

Zeroth law of black hole thermodynamics for higher derivative Proca theories

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We consider diffeomorphism invariant theories of gravity with arbitrary higher derivative corrections to the two derivative Einstein's theory in the presence of Proca fields—vector fields without $U(1)$ gauge invariance. We prove that, in such theories, the surface gravity remains constant across the Killing horizon of stationary black hole solutions, without requiring the existence of a bifurcation surface. Our analysis is carried out within an effective field theory framework, where higher derivative contributions are treated perturbatively around the two derivative theory. The proof relies on specific components of the coupled gravity—Proca field equations together with constraints implied by the near horizon boost symmetry of stationary black hole geometries, analyzed using boost-weight arguments. Implementing these arguments in the presence of Proca fields introduces new subtleties beyond those encountered in the pure gravity case. We provide a systematic resolution of these subtleties and demonstrate that the surface gravity remains constant to all orders in the perturbative expansion, thereby establishing the zeroth law. Finally, we outline how this framework can be generalized to broader classes of vector theories with four propagating degrees of freedom, which exhibit qualitative differences from both the Proca and gauge invariant vector theories.

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I. INTRODUCTION

Black holes are fascinating objects, yet in one respect they behave like ordinary matter: the classical properties of their horizons admit a thermodynamic interpretation, where geometric quantities are identified with thermodynamic ones. This correspondence has been rigorously established within Einstein's general relativity (GR) [1–3]. Since GR is not a UV-complete theory of gravity, it is natural to examine black hole thermodynamics including higher derivative corrections to the Einstein-Hilbert action. These terms generally modify black hole solutions and, consequently, alter the geometric quantities entering thermodynamic relations—thus providing an intriguing setting to test the validity of the laws of black hole thermodynamics beyond GR.

In [4,5], the authors demonstrated that *the equilibrium state version* of the first law holds universally for arbitrary

diffeomorphism invariant higher curvature theories of gravity using the Noether charge method without relying on effective field theory (EFT) approximation. They identified entropy with a geometric quantity which works perfectly well for stationary situations but suffers from well-known Jacobson-Kang-Myers (JKM) ambiguities [6] in the dynamical situation. Progress was made in certain restricted cases [7–9], yet a general proof of the zeroth or second law remained absent. In recent years, significant advances have been made. By treating the dynamics as small perturbations around stationary black hole solutions, the authors of [10–12] have proven the second law for arbitrary higher-curvature theories of gravity, without relying on EFT approximations. These results have since been significantly extended in [13–23], broadening the scope and applicability of the second law in increasingly general settings.

This article, on the other hand, focuses on the zeroth law of black hole thermodynamics. The zeroth law of black hole thermodynamics states that *the surface gravity and hence the temperature of any stationary black hole is uniform across its event horizon*. This was proven for Einstein's GR in [2]. A fully nonperturbative proof of the zeroth law in arbitrary higher curvature theories of gravity remains unavailable. Approaches based on small dynamical perturbations [10–12] are ill suited, since the law pertains exclusively to strictly stationary solutions. Instead, one can invoke an effective field

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theory expansion, treating higher derivative terms as perturbative corrections to the two derivative Einstein-Hilbert action. Within this EFT framework, Bhattacharyya *et al.* have recently established the zeroth law for general higher curvature gravities in vacuum [24], and their proof has been extended to include scalar and $U(1)$ gauge fields in [25].

The proofs in [24,25] rely heavily on the “boost-weight” arguments developed in [10–12]. These arguments work nicely for pure gravity and for theories including scalar or $U(1)$ gauge fields—but become a bit subtle when applied to Proca fields (massive vector fields). In this article, we address the subtlety and thereby establish the zeroth law for theories that include Proca fields.

In this article, we aim to establish the zeroth law of black hole thermodynamics for the Einstein-Proca theory, incorporating higher curvature corrections within the framework of EFT. If we consider a *gauge field* B_μ in D dimensions, it has $D - 2$ propagating degrees of freedom. Whereas, if we consider a *Proca field* A_μ that does not enjoy $U(1)$ gauge invariance, *one more* degree of freedom starts propagating, therefore in D dimension, Proca field A_μ has $D - 1$ propagating degrees of freedom—the time component does not propagate [26–31]. Massive gravity and its ghost-free formulations have been extensively explored as consistent extensions of general relativity [32]. There has been increasing recent interest in nongauge vector fields, ranging from the consistency of self-interacting massive vectors [33] to the cosmological role of Proca fields with nonminimal couplings [34–37].

To prove the zeroth law, we work in a coordinate system adapted to the near-horizon geometry. We demonstrate the constancy of surface gravity on a Killing horizon, defined as a codimension-one null hypersurface whose normal vector is Killing, without assuming the horizon to be of bifurcate type. Our strategy involves organizing a specific component of the equations of motion (EOM) order by order in the EFT expansion. We show that the off-shell structure of this component can be expressed in terms of derivatives of the surface gravity with respect to horizon coordinates. Although higher curvature terms can be intricate, we do not require their explicit forms. Instead, by employing “boost-weight” arguments as developed in [24], we abstractly show this particular structure of the off shell component of the equation of motion within the EFT framework. Finally, imposing the vanishing of the EOM yields the desired result: the surface gravity remains constant across the Killing horizon.

We can consider a broader class of vector fields C_μ , in which all D degrees of freedom are dynamical.¹

¹Throughout this article, we use the notation C_μ to denote a vector field whose components each correspond to propagating degrees of freedom. In contrast, B_μ refers to a standard $U(1)$ gauge field, and A_μ denotes a Proca field. In D dimensional spacetime, for example, the field C_μ carries D propagating degrees of freedom, whereas B_μ and A_μ possess $D - 2$ and $D - 1$ propagating degrees of freedom, respectively.

This generalization allows for additional terms in the Lagrangian. Within this extended framework, we have been able to prove the zeroth law of black hole thermodynamics only for a restricted class of Lagrangians. To avoid confusion and maintain clarity, we refrain from discussing this broader class of theories outside Sec. IV, where we elaborate on these issues in detail.

The remainder of this paper is organized as follows. In Sec. II, we present the effective field theory framework, introduce the near-horizon coordinate system, and state our main result. Section III, which forms the core of this article, contains the inductive argument establishing the zeroth law to arbitrary perturbative order. In Sec. IV, we discuss the broader class of vector theories involving C_μ and offer related observations. Section V summarizes our findings and outlines possible directions for future research. Finally, Appendixes A, B, and C provide additional computational details and a summary of the “boost-weight” arguments.

II. SETUP AND STATEMENT OF THE PROBLEM

We consider arbitrary higher curvature theories of gravity coupled to a Proca field A_μ . Schematically, we can write our full effective field theory action as follows,

$$S = \int d^D x \sqrt{-g} \left(\sum_{n=0}^{\infty} \alpha^n \mathcal{L}_{(n)} \right), \quad (2.1)$$

where α is the parameter that controls the effective field theory expansion. The leading-order Lagrangian $\mathcal{L}_{(0)}$ has the standard Einstein-Proca theory form

$$\begin{aligned} S &= \int d^D x \sqrt{-g} \mathcal{L}_{(0)} \\ &= \int d^D x \sqrt{-g} \left(R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \right), \end{aligned} \quad (2.2)$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad (2.3)$$

and we work in a system of units in which $16\pi G_N$ is set to unity. All the terms in Lagrangian $\mathcal{L}_{(n)}$ should be of the same “strength” in effective field theory expansion. In our effective field theory expansion, we organize terms according to their number of derivatives: the more derivatives a term has, the weaker its strength. Which implies that in our starting Lagrangian, the $m^2 A_\mu A^\mu$ term is of the same order as that of R and $F_{\mu\nu} F^{\mu\nu}$; therefore, we will treat $m^2 A_\mu A^\mu$ effectively as a two derivative term.

Let us discuss briefly possible higher-order terms. In pure gravity, every term in the action involves an even number of derivatives—for example, the lowest-order contribution is the standard two-derivative Einstein-Hilbert

term, with corrections appearing at the four-derivative level, six-derivative level, and so on. However, when a vector field without gauge symmetry is introduced, odd-derivative terms can also arise. In our case $\mathcal{L}_{(1)}$ is a three derivative term which in general will contain the following terms [28,29,38]:

$$\begin{aligned} \mathcal{L}_{(1)} = & a_1 m^2 A^2 (\nabla_\mu A^\mu) + a_2 \tilde{F}^{\mu\alpha} \tilde{F}^\nu{}_\alpha \nabla_\mu A_\nu \\ & + a_3 R (\nabla_\mu A^\mu) + a_4 R_{\mu\nu} \nabla^\mu A^\nu, \end{aligned} \quad (2.4)$$

where the coefficients a_i are dimensionless constants and $\tilde{F}_{\mu\nu}$ is dual to Maxwell field strength $F_{\mu\nu}$.

In the next order, the expansion will include terms with up to four derivatives. Specifically, the gravitational sector will include contributions such as

$$R^2, \quad R^{\mu\nu} R_{\mu\nu}, \quad R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}, \quad (2.5)$$

while the vector field sector will include terms like [28,29,38]

$$m^4 A^4, \quad m^2 A^2 F^{\mu\nu} F_{\mu\nu}, \quad m^2 A^2 [(\nabla \cdot A)^2 - (\nabla_\alpha A_\beta)(\nabla^\beta A^\alpha)], \dots \quad (2.6)$$

along with mixed contributions involving both the curvature tensor $R_{\mu\nu\alpha\beta}$ and the vector field A_μ . We do not aim to provide an exhaustive list here, as the explicit forms of these terms are not essential for the present discussion.

If we compute the equations of motion from (2.1), they will take the form of a perturbative expansion in powers of α . Let us schematically write the equations of motion for the metric and the vector field as follows:

$$\begin{aligned} E_{\mu\nu} = & E_{\mu\nu}^{(0)} + \alpha E_{\mu\nu}^{(1)} + \alpha^2 E_{\mu\nu}^{(2)} + \dots, \\ E_\mu = & E_\mu^{(0)} + \alpha E_\mu^{(1)} + \alpha^2 E_\mu^{(2)} + \dots, \end{aligned} \quad (2.7)$$

where $E_{\mu\nu}^{(0)}$ and $E_\mu^{(0)}$ are the part of the equations of motion coming from $\mathcal{L}_{(0)}$ and similarly for all the other $E_{\mu\nu}^{(n)}$ and $E_\mu^{(n)}$.

As mentioned earlier in the Introduction, the zeroth law of black hole thermodynamics pertains to stationary solutions. Suppose $g_{\mu\nu}^{(bh)}$ and A_μ is a stationary solution of $E_{\mu\nu} = 0$ and $E_\mu = 0$. Therefore, $g_{\mu\nu}^{(bh)}$ and A_μ will also admit expansion in terms of effective field theory parameter α ,

$$\begin{aligned} g_{\mu\nu}^{(bh)} = & g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)} + \dots, \\ A_\mu = & A_\mu^{(0)} + \alpha A_\mu^{(1)} + \dots. \end{aligned} \quad (2.8)$$

As the metric $g_{\mu\nu}^{(bh)}$ is stationary, it admits an asymptotically timelike Killing vector field, which we will denote by ξ^μ .

Let us further assume that the event horizon Σ of $g_{\mu\nu}^{(bh)}$ is Killing (but not assumed to be of *bifurcate* type) for the Killing vector field ξ^μ . As discussed in [12,24], it is always possible to choose a coordinate system in which the metric near the Killing horizon Σ takes the following form:

$$\begin{aligned} ds^2 = & g_{\mu\nu}^{(bh)} dX^\mu dX^\nu \\ = & 2d\tau d\rho - \rho X(\rho, x^i) d\tau^2 \\ & + 2\rho\omega_i(\rho, x^i) d\tau dx^i + h_{ij}(\rho, x^i) dx^i dx^j. \end{aligned} \quad (2.9)$$

Here, $\xi^\mu = \partial_\tau$ is the Killing vector, implying that none of the metric components depend on τ . In the coordinate system (τ, ρ, x^i) , the Killing horizon Σ is located at $\rho = 0$, where the norm of the Killing vector vanishes. The vector potential A_μ must also satisfy the stationarity condition $\mathcal{L}_\xi A_\mu = 0$, which leads to

$$\mathcal{L}_\xi A_\mu = 0 \Rightarrow \partial_\tau A_\mu = 0. \quad (2.10)$$

Thus, the components of A_μ are independent of τ , and can be expressed as functions of ρ and x^i

$$A_\mu = A_\mu(\rho, x^i). \quad (2.11)$$

To prove that the surface gravity is constant over the Killing horizon, we start with its definition:

$$\kappa = \sqrt{-\frac{1}{2} (\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)|_{\rho=0}}. \quad (2.12)$$

By evaluating κ for the metric (2.9), it is found to be

$$\kappa = \frac{1}{2} X(\rho = 0, x^i). \quad (2.13)$$

To establish the constancy of the surface gravity over the horizon in our coordinate system, we need to show that

$$\partial_\tau \kappa = 0 \quad \text{and} \quad \partial_i \kappa = 0. \quad (2.14)$$

Since ∂_τ is a Killing vector, none of the metric components depend on τ . Consequently, X has no dependence on τ , and hence $\partial_\tau \kappa = 0$ is satisfied trivially. To complete the proof of the zeroth law, it remains to show that $\partial_i \kappa = 0$; in our coordinate system we have to show that

$$\partial_i X|_{\rho=0} = 0. \quad (2.15)$$

We will demonstrate this by utilizing two specific components $E_{\tau i}$ and E_τ of the equations of motion, analyzed order by order within the framework of the effective field theory expansion.

III. PROOF OF THE ZEROETH LAW

As we have discussed in Sec. II, our goal is to prove Eq. (2.15) using equation of motion within the framework of effective field theory. We write our equation of motion in effective field theory expansion as written in Eq. (2.7),

$$\begin{aligned} E_{\mu\nu} &= E_{\mu\nu}^{(0)} + \alpha E_{\mu\nu}^{(1)} + \alpha^2 E_{\mu\nu}^{(2)} + \dots, \\ E_\mu &= E_\mu^{(0)} + \alpha E_\mu^{(1)} + \alpha^2 E_\mu^{(2)} + \dots. \end{aligned} \quad (3.1)$$

The solution of $E_{\mu\nu} = 0$ and $E_\mu = 0$ which we denote by $g_{\mu\nu}^{(bh)}$ and A_μ can also be written in effective field theory expansion as we have written in (2.8),

$$\begin{aligned} g_{\mu\nu}^{(bh)} &= g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)} + \dots, \\ A_\mu &= A_\mu^{(0)} + \alpha A_\mu^{(1)} + \dots. \end{aligned} \quad (3.2)$$

If we write our metric in the horizon adapted coordinate system as written in (2.9) then Eq. (3.2) implies that X , ω_i , h_{ij} , and also A_μ will admit expansion in terms of α ,

$$\begin{aligned} X(\rho, x^i) &= X^{(0)}(\rho, x^i) + \alpha X^{(1)}(\rho, x^i) + \alpha^2 X^{(2)}(\rho, x^i) + \dots, \\ \omega_i(\rho, x^i) &= \omega_i^{(0)}(\rho, x^i) + \alpha \omega_i^{(1)}(\rho, x^i) + \alpha^2 \omega_i^{(2)}(\rho, x^i) + \dots, \\ h_{ij}(\rho, x^i) &= h_{ij}^{(0)}(\rho, x^i) + \alpha h_{ij}^{(1)}(\rho, x^i) + \alpha^2 h_{ij}^{(2)}(\rho, x^i) + \dots, \\ A_\mu(\rho, x^i) &= A_\mu^{(0)}(\rho, x^i) + \alpha A_\mu^{(1)}(\rho, x^i) + \alpha^2 A_\mu^{(2)}(\rho, x^i) + \dots. \end{aligned} \quad (3.3)$$

At the leading order, i.e., at $\mathcal{O}(\alpha)^0$, the metric $g_{\mu\nu}^{(0)}$ is an exact solution to $E_{\mu\nu}^{(0)} = 0$. Assuming that the equation of motion at the leading order, $E_{\mu\nu}^{(0)} = 0$, has been solved, it follows that $g_{\mu\nu}^{(0)}$ is known exactly. At the first subleading order, i.e., at $\mathcal{O}(\alpha)^1$, the equation of motion is

$$E_{\mu\nu}^{(0)} + \alpha E_{\mu\nu}^{(1)} = \mathcal{O}(\alpha)^2. \quad (3.4)$$

At this order the metric also has the following form:

$$g_{\mu\nu}^{(bh)} = g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)} + \mathcal{O}(\alpha)^2. \quad (3.5)$$

Here $g_{\mu\nu}^{(0)}$ are some known functions; therefore, the variables of the equation of motion at the first subleading order are $g_{\mu\nu}^{(1)}$. The equation we have to solve to determine $g_{\mu\nu}^{(1)}$ is

$$[E_{\mu\nu}^{(0)} + \alpha E_{\mu\nu}^{(1)}][g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)}] = \mathcal{O}(\alpha)^2. \quad (3.6)$$

All the terms in the above equation are $\mathcal{O}(\alpha)$ terms as we have solved the equation of motion at the leading $\mathcal{O}(\alpha)^0$ order. The above equation can be written as

$$E_{\mu\nu}^{(0)}[g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)}] + \alpha E_{\mu\nu}^{(1)}[g_{\mu\nu}^{(0)}] = \mathcal{O}(\alpha)^2. \quad (3.7)$$

The first part in the above equation represents the linearization of $E_{\mu\nu}^{(0)}$ around $g_{\mu\nu}^{(0)}$, treating $g_{\mu\nu}^{(1)}$ as a small perturbation. We will refer to this as the *homogeneous part* of the differential equation. As we will explain below, this part of the equation is universal when solving the equations of motion at any arbitrary higher order. However, the homogeneous part depends on the specific form of $\mathcal{L}_{(0)}$. It is universal with respect to effective field theory perturbations, meaning that once $\mathcal{L}_{(0)}$ is specified, the form of the homogeneous part will remain the same across all higher orders.

The term $E_{\mu\nu}^{(1)}$, on the other hand, depends on the higher curvature contributions $\mathcal{L}_{(1)}$ in the Lagrangian. Once $\mathcal{L}_{(1)}$ is specified, $E_{\mu\nu}^{(1)}$ becomes a known covariant tensor, which we have to evaluate on the zeroth-order solution $g_{\mu\nu}^{(0)}$. This part of the equation will be referred to as the *source part* which acts like source in the inhomogeneous partial differential equations for $g_{\mu\nu}^{(1)}$. At any arbitrary higher order, the equation of motion can always be decomposed into two parts: a homogeneous part and a source part. Unlike the homogeneous part, the source part is not universal—it varies depending on the specific theory and the order being considered. As we will demonstrate, there is no need to compute the source term directly: it can instead be addressed using abstract arguments, as developed in [24].

The proof can be systematically extended to arbitrary higher order using *the method of induction*. According to this method, if the proof holds at the leading order, then it will continue to hold at arbitrary higher orders—provided we can demonstrate its validity at the $(n+1)$ th order assuming it holds at the n th order. Notably, this method does not require us to explicitly verify the proof at the first subleading order. However, for illustrative purposes, we will present the proof at the first subleading order, which captures all essential components of our argument in a simpler setting.

Let us assume that the equations of motion for both the metric and the vector field have been solved up to order $\mathcal{O}(\alpha)^n$. At the next order, $\mathcal{O}(\alpha)^{n+1}$, the unknown functions are $g_{\mu\nu}^{(n+1)}$ and $A_\mu^{(n+1)}$. At this order, the equations of motion can be systematically decomposed into two parts—one involving the unknown functions $g_{\mu\nu}^{(n+1)}$ and $A_\mu^{(n+1)}$, which possesses a universal structure across all orders and is referred to as the *homogeneous part*, and another that depends on the specific form of the higher-curvature gravity theory, rendering it nonuniversal and termed as the *source part*. It is important to note that the source part is independent of the unknown functions $g_{\mu\nu}^{(n+1)}$ and $A_\mu^{(n+1)}$. The homogeneous part at the order $\mathcal{O}(\alpha)^{n+1}$ has the following structure:

$$E_{\mu\nu}^{(0)}[g_{\mu\nu}^{(0)} + \alpha^{n+1}g_{\mu\nu}^{(n+1)}, A_\mu^{(0)} + \alpha^{n+1}A_\mu^{(n+1)}], \quad (3.8)$$

which is essentially the linearization of $E_{\mu\nu}^{(0)}$ around $(g_{\mu\nu}^{(0)}, A_\mu^{(0)})$ assuming $\alpha^{n+1}g_{\mu\nu}^{(n+1)}$ and $\alpha^{n+1}A_\mu^{(n+1)}$ as linearized fluctuations.

On the other hand, the source part has the following structure:

$$[E_{\mu\nu}^{(0)} + \dots + \alpha^{n+1}E_{\mu\nu}^{(n+1)}] \\ [g_{\mu\nu}^{(0)} + \dots + \alpha^n g_{\mu\nu}^{(n)}, A_\mu^{(0)} + \dots + \alpha^n A_\mu^{(n)}]. \quad (3.9)$$

At the current order $\mathcal{O}(\alpha)^{n+1}$, all metric components up to $g_{\mu\nu}^{(n)}$ and vector field components up to $A_\mu^{(n)}$ are known functions. Consequently, once the higher-curvature theory is specified, the source part is, in principle, determined. However, explicitly computing these source terms can be exceedingly tedious. Following [24], we will show that such a computation is unnecessary; instead, the source part can be treated abstractly using boost weight arguments.

A. Proof at the leading order

Now, we will prove constancy of surface gravity at the leading order that is for Einstein-Proca theory defined by the Lagrangian (2.2). If we compute equations of motion from (2.2) we get

$$E_{\mu\nu}^{(0)} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{8}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} \\ + \frac{1}{4}m^2g_{\mu\nu}A^\alpha A_\alpha - \frac{1}{2}m^2A_\mu A_\nu = 0, \\ E_\mu^{(0)} \equiv \nabla^\nu F_{\nu\mu} - m^2A_\mu = 0. \quad (3.10)$$

Following [24], we now consider the τi component of the equation of motion. In our horizon adapted coordinate system the zeroth-order metric can be written as

$$ds^2 = 2d\tau d\rho - \rho X^{(0)}(\rho, x^i)d\tau^2 + 2\rho\omega_i^{(0)}(\rho, x^i)d\tau dx^i \\ + h_{ij}^{(0)}(\rho, x^i)dx^i dx^j. \quad (3.11)$$

Now, we would compute the τi component of $E_{\mu\nu}^{(0)}$ on the horizon $\rho = 0$,

$$E_{\tau i}^{(0)}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}]_{\rho=0} \\ = \left(R_{\tau i}^{(0)} - \frac{1}{2}g^{\alpha\beta}F_{\tau\alpha}^{(0)}F_{i\beta}^{(0)} - \frac{1}{2}m^2A_\tau^{(0)}A_i^{(0)} \right)_{\rho=0}. \quad (3.12)$$

We can compute $R_{\tau i}|_{\rho=0}$ on the metric (3.11). The answer is

$$R_{\tau i}[g_{\mu\nu}^{(0)}]_{\rho=0} = -\frac{1}{2}\partial_i X^{(0)}. \quad (3.13)$$

See Appendix A 1 for details. Using (3.13) and the stationarity condition (2.10) we get

$$E_{\tau i}^{(0)}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}]_{\rho=0} \\ = -\frac{1}{2}[\partial_i X^{(0)} - g^{\alpha\beta}(\partial_\alpha A_\tau^{(0)})F_{i\beta}^{(0)} + m^2A_\tau^{(0)}A_i^{(0)}]_{\rho=0}. \quad (3.14)$$

Using the expressions of inverse metric (Appendix A 1), we get

$$E_{\tau i}^{(0)}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}]_{\rho=0} = -\frac{1}{2}[\partial_i X^{(0)} - (\partial_\rho A_\tau^{(0)})(\partial_i A_\tau^{(0)}) - h^{jk}(\partial_j A_\tau^{(0)})F_{ik}^{(0)} + m^2A_\tau^{(0)}A_i^{(0)}]_{\rho=0}. \quad (3.15)$$

To establish the zeroth law τi component of the metric equation of motion alone is not enough; it turns out that we also have to consider the τ component of the vector equation of motion [second equation in (3.10)]

$$E_\tau^{(0)}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}]_{\rho=0} = (h_{(0)}^{ij}\mathcal{D}_i^{(0)}F_{j\tau}^{(0)} - m^2A_\tau^{(0)})_{\rho=0}, \quad (3.16)$$

where $\mathcal{D}_i^{(0)}$ is the covariant derivative with respect to the metric $h_{ij}^{(0)}$ (see Appendix A 2 for the derivation of the above equation). Using $\partial_\tau A_j = 0$ we can write

$$F_{j\tau}^{(0)} = \partial_j A_\tau^{(0)} \\ = \mathcal{D}_j^{(0)}A_\tau^{(0)}. \quad (3.17)$$

We now multiply (3.16) with $\sqrt{h^{(0)}}A_\tau^{(0)}$ and integrate over a $D-2$ dimensional compact spatial manifold C

$$\int_C d^{D-2}x \sqrt{h^{(0)}}A_\tau^{(0)}[h_{(0)}^{ij}\mathcal{D}_i^{(0)}\mathcal{D}_j^{(0)}A_\tau^{(0)} - m^2A_\tau^{(0)}]_{\rho=0} = 0 \\ \Rightarrow \int_C d^{D-2}x \sqrt{h^{(0)}}[-h_{(0)}^{ij}(\mathcal{D}_i^{(0)}A_\tau^{(0)})(\mathcal{D}_j^{(0)}A_\tau^{(0)}) - m^2(A_\tau^{(0)})^2]_{\rho=0} = 0 \quad (3.18)$$

where we have used the fact that the total derivative term vanishes as we are integrating on a compact manifold. Now, both terms are full squared terms; therefore, both of them have to vanish separately. Which gives $A_\tau^{(0)}|_{\rho=0} = 0$ which in turn implies $A_\tau^{(0)}$ can be written as

$$A_\tau^{(0)} = \rho \psi_\tau^{(0)}(\rho, x^i). \quad (3.19)$$

Similar computations have been used for the proof of the zeroth law in the context of gauge fields [25]. Substituting Eq. (3.19) into (3.15), we obtain

$$\partial_i X^{(0)}|_{\rho=0} = 0, \quad (3.20)$$

$$ds^2 = 2d\tau d\rho - \rho(X^{(0)}(\rho, x^i) + \alpha X^{(1)}(\rho, x^i))d\tau^2 + 2\rho(\omega_i^{(0)}(\rho, x^i) + \alpha\omega_i^{(1)}(\rho, x^i))d\tau dx^i + (h_{ij}^{(0)}(\rho, x^i) + \alpha h_{ij}^{(1)}(\rho, x^i))dx^i dx^j. \quad (3.21)$$

Now, we would compute the τi component of $E_{\mu\nu}^{(0)}$ on the horizon $\rho = 0$,

$$\begin{aligned} E_{\tau i}^{(0)}[g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)}, A_\mu^{(0)} + \alpha A_\mu^{(1)}]_{\rho=0} &= \alpha \partial_i X^{(1)} - \frac{1}{2} \alpha g^{\alpha\beta} F_{\tau\alpha}^{(0)} F_{i\beta}^{(1)} - \frac{1}{2} \alpha g^{\alpha\beta} F_{\tau\alpha}^{(1)} F_{i\beta}^{(0)} - \frac{1}{2} \alpha m^2 A_\tau^{(0)} A_i^{(1)} - \frac{1}{2} \alpha m^2 A_\tau^{(1)} A_i^{(0)} + \mathcal{O}(\alpha)^2 \\ &= \alpha \partial_i X^{(1)} + \frac{1}{2} \alpha (\partial_\rho A_\tau^{(0)}) (\partial_i A_\tau^{(1)}) + \frac{1}{2} \alpha h^{jk} (\partial_j A_\tau^{(1)}) F_{ik}^{(0)} - \frac{1}{2} \alpha m^2 A_\tau^{(1)} A_i^{(0)} + \mathcal{O}(\alpha)^2, \end{aligned} \quad (3.22)$$

where in the second line we have used the expression of $R_{\tau i}$ from Appendix A 1. In the third line, we have used Eq. (2.10) and Eq. (3.19). This will be the expression of the homogeneous part at any arbitrary k th order; we just have to replace the $A_\tau^{(1)}$ with $A_\tau^{(k)}$, that is the reason we call the homogeneous part universal.

Now, we turn to the computation of the source term. As previously noted, its structure depends on the details of the underlying theory. However, by employing abstract arguments based on boost weight, we will demonstrate that the source term vanishes. In this subsection, we will illustrate this result at the first subleading order. In the following subsection, we will extend the proof to arbitrary higher orders.

Although it is possible to directly present the general analysis for arbitrary higher orders without discussing the first subleading case, we choose to begin with the first subleading order for clarity and explicitness. For this discussion, we will closely follow the approach of [24]. Now, from (3.20) we can write $X^{(0)}$ as

$$X^{(0)}(\rho, x^i) = C^{(0)} + \rho F^{(0)}(\rho, x^i), \quad (3.23)$$

where $C^{(0)}$ is some constant. Therefore the zeroth-order metric becomes

$$\begin{aligned} ds^2 &= 2d\tau d\rho - \rho(C^{(0)} + \rho F^{(0)}(\rho, x^i))d\tau^2 \\ &\quad + 2\rho\omega_i^{(0)}(\rho, x^i)d\tau dx^i + h_{ij}^{(0)}(\rho, x^i)dx^i dx^j. \end{aligned} \quad (3.24)$$

which completes our proof of the zeroth law at the leading order.

B. Proof at the first subleading order

As we have discussed before, at any particular order, we will always divide the equation of motion into two parts: an *universal homogeneous part* and a *theory dependent source part*. Let us first compute the homogeneous part. From Eq. (3.7), we know that the homogeneous part is given by $E_{\mu\nu}^{(0)}[g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)}]$. To evaluate this, we will assume that $g_{\mu\nu}^{(0)}$ satisfies the equation of motion $E_{\mu\nu}^{(0)} = 0$. In our horizon adapted coordinate system the metric up to first subleading order can be written as

Now following [11,24], we will do the following coordinate transformation from $x^\mu = \{\tau, \rho, x^i\}$ to $\tilde{x}^\mu = \{v, r, x^i\}$:

$$v = \frac{2}{C^{(0)}} \exp\left(\frac{C^{(0)}}{2} \tau\right), \quad r = \rho \exp\left(-\frac{C^{(0)}}{2} \tau\right), \quad (3.25)$$

where v is the affine parameter along the null generator of the Killing horizon. The inverse transformation is the following:

$$\tau = \frac{2}{C^{(0)}} \log\left(\frac{C^{(0)}}{2} v\right), \quad \rho = \frac{C^{(0)}}{2} r v. \quad (3.26)$$

The infinitesimal distance in this new coordinate take the following form:

$$\begin{aligned} ds^2 &= 2dvdr - r^2 F^{(0)}(C^{(0)} r v / 2) dv^2 \\ &\quad + r \omega_i^{(0)}(C^{(0)} r v / 2) dv dx^i + h_{ij}^{(0)}(C^{(0)} r v / 2) dx^i dx^j. \end{aligned} \quad (3.27)$$

The vector field A_μ also transforms under this coordinate change according to

$$\tilde{A}_\mu^{(0)}(\tilde{x}^\alpha) = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} A_\nu^{(0)}(x^\alpha). \quad (3.28)$$

Component-wise this yields

$$\begin{aligned}
\tilde{A}_v^{(0)} &= \frac{2}{C^{(0)}v} A_\tau^{(0)}(C^{(0)}rv/2, x^i) + \frac{C^{(0)}}{2} r A_\rho^{(0)}(C^{(0)}rv/2, x^i), \\
\tilde{A}_r^{(0)} &= \frac{C^{(0)}}{2} v A_\rho^{(0)}(C^{(0)}rv/2, x^i), \\
\tilde{A}_i^{(0)} &= A_i^{(0)}(C^{(0)}rv/2, x^i).
\end{aligned} \tag{3.29}$$

From (3.7), we know that to compute the source part at first subleading order, we have to compute $\alpha E_{\mu\nu}^{(1)}[g_{\mu\nu}^{(0)}]$. We have brought the zeroth-order metric in the particular form (3.27). $E_{\mu\nu}^{(1)}$ denotes a covariant tensor constructed from the Riemann tensor, along with its derivatives and contractions. It will transform according to the way a two index tensor would transform. Therefore, we have

$$\begin{aligned}
E_{\mu\nu}^{(1)} &= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{E}_{\alpha\beta}^{(1)} \\
\Rightarrow E_{\tau i}^{(1)} &= \exp\left(\frac{C^{(0)}}{2}\tau\right) \tilde{E}_{vi}^{(1)} - \rho \frac{C^{(0)}}{2} \exp\left(-\frac{C^{(0)}}{2}\tau\right) \tilde{E}_{ri}^{(1)} \\
\Rightarrow E_{\tau i}^{(1)}|_{\rho=0} &= \exp\left(\frac{C^{(0)}}{2}\tau\right) \tilde{E}_{vi}^{(1)}|_{r=0}.
\end{aligned} \tag{3.30}$$

From the boost-weight arguments (see Appendix B), we know that $\tilde{E}_{vi}^{(1)}$ carries a boost weight of +1, which in principle can arise either from a ∂_v derivative or from the presence of $\tilde{A}_v$. Now using (3.19), at the relevant order we are working here, $\tilde{A}_v$ has the following structure:

$$\begin{aligned}
\tilde{A}_v^{(0)} &= r \chi_\tau^{(0)}(C^{(0)}rv/2, x^i) + \frac{C^{(0)}}{2} r A_\rho^{(0)}(C^{(0)}rv/2, x^i), \\
&\equiv r Z^{(0)}(rv, x^i).
\end{aligned} \tag{3.31}$$

Therefore, the boost weight of $\tilde{A}_v$ is effectively carried by the overall factor of r . Although r carries positive boost weight +1, it can not account for positive boost weight when we are computing some quantities on the horizon $r = 0$. Consequently, the positive boost weight +1 of $\tilde{E}_{vi}^{(1)}$ must be attributed to the derivative ∂_v . Therefore all the boost-weight arguments will proceed as [24].

Therefore, $\tilde{E}_{vi}^{(1)}$ contains one extra ∂_v derivative with respect to ∂_r or $\tilde{A}_r$. This extra ∂_v acts on quantities that depend on v through the product rv ; therefore we will bring a factor of r which will vanish once evaluated on $r = 0$.² Hence we conclude that $\tilde{E}_{vi}^{(1)}|_{r=0} = 0$. Therefore, from (3.30), we obtain:

$$E_{\tau i}^{(1)}|_{\rho=0} = 0. \tag{3.32}$$

²If we want to construct a positive boost weight +1 term by acting ∂_v on $\tilde{A}_r$, we have to act with two ∂_v derivatives which will again bring an overall factor of r .

Using this the total equation of motion at first subleading order becomes

$$\begin{aligned}
E_{\tau i}^{(0)}[g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)}, A_\mu^{(0)} + \alpha A_\mu^{(1)}]_{\rho=0} &+ \alpha E_{\tau i}^{(1)}[g_{\mu\nu}^{(0)}]_{\rho=0} \\
&= \alpha \partial_i X^{(1)} + \frac{1}{2} \alpha \partial_\alpha A_\tau^{(1)} F_i^{(0)\alpha} - \frac{1}{2} \alpha m^2 A_\tau^{(1)} A_i^{(0)} + \mathcal{O}(\alpha)^2. \\
&= \alpha \partial_i X^{(1)} + \frac{1}{2} \alpha (\partial_\rho A_\tau^{(0)})(\partial_i A_\tau^{(1)}) + \frac{1}{2} \alpha h^{jk} (\partial_j A_\tau^{(1)}) F_{ik}^{(0)} \\
&\quad - \frac{1}{2} \alpha m^2 A_\tau^{(1)} A_i^{(0)} + \mathcal{O}(\alpha)^2.
\end{aligned} \tag{3.33}$$

We will now turn to the vector equation of motion E_μ . Similar to the metric equation of motion, we can decompose E_μ into two parts: an *universal homogeneous part* and a *theory dependent source part*. Up to first subleading order it will be

$$E_\mu = E_\mu^{(0)}[g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)}, A_\mu^{(0)} + \alpha A_\mu^{(1)}] + \alpha E_\mu^{(1)}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}]. \tag{3.34}$$

We have to analyze the E_τ component of the above equation at $\rho = 0$. Let us first consider the homogeneous part

$$\begin{aligned}
E_\tau^{(0)}[g_{\mu\nu}^{(0)} + \alpha g_{\mu\nu}^{(1)}, A_\mu^{(0)} + \alpha A_\mu^{(1)}]_{\rho=0} \\
= (\alpha h_{(0)}^{ij} \mathcal{D}_i^{(0)} F_{j\tau}^{(1)} - \alpha m^2 A_\tau^{(1)})_{\rho=0}
\end{aligned} \tag{3.35}$$

where we have used the leading-order equation of motion (3.19) i.e., $A_\tau^{(0)} = 0$. See Appendix A 2 [specifically Eq. (A27)] for computational details.

Now, let us consider the source part

$$\alpha E_\tau^{(1)}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}]_{\rho=0}. \tag{3.36}$$

Using the boost-weight argument (see Appendix B), we will demonstrate that the source term in Eq. (3.36) vanishes. The reasoning closely parallels the derivation of Eq. (3.32), so we will not elaborate in details. Now, we will do the coordinate transformation (3.25). Under this coordinate transformation $E_\mu^{(1)}$ will transform as

$$\begin{aligned}
E_\mu^{(1)} &= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \tilde{E}_\alpha^{(1)} \\
\Rightarrow E_\tau^{(1)} &= \exp\left(\frac{C^{(0)}}{2}\tau\right) \tilde{E}_v^{(1)} - \rho \frac{C^{(0)}}{2} \exp\left(-\frac{C^{(0)}}{2}\tau\right) \tilde{E}_r^{(1)} \\
\Rightarrow E_\tau^{(1)}|_{\rho=0} &= \exp\left(\frac{C^{(0)}}{2}\tau\right) \tilde{E}_v^{(1)}|_{r=0}.
\end{aligned} \tag{3.37}$$

Again, $\tilde{E}_v^{(1)}$ carries a positive boost weight of +1. Therefore, every term in $\tilde{E}_v^{(1)}$ includes an additional ∂_v derivative. As a result, these terms vanish when evaluated at the horizon $\rho = 0$, as previously argued. Thus, by using the τ -component of the vector equation of motion, $E_\tau = 0$, we obtain

$$(\alpha h_{(0)}^{ij} \mathcal{D}_i^{(0)} F_{j\tau}^{(1)} - \alpha [m^{(0)}]^2 A_\tau^{(1)})_{\rho=0} = 0. \quad (3.38)$$

Now, we will use steps similar to (3.17) and (3.18). Using the stationarity condition we get

$$\begin{aligned} \int_C d^{D-2}x \sqrt{h^{(0)}} A_\tau^{(1)} [h_{(0)}^{ij} \mathcal{D}_i^{(0)} \mathcal{D}_j^{(0)} A_\tau^{(1)} - [m^{(0)}]^2 A_\tau^{(1)}]_{\rho=0} &= 0 \\ \Rightarrow \int_C d^{D-2}x \sqrt{h^{(0)}} [-h_{(0)}^{ij} (\mathcal{D}_i^{(0)} A_\tau^{(1)}) (\mathcal{D}_j^{(0)} A_\tau^{(1)}) - m^2 (A_\tau^{(1)})^2]_{\rho=0} &= 0. \end{aligned} \quad (3.40)$$

Since both terms are manifestly positive definite, they must vanish separately. This leads to the condition

$$A_\tau^{(1)}|_{\rho=0} = 0. \quad (3.41)$$

Using (3.41) in (3.33) we finally get

$$\partial_i X^{(1)}|_{\rho=0} = 0, \quad (3.42)$$

which completes the proof of the zeroth law at first subleading order.

C. Proof at arbitrary higher order

In this subsection, we extend our proof to arbitrary orders in the effective field theory expansion using the *method of induction*. According to this method, if the proof holds at the leading order, and we can demonstrate that it holds at order $(n+1)$ assuming it holds at order n , then the proof is valid at all higher orders.

As we have already shown that the proof works at the leading order in Sec. III A, our focus here will be on establishing this inductive step—namely, showing that the zeroth law holds at the $(n+1)$ th order, provided it holds at the n th order. Strictly speaking, the proof at the first subleading order in Sec. III B is not required for the induction to work. However, we include it as a clear and illustrative example of how the argument proceeds in practice. The essential framework required for generalization to higher perturbative orders is already embedded within the first subleading order proof. Therefore, we will keep this subsection brief.

Under our gauge choice, the metric takes the form given in Eq. (2.9). For convenience, we explicitly write it out below:

$$\begin{aligned} ds^2 &= 2d\tau dp - \rho X(\rho, x^i) d\tau^2 \\ &\quad + 2\rho \omega_i(\rho, x^i) d\tau dx^i + h_{ij}(\rho, x^i) dx^i dx^j, \end{aligned} \quad (3.43)$$

where X , ω_i , h_{ij} and also the vector field A_μ admit the effective field theory expansion written in Eq. (3.3). We will assume that we have solved both the equations of

$$F_{j\tau}^{(1)} = \partial_j A_\tau^{(1)} = \mathcal{D}_j^{(0)} A_\tau^{(1)}. \quad (3.39)$$

We now multiply (3.30) with $\sqrt{h^{(0)}} A_\tau^{(1)}$ and integrate over a $D-2$ dimensional compact spatial manifold C as we did in (3.18),

motion for the metric and also for the vector field up to order $\mathcal{O}(\alpha)^n$ implying

$$\begin{aligned} E_{\mu\nu}^{(0)} + \alpha E_{\mu\nu}^{(1)} + \cdots + \alpha^n E_{\mu\nu}^{(n)} &= \mathcal{O}(\alpha)^{n+1}, \\ E_\mu^{(0)} + \alpha E_\mu^{(1)} + \cdots + \alpha^n E_\mu^{(n)} &= \mathcal{O}(\alpha)^{n+1}. \end{aligned} \quad (3.44)$$

We will assume that we have already shown that the extrinsic curvature is constant up to order $\mathcal{O}(\alpha)^n$ which implies

$$\partial_i (X^{(0)} + \alpha X^{(1)} + \cdots + \alpha^n X^{(n)}) = \mathcal{O}(\alpha)^{n+1}. \quad (3.45)$$

Now, we will consider the equations of motion at the order $\mathcal{O}(\alpha)^{n+1}$. As we have discussed at the beginning of Sec. III, we can divide the equations of motion into two parts—homogeneous part and the source part. Let us first consider the homogeneous part,

$$E_{\tau i}^{(0)} [g_{\mu\nu}^{(0)} + \alpha^{n+1} g_{\mu\nu}^{(n+1)}, A_\mu^{(0)} + \alpha^{n+1} A_\mu^{(n+1)}]_{\rho=0}. \quad (3.46)$$

The homogeneous part by definition contains only and all the terms involving $g_{\mu\nu}^{(n+1)}$ and $A_\mu^{(n+1)}$. Therefore it has to have the same structure as that of (3.22) (see also Appendix D of [24]),

$$\begin{aligned} E_{\tau i}^{(0)} [g_{\mu\nu}^{(0)} + \alpha^{n+1} g_{\mu\nu}^{(n+1)}, A_\mu^{(0)} + \alpha^{n+1} A_\mu^{(n+1)}]_{\rho=0} \\ = \alpha^{n+1} \left[\partial_i X^{(n+1)} + \frac{1}{2} (\partial_\rho A_\tau^{(0)}) (\partial_i A_\tau^{(n+1)}) \right. \\ \left. + \frac{1}{2} h^{jk} (\partial_j A_\tau^{(n+1)}) F_{ik}^{(0)} - \frac{1}{2} m^2 A_\tau^{(n+1)} A_i^{(0)} \right] + \mathcal{O}(\alpha)^{n+2}. \end{aligned} \quad (3.47)$$

Similarly, the homogeneous part of the vector equation has the following form (3.35):

$$\begin{aligned} E_\tau^{(0)} [g_{\mu\nu}^{(0)} + \alpha^{n+1} g_{\mu\nu}^{(n+1)}, A_\mu^{(0)} + \alpha^{n+1} A_\mu^{(n+1)}]_{\rho=0} \\ = \alpha^{n+1} (h_{(0)}^{ij} \mathcal{D}_i^{(0)} F_{j\tau}^{(n+1)} - m^2 A_\tau^{(n+1)})_{\rho=0} + \mathcal{O}(\alpha)^{n+2}. \end{aligned} \quad (3.48)$$

We now turn to the computation of the source term. Before proceeding, we introduce a notation following [24]. Let $Y^{(n)}$ denote the coefficient of α^n in the effective field theory expansion of Y . We will denote the series correct up to order $\mathcal{O}(\alpha)^n$ by

$$Y^{[n]} = \sum_{k=0}^n \alpha^k Y^{(k)}. \quad (3.49)$$

In this notation, our assumption (3.44) implies

$$E_{\mu\nu}^{[n]}[g_{\mu\nu}^{[n]}, A_\mu^{[n]}] = \mathcal{O}(\alpha)^{n+1}, \quad E_\mu^{[n]}[g_{\mu\nu}^{[n]}, A_\mu^{[n]}] = \mathcal{O}(\alpha)^{n+1}. \quad (3.50)$$

The source parts in the $E_{\tau i}$ and E_τ equations of motion are respectively

$$E_{\tau i}^{[n+1]}[g_{\mu\nu}^{[n]}, A_\mu^{[n]}]_{\rho=0}, \quad \text{and} \quad E_\tau^{[n+1]}[g_{\mu\nu}^{[n]}, A_\mu^{[n]}]_{\rho=0}, \quad (3.51)$$

and Eq. (3.50) implies both Eqs. (3.51) are order $\mathcal{O}(\alpha)^{n+1}$ terms.

The truncated metric $g_{\mu\nu}^{[n]}$ has the following form:

$$ds^2 = 2d\tau d\rho - \rho X^{[n]}(\rho, x^i) d\tau^2 + 2\rho \omega_i^{[n]}(\rho, x^i) d\tau dx^i + h_{ij}^{[n]}(\rho, x^i) dx^i dx^j. \quad (3.52)$$

The condition (3.45) implies

$$X^{[n]}(\rho, x^i) = C^{[n]} + \rho F^{[n]}(\rho, x^i) \quad (3.53)$$

where $C^{[n]}$ is a constant and $F^{[n]}(\rho, x^i)$ denotes an arbitrary function of ρ and x^i .

Now following Sec. III B, we will do the following coordinate transformation from $x^\mu = \{\tau, \rho, x^i\}$ to $\tilde{x}^\mu = \{v, r, x^i\}$:

$$v = \frac{2}{C^{[n]}} \exp\left(\frac{C^{[n]}}{2} \tau\right), \quad r = \rho \exp\left(-\frac{C^{[n]}}{2} \tau\right), \quad (3.54)$$

where v is the affine parameter along the null generator of the Killing horizon. The infinitesimal distance in this new coordinate take the form

$$ds^2 = 2dvdr - r^2 F^{[n]}(C^{[n]}rv/2) dv^2 + r\omega_i^{[n]}(C^{[n]}rv/2) dv dx^i + h_{ij}^{[n]}(C^{[n]}rv/2) dx^i dx^j. \quad (3.55)$$

Now, $E_{\mu\nu}^{[n]}$ and $E_\mu^{[n]}$ are covariant tensors of rank two and one, respectively. Hence, they transform as (3.37) and (3.30)

$$E_\mu^{[n]} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \tilde{E}_\alpha^{[n]} \quad \text{and} \quad E_{\mu\nu}^{[n]} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{E}_{\alpha\beta}^{[n]}, \quad (3.56)$$

which gives

$$E_\tau^{[n]}|_{\rho=0} = \exp\left(\frac{C^{(0)}}{2} \tau\right) \tilde{E}_v^{[n]}|_{r=0} \quad \text{and} \quad E_{\tau i}^{[n]}|_{\rho=0} = \exp\left(\frac{C^{(0)}}{2} \tau\right) \tilde{E}_{vi}^{[n]}|_{r=0}. \quad (3.57)$$

Since both $\tilde{E}_v^{[n]}$ and $\tilde{E}_{vi}^{[n]}$ carry a positive boost weight of +1, the boost-weight argument from Sec. III B (see also Appendix B for further details) implies that they must vanish. Consequently, we have

$$E_\tau^{[n]}|_{\rho=0} = 0, \quad \text{and} \quad E_{\tau i}^{[n]}|_{\rho=0} = 0. \quad (3.58)$$

The source terms in both equations of motion vanish, which implies that the homogeneous parts (3.48) and (3.47) must vanish independently. Vanishing of (3.48) implies

$$\alpha^{n+1} (h_{(0)}^{ij} \mathcal{D}_i^{(0)} F_{j\tau}^{(n+1)} - m^2 A_\tau^{(n+1)})_{\rho=0} = 0. \quad (3.59)$$

Using steps similar to (3.39) and (3.40) we get

$$A_\tau^{(n+1)}|_{\rho=0} = 0. \quad (3.60)$$

Using (3.60) in (3.47) we get

$$\partial_i X^{(n+1)}|_{\rho=0} = 0. \quad (3.61)$$

This completes the proof that if we assume the constancy of surface gravity at order $\mathcal{O}(\alpha)^n$, then it also holds at order $\mathcal{O}(\alpha)^{n+1}$. Therefore, by the principle of *the method of induction*, the surface gravity remains constant to all orders in the effective field theory expansion.

IV. SOME OBSERVATIONS ON MORE GENERAL VECTOR THEORIES

We now turn to a broader class of theories where gravity is coupled to a vector field C_μ that has all D degrees of freedom propagating. Schematically, the effective field theory action can be written like that in (2.1). However, the leading-order Lagrangian $\mathcal{L}_{(0)}$ contains all possible kinetic terms that ensure D propagating degrees of freedom [28,32]. Explicitly, one may write

$$\mathcal{L}_{(0)} = R - \frac{1}{2} m^2 C_\mu C^\mu + c_1 (\nabla_\mu C_\nu) (\nabla^\mu C^\nu) + c_2 (\nabla_\mu C^\mu) (\nabla_\nu C^\nu) + c_3 (\nabla_\mu C_\nu) (\nabla^\nu C^\mu). \quad (4.1)$$

This can be further modified to (for a detailed derivation, see Appendix C 1)

$$\mathcal{L}_{(0)} = R + \frac{c_1}{2} \chi_{\mu\nu} \chi^{\mu\nu} - \frac{1}{2} m^2 C_\mu C^\mu + (c_1 + c_2 + c_3) (\nabla_\mu C^\mu)^2 - (c_1 + c_3) R_{\mu\nu} C^\mu C^\nu, \quad (4.2)$$

with $\chi_{\mu\nu} = \nabla_\mu C_\nu - \nabla_\nu C_\mu$. From now on, let us set $c_1 = -1/2$ so that, in analogy with the Maxwell case, the coefficient of the $\chi_{\mu\nu} \chi^{\mu\nu}$ term becomes $-1/4$.

Unlike the gauge-invariant case, these additional terms already appear at the same EFT order as the mass and Maxwell-like kinetic terms. Their presence generically renders *all* D components of C_μ dynamical, so that in four spacetime dimensions the theory typically propagates four degrees of freedom. In what follows, we examine the implications of such generalized vector theories for the zeroth law of black hole mechanics, and identify the restricted subclasses for which the law continues to hold consistently.

In order to examine the robustness of the zeroth law in this generalized setting, we adopt the following strategy: starting from the full Lagrangian (4.2), we selectively switch off certain terms and analyze the resulting field equations near the horizon. This procedure isolates the role of individual terms in enforcing the constraint $\partial_i X|_{\rho=0} = 0$, which guarantees the constancy of the surface gravity. In particular, we focus on two cases.

A. Case I: $c_3 = \frac{1}{2}$

In this case, theory reduces to the Einstein-Proca sector supplemented solely by the divergence term $(\nabla_\mu C^\mu)^2$

$$\mathcal{L}_{(0)} = R - \frac{1}{4} \chi_{\mu\nu} \chi^{\mu\nu} - \frac{1}{2} m^2 C_\mu C^\mu + c_2 (\nabla_\mu C^\mu)^2. \quad (4.3)$$

Evaluating the τi component of the corresponding field equations on the horizon (see Appendix C 2), we get

$$E_{\tau i}^{(0)}|_{\rho=0} = R_{\tau i}^{(0)} - \frac{1}{2} g^{\alpha\beta} (\chi_{\tau\alpha}^{(0)} \chi_{i\beta}^{(0)}) - \frac{1}{2} m^2 C_\tau^{(0)} C_i^{(0)} - c_2 C_\tau^{(0)} \partial_i (\nabla \cdot C)^{(0)}. \quad (4.4)$$

Following the discussion in Sec. III, we now turn to the τ component of the vector equation of motion. The equation of motion for the vector field C_μ is

$$E_\mu^{(0)} = (\nabla^\nu \chi_{\nu\mu})^{(0)} - m^2 C_\mu^{(0)} - 2c_2 \nabla_\mu (\nabla_\nu C^\nu)^{(0)}, \quad (4.5)$$

which implies

$$E_\tau^{(0)}|_{\rho=0} = (\nabla^\nu \chi_{\nu\tau})^{(0)} - m^2 C_\tau^{(0)} - 2c_2 \nabla_\tau (\nabla \cdot C)^{(0)}, \\ = h_{(0)}^{ij} \mathcal{D}_i^{(0)} \chi_{j\tau}^{(0)} - m^2 C_\tau^{(0)} - 2c_2 \partial_\tau (\nabla \cdot C)^{(0)}. \quad (4.6)$$

The last term does not contribute, as the Killing vector for our gauge choice is ∂_τ . Hence, we obtain an equation similar to the Einstein-Proca case. We multiply (4.6) by $\sqrt{h} C_\tau^{(0)}$ and integrating over the compact horizon cross-section C , total derivatives drop out, and we obtain a sum of squares

$$\int_C d^{D-2} x \sqrt{h^{(0)}} [-h_0^{ij} (\mathcal{D}_i C_\tau^{(0)}) (\mathcal{D}_j C_\tau^{(0)}) - m^2 (C_\tau^{(0)})^2]_{\rho=0} = 0, \quad (4.7)$$

implying

$$C_\tau^{(0)}|_{\rho=0} = 0. \quad (4.8)$$

Inserting (4.8) into (4.4) leaves $E_{\tau i}^{(0)}|_{\rho=0} = \frac{1}{2} \partial_i X^{(0)}$, so $E_{\tau i}^{(0)} = 0$ implies

$$\partial_i X^{(0)}|_{\rho=0} = 0 \Rightarrow \partial_i \kappa = 0,$$

which establishes the constancy of the surface gravity on the horizon. Hence, the zeroth law continues to hold in this case.

B. Case II: $c_3 = \text{arbitrary}$

Keeping the nonminimal coupling at leading order, the quadratic Lagrangian reads

$$\mathcal{L}_{(0)} = R - \frac{1}{4} \chi_{\mu\nu} \chi^{\mu\nu} - \frac{1}{2} m^2 C_\mu C^\mu + c' (\nabla_\mu C^\mu)^2 - d' R_{\mu\nu} C^\mu C^\nu. \quad (4.9)$$

The mixed Einstein equation then takes the form (for detailed calculation see Appendix C 2).

$$E_{\tau i}^{(0)}|_{\rho=0} = -\frac{\partial_i X^{(0)}}{2} - \frac{1}{2} g^{\alpha\beta} \chi_{\tau\alpha}^{(0)} \chi_{i\beta}^{(0)} - \frac{1}{2} m^2 C_\tau^{(0)} C_i^{(0)} - c' C_\tau^{(0)} [\partial_i (\nabla \cdot C)^{(0)}] - d' (2C_i C^j R_{\tau j}^{(0)} + P C_\tau^{(0)} + Q \partial_i C_\tau^{(0)}) \\ = -\frac{\partial_i X^{(0)}}{2} - \frac{1}{2} g^{\alpha\beta} \chi_{\tau\alpha}^{(0)} \chi_{i\beta}^{(0)} - \frac{1}{2} m^2 C_\tau^{(0)} C_i^{(0)} - c' C_\tau^{(0)} [\partial_i (\nabla \cdot C)^{(0)}] - d' (-\partial_j X^{(0)} C_i C^j + P C_\tau^{(0)} + Q \partial_i C_\tau^{(0)}). \quad (4.10)$$

Thus, for the zeroth law to hold, all terms except $\partial_i X^{(0)}$ in the above equation must vanish. This requires imposing the horizon conditions $C_\tau|_{\rho=0} = 0$. We try to derive this condition by manipulating the τ -component of the vector equation of motion as we did previously. The equation of motion for the vector field is

$$E_\mu^{(0)} = (\nabla^\nu \chi_{\nu\mu})^{(0)} - m^2 C_\mu^{(0)} - 2c' \nabla_\mu (\nabla_\nu C^\nu)^{(0)} - 2d' R_{\mu\nu}^{(0)} C_{(0)}^\nu. \quad (4.11)$$

Its τ component is

$$E_\tau^{(0)}|_{\rho=0} = h_{(0)}^{ij} \mathcal{D}_i^{(0)} \chi_{j\tau}^{(0)} - m^2 C_\tau^{(0)} - 2d' \left[C_\tau^{(0)} \mathcal{M} - \frac{1}{2} h^{ij} \partial_i X^{(0)} C_j^{(0)} \right], \quad (4.12)$$

where

$$\mathcal{M} = -\partial_\rho X^{(0)} - \frac{1}{2} \omega_i^{(0)} \omega_{(0)}^i + \frac{1}{2} \partial_i \omega_{(0)}^i - \frac{1}{4} \partial_\rho h_{ij} h^{ij} X^{(0)} + \frac{1}{2} \Gamma_{jk}^j \omega_{(0)}^k. \quad (4.13)$$

Multiplying (4.12) by $\sqrt{h} C_\tau^{(0)}$ and integrating over the compact cross-section C , we are unable to rearrange the integrand into whole square terms and, therefore, we are not able to derive the required constraint on C_τ . Thus, we could not establish the $\partial_i X^{(0)}$ in full generality for case II. In short, fully dynamical vector fields C_μ differ qualitatively from gauge fields B_μ and massive Proca fields A_μ , and demanding universal horizon thermodynamics for higher curvature theories of gravity coupled to vector fields C_μ might impose nontrivial constraints on admissible higher-derivative couplings.

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this article, we have shown that the surface gravity across a Killing horizon (not necessarily of bifurcate type) remains constant for stationary solutions of Einstein-Proca theory, including higher-order corrections within a controlled EFT expansion. Since surface gravity can be identified with temperature, this result implies the constancy of temperature for such solutions in the presence of a Killing horizon—precisely the statement of the zeroth law. In pure gravity, one demonstrates this by examining the τi component of the equations of motion in a specific near horizon coordinate system (2.9), as described in [24]. When a vector field is present, however, the τi component alone is insufficient; one must also include the τ component of the vector equation of motion [25].

The proof is manifestly perturbative: we expand the metric and the Proca field in the EFT parameter (2.8) and isolate at each order of the EOM a universal homogeneous part and a theory-dependent source part. The homogeneous part is simple as it has the same universal structure at all orders, whereas the source part has different complicated forms in different perturbative orders as it depends on the details of the particular higher curvature terms. By applying “boost-weight” arguments, we have shown abstractly that the source term vanishes at each order (see Sec. III). Implementing these arguments in the presence of a Proca field introduces subtleties beyond those encountered in the pure-gravity case [10,12,24], which we now briefly discuss.

Consider the metric (3.27) and (3.55) which are written in the horizon adapted coordinate system $\tilde{x}^\mu = \{v, r, x^i\}$ where ∂_v is the null generator of the Killing horizon with v being the affine parameter. We write the metric here for ease of referencing in a slightly notationally convenient form

$$ds^2 = 2dvdr - r^2 X(rv, x^i) dv^2 + 2r\omega_i(rv, x^i) dv dx^i + h_{ij}(rv, x^i) dx^i dx^j. \quad (5.1)$$

This metric is invariant under the following boost transformation as discussed in Appendix B,

$$r \rightarrow \lambda r, \quad v \rightarrow \frac{v}{\lambda}, \quad (5.2)$$

where λ is a constant. Then we can classify different terms according to their boost weight, e.g., if we consider the $\tilde{E}_{vi}$ component of the equation of motion, it has boost weight +1. In pure gravity, only ∂_v carries positive boost weight, which implies that on the Killing horizon ($r = 0$), $\tilde{E}_{vi}$ must include one extra ∂_v compared to ∂_r . Therefore, for a stationary solution, since $\tilde{E}_{vi}$ depends on r only through the product rv , it vanishes once evaluated at $r = 0$. This observation forms the core of the abstract argument presented in [24,25] for the vanishing of the source term. In the presence of a Proca field, one might naively expect that the component A_v itself can carry positive boost weight, thereby invalidating the straightforward logic outlined above. However, we have shown that even with a Proca field, in the stationary scenario relevant for our discussion, it is only ∂_v that carries positive boost weight (see Sec. III).

In Sec. IV, we have considered an even broader class of theories involving vector field C_μ , where unlike the Proca theory, the temporal component of the vector field also propagates. For this general class, we are unable to prove the zeroth law for the most general Lagrangian, and succeeded only for a restricted subclass. Our analysis of the two diagnostic cases discussed in Sec. IV highlights the central role played by the curvature coupling in this generalized setup. In Case I, where the nonminimal term $d' R_{\mu\nu} C^\mu C^\nu$ is absent, the zeroth law remains intact: the

surface gravity is constant across the horizon once the standard regularity conditions are imposed. In Case II, however, where curvature coupling is retained at leading order, we are unable to demonstrate that $C_\tau = 0$, and therefore cannot establish the zeroth law.

Taken together, these results suggest that the zeroth law can serve as a practical diagnostic tool for constraining higher derivative terms. Requiring universal horizon thermodynamics may act as a filter, ruling out certain higher curvature contributions. Pursuing this perspective more systematically—potentially extending it to higher-spin EFTs—is an important direction for future work. It would be interesting to investigate whether similar obstructions arise in massive spin-2 or higher-spin EFTs, and how these connect to positivity bounds and other consistency conditions.

It is also well known that these more general theories typically exhibit Ostrogradsky (ghost) instabilities, but certain special combinations of Lagrangian terms can cancel the ghost [28]. It would therefore be interesting to investigate whether the zeroth law can be established under those ghost-free restrictions; we leave this question for future work.

As we have emphasized several times, our proof is inherently perturbative. A nonperturbative proof of the zeroth law would be a significant advancement and is left for future investigation.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: SOME COMPUTATIONAL DETAILS

1. Inverse metric, Christoffel symbols, and Ricci tensor

In this appendix we will compute Christoffel symbols and Ricci tensor for the metric

$$ds^2 = 2d\tau d\rho - \rho X(\rho, x^i) d\tau^2 + 2\rho\omega_i(\rho, x^i) d\tau dx^i + h_{ij}(\rho, x^i) dx^i dx^j. \quad (\text{A1})$$

The inverse metric components are

$$\begin{aligned} g^{\tau\tau} &= 0, & g^{\tau\rho} &= 1, & g^{\tau i} &= 0, & g^{\rho\rho} &= \rho X + \rho^2 \omega^2, \\ g^{\rho i} &= -\rho\omega^i, & g^{ij} &= h^{ij}, \end{aligned} \quad (\text{A2})$$

where h^{ij} is defined as $h^{ik}h_{kj} = \delta^i_j$, ω^i is defined as $\omega^i = h^{ij}\omega_j$, and ω^2 is defined as $\omega^2 = \omega^i\omega_i$.

We now proceed to the computation of Christoffel symbols. Different components of the Christoffel symbols on the horizon ($\rho = 0$) are

$$\begin{aligned} \Gamma_{\rho\rho}^\rho &= 0, & \Gamma_{\tau\rho}^\rho &= -\frac{1}{2}X, & \Gamma_{\rho j}^\rho &= \frac{1}{2}\omega_j, & \Gamma_{i\tau}^\rho &= 0, & \Gamma_{ij}^\rho &= 0, & \Gamma_{\tau\tau}^\rho &= 0, & \Gamma_{\tau\tau}^\tau &= \frac{1}{2}X, & \Gamma_{\tau\rho}^\tau &= 0, \\ \Gamma_{\rho\rho}^\tau &= 0, & \Gamma_{\tau i}^\tau &= -\frac{1}{2}\omega_i, & \Gamma_{ij}^\tau &= -\frac{1}{2}\partial_\rho h_{ij}, & \Gamma_{i\rho}^\tau &= 0, & \Gamma_{i\tau}^j &= 0, & \Gamma_{\tau\tau}^j &= 0, & \Gamma_{j\rho}^i &= \frac{1}{2}h^{ik}\partial_\rho h_{jk}, \\ \Gamma_{jk}^i &\equiv \hat{\Gamma}_{jk}^i = \frac{1}{2}h^{im}(\partial_j h_{km} + \partial_k h_{jm} - \partial_m h_{jk}), & \Gamma_{\tau\rho}^i &= \frac{1}{2}h^{ij}\omega_j, & \Gamma_{\rho\rho}^i &= 0. \end{aligned} \quad (\text{A3})$$

We will require the following components of the Christoffel symbols off the horizon up to order $\mathcal{O}(\rho)$:

$$\begin{aligned} \Gamma_{i\tau}^\rho &= -\frac{\rho}{2}[X\omega_i + \partial_i X], & \Gamma_{\tau i}^\tau &= -\frac{1}{2}[\omega_i + \rho\partial_\rho\omega_i], & \Gamma_{i\tau}^j &= \frac{1}{2}\rho\omega^j\omega_i - \frac{\rho}{2}h^{jk}[\partial_k\omega_i - \partial_i\omega_k], \\ \Gamma_{\tau\tau}^\tau &= \frac{1}{2}[X + \rho\partial_\rho X], & \Gamma_{\tau\tau}^\rho &= \frac{1}{2}\rho X^2, & \Gamma_{\tau\tau}^j &= -\frac{\rho}{2}Xh^{jk}\omega_k + \frac{\rho}{2}h^{jk}\partial_k X. \end{aligned} \quad (\text{A4})$$

With this, we can compute Ricci tensors, relevant for our analysis.

Evaluation of $R_{\tau i}$ on the horizon reads

$$R_{\tau i} = R^\tau_{\tau\tau i} + R^\rho_{\tau\rho i} + R^j_{\tau j i}, \quad (\text{A5})$$

$$\begin{aligned} R^\tau_{\tau\tau i}|_{\rho=0} &= \partial_\tau \Gamma^\tau_{i\tau} - \partial_i \Gamma^\tau_{\tau\tau} + \Gamma^\tau_{\tau\lambda} \Gamma^\lambda_{i\tau} - \Gamma^\tau_{i\lambda} \Gamma^\lambda_{\tau\tau} \\ &= -\frac{1}{2} \partial_i X, \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} R^\rho_{\tau\rho i}|_{\rho=0} &= \partial_\rho \Gamma^\rho_{i\tau} - \partial_i \Gamma^\rho_{\rho\tau} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{i\tau} - \Gamma^\rho_{i\lambda} \Gamma^\lambda_{\rho\tau} \\ &= 0, \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} R^j_{\tau j i}|_{\rho=0} &= \partial_j \Gamma^j_{i\tau} - \partial_i \Gamma^j_{j\tau} + \Gamma^j_{j\lambda} \Gamma^\lambda_{i\tau} - \Gamma^j_{i\lambda} \Gamma^\lambda_{j\tau} \\ &= 0. \end{aligned} \quad (\text{A8})$$

Finally we get

$$R_{\tau i}|_{\rho=0} = -\frac{1}{2} \partial_i X|_{\rho=0}. \quad (\text{A9})$$

Evaluation of $R_{\tau\rho}$ on the horizon reads

$$R_{\tau\rho} = R^\tau_{\tau\tau\rho} + R^\rho_{\tau\rho\rho} + R^j_{\tau j\rho}. \quad (\text{A10})$$

Let us first compute different components of the Riemann tensor

$$\begin{aligned} R^\tau_{\tau\tau\rho}|_{\rho=0} &= \partial_\tau \Gamma^\tau_{\rho\tau} - \partial_\rho \Gamma^\tau_{\tau\tau} + \Gamma^\tau_{\tau\lambda} \Gamma^\lambda_{\rho\tau} - \Gamma^\tau_{\rho\lambda} \Gamma^\lambda_{\tau\tau} \\ &= -\partial_\rho X - \frac{1}{4} \omega_i \omega^i, \end{aligned} \quad (\text{A11})$$

$$\begin{aligned} R^\rho_{\tau\rho\rho}|_{\rho=0} &= \partial_\rho \Gamma^\rho_{\rho\tau} - \partial_\tau \Gamma^\rho_{\rho\rho} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\rho\tau} - \Gamma^\rho_{\tau\lambda} \Gamma^\lambda_{\rho\rho} \\ &= 0, \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} R^j_{\tau j\rho}|_{\rho=0} &= \partial_j \Gamma^j_{\rho\tau} - \partial_\rho \Gamma^j_{j\tau} + \Gamma^j_{j\lambda} \Gamma^\lambda_{\rho\tau} - \Gamma^j_{\rho\lambda} \Gamma^\lambda_{j\tau} \\ &= -\frac{1}{4} \omega_i \omega^i - \frac{1}{4} \partial_\rho h_{ij} h^{ij} X + \frac{1}{2} \partial_i \omega^i + \frac{1}{2} \Gamma^j_{jk} \omega^k. \end{aligned} \quad (\text{A13})$$

Finally we get

$$R_{\tau\rho}|_{\rho=0} = -\partial_\rho X - \frac{1}{2} \omega^i \omega_i - \frac{1}{4} \partial_\rho h_{ij} h^{ij} X + \frac{1}{2} \partial_i \omega^i + \frac{1}{2} \Gamma^j_{jk} \omega^k. \quad (\text{A14})$$

In this appendix, we have thus far refrained from performing any expansion in the effective field theory parameter α . In principle, any of the expressions above can be expanded to the desired order. For instance, let us compute h^{ij} up to order $\mathcal{O}(\alpha)^2$. To do this we have to expand $X(\rho, x^i)$, $\omega_i(\rho, x^i)$ and $h_{ij}(\rho, x^i)$ up to order $\mathcal{O}(\alpha)^2$,

$$\begin{aligned} X(\rho, x^i) &= X^{(0)}(\rho, x^i) + \alpha X^{(1)}(\rho, x^i) + \alpha^2 X^{(2)}(\rho, x^i), \\ \omega_i(\rho, x^i) &= \omega_i^{(0)}(\rho, x^i) + \alpha \omega_i^{(1)}(\rho, x^i) + \alpha^2 \omega_i^{(2)}(\rho, x^i), \\ h_{ij}(\rho, x^i) &= h_{ij}^{(0)}(\rho, x^i) + \alpha h_{ij}^{(1)}(\rho, x^i) + \alpha^2 h_{ij}^{(2)}(\rho, x^i). \end{aligned} \quad (\text{A15})$$

Then we can compute h^{ij} which has the following expression:

$$h^{ij} = [h^{(0)}]^{ij} + \alpha [h^{(1)}]^{ij} + \alpha^2 [h^{(2)}]^{ij} \quad (\text{A16})$$

where $[h^{(0)}]^{ij}$ is the inverse of $h_{ij}^{(0)}$ and

$$\begin{aligned} [h^{(1)}]^{ij} &= -[h^{(0)}]^{im} h_{mn}^{(1)} [h^{(0)}]^{nj}, \\ [h^{(2)}]^{ij} &= [h^{(0)}]^{im} h_{mk}^{(1)} [h^{(0)}]^{k\ell} h_{\ell n}^{(1)} [h^{(0)}]^{nj} - [h^{(0)}]^{im} h_{mn}^{(2)} [h^{(0)}]^{nj}. \end{aligned} \quad (\text{A17})$$

In a similar manner, we can compute other quantities as well. Next, we provide the expression of $\hat{\Gamma}_{jk}^i$ up to order $\mathcal{O}(\alpha)$, as this will be necessary for our further computation,

$$\hat{\Gamma}_{jk}^i = [\hat{\Gamma}^{(0)}]_{jk}^i + \alpha [\hat{\Gamma}^{(1)}]_{jk}^i, \quad (\text{A18})$$

where

$$\begin{aligned} [\hat{\Gamma}^{(0)}]_{jk}^i &= \frac{1}{2} h_{(0)}^{i\ell} [\partial_j h_{k\ell}^{(0)} + \partial_k h_{j\ell}^{(0)} - \partial_\ell h_{jk}^{(0)}], \\ [\hat{\Gamma}^{(1)}]_{jk}^i &= \frac{1}{2} h_{(0)}^{i\ell} [\partial_j h_{k\ell}^{(1)} + \partial_k h_{j\ell}^{(1)} - \partial_\ell h_{jk}^{(1)}] \\ &\quad - \frac{1}{2} h_{(1)}^{i\ell} [\partial_j h_{k\ell}^{(0)} + \partial_k h_{j\ell}^{(0)} - \partial_\ell h_{jk}^{(0)}]. \end{aligned} \quad (\text{A19})$$

From (A9) we can write $R_{\tau i}|_{\rho=0}$ up to any arbitrary order as

$$\begin{aligned} R_{\tau i}|_{\rho=0} &= -\frac{1}{2} \partial_i (X^{(0)}(\rho, x^j) + \alpha X^{(1)}(\rho, x^j) \\ &\quad + \alpha^2 X^{(2)}(\rho, x^j) + \dots)_{\rho=0}. \end{aligned} \quad (\text{A20})$$

2. Simplification of $E_\tau^{(0)}$

$E_\mu^{(0)}$ has the following structure:

$$E_\mu^{(0)} = \nabla^\nu F_{\nu\mu} - m^2 A_\mu. \quad (\text{A21})$$

In this appendix, we will explicitly compute $E_\tau^{(0)}[g_{\alpha\beta}^{(0)} + \alpha g_{\alpha\beta}^{(1)}, A_\alpha^{(0)} + \alpha A_\alpha^{(1)}]$ up to $\mathcal{O}(\alpha)$,

$$\begin{aligned} E_\tau^{(0)} &= \nabla^\nu F_{\nu\tau} - m^2 A_\tau \\ &= g^{\nu\lambda} \nabla_\lambda F_{\nu\tau} - m^2 A_\tau. \end{aligned} \quad (\text{A22})$$

Here, the coordinate system is $x^\mu = \{\tau, \rho, x^i\}$. Therefore, we get

$$E_\tau^{(0)} = g^{\rho\lambda} \nabla_\lambda F_{\rho\tau} + g^{i\lambda} \nabla_\lambda F_{i\tau} - m^2 A_\tau. \quad (\text{A23})$$

Now, we evaluate the above equation at $\rho = 0$,

$$\begin{aligned} E_\tau^{(0)}|_{\rho=0} &= (\nabla_\tau F_{\rho\tau} + h^{ij} \nabla_j F_{i\tau} - m^2 A_\tau)|_{\rho=0} \\ &= [\partial_\tau F_{\rho\tau} - \Gamma_{\tau\rho}^\lambda F_{\lambda\tau} - \Gamma_{\tau\tau}^\lambda F_{\rho\lambda} + h^{ij}(\partial_j F_{i\tau} - \Gamma_{ji}^\lambda F_{\lambda\tau} - \Gamma_{j\tau}^\lambda F_{i\lambda}) - m^2 A_\tau]|_{\rho=0} \\ &= [-\Gamma_{\tau\rho}^\rho F_{\rho\tau} - \Gamma_{\tau\rho}^i F_{i\tau} - \Gamma_{\tau\tau}^\tau F_{\rho\tau} + h^{ij}(\partial_j F_{i\tau} - \hat{\Gamma}_{ji}^k F_{k\tau} - \Gamma_{j\tau}^\tau F_{i\tau}) - m^2 A_\tau]|_{\rho=0} \\ &= [-\Gamma_{\tau\rho}^i F_{i\tau} + h^{ij}(\partial_j F_{i\tau} - \hat{\Gamma}_{ji}^k F_{k\tau} - \Gamma_{j\tau}^\tau F_{i\tau}) - m^2 A_\tau]|_{\rho=0} \\ &= [h^{ij}(\partial_j F_{i\tau} - \hat{\Gamma}_{ji}^k F_{k\tau}) - m^2 A_\tau]|_{\rho=0}. \end{aligned} \quad (\text{A24})$$

Now, we will use EFT expansion for the metric and the vector field and expand $E_\tau^{(0)}$ in powers of α . Schematically, we represent this as

$$E_\tau^{(0)}[g_{\alpha\beta}^{(0)} + \alpha g_{\alpha\beta}^{(1)} + \dots, A_\alpha^{(0)} + \alpha A_\alpha^{(1)} + \dots]_{\rho=0} = (E_\tau^{(0,0)} + \alpha E_\tau^{(0,1)} + \alpha^2 E_\tau^{(0,2)} + \dots)_{\rho=0}. \quad (\text{A25})$$

Now, we will explicitly compute $E_\tau^{(0)}$ up to order $\mathcal{O}(\alpha)$,

$$\begin{aligned} E_\tau^{(0)}[g_{\alpha\beta}^{(0)} + \alpha g_{\alpha\beta}^{(1)} + \dots, A_\alpha^{(0)} + \alpha A_\alpha^{(1)} + \dots]_{\rho=0} \\ &= [[h^{(0)}]^{ij} + \alpha[h^{(1)}]^{ij}][\partial_j F_{i\tau}^{(0)} + \alpha\partial_j F_{i\tau}^{(1)} - ([\hat{\Gamma}^{(0)}]_{jk}^i + \alpha[\hat{\Gamma}^{(1)}]_{jk}^i)(F_{k\tau}^{(0)} + \alpha F_{k\tau}^{(1)})]_{\rho=0} - (m^{(0)} + \alpha m^{(1)})^2(A_\tau^{(0)} + \alpha A_\tau^{(1)}) \\ &= [h^{(0)}]^{ij}[\partial_j F_{i\tau}^{(0)} - [\hat{\Gamma}^{(0)}]_{jk}^i F_{k\tau}^{(0)}]_{\rho=0} - [m^{(0)}]^2 A_\tau^{(0)} + \alpha[h^{(1)}]^{ij}[\partial_j F_{i\tau}^{(0)} - [\hat{\Gamma}^{(0)}]_{jk}^i F_{k\tau}^{(0)}]_{\rho=0} \\ &\quad + \alpha[h^{(0)}]^{ij}(\partial_j F_{i\tau}^{(1)} - [\hat{\Gamma}^{(0)}]_{jk}^i F_{k\tau}^{(1)} - [\hat{\Gamma}^{(1)}]_{jk}^i F_{k\tau}^{(0)}) - 2\alpha m^{(0)} m^{(1)} A_\tau^{(0)} - \alpha[m^{(0)}]^2 A_\tau^{(1)}. \end{aligned} \quad (\text{A26})$$

Therefore we finally get

$$\begin{aligned} E_\tau^{(0,0)} &= [h^{(0)}]^{ij}[\partial_j F_{i\tau}^{(0)} - [\hat{\Gamma}^{(0)}]_{jk}^i F_{k\tau}^{(0)}]_{\rho=0} - [m^{(0)}]^2 A_\tau^{(0)} \\ &= [h^{(0)}]^{ij} \mathcal{D}_j^{(0)} F_{i\tau}^{(0)} - [m^{(0)}]^2 A_\tau^{(0)}, \\ E_\tau^{(0,1)} &= [h^{(1)}]^{ij}[\partial_j F_{i\tau}^{(0)} - [\hat{\Gamma}^{(0)}]_{jk}^i F_{k\tau}^{(0)}]_{\rho=0} + [h^{(0)}]^{ij}(\partial_j F_{i\tau}^{(1)} - [\hat{\Gamma}^{(0)}]_{jk}^i F_{k\tau}^{(1)} - [\hat{\Gamma}^{(1)}]_{jk}^i F_{k\tau}^{(0)}) - 2m^{(0)} m^{(1)} A_\tau^{(0)} - [m^{(0)}]^2 A_\tau^{(1)} \\ &= [h^{(1)}]^{ij} \mathcal{D}_j^{(0)} F_{i\tau}^{(0)} + [h^{(0)}]^{ij} \mathcal{D}_j^{(1)} F_{i\tau}^{(1)} - [h^{(0)}]^{ij} [\hat{\Gamma}^{(1)}]_{jk}^i F_{k\tau}^{(0)} - 2m^{(0)} m^{(1)} A_\tau^{(0)} - [m^{(0)}]^2 A_\tau^{(1)}, \end{aligned} \quad (\text{A27})$$

where $\mathcal{D}_j^{(0)}$ is a covariant derivative with respect to $h_{ij}^{(0)}$.

$C = 2\kappa$. With this (2.9) becomes

APPENDIX B: SUMMARY OF BOOST-WEIGHT ARGUMENTS

The most general near-horizon metric of a stationary black hole is given by (2.9). If the zeroth law is satisfied, that is $\partial_i X|_{\rho=0} = 0$, we can obtain a general solution of the metric coefficient as $X|_{\rho=0}$,

$$X(\rho, x^i) = C + \rho F(\rho, x^i), \quad (\text{B1})$$

where C is an integration constant and $F(\rho, x^i)$ is an arbitrary function of (ρ, x^i) . The condition $\partial_i X|_{\rho=0}$ fixes

$$\begin{aligned} ds^2 &= 2d\tau d\rho - \rho(C + \rho F(\rho, x^i))d\tau^2 \\ &\quad + 2\rho\omega_i(\rho, x^i)d\tau dx^i + h_{ij}(\rho, x^i)dx^i dx^j. \end{aligned} \quad (\text{B2})$$

Here, we can perform a particular coordinate transformation from (ρ, τ, x^i) to (r, v, x^i) by

$$\begin{aligned} v &= \frac{2}{C} \exp\left(\frac{C}{2}\tau\right), \\ r &= \rho \exp\left(-\frac{C}{2}\tau\right), \end{aligned} \quad (\text{B3})$$

to obtain

$$ds^2 = 2dvdr - r^2 F(Crv/2) dv^2 + r\omega_i(Crv/2) dv dx^i + h_{ij}(Crv/2) dx^i dx^j. \quad (\text{B4})$$

The horizon is now located at $r = 0$. This choice of metric is different from the previous one because here ∂_v is the affinely parametrized null generator of the horizon. In the metric (2.9), the metric components were independent of the coordinate τ , but here in (B4), they are functions of v , specifically functions of the product rv . This form of the metric is useful because, despite fixing the gauge, there is still some residual freedom to perform a scaling transformation of the following form (for a more detailed discussion, refer to Sec. 3 of [14]):

$$r \rightarrow \lambda r \quad \text{along with} \quad v \rightarrow \frac{v}{\lambda}, \quad (\text{B5})$$

where λ is a constant parameter. This particular transformation keeps (B4) invariant, which we call boost symmetry. This information about the stationary black hole metric helps us to understand how any covariant tensor will transform under this boost transformation. Therefore, the metric components in a stationary or equilibrium configuration will be of the form $g_{\mu\nu}^{bh}(rv, x^i)$, because it respects the boost symmetry. Any quantity outside this function is out of equilibrium, and the boost weight explains their stationarity deviation. Any covariant tensor $\mathcal{A}$ will transform in the following way:

$$\mathcal{A} \rightarrow \tilde{\mathcal{A}} = \lambda^w \mathcal{A}, \quad (\text{B6})$$

where w is the boost weight of $\mathcal{A}$. $\mathcal{A}$ is the positive boost-weight quantity if

$$A(rv, x^i) \rightarrow \partial_r^{m_r} \partial_v^{m_v} \tilde{A}(rv, x^i)|_{r=0} \quad \text{whenever } m_v > m_r. \quad (\text{B7})$$

$\mathcal{A}$ being a positive boost-weight quantity of boost-weight $m_v - m_r$ means if one operates $\partial_r^{m_r} \partial_v^{m_v}$ on $\tilde{A}(rv, x^i)$, it will leave behind an extra r , and hence it will vanish on the horizon $r = 0$. Thus, positive boost weight quantities will vanish on a stationary horizon. Thus, we can use this useful result if we can write in this coordinate system (B4).

Similar discussions have been done for the case of gauged vector fields in Sec. 4 of [14].

APPENDIX C: CALCULATIONS OF MORE GENERAL VECTOR THEORIES

1. Derivation of the general non-Proca Lagrangian

The leading-order terms in the most general higher-curvature gravity theories with nonminimal couplings to vector fields, particularly nongauge vector fields, can be represented by

$$\mathcal{L}_{(0)} = R - \frac{1}{2} m^2 C_\mu C^\mu + c_1 (\nabla_\mu C_\nu) (\nabla^\mu C^\nu) + c_2 (\nabla_\mu C^\mu) (\nabla_\nu C^\nu) + c_3 (\nabla_\mu C_\nu) (\nabla^\nu C^\mu). \quad (\text{C1})$$

The various derivative structures can be simplified. For instance,

$$c_1 (\nabla_\mu C_\nu) (\nabla^\mu C^\nu) = \frac{c_1}{2} \chi_{\mu\nu} \chi^{\mu\nu} + c_1 \nabla^\mu C_\nu \nabla^\nu C_\mu, \quad (\text{C2})$$

while

$$c_3 (\nabla_\mu C_\nu) (\nabla^\nu C^\mu) = c_3 (\nabla_\mu C^\mu) (\nabla_\nu C^\nu) - c_3 C^\mu R_{\mu\nu} C^\nu. \quad (\text{C3})$$

Combining these relations and absorbing coefficients into redefinitions, the final form of the Lagrangian can be written as

$$\mathcal{L}_{(0)} = R + \frac{c_1}{2} \chi_{\mu\nu} \chi^{\mu\nu} - \frac{1}{2} m^2 C_\mu C^\mu + (c_1 + c_2 + c_3) (\nabla_\mu C^\mu)^2 - (c_1 + c_3) R_{\mu\nu} C^\mu C^\nu \quad (\text{C4})$$

which is the compact form quoted in Eq. (4.2) of the main text.

2. Computational details for case I

$$\mathcal{L}_{(0)} = R - \frac{1}{4} \chi_{\mu\nu} \chi^{\mu\nu} - \frac{1}{2} m^2 C_\mu C^\mu + c_2 (\nabla_\mu C^\mu)^2. \quad (\text{C5})$$

Equation of motion with respect to the metric $g_{\mu\nu}$,

$$E_{\mu\nu}^{(0)} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{8} g_{\mu\nu} \chi_{\alpha\beta} \chi^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} \chi_{\mu\alpha} \chi_{\nu\beta} + \frac{1}{4} m^2 g_{\mu\nu} C^\alpha C_\alpha - \frac{1}{2} m^2 C_\mu C_\nu + c_2 H_{\mu\nu}, \quad (\text{C6})$$

where

$$H_{\mu\nu}^{(0)} = -\frac{1}{2} g_{\mu\nu} \nabla_\alpha C^\alpha \nabla_\beta C^\beta + (\nabla_\mu C_\nu + \nabla_\nu C_\mu) \nabla_\alpha C^\alpha - \nabla_\mu (C_\nu \nabla_\alpha C^\alpha) - \nabla_\nu (C_\mu \nabla_\beta C^\beta) + g_{\mu\nu} \nabla^\sigma (C_\sigma \nabla_\beta C^\beta). \quad (\text{C7})$$

Equation of motion with respect to the vector field C_μ ,

$$E_\mu^{(0)} = \nabla^\nu \chi_{\nu\mu} - m^2 C_\mu - 2c_2 \nabla_\mu (\nabla_\nu C^\nu). \quad (\text{C8})$$

For the zeroth law, the relevant component of the EOM is the E_{ti} component,

$$E_{ti}^{(0)} = R_{ti} - \frac{1}{2}g_{ti}R + \frac{1}{8}g_{ti}\chi_{\alpha\beta}\chi^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\chi_{\tau\alpha}\chi_{i\beta} + \frac{1}{4}m^2g_{ti}C^\alpha C_\alpha - \frac{1}{2}m^2C_\tau C_i + c_2H_{ti} \quad (C9)$$

$$\begin{aligned} H_{ti}^{(0)} &= -\frac{1}{2}g_{ti}\nabla_\beta C^\beta \nabla_\alpha C^\alpha + (\nabla_\tau C_i + \nabla_i C_\tau)\nabla_b C^b + [-\nabla_\tau(C_i \nabla_b C^b) - \nabla_i(C_\tau \nabla_b C^b) + g_{\tau i}\nabla^\sigma(\nabla_\sigma C^\alpha C_\alpha)] \\ &= (\nabla_\tau C_i + \nabla_i C_\tau)\nabla_b C^b + [-\nabla_\tau(\nabla_b C^\beta C_i) - \nabla_i(C_\tau \nabla_b C^\beta)] \\ &= (\partial_i C_\tau - 2\Gamma_{i\tau}^\tau C_\tau)\nabla_a C^a + \Gamma_{ti}^\tau C_\tau \nabla_b C^b - (\partial_i(C_\tau \nabla_b C^b) - \Gamma_{i\tau}^\tau C_\tau \nabla_b C^b) \\ &= -C_\tau[\partial_i(\nabla \cdot C)]. \end{aligned} \quad (C10)$$

3. Computational details for case II

$$\mathcal{L}_{(0)} = R - \frac{1}{4}F_{\mu\nu}\chi^{\mu\nu} - \frac{1}{2}m^2C_\mu C^\mu + c'(\nabla_\mu C^\mu)^2 - d'R_{\mu\nu}C^\mu C^\nu. \quad (C11)$$

Equation of motion with respect to the metric $g_{\mu\nu}$,

$$E_{\mu\nu}^{(0)} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{8}g_{\mu\nu}\chi_{\alpha\beta}\chi^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\chi_{\mu\alpha}\chi_{\nu\beta} + \frac{1}{4}m^2g_{\mu\nu}C^\alpha C_\alpha - \frac{1}{2}m^2C_\mu C_\nu + c'H_{\mu\nu} - d'D_{\mu\nu} \quad (C12)$$

where

$$H_{\mu\nu}^{(0)} = -\frac{1}{2}g_{\mu\nu}\nabla_\alpha C^\alpha \nabla_\beta C^\beta + (\nabla_\mu C_\nu + \nabla_\nu C_\mu)\nabla_\alpha C^\alpha - \nabla_\mu(C_\nu \nabla_\alpha C^\alpha) - \nabla_\nu(C_\mu \nabla_\alpha C^\alpha) + g_{\mu\nu}\nabla^\sigma(C_\sigma \nabla_\beta C^\beta) \quad (C13)$$

and

$$\begin{aligned} D_{\mu\nu}^{(0)} &= \frac{1}{2}[-C^\alpha C^\beta g_{\mu\nu}R_{\alpha\beta} - (\nabla_\mu C_\nu + \nabla_\nu C_\mu)(\nabla_\alpha C^\alpha) - C^\alpha(\nabla_\alpha \nabla_\mu C_\nu) - C^\alpha(\nabla_\alpha \nabla_\nu C_\mu) \\ &\quad + C_\nu(2C^\alpha R_{\mu\alpha} - \nabla_\alpha \nabla_\mu C^\alpha + \nabla^\alpha \nabla_\alpha C_\mu) + C_\mu(2C^\alpha R_{\nu\alpha} - \nabla_\alpha \nabla_\nu C^\alpha + \nabla^\alpha \nabla_\alpha C_\nu) \\ &\quad + C^\alpha g_{\mu\nu}(\nabla_\alpha \nabla_\beta C^\beta) - (\nabla_\nu C_\alpha - 2\nabla_\alpha C_\nu)(\nabla^\alpha C_\mu) - (\nabla_\mu C_\alpha)(\nabla^\alpha C_\nu) \\ &\quad + g_{\mu\nu}(\nabla_\alpha C^\alpha)(\nabla_\beta C^\beta) + C^\alpha g_{\mu\nu}(\nabla_\beta \nabla_\alpha C^\beta) + g_{\mu\nu}(\nabla_\alpha C_\beta)(\nabla^\beta C^\alpha)] \end{aligned} \quad (C14)$$

Evaluating the E_{ti} component, we get

$$\begin{aligned} E_{ti}^{(0)} &= R_{ti} - \frac{1}{2}g_{ti}R + \frac{1}{8}g_{ti}\chi_{\alpha\beta}\chi^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\chi_{\tau\alpha}\chi_{i\beta} + \frac{1}{4}m^2g_{ti}C^\alpha C_\alpha - \frac{1}{2}m^2C_\tau C_i + c'C_{ti} - d'D_{ti} \\ &= R_{ti} - \frac{1}{2}g^{\alpha\beta}\chi_{\tau\alpha}\chi_{i\beta} - \frac{1}{2}m^2C_\tau C_i + c'C_{ti} - d'D_{ti} \\ &= -\frac{\partial_i X^{(0)}}{2} - \frac{1}{2}g^{\alpha\beta}\chi_{\tau\alpha}\chi_{i\beta} - \frac{1}{2}m^2C_\tau C_i + c'C_{ti} - d'D_{ti}. \end{aligned} \quad (C15)$$

So, for the validity of the zeroth law, all the terms except the first must vanish for some constraint on the vector field C_μ . From (C10)

$$H_{ti}^{(0)} = -C_\tau[\partial_i(\nabla \cdot C)] \quad (C16)$$

$$\begin{aligned}
D_{\tau i}^{(0)} &= \frac{1}{2} [C_\tau (2C^\alpha R_{i\alpha} + \nabla_\alpha \nabla^\alpha C_i - \nabla_\alpha \nabla_i C^\alpha) - C^\alpha \nabla_\alpha \nabla_i C_\tau + C_i (2C^\alpha R_{\tau\alpha} + \nabla_\alpha \nabla^\alpha C_\tau - \nabla_\alpha \nabla_\tau C^\alpha) \\
&\quad - C^\alpha \nabla_\alpha \nabla_\tau C_i - (\nabla^\alpha C_\tau)(-2\nabla_\alpha C_i + \nabla_i C_\alpha) - (\nabla^\alpha C_i)(\nabla_\tau C_\alpha) - (\nabla_\alpha C^\alpha)(\nabla_i C_\tau + \nabla_\tau C_i)] \\
&= \frac{1}{2} [C_\tau (2C^\alpha R_{i\alpha} - \nabla_\alpha \nabla_i C^\alpha + \nabla^\alpha \nabla_\alpha C_i) + C_i (2(C_\tau R_{\tau\rho} + C^j R_{\tau j}) + \partial_j (-\nabla_\tau C^j + \nabla^j C_\tau) \\
&\quad + \Gamma_{j\tau}^\tau (\nabla_\tau C^j - \nabla^j C_\tau)) - C_\tau (\nabla_\rho \nabla_i C_\tau + \nabla_\rho \nabla_\tau C_i) - C_\rho (-\Gamma_{\tau\tau}^\tau \nabla_\tau C_i - \Gamma_{\tau i}^\tau \nabla_\tau C_\tau - \Gamma_{\tau\tau}^\tau \nabla_i C_\tau - \Gamma_{\tau i}^\tau \nabla_\tau C_\tau) \\
&\quad - C^j (\partial_j (\nabla_\tau C_i) - \Gamma_{j\tau}^\tau (\nabla_\tau C_i) - \Gamma_{ji}^\tau (\nabla_\tau C_\tau) + \partial_j (\nabla_i C_\tau) - \Gamma_{j\tau}^\tau (\nabla_i C_\tau)) \\
&\quad - (\nabla_i C_\tau \nabla^\tau C_\tau + \nabla_i C_\rho \nabla^\rho C_\tau + \nabla_i C_j \nabla^j C_\tau - 2(\nabla_\tau C_i \nabla^\tau C_\tau + \nabla_\rho C_i \nabla_\tau C_\tau + \nabla_j C_i \nabla^j C_\tau)) \\
&\quad - ((\nabla_\tau C_\tau)(\nabla^\tau C_i) + (\nabla_\tau C_\rho)(\nabla^\tau C_i) + (\nabla_\tau C_j)(\nabla^j C_i)) - (\nabla_\tau C_i + \nabla_i C_\tau)(\nabla_\alpha C^\alpha)]. \tag{C17}
\end{aligned}$$

Now, if we look at all the terms of $D_{\tau i}^{(0)}$, all the terms except $C_i C^j R_{\tau j}$ have one of the below listed terms:

- (i) C_τ ,
- (ii) $\nabla_\tau C_i = \partial_\tau C_i^0 - \Gamma_{i\tau}^\tau C_\tau$,
- (iii) $\nabla_i C_\tau = \partial_i C_\tau - \Gamma_{i\tau}^\tau C_\tau$,
- (iv) $\nabla_\tau C_\tau = \partial_\tau C_\tau^0 - \Gamma_{\tau\tau}^\tau C_\tau$.

Therefore, $D_{\tau i}$ can be written schematically as a combination of three types of terms,

$$D_{\tau i}^{(0)} = 2C_i C^j R_{\tau j}^{(0)} + P C_\tau^{(0)} + Q \partial_i C_\tau^{(0)} \tag{C18}$$

where P and Q denote, respectively, the coefficients of C_τ and $\partial_i C_\tau$, whose explicit forms are algebraically involved. Therefore, total $E_{\tau i}$ becomes

$$\begin{aligned}
E_{\tau i}^{(0)} &= -\frac{\partial_i X^{(0)}}{2} - \frac{1}{2} g^{\alpha\beta} \chi_{\tau\alpha}^{(0)} \chi_{i\beta}^{(0)} - \frac{1}{2} m^2 C_\tau^{(0)} C_i^{(0)} - c' C_\tau^{(0)} [\partial_i (\nabla \cdot C)]^{(0)} - d' (2C_i C^j R_{\tau j}^{(0)} + P C_\tau^{(0)} + Q \partial_i C_\tau^{(0)}) \\
&= -\frac{\partial_i X^{(0)}}{2} - \frac{1}{2} g^{\alpha\beta} \chi_{\tau\alpha}^{(0)} \chi_{i\beta}^{(0)} - \frac{1}{2} m^2 C_\tau^{(0)} C_i^{(0)} - c' C_\tau^{(0)} [\partial_i (\nabla \cdot C)]^{(0)} - d' (-\partial_j X^{(0)} C_i^{(0)} C_{(0)}^j + P C_\tau^{(0)} + Q \partial_i C_\tau^{(0)}). \tag{C19}
\end{aligned}$$

The equation of motion with respect to the vector field becomes

$$E_\mu^{(0)} = \nabla^\nu \chi_{\nu\mu} - m^2 C_\mu - 2c' \nabla_\mu (\nabla_\nu C^\nu) - 2d' R_{\mu\nu} C^\nu. \tag{C20}$$

If we evaluate the τ component of the above equation, we obtain

$$\begin{aligned}
E_\tau^{(0)} &= \nabla^\nu \chi_{\nu\tau}^{(0)} - b_1 m^2 C_\tau^{(0)} - \underline{2c' \nabla_\tau (\nabla_\nu C^\nu)} - 2d' R_{\tau\nu}^{(0)} C_{(0)}^\nu \\
&= h_{(0)}^{ij} \mathcal{D}_i^{(0)} \chi_{j\tau}^{(0)} - b_1 m^2 C_\tau^{(0)} - 2d' \left[C_\tau^{(0)} \left(-\partial_\rho X^{(0)} - \frac{1}{2} \omega_i^{(0)} \omega_{(0)}^i + \frac{1}{2} \partial_i \omega_{(0)}^i - \frac{1}{4} \partial_\rho h_{ij} h^{ij} X^{(0)} + \frac{1}{2} \Gamma_{jk}^j \omega_{(0)}^k \right) - \frac{1}{2} h^{ij} \partial_i X^{(0)} C_j^{(0)} \right] \\
&= h_{(0)}^{ij} \mathcal{D}_i^{(0)} \chi_{j\tau}^{(0)} - b_1 m^2 C_\tau^{(0)} - 2d' \left[C_\tau^{(0)} \mathcal{M} - \frac{1}{2} h^{ij} \partial_i X^{(0)} C_j^{(0)} \right] \tag{C21}
\end{aligned}$$

where we have defined

$$\mathcal{M} = -\partial_\rho X^{(0)} - \frac{1}{2} \omega_i^{(0)} \omega_{(0)}^i + \frac{1}{2} \partial_i \omega_{(0)}^i - \frac{1}{4} \partial_\rho h_{ij} h^{ij} X^{(0)} + \frac{1}{2} \Gamma_{jk}^j \omega_{(0)}^k \tag{C22}$$

and we have computed $R_{\tau\nu} C^\nu$ using (A9) and (A14)

$$\begin{aligned}
R_{\tau\nu} C^\nu &= R_{\tau\tau} C^\tau + R_{\tau\rho} C^\rho + R_{\tau i} C^i \\
&= \left(-\partial_\rho X - \frac{1}{2} \omega^i \omega_i - \frac{1}{4} \partial_\rho h_{ij} h^{ij} X + \frac{1}{2} \partial_i \omega^i + \frac{1}{2} \Gamma_{jk}^j \omega^k \right) C_\tau - \frac{1}{2} \partial_i X C^i. \tag{C23}
\end{aligned}$$

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