - 2n_f/3)$ and $\beta_1^V = \beta_1^{\overline{\text{MS}}} = \frac{1}{4^2}(102 - 38n_f/3)$ with n_f the number of quark flavors. At high orders ($i \geq 2$), differences arise between the two β -functions, with the β^V -functions expressible via by $\beta^{\overline{\text{MS}}}$ -functions [73,74]. For present calculations, we need the four-loop QCD β -function under V -scheme [73,74] and $\overline{\text{MS}}$ -scheme [102,103], where the third and fourth terms are

$$\beta_2^{\overline{\text{MS}}} = \frac{1}{4^3} \left(\frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right), \quad (\text{A2})$$

$$\begin{aligned} \beta_3^{\overline{\text{MS}}} = \frac{1}{4^4} & \left[\frac{149753}{6} + 3564\zeta_3 - \left(\frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f \right. \\ & \left. + \left(\frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 + \frac{1093}{729} n_f^3 \right], \end{aligned} \quad (\text{A3})$$

$$\beta_2^V = \beta_2^{\overline{\text{MS}}} - a_1\beta_1 + (a_2 - a_1^2)\beta_0, \quad (\text{A4})$$

$$\beta_3^V = \beta_3^{\overline{\text{MS}}} - 2a_1\beta_2^{\overline{\text{MS}}} + a_1^2\beta_1 + (2a_3 - 6a_1a_2 + 4a_1^3)\beta_0, \quad (\text{A5})$$

where the known coefficients a_i [73,74] with quark flavor n_f and Riemann zeta function ζ_3 are

$$a_1 = \frac{1}{4} \left(\frac{31}{3} - \frac{10}{9} n_f \right), \quad (\text{A6})$$

$$\begin{aligned} a_2 = \frac{1}{4^2} & \left[\frac{4343}{18} + 36\pi^2 - \frac{9}{4}\pi^4 + 66\zeta_3 - \left(\frac{1229}{27} + \frac{52\zeta_3}{3} \right) n_f \right. \\ & \left. + \frac{100}{81} n_f^2 \right], \end{aligned} \quad (\text{A7})$$

$$a_3 = 209.884 - 51.405n_f + 2.906n_f^2 - 0.021n_f^3. \quad (\text{A8})$$

The transformation relation between α_s^V and $\alpha_s^{\overline{\text{MS}}}$ is known as [74,102–104]

³The combination of Eqs. (A1) and (A9) yields the relations between the $\{\beta_i^V\}$ -coefficients and $\{\beta_i^{\overline{\text{MS}}}\}$ -coefficients, with the first two coefficients being identical.

$$a_s^V(\Lambda_V) = a_s^{\overline{\text{MS}}}(\Lambda_V)[1 + a_1 a_s^{\overline{\text{MS},1}}(\Lambda_V) + a_2 a_s^{\overline{\text{MS},2}}(\Lambda_V) + a_3 a_s^{\overline{\text{MS},3}}(\Lambda_V) + \dots], \quad (\text{A9})$$

where the strong coupling a_s is a function of Q and Λ_{QCD} in Eq. (29). The symbol Λ_V is the V -scheme asymptotic scale which be defined as $\Lambda_V^2 = \Lambda_{\overline{\text{MS}}}^2 \exp(a_1/\beta_0^{\overline{\text{MS}}})$ with $\overline{\text{MS}}$ -scheme asymptotic scale $\Lambda_{\overline{\text{MS}}}$.

The effective coupling $a_s^{g_1}(Q)$ in Eq. (3) under $\overline{\text{MS}}$ -scheme includes the nonsinglet and singlet contributions to the pQCD series up to N^3LO [16–20] and the decoupling mass corrections at $\mathcal{O}(a_s^{\overline{\text{MS},2}})$ [21]. After complicating the transform into the V -scheme, the reduced pQCD coefficients $\hat{r}_{i,j}^V$ up to N^3LO in Eq. (9) are

$$\hat{r}_{1,0}^V = \frac{3}{4}\gamma_1^{ns}, \quad (\text{A10})$$

$$\hat{r}_{2,0}^V = \gamma_2^m + \frac{3}{2}\gamma_1^{ns} + \frac{3}{4}\gamma_2^{ns} - \frac{9}{16}(\gamma_1^{ns})^2, \quad (\text{A11})$$

$$\hat{r}_{2,1}^V = -\frac{5}{4}\gamma_1^{ns} + K_1^{ns} + \frac{3}{4}\Pi_1^{ns}, \quad (\text{A12})$$

$$\hat{r}_{3,0}^V = 4\gamma_2^m + 3\gamma_2^{ns} + \frac{3}{4}\gamma_3^{ns} + 3 \times 10^{-15}K_1^{ns} + \gamma_1^{ns} \left[11.796 - \frac{9}{4}\gamma_1^{ns} - \frac{9}{8}\gamma_2^{ns} + \frac{27}{64}(\gamma_1^{ns})^2 \right], \quad (\text{A13})$$

$$\hat{r}_{3,1}^V = -\frac{5}{3}\gamma_2^m - \frac{5}{4}\gamma_2^{ns} + 2K_1^{ns} + \frac{K_2^{ns}}{2} + \frac{3}{2}\Pi_1^{ns} + \frac{3}{4}\Pi_2^{ns} - \gamma_1^{ns} \left(5.633 - \frac{15}{16}\gamma_1^{ns} + \frac{3}{8}K_1^{ns} + \frac{9}{16}\Pi_1^{ns} \right), \quad (\text{A14})$$

$$\hat{r}_{3,2}^V = \frac{25}{12}\gamma_1^{ns} - \frac{10}{3}K_1^{ns} - \frac{5}{2}\Pi_1^{ns}, \quad (\text{A15})$$

$$\begin{aligned} \hat{r}_{4,0}^V = & -10 + 35.456\gamma_2^m + 26.592\gamma_2^{ns} + \frac{9}{2}\gamma_3^{ns} + \frac{3}{4}\gamma_4^{ns} \\ & - 4 \times 10^{-14}K_1^{ns} - 2 \times 10^{-14}\Pi_1^{ns} \\ & + \gamma_1^{ns} \left[6.832 - \frac{27}{4}\gamma_2^{ns} - \frac{9}{8}\gamma_3^{ns} \right] - \frac{9}{16}(\gamma_2^{ns})^2 \\ & - (\gamma_1^{ns})^2 \left[19.944 - \frac{81}{64}\gamma_2^{ns} - \frac{81}{32}\gamma_1^{ns} + \frac{81}{256}(\gamma_1^{ns})^2 \right], \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} \hat{r}_{4,1}^V = & -12.237\gamma_2^m - 9.178\gamma_2^{ns} - \frac{5}{4}\gamma_3^{ns} + 2K_2^{ns} + \frac{K_3^{ns}}{3} \\ & + 3\Pi_2^{ns} + \frac{3}{4}\Pi_3^{ns} - \gamma_1^{ns} \left(16.163 - \frac{15}{8}\gamma_2^{ns} + \frac{3}{2}K_1^{ns} \right. \\ & + \frac{K_2^{ns}}{4} + \frac{9}{4}\Pi_1^{ns} + \frac{3}{4}\Pi_2^{ns} \left. \right) + (\gamma_1^{ns})^2 \left[6.883 + \frac{3}{16}K_1^{ns} \right. \\ & + \frac{27}{64}\Pi_1^{ns} - \frac{45}{64}\gamma_1^{ns} \left. \right] + K_1^{ns} \left(15.728 - \frac{\gamma_2^{ns}}{4} \right) \\ & + \Pi_1^{ns} \left(11.796 - \frac{3}{8}\gamma_2^{ns} \right), \end{aligned} \quad (\text{A17})$$

$$\begin{aligned} \hat{r}_{4,2}^V = & \frac{25}{9}\gamma_2^m + \frac{25}{12}\gamma_2^{ns} - 15.022K_1^{ns} - \frac{5}{3}K_2^{ns} - \frac{5}{2}\Pi_2^{ns} \\ & + \gamma_1^{ns} \left[13.939 - \frac{25}{16}\gamma_1^{ns} + \frac{5}{4}K_1^{ns} + \frac{15}{8}\Pi_1^{ns} \right] \\ & - \Pi_1^{ns} \left[11.266 + \frac{K_1^{ns}}{4} + \frac{3}{16}\Pi_1^{ns} \right], \end{aligned} \quad (\text{A18})$$

$$\hat{r}_{4,3}^V = -\frac{125}{36}\gamma_1^{ns} + \frac{25}{3}K_1^{ns} + \frac{25}{4}\Pi_1^{ns}. \quad (\text{A19})$$

where γ_i^{ns} -, K_i^{ns} -, and Π_i^{ns} -functions defined in Refs. [18,20]. It has been emphasized that only the RGE-involved n_f -terms should be adopted for fixing the magnitude of α_s , and for the present case, the n_f -terms pertained to anomalous dimension should be teated as conformal terms [64,105].⁴ And the decoupling mass corrections [21] are

$$\gamma_2^m = -C_{\text{pBJ}}^m(\xi_c) - C_{\text{pBJ}}^m(\xi_b), \quad (\text{A20})$$

$$\begin{aligned} C_{\text{pBJ}}^m = & \frac{2}{3} \left[\frac{6\xi_x^2 + 2735\xi + 11724}{5040\xi_x} - \frac{\sqrt{\xi_x + 4}}{5040\xi_x^{3/2}} (3\xi_x^3 \right. \\ & + 106\xi_x^2 + 1054\xi_x + 4812) \log \left(\frac{\sqrt{\xi_x + 4} + \sqrt{\xi_x}}{\sqrt{\xi_x + 4} - \sqrt{\xi_x}} \right) \\ & - \frac{5}{12\xi_x^2} \log^2 \left(\frac{\sqrt{\xi_x + 4} + \sqrt{\xi_x}}{\sqrt{\xi_x + 4} - \sqrt{\xi_x}} \right) \\ & \left. + \frac{3\xi_x^2 + 112\xi_x + 1260}{5040} \log(\xi_x) \right], \end{aligned} \quad (\text{A21})$$

where $\xi_c = Q^2/m_c^2$ and $\xi_b = Q^2/m_b^2$.

⁴There are some wrong comments on PMC which are based on using all n_f -terms. Practically, when we cannot do strict distinguishing of whether the n_f -terms are RGE-involved or uninvolved, such a simple treatment could be treated as an effective PMC treatment, which may also work well.

- [1] D. J. Gross and F. Wilczek, *Phys. Rev. Lett.* **30**, 1343 (1973).
- [2] H. D. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973).
- [3] G. 't Hooft, *Nucl. Phys.* **B61**, 455 (1973).
- [4] S. Weinberg, *Phys. Rev. D* **8**, 3497 (1973).
- [5] J. D. Bjorken, *Phys. Rev.* **148**, 1467 (1966).
- [6] J. D. Bjorken, *Phys. Rev. D* **1**, 1376 (1970).
- [7] K. Ackerstaff *et al.* (HERMES Collaboration), *Phys. Lett. B* **404**, 383 (1997); **444**, 531 (1998); A. Airapetian *et al.* (HERMES Collaboration), *Phys. Lett. B* **442**, 484 (1998); *Phys. Rev. Lett.* **90**, 092002 (2003); *Phys. Rev. D* **75**, 012007 (2007).
- [8] V. Y. Alexakhin *et al.* (COMPASS Collaboration), *Phys. Lett. B* **647**, 8 (2007); M. G. Alekseev *et al.* (COMPASS Collaboration), *Phys. Lett. B* **690**, 466 (2010); C. Adolph *et al.* (COMPASS Collaboration), *Phys. Lett. B* **753**, 18 (2016).
- [9] P. L. Anthony *et al.* (E142 Collaboration), *Phys. Rev. Lett.* **71**, 959 (1993); *Phys. Rev. D* **54**, 6620 (1996); K. Abe *et al.* (E143 Collaboration), *Phys. Rev. Lett.* **74**, 346 (1995); **75**, 25 (1995); **76**, 587 (1996); *Phys. Lett. B* **364**, 61 (1995); *Phys. Rev. D* **58**, 112003 (1998).
- [10] A. Deur *et al.*, *Phys. Rev. Lett.* **93**, 212001 (2004).
- [11] A. Deur, V. Burkert, J. P. Chen, and W. Korsch, *Phys. Lett. B* **650**, 244 (2007).
- [12] A. Deur *et al.*, *Phys. Rev. D* **78**, 032001 (2008).
- [13] A. Deur, Y. Prok, V. Burkert, D. Crabb, F.-X. Girod, K. A. Griffioen, N. Guler, S. E. Kuhn, and N. Kvaltine, *Phys. Rev. D* **90**, 012009 (2014).
- [14] A. Deur, V. Burkert, J. P. Chen, and W. Korsch, *Particles* **5**, 171 (2022).
- [15] S. Navas *et al.* (Particle Data Group), *Phys. Rev. D* **110**, 030001 (2024).
- [16] S. G. Gorishnii and S. A. Larin, *Phys. Lett. B* **172**, 109 (1986).
- [17] S. A. Larin and J. A. M. Vermaseren, *Phys. Lett. B* **259**, 345 (1991).
- [18] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, *Phys. Rev. Lett.* **104**, 132004 (2010).
- [19] S. A. Larin, *Phys. Lett. B* **723**, 348 (2013).
- [20] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Nucl. Part. Phys. Proc.* **261–262**, 3 (2015).
- [21] J. Blümlein, G. Falcioni, and A. De Freitas, *Nucl. Phys. B* **910**, 568 (2016).
- [22] C. Ayala, G. Cvetič, A. V. Kotikov, and B. G. Shaikhhatdenov, *Eur. Phys. J. C* **78**, 1002 (2018).
- [23] Q. Yu, X. G. Wu, H. Zhou, and X. D. Huang, *Eur. Phys. J. C* **81**, 690 (2021).
- [24] C. Ayala, C. Castro-Arriaza, and G. Cvetič, *Nucl. Phys. B* **1007**, 116668 (2024).
- [25] S. J. Brodsky, G. F. de Teramond, H. G. Dosch, and J. Erlich, *Phys. Rep.* **584**, 1 (2015).
- [26] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).
- [27] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Phys. Lett. B* **428**, 105 (1998).
- [28] E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
- [29] P. A. M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949).
- [30] S. J. Brodsky, H. C. Pauli, and S. S. Pinsky, *Phys. Rep.* **301**, 299 (1998).
- [31] C. D. Roberts and A. G. Williams, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
- [32] D. Binosi, C. Mezrag, J. Papavassiliou, C. D. Roberts, and J. Rodríguez-Quintero, *Phys. Rev. D* **96**, 054026 (2017).
- [33] Z. F. Cui, J. L. Zhang, D. Binosi, F. de Soto, C. Mezrag, J. Papavassiliou, C. D. Roberts, J. Rodríguez-Quintero, J. Segovia, and S. Zafeiropoulos, *Chin. Phys. C* **44**, 083102 (2020).
- [34] J. M. Cornwall, *Phys. Rev. D* **26**, 1453 (1982).
- [35] J. M. Cornwall and J. Papavassiliou, *Phys. Rev. D* **40**, 3474 (1989).
- [36] D. Binosi and J. Papavassiliou, *Phys. Rep.* **479**, 1 (2009).
- [37] A. Pilaftsis, *Nucl. Phys.* **B487**, 467 (1997).
- [38] L. F. Abbott, *Nucl. Phys.* **B185**, 189 (1981).
- [39] A. Deur, S. J. Brodsky, and C. D. Roberts, *Prog. Part. Nucl. Phys.* **134**, 104081 (2024).
- [40] S. J. Brodsky, A. Deur, and C. D. Roberts, *Sci. Am.*, **330**, 32 (2024).
- [41] A. Petermann, *Helv. Phys. Acta* **26**, 499 (1953).
- [42] C. G. Callan, Jr., *Phys. Rev. D* **2**, 1541 (1970).
- [43] K. Symanzik, *Commun. Math. Phys.* **18**, 227 (1970).
- [44] A. Peterman, *Phys. Rep.* **53**, 157 (1979).
- [45] Q. Yu, H. Zhou, X. D. Huang, J. M. Shen, and X. G. Wu, *Chin. Phys. Lett.* **39**, 071201 (2022).
- [46] S. J. Brodsky and X. G. Wu, *Phys. Rev. D* **85**, 034038 (2012).
- [47] S. J. Brodsky and X. G. Wu, *Phys. Rev. Lett.* **109**, 042002 (2012).
- [48] S. J. Brodsky and L. Di Giustino, *Phys. Rev. D* **86**, 085026 (2012).
- [49] M. Mojaza, S. J. Brodsky, and X. G. Wu, *Phys. Rev. Lett.* **110**, 192001 (2013).
- [50] S. J. Brodsky, M. Mojaza, and X. G. Wu, *Phys. Rev. D* **89**, 014027 (2014).
- [51] D. J. Gross and A. Neveu, *Phys. Rev. D* **10**, 3235 (1974).
- [52] B. E. Lautrup, *Phys. Lett.* **69B**, 109 (1977).
- [53] M. Beneke, *Phys. Rep.* **317**, 1 (1999).
- [54] F. Strocchi, *Int. Ser. Monogr. Phys.* **158**, 1 (2013).
- [55] X. G. Wu, Y. Ma, S.-Q. Wang, H.-B. Fu, H.-H. Ma, S. J. Brodsky, and M. Mojaza, *Rep. Prog. Phys.* **78**, 126201 (2015).
- [56] X. G. Wu, J. M. Shen, B. L. Du, and S. J. Brodsky, *Phys. Rev. D* **97**, 094030 (2018).
- [57] X. G. Wu, J. M. Shen, B. L. Du, X. D. Huang, S. Q. Wang, and S. J. Brodsky, *Prog. Part. Nucl. Phys.* **108**, 103706 (2019).
- [58] Z. F. Wu, X. G. Wu, J. Yan, X. D. Huang, and J. M. Shen, *arXiv:2505.05131*.
- [59] X. C. Zheng, X. G. Wu, S. Q. Wang, J. M. Shen, and Q. L. Zhang, *J. High Energy Phys.* **10** (2013) 117.
- [60] S. J. Brodsky, G. F. de Teramond, and A. Deur, *Phys. Rev. D* **81**, 096010 (2010).
- [61] A. Deur, S. J. Brodsky, and G. F. de Teramond, *Phys. Lett. B* **750**, 528 (2015).
- [62] A. Deur, S. J. Brodsky, and G. F. de Teramond, *Phys. Lett. B* **757**, 275 (2016).
- [63] A. Deur, J. M. Shen, X. G. Wu, S. J. Brodsky, and G. F. de Teramond, *Phys. Lett. B* **773**, 98 (2017).
- [64] J. M. Shen, X. G. Wu, B. L. Du, and S. J. Brodsky, *Phys. Rev. D* **95**, 094006 (2017).

- [65] J. Yan, Z. F. Wu, J. M. Shen, and X. G. Wu, *J. Phys. G* **50**, 045001 (2023).
- [66] L. Di Giustino, S. J. Brodsky, S. Q. Wang, and X. G. Wu, *Phys. Rev. D* **102**, 014015 (2020).
- [67] M. Peter, *Phys. Rev. Lett.* **78**, 602 (1997).
- [68] Y. Schroder, *Phys. Lett. B* **447**, 321 (1999).
- [69] T. Appelquist, M. Dine, and I. J. Muzinich, *Phys. Lett.* **69B**, 231 (1977).
- [70] W. Fischler, *Nucl. Phys.* **B129**, 157 (1977).
- [71] A. Billoire, *Phys. Lett.* **92B**, 343 (1980).
- [72] X. D. Huang, J. Yan, H. H. Ma, L. Di Giustino, J. M. Shen, X. G. Wu, and S. J. Brodsky, *Nucl. Phys.* **B989**, 116150 (2023).
- [73] A. L. Kataev and V. S. Molokoedov, *Phys. Rev. D* **92**, 054008 (2015).
- [74] A. L. Kataev and V. S. Molokoedov, *Theor. Math. Phys.* **217**, 1459 (2023).
- [75] S. J. Brodsky and H. J. Lu, *Phys. Rev. D* **51**, 3652 (1995).
- [76] J. L. Basdevant, *Fortschr. Phys.* **20**, 283 (1972).
- [77] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, *Phys. Rev. Lett.* **79**, 2184 (1997).
- [78] B. L. Du, X. G. Wu, J. M. Shen, and S. J. Brodsky, *Eur. Phys. J. C* **79**, 182 (2019).
- [79] C. Ayala, C. Castro-Arriaza, and G. Cvetič, *Phys. Lett. B* **848**, 138386 (2024).
- [80] G. M. Prospero, M. Raciti, and C. Simolo, *Prog. Part. Nucl. Phys.* **58**, 387 (2007).
- [81] A. Deur, S. J. Brodsky, and G. F. de Téramond, *Prog. Part. Nucl. Phys.* **90**, 1 (2016).
- [82] S. J. Brodsky, G. F. de Téramond, H. G. Dosch, and C. Lorcé, *Phys. Lett. B* **759**, 171 (2016).
- [83] K. Ackerstaff *et al.* (OPAL Collaboration), *Eur. Phys. J. C* **7**, 571 (1999).
- [84] S. J. Brodsky, S. Menke, C. Merino, and J. Rathsmann, *Phys. Rev. D* **67**, 055008 (2003).
- [85] D. J. Gross and C. H. Llewellyn Smith, *Nucl. Phys.* **B14**, 337 (1969).
- [86] J. H. Kim *et al.*, *Phys. Rev. Lett.* **81**, 3595 (1998).
- [87] K. Abe *et al.* (E154 Collaboration), *Phys. Rev. Lett.* **79**, 26 (1997); *Phys. Lett. B* **404**, 377 (1997); **405**, 180 (1997).
- [88] P. L. Anthony *et al.* (E155 Collaboration), *Phys. Lett. B* **458**, 529 (1999); **463**, 339 (1999); **493**, 19 (2000); **553**, 18 (2003).
- [89] B. Adeva *et al.* (Spin Muon Collaboration), *Phys. Rev. D* **58**, 112001 (1998).
- [90] M. Schmelling, *Phys. Scr.* **51**, 676–679 (1995).
- [91] A. Deur, J. P. Chen, S. E. Kuhn, C. Peng, M. Ripani, V. Sulkosky, K. Adhikari, M. Battaglieri, V. D. Burkert, G. D. Cates *et al.*, *Phys. Lett. B* **825**, 136878 (2022).
- [92] H. Kawamura, T. Uematsu, J. Kodaira, and Y. Yasui, *Mod. Phys. Lett. A* **12**, 135 (1997).
- [93] X. D. Ji and P. Unrau, *Phys. Lett. B* **333**, 228 (1994).
- [94] E. V. Shuryak and A. I. Vainshtein, *Nucl. Phys.* **B201**, 141 (1982).
- [95] S. Zafeiropoulos, P. Boucaud, F. De Soto, J. Rodríguez-Quintero, and J. Segovia, *Phys. Rev. Lett.* **122**, 162002 (2019).
- [96] D. d’Enterria, S. Kluth, S. Alekhin, P. A. Baikov, A. Banfi, F. Barreiro, A. Bazavov, S. Bethke, J. Blümlein, D. Boito *et al.*, [arXiv:1907.01435](https://arxiv.org/abs/1907.01435).
- [97] D. d’Enterria, S. Kluth, G. Zanderighi, C. Ayala, M. A. Benitez-Rathgeb, J. Bluemlein, D. Boito, N. Brambilla, D. Britzger, S. Camarda *et al.*, *J. Phys. G* **51**, 090501 (2024).
- [98] J. M. Shen, B. H. Qin, J. Yan, S. Q. Wang, and X. G. Wu, *J. High Energy Phys.* **07** (2023) 109.
- [99] A. Deur, S. J. Brodsky, and G. F. de Téramond, *J. Phys. G* **44**, 105005 (2017).
- [100] G. F. de Téramond, A. Paul, S. J. Brodsky, A. Deur, H. G. Dosch, T. Liu, and R. S. Sufian (HLFHS Collaboration), *Phys. Rev. Lett.* **133**, 181901 (2024).
- [101] J. de Blas, M. Pierini, L. Reina, and L. Silvestrini, *Phys. Rev. Lett.* **129**, 271801 (2022).
- [102] K. G. Chetyrkin, *Nucl. Phys.* **B710**, 499 (2005).
- [103] M. Czakon, *Nucl. Phys.* **B710**, 485 (2005).
- [104] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, *Phys. Rev. Lett.* **118**, 082002 (2017).
- [105] J. Yan, S. J. Brodsky, L. Di Giustino, P. G. Ratcliffe, S. Q. Wang, and X. G. Wu, *Symmetry* **17**, 411 (2025).