

Black hole thermodynamics at null infinity. I. Dual generalized second law

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The generalized second law (GSL) of black hole thermodynamics asserts the monotonic increase of the generalized entropy combining the black hole area and the entropy of quantum fields outside the horizon. Modern proofs of the GSL rely on information-theoretic methods and are typically formulated using algebras of observables defined on the event horizon together with a vacuum state invariant under horizon symmetries, inducing a geometric modular flow. In this work, we formulate a dual version of the generalized second law from the perspective of asymptotic observers at future null infinity, who do not have access to the black hole area. Our approach exploits the dependence of the second law on the choice of algebra of observables and of a reference state invariant under suitable symmetries, in close analogy with open quantum thermodynamics. Using algebraic quantum field theory and modular theory, we analyze several physically motivated vacuum states, including the Hartle-Hawking state and two classes of regularized vacua. We show that, at null infinity, the monotonic quantity governing an irreversible evolution is no longer the generalized entropy but rather a thermodynamic potential constructed from asymptotic observables. Depending on the chosen vacuum, this potential takes the form of the free energy or of a generalized grand potential built from the Bondi mass and additional (angular) mode-dependent chemical potentials. The resulting inequalities define a dual generalized second law at future null infinity, which can be consistently combined with the standard GSL involving variations of the black hole area.

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I. INTRODUCTION

For more than two centuries, thermodynamics, and in particular its second law, has been one of the cornerstones of physics. It has repeatedly guided the development of fundamental theories, from the birth of quantum mechanics—which resolved the ultraviolet catastrophe arising from the equipartition theorem for the electromagnetic field—to the modern search for a quantum theory of gravity [1,2]. A remarkable indication of this connection is that black holes behave as thermodynamic systems and possess an entropy given by the celebrated Bekenstein-Hawking formula [3–5]

$$S_{\text{BH}} = \frac{A}{4G\hbar}, \quad (1.1)$$

where A denotes the area of a spacelike section of the event horizon.

Unlike in classical gravity, however, quantum effects such as Hawking evaporation allow the horizon area to decrease, apparently violating the classical area theorem [6,7]. This observation led Bekenstein to propose the generalized entropy

$$S_{\text{gen}} = \frac{A}{4G\hbar} + S_{\text{out}} \quad (1.2)$$

whose monotonic increase constitutes the generalized second law (GSL) [5,8,9]. Understanding the microscopic origin of this law remains one of the central questions in black hole thermodynamics [10–24].

Modern proofs of the GSL [20,25–30] rely on the monotonicity of the relative entropy in quantum field theory [31–35]. In its most systematic form, due to Wall [28], one compares an arbitrary state with a preferred vacuum and exploits the monotonicity of the relative entropy under restriction to nested algebras of observables. Through the geometric interpretation of the modular Hamiltonian and the semiclassical Einstein equations, this

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information-theoretic statement becomes precisely the generalized second law.

Besides providing elegant proofs of the GSL, the relative entropy [33,36] has emerged as one of the fundamental quantities of quantum field theory. Unlike the von Neumann entropy, it remains finite for local algebras and has played a central role in subjects ranging from the Bekenstein bound to quantum energy inequalities and semiclassical gravity [37–45]. These developments suggest that the second law should not be regarded as a peculiarity of a particular spacetime construction but rather as a general consequence of the algebraic structure of quantum field theory.

A useful perspective on the GSL comes from open quantum thermodynamics [46–52]. There, changing the environment with which a system exchanges conserved quantities modifies the stationary state of the dynamics and changes the appropriate thermodynamic potential. An isolated system evolves toward states of increasing entropy, whereas a system coupled to a thermal reservoir [53,54] is governed instead by the monotonic decrease of the Helmholtz free energy [55]. More general reservoirs exchanging particles or other conserved quantities lead naturally to grand potentials and generalized Gibbs ensembles [56–61]. From this viewpoint, the freedom in choosing the vacuum state in quantum field theory strongly resembles the freedom of choosing the thermodynamic environment. Different vacua should therefore be expected to define different thermodynamic potentials and, consequently, different formulations of the second law.

An interesting setting in which to explore these ideas is future null infinity. Although future null infinity and the future event horizon are both nonexpanding null hypersurfaces and admit closely related algebraic descriptions [62–69], they correspond to different physical observers and therefore naturally suggest different thermodynamic descriptions. While the generalized entropy governs the evolution from the viewpoint of horizon observers, asymptotic observers only have access to the Bondi mass and the outgoing Hawking radiation. This raises the question of whether a second law can be formulated entirely in terms of asymptotic observables.

A first attempt was proposed by one of us in Ref. [65], where it was argued that asymptotic observers satisfy a *dual generalized second law*. In the idealized limit of a perfectly thermal Hawking flux, the relevant thermodynamic potential is the Helmholtz free energy built from the Bondi mass. More generally, graybody factors modify the stationary state seen at null infinity, producing additional nongeometric contributions that naturally take the form of effective chemical potentials. From the perspective advocated here, these corrections simply reflect that the asymptotic vacuum defines a thermodynamic environment which differs from an ideal thermal reservoir.

A realistic description of Hawking radiation therefore requires going beyond the Hartle-Hawking vacuum

[70–72]. Indeed, this state, which describes a black hole in thermal equilibrium, is incompatible with asymptotic flatness, as it corresponds to a thermal bath extending all the way to spatial infinity and carrying an infinite energy flux. Moreover, although the quantum field is thermal near the horizon, the effective potential surrounding the black hole backscatters part of the outgoing radiation, so the spectrum observed at future null infinity is no longer exactly thermal. In particular, high-angular-momentum modes are strongly suppressed. Motivated by these observations, we introduce two regularizations of the Hartle-Hawking vacuum. A hard regularization suppresses all modes above a prescribed angular momentum cutoff, whereas a soft regularization assigns an effective temperature to each angular momentum mode, thereby mimicking the angular momentum dependent transmission probabilities associated with graybody factors.

Within these three frameworks, i.e., the Hartle-Hawking vacuum and its hard and soft regularizations, we derive the corresponding dual generalized second laws. For the hard regularization, the relevant thermodynamic potential is the Helmholtz free energy,

$$\Delta M - T_H \Delta S \leq 0, \quad (1.3)$$

where M denotes the Bondi mass, S is the entropy of the quantum fields between two cuts of future null infinity, and T_H is the Hawking temperature. Combining this inequality with the generalized second law on the horizon yields

$$\Delta M - T_H \Delta \left(\frac{A}{4G} + S \right) \leq 0, \quad (1.4)$$

which reduces to the ordinary generalized second law when the asymptotic slices terminate on the same cut. For the soft regularization, the effective chemical potentials generated by graybody factors replace the Helmholtz free energy by a generalized grand potential,

$$\Delta M - \sum_{lm} \int_0^{+\infty} \mu_{\omega l} \Delta n_{\omega lm} d\omega - T_H \Delta S \leq 0, \quad (1.5)$$

whose precise form depends on the chosen regularization. The detailed assumptions underlying these inequalities are discussed in the main text.

In particular, compared with Ref. [65], the present work develops the dual generalized second law from a fully algebraic perspective. We construct explicitly the relevant algebras of observables, identify the corresponding vacuum states, derive their modular Hamiltonians, and establish the relation between the resulting modular energies and the geometric quantities entering the semiclassical Einstein equations. In addition, we introduce physically motivated regularizations of the Hartle-Hawking vacuum that account

for graybody effects while preserving the algebraic structure of the proof.

To keep the paper reasonably self-contained, we complement the main text with six Appendixes reviewing material scattered throughout the literature. Appendix A proves the Unruh effect for arbitrary nonexpanding null hypersurfaces, making it applicable both to horizons and future null infinity. Appendix B reviews the construction of vacuum states and one-particle Hilbert spaces from positive-frequency solutions. Appendix C introduces the basics of algebraic quantum field theory. Appendix D reviews von Neumann algebras and modular theory, while Appendix E relates the modular Hamiltonian to the integral of the normal-ordered stress-energy tensor. Finally, Appendix F complements the analysis of the paper by proving the dual GSL in the Minkowski vacuum, thereby connecting our framework with the results of Refs. [73–75].

The paper is organized as follows. Section II reviews the Schwarzschild geometry, introduces the matter content (namely, a scalar field) on the horizon and at future null infinity, and derives the corresponding semiclassical Einstein equations. Section III constructs the algebra of observables at null infinity, quantizes the field, and studies the Hartle-Hawking vacuum together with its hard and soft regularizations. Section IV establishes the dual generalized second law from the monotonicity of the relative entropy and derives the associated thermodynamic potentials in each of these settings. Throughout this work, we set $c = k_B = \hbar = G = 1$.

II. SETTING THE STAGE: KINEMATICS AND SEMICLASSICAL DYNAMICS

We consider in this work the maximal extension of an eternal Schwarzschild black hole in D -dimensions, with $D \geq 2$. Most of the time in this work, we will be in $D = 4$ dimensions. Upon conformal compactification, one gets the Penrose diagram depicted in Fig. 1. While regions I and II represent the space outside the event horizons, regions III and IV are, respectively, the black and white hole. Region I is bounded by the white hole horizon $\mathcal{H}_I^-$, the black hole horizon $\mathcal{H}_I^+$, past $\mathcal{I}_I^-$, and future $\mathcal{I}_I^+$ null infinity (and similarly for region II). The reunion of several of these regions will be widely used in the remaining of the paper, so we define them now once and for all:

- (i) The reunion $\mathcal{H}_I^- \cup \mathcal{H}_{II}^+$ will be called *the right horizon* and denoted $\mathcal{H}_R$.
- (ii) The reunion $\mathcal{H}_{II}^- \cup \mathcal{H}_I^+$ will be called *the left horizon* and denoted $\mathcal{H}_L$.
- (iii) The reunion $\mathcal{I}_I^+ \cup \mathcal{I}_{II}^-$ will be called *right maximally extended null infinity* and denoted $\mathcal{I}_R$.
- (iv) The reunion $\mathcal{I}_{II}^+ \cup \mathcal{I}_I^-$ will be called *left maximally extended null infinity* and denoted $\mathcal{I}_L$.

The reason why these regions are introduced will become clearer in the next paragraph. For the time being, we remark that Kruskal coordinates, which we denote $(\tilde{U}, \tilde{V})$, are a

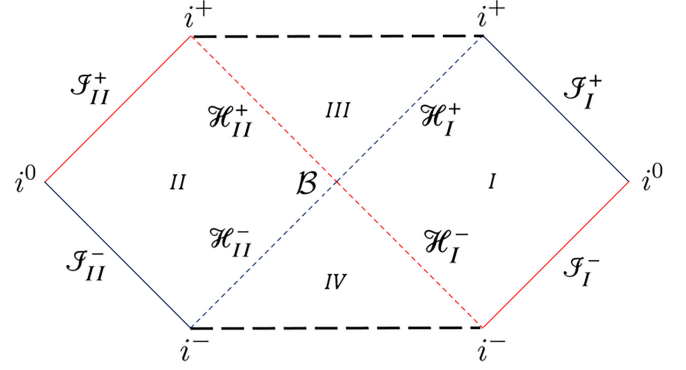


FIG. 1. Penrose diagram of the maximal extension of the Schwarzschild solution. The solid blue (respectively, red) regions depict $\mathcal{I}_R$ (respectively, $\mathcal{I}_L$), whereas the dashed blue (respectively, red) regions depict $\mathcal{H}_L$ (respectively, $\mathcal{H}_R$). Hence, the whole blue (respectively, red) region represents Σ_f (respectively, Σ_i).

suitable pair of coordinates from which we are able to describe the entire spacetime [we also have to complete these coordinates with a set of angular coordinates (x^A) , $A = 2, \dots, D$ in $D > 2$]. These coordinates, which are inertial and affine on $\mathcal{H}_R$ and $\mathcal{H}_L$ respectively, go to infinity at spacelike infinity i^0 and vanish at the bifurcation surface $\mathcal{B}$. Also, on $\mathcal{I}_R$, we have that $\tilde{V} = \pm\infty$, and on $\mathcal{I}_L$, we have $\tilde{U} = \pm\infty$. The affine coordinates on $\mathcal{I}_I^+$ and $\mathcal{I}_{II}^-$ are denoted u_+ and u_- , while v_+ and v_- are, respectively, affine on $\mathcal{I}_I^-$ and $\mathcal{I}_{II}^+$. The relations between the affine and Kruskal coordinates are

$$\tilde{U} = \begin{cases} -e^{-\kappa u_+} & \text{on } \mathcal{I}_I^+ \\ e^{\kappa u_-} & \text{on } \mathcal{I}_{II}^- \end{cases} \quad \text{and} \quad \tilde{V} = \begin{cases} e^{\kappa v_+} & \text{on } \mathcal{I}_I^- \\ -e^{-\kappa v_-} & \text{on } \mathcal{I}_{II}^+ \end{cases}, \quad (2.1)$$

where $\kappa = \frac{1}{4M}$ is the surface gravity.

This spacetime exhibits four isometries, one of which is generated by a Killing vector that is timelike in the regions I and II and can be written in terms of the Kruskal coordinates as

$$\xi = \kappa(\tilde{V}\partial_{\tilde{V}} - \tilde{U}\partial_{\tilde{U}}). \quad (2.2)$$

This Killing field generates the natural time translations in the aforementioned regions. Moreover, the left horizon $\mathcal{H}_L$ of the black hole is a Killing horizon for that vector (evaluated at $\tilde{U} = 0$), while $\mathcal{H}_R$ is Killing for the same vector but evaluated at $\tilde{V} = 0$.

A. Scalar field on the Schwarzschild background

We study the propagation and properties of a massless quantum field on such a background. For a massless scalar

field $\Phi = \frac{\phi}{r}$ the Klein-Gordon equation reads in the external regions

$$(\partial_{u_{\pm}} \partial_{v_{\pm}} + V(r, x^A)) \phi(u_{\pm}, v_{\pm}, x^A) = 0, \quad (2.3)$$

where $V(r, x^A)$ is a linear operator that vanishes in a Schwarzschild background at $r = 2M$ and $r = +\infty$, i.e., on the horizon and at null infinity. Also, $u_{\pm} = t_{\pm} - r_{\pm}^*$ and $v_{\pm} = t_{\pm} + r_{\pm}^*$ are¹ the retarded and advanced time coordinates which are finite in regions I and II. Note that a similar equation holds for massless fields with higher spins, especially for $s = 1, 2$, even though we focus in this work on scalar fields. Solving the equations of motion (2.3) requires specifying an initial Cauchy slice as well as a final one. Since in Schwarzschild spacetime, there are no stable bounded solutions for massless fields (they either fall into the black hole or escape at null infinity)² [76–79]; a good example of complete initial data surface on which we can specify the massless field's initial data is

$$\Sigma_i = \mathcal{H}_R \cup \mathcal{I}_L, \quad (2.4)$$

while a final null data surface can be given equivalently by

$$\Sigma_f = \mathcal{H}_L \cup \mathcal{I}_R \quad (2.5)$$

(see Fig. 1).

Since the linear operator V vanishes on the horizon and null infinity, it vanishes on Σ_i and Σ_f , so (2.3) reads

$$\partial_{u_{\pm}} \partial_{v_{\pm}} (\phi(u_{\pm}, v_{\pm}, x^A)) = 0 \quad \text{on } \Sigma_i \quad \text{and} \quad \Sigma_f. \quad (2.6)$$

The solutions of (2.6) on $\Sigma_{i,f}$ can be written as a sum separating left- and right-moving modes

$$\phi(u_{\pm}, v_{\pm}, x^A) = f(u_{\pm}, x^A) + g(v_{\pm}, x^A), \quad (2.7)$$

where $f(u_{\pm}, x^A)$ and $g(v_{\pm}, x^A)$ are arbitrary smooth functions. In the two-dimensional case, this separation occurs at any point, but in the four-dimensional case, we only have it on the horizon or at null infinity, because the potential vanishes there. These regions can be covered both by the affine or Kruskal coordinates (2.1). Therefore, we can express the arbitrary functions $f(u_{\pm}, x^A)$ [respectively, $g(v_{\pm}, x^A)$] in terms of $\tilde{U}$ (respectively, $\tilde{V}$) instead of $u_{\pm}$ (respectively, $v_{\pm}$). In fact, we can perform any smooth arbitrary coordinate transformations $U = U(u_{\pm})$ and $V = V(v_{\pm})$ and still have a valid solution of the equations of motion on $\Sigma_{i,f}$. This is reminiscent of a two-dimensional conformal field theory, that is classically invariant under any holomorphic (respectively antiholomorphic),

transformation $Z = Z(z)$ and $\bar{Z} = \bar{Z}(\bar{z})$. In particular, we can expand the solutions on $\Sigma_{i,f}$ in terms of the (yet unnormalized) modes

$$\phi_{\Omega lm} = e^{-i\Omega U(u_{\pm})} Y_m^l(x^A), \quad \phi_{\Omega lm} = e^{-i\Omega V(v_{\pm})} Y_m^l(x^A), \quad (2.8)$$

where $U(u_{\pm})$ and $V(v_{\pm})$ denote arbitrary choices of time coordinates. Although these different mode decompositions are classically equivalent, they generally lead to inequivalent notions of positive frequency in quantum theory, making the choice of time coordinate physically significant.

B. Maximally extended null infinity

The region that will be of most interest in the remaining of the paper is the conformal infinity appearing in the final null data surface (2.5), namely, right maximally extended null infinity $\mathcal{I}_R$. It can be covered by the coordinates (U, x^A) where U is defined to be

$$U := -\frac{1}{\tilde{U}} = \begin{cases} e^{\kappa u_+} & \text{on } \mathcal{I}_R^+ := \mathcal{I}_I^+ \\ -e^{-\kappa u_-} & \text{on } \mathcal{I}_R^- := \mathcal{I}_{II}^-, \end{cases} \quad (2.9)$$

that is, the inverse of the Kruskal coordinate. The asymptotic Killing field (2.2) reduces to $\xi = \kappa U \partial_U = \partial_{u_+}$ (respectively, $\xi = \partial_{u_-}$) on $\mathcal{I}_R^+$ (respectively, $\mathcal{I}_R^-$). The advantage of these new coordinates is that we cover $\mathcal{I}_R$ using a single coordinate U instead of the pair (u_+, u_-) so that it is more natural to use it for a maximally extended black hole, and contrary to the Kruskal coordinate $\tilde{U}$, the generator of the “time” translation with respect to the coordinate U is future oriented on $\mathcal{I}_R^+$, while $\partial_{\tilde{U}}$ would have been past oriented. Similarly, we chart left maximally extended null infinity with

$$V := -\frac{1}{\tilde{V}} = \begin{cases} -e^{-\kappa v_+} & \text{on } \mathcal{I}_L^- := \mathcal{I}_I^- \\ e^{\kappa v_-} & \text{on } \mathcal{I}_L^+ := \mathcal{I}_{II}^+, \end{cases} \quad (2.10)$$

and therefore (2.2) can also be expressed in the bulk as

$$\xi = \kappa(U \partial_U - V \partial_V). \quad (2.11)$$

Since $\mathcal{I}_R$ is the union of two null hypersurfaces (in the conformally compactified spacetime), it is also a null hypersurface, and it consists of two copies of $\mathbb{R} \times S^2$, i.e., two three-dimensional cylinders. One of these cylinders is covered by the patch $U > 0$, while the other one is covered by the patch $U < 0$. Therefore, even if the locus $U = 0$ has not been well defined yet, we add it to the description of $\mathcal{I}_R$. This means that we glue $\mathcal{I}_I^+$ to $\mathcal{I}_{II}^-$ together through the sphere at $U = 0$, that we must morally

¹Recall that $r^* = r + 2M \ln\left(\frac{|r-2M|}{2M}\right)$.

²However, there are unstable ones, for instance, the photon sphere at $r = 3M$.

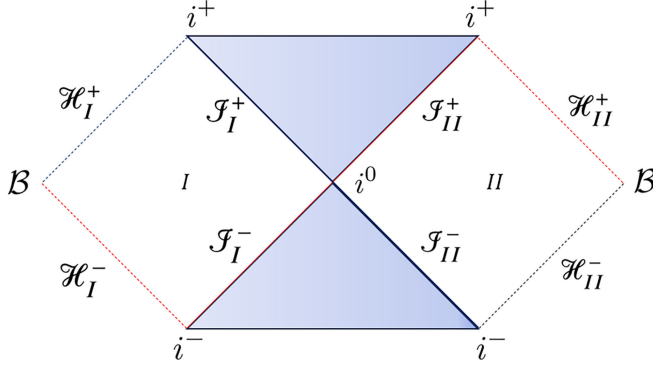


FIG. 2. Conformal extension of the black hole spacetime centered on spacelike infinity. Regions III and IV (respectively, the black and white hole regions) are not depicted here, since they lie beyond the horizons. The region shaded in blue is called $\mathcal{R}_\infty$ (see Ref. [20]) and does not exist in the physical spacetime, but can be constructed in the conformal spacetime, although it may contain singularities. It is the dual of the black hole regions in the conformal spacetime.

identify to spacelike infinity i^0 . The final configurations of the field should now be specified on this null hypersurface (that we continue to call $\mathcal{I}_R$, since it is different from what we previously called $\mathcal{I}_R$ through a null measure set). Once we have done it, the topology of $\mathcal{I}_R$ remains the topology of the three-dimensional cylinder $\mathbb{R} \times S^2$. Analogous considerations apply also to $\mathcal{I}_L$ for the initial configurations of the bulk field.

The set of coordinates (U, V, x^A) describes regions I and II, so spacelike infinity (on both regions) is located at $U = V = 0$ and the bifurcation surface is now at $U = V = \infty$; see the Penrose diagram in Fig. 2. Therefore, it is natural to identify these two regions and see $\mathcal{I}_R$ and $\mathcal{I}_L$ as complete null hypersurfaces in the conformal compactification, but now centered on i^0 (see Fig. 2). This construction is very similar to the conformal extension of Schwarzschild spacetime by Faulkner and Speranza [20]. The regions $\mathcal{R}_\infty$ (located beyond null infinity in the conformal spacetime) that they consider in their work would be covered by the coordinates $U > 0, V < 0$, and $U < 0, V > 0$.³ However, notice that the coordinates U and V cannot cover the black hole and white hole interiors. Similarly, the Kruskal coordinates $(\tilde{U}, \tilde{V})$ did not cover their conformally extended region $\mathcal{R}_\infty$. It was also outlined in Ref. [20] that the algebra of observables at $\mathcal{I}_I^+$ was entangled with the algebra of observables at $\mathcal{I}_{II}^-$ and that the restriction of the Hartle-Hawking state was a pure

³The authors of Ref. [20] needed these regions so that they could smear the field around infinity and get a well-defined algebra of observables at infinity. Here, alternatively, we will bypass this step by quantizing the field directly on the three-dimensional null hypersurface $\mathcal{I}_R$ to obtain a suitable algebra.

state when restricted to this algebra.⁴ These considerations align with ours, as we will see in the next sections of this paper. The take-home message of our construction is that maximally extended null infinity $\mathcal{I}_R$ and $\mathcal{I}_L$ are suitable null initial data surfaces in a black hole background and that it makes sense to identify them as overcomplete null hypersurfaces with the topology of the cylinder in the conformal spacetime. It renders the situation symmetric with respect to the horizon. However, it is possible to think about the later constructions from a purely algebraic point of view and define the algebra of observable on $\mathcal{I}_R$ (respectively, $\mathcal{I}_L$) as the algebraic union of the algebra on $\mathcal{I}_I^+$ and the algebra on $\mathcal{I}_{II}^-$ (respectively, the union of the algebra on $\mathcal{I}_{II}^+$ and the one on $\mathcal{I}_I^-$) without referring to the gluing.

C. Semiclassical Einstein equations

Having defined the kinematics of our problem, we now focus on dynamics, both on the future horizon $\mathcal{H}_I^+$ charted by the patch $\tilde{V} > 0$ and at $\mathcal{I}_I^+$ charted by $U > 0$ or the affine u_+ (plus angles). These are the two regions of interest from the point of view of an external observer in region I who studies the radiation falling across the horizon or escaping to infinity. The aim of this paragraph is to relate the variation of the canonical charges of interest, namely, the area of the horizon A or the Bondi mass M , to an integral of the matter stress-energy tensor.

1. On the horizon

The dynamics of a black hole horizon is contained in the completely longitudinal and the mixed projections of Einstein's equations, dubbed, respectively, the Raychaudhuri [80] and Damour [81] equations. When the horizon is perturbed at order $\mathcal{O}(\epsilon)$ by matter (with $\epsilon \ll 1$), we neglect any higher-order contributions like, e.g., the shear square σ^2 and the (affine) expansion square θ_I^2 . Under this hypotheses, the Raychaudhuri equation becomes

$$\frac{d\theta_I}{d\tilde{V}} = -8\pi T_{\tilde{V}\tilde{V}} + \mathcal{O}(\epsilon^2), \quad (2.12)$$

where $\mathbf{l} = \partial_{\tilde{V}}$ is an affine normal to the horizon. If we go to the semiclassical regime, the perturbation in stress energy is of order $\mathcal{O}(\hbar)$, and we have to replace the classical stress-energy component $T_{\tilde{V}\tilde{V}}$ by the average in the quantum state $|\Psi\rangle$ of its (properly renormalized) stress-energy tensor $\langle T_{\tilde{V}\tilde{V}} \rangle_\Psi$ so that we can rewrite (2.12) as

⁴Therefore, the union of the algebra of observables at $\mathcal{I}_I^+$ and at $\mathcal{I}_{II}^-$ is type I von Neumann algebra; see Appendix D for some basics of von Neumann algebras.

$$\frac{d^2 \epsilon_S}{d\tilde{V}^2} = -8\pi \langle T_{\tilde{V}\tilde{V}} \rangle_\Psi \epsilon_S + O(\hbar^2), \quad (2.13)$$

where ϵ_S is the volume form of a codimension-1 cross section of the horizon.⁵ Then, for an arbitrary cross section at $\tilde{V} = \tilde{V}_0$, we can multiply both sides of (2.13) by the factor $-(\tilde{V} - \tilde{V}_0)$, and after a partial integration on a cross section S of constant affine parameter $\tilde{V}$, we get

$$\frac{d}{d\tilde{V}} \left[\frac{1}{4} \left(A - (\tilde{V} - \tilde{V}_0) \frac{dA}{d\tilde{V}} \right) \right] = 2\pi \int_S (\tilde{V} - \tilde{V}_0) \langle T_{\tilde{V}\tilde{V}} \rangle_\Psi \epsilon_S, \quad (2.15)$$

where A is the area of the cross section S . The quantity

$$S_D := \frac{1}{4} \left(A - (\tilde{V} - \tilde{V}_0) \frac{dA}{d\tilde{V}} \right) \quad (2.16)$$

has been referred as the Dirichlet entropy in Ref. [82] and as the *dynamical entropy* of the Killing horizon in the case where $\tilde{V}_0 = 0$ [83–86]. It is the quantity that satisfies a local version of the physical process first law [82–84,87]. Consider now the three-dimensional domain $\mathcal{D}_0^{\mathcal{H}} = (V_0, +\infty) \times S^2$. The variation of S_D between the cut $\tilde{V} = \tilde{V}_0$ and $\tilde{V} \rightarrow +\infty$ is given by the null time integration of (2.15)

$$\Delta S_D = 2\pi \int_{\mathcal{D}_0^{\mathcal{H}}} (\tilde{V} - \tilde{V}_0) \langle T_{\tilde{V}\tilde{V}} \rangle_\Psi d\tilde{V} \wedge \epsilon_S \quad (2.17)$$

that is positive if the null energy conditions are satisfied.⁶ Assuming⁷ $\lim_{\tilde{V} \rightarrow +\infty} \tilde{V} \theta_l = 0$, we get

$$\begin{aligned} & \frac{1}{4} (A|_{+\infty} - A|_{\tilde{V}=\tilde{V}_0}) \\ &= 2\pi \int_{\mathcal{D}_0^{\mathcal{H}}} (\tilde{V} - \tilde{V}_0) \langle T_{\tilde{V}\tilde{V}} \rangle_\Psi dV \wedge \epsilon_S := 2\pi \langle K_{\tilde{V}_0} \rangle_\Psi, \end{aligned} \quad (2.18)$$

and we see that the quantity that appears inside the integral is the local boost energy above the cut $\tilde{V} = \tilde{V}_0$. Indeed, the boost field preserving the region delimited by the cut $\tilde{V} = \tilde{V}_0$ of the horizon is

⁵The latter is obtained via the pullback of the ambient space volume form $\epsilon_{\mathcal{M}}$,

$$\epsilon_S = \iota_k \epsilon_{\mathcal{M}}, \quad (2.14)$$

where $k = k^{\tilde{V}} \partial_{\tilde{V}} + k^A \partial_A$ is an auxiliary vector field such that its associated 1-form via the ambient space musical isomorphism reads $k^b = -d\tilde{V}$.

⁶They are usually satisfied classically but not in quantum field theory, where the energy density can be in general arbitrary low.

⁷Wall qualified in Ref. [28] this late-time boundary condition as “theological.”

$$\xi = (\tilde{V} - \tilde{V}_0) \partial_{\tilde{V}} \quad (2.19)$$

and is different from the action of the Killing field on the horizon (i.e., $\tilde{V} \partial_{\tilde{V}}$), that corresponds to $\tilde{V}_0 = 0$ (since the bifurcation surface is located at $\tilde{V} = 0$). Note that the right-hand side of (2.18) is the quantity that appears in Ref. [28] and is needed to prove the generalized second law on the horizon.

The quantity denoted $K_{\tilde{V}_0}$, the boost energy of the region $\tilde{V} > \tilde{V}_0$ of the horizon, is closely related to the notion of modular Hamiltonian coming from Tomita-Takesaki’s theory, reviewed in Appendixes D and E. Note, however, that we are being slightly imprecise here by omitting the reference to any notion of algebra of observables and Hilbert space representation built upon a reference vacuum state $|\Omega\rangle$, to which our state $|\Psi\rangle$ belongs. Everything will become clearer in the next section, but for the time being, it is important to observe that (2.18) is written in terms of the covariant stress-energy tensor $T_{\tilde{V}\tilde{V}}$, while the modular Hamiltonian of Tomita-Takesaki involves the normal-ordered version (with respect to $|\Omega\rangle$). This will play a prominent role in our proof of the dual GSL in Sec. IV.

Considering now two cuts $\tilde{V} = \tilde{V}_1$ and $\tilde{V} = \tilde{V}_2 > \tilde{V}_1$, we can write the variation of area as

$$\frac{1}{4} (A_{\tilde{V}_2} - A_{\tilde{V}_1}) = -2\pi (\langle K_{\tilde{V}_2} \rangle_\Psi - \langle K_{\tilde{V}_1} \rangle_\Psi). \quad (2.20)$$

At this point, since we look at a *difference* of boost energies $\langle K_{\tilde{V}_i} \rangle_\Psi$, the condition $\tilde{V} \theta_l \rightarrow_{\tilde{V} \rightarrow +\infty} 0$ is not strictly necessary; all we need is that it converges to a constant.⁸

2. At null infinity

A analogous story runs at $\mathcal{I}_R^+$ where Raychaudhuri’s and Damour’s equations are replaced by the Bondi-Metzner-Sachs equations for the mass and angular momentum [88–90]. The mass equation can be written as

$$\frac{dM}{du} = - \int_S T_{uu} \epsilon_S, \quad (2.21)$$

where M is the integrated Bondi mass,⁹ $u = u_+$ is the affine time on $\mathcal{I}_R^+$, and S is a cross section of the latter. Here, $T_{uu} = T_{uu}^{\text{matter}} + \frac{1}{32\pi} N_{AB} N^{AB}$ is the total stress-energy tensor at null infinity, including also the contribution of the gravitons through the new tensor N_{AB} . Now, we are looking for an equivalent of (2.15) at null infinity when we consider perturbations with respect to the black hole background. First, the semiclassical version of (2.21) is obtained by

⁸However, if we consider that the semiclassical equations apply up to $\tilde{V} = +\infty$, then we expect that caustics form at a finite affine parameter $\tilde{V}$ if we do not assume that $\tilde{V} \theta_l \rightarrow_{\tilde{V} \rightarrow +\infty} 0$.

⁹This quantity is related to the integral on a compact cross section S of the Weyl curvature coefficient ψ_2 and the local shear.

replacing the stress energy T_{uu} by the average of the (properly renormalized) stress-energy tensor operator evaluated in the quantum state $|\Psi\rangle$, so that (2.21) is now

$$\frac{dM}{du} = - \int_S \langle T_{uu} \rangle_\Psi \epsilon_S. \quad (2.22)$$

Second, we write the above equation in terms of the coordinate $U = e^{\kappa u}$ that covers $\mathcal{I}_R$ (exactly as the coordinate $\tilde{V}$ was covering $\mathcal{H}_L$ in the previous paragraph), which will turn out to be very practical in the following. Therefore, we change $\frac{dM}{du}$ into $\kappa U \frac{dM}{dU}$ and take its derivative with respect to ∂_u . We get

$$\kappa U^2 \frac{d^2 M}{dU^2} = \int_S (-\kappa^{-1} \partial_u \langle T_{uu} \rangle_\Psi + \langle T_{uu} \rangle_\Psi) \epsilon_S, \quad (2.23)$$

and now we assume to have

$$\kappa^{-1} \partial_u \langle T_{uu} \rangle_\Psi \ll \langle T_{uu} \rangle_\Psi, \quad (2.24)$$

that is the equivalent on a black hole background of the condition $\theta_l^2 \ll \langle T_{\tilde{V}\tilde{V}} \rangle_\Psi$. Indeed, if we assume that the small perturbations do not perturb the horizon a lot, in a black hole background, we have a constant flux of particle at future null infinity that carries stress energy. The condition (2.24) is verified in most of the known vacuum states one deals with in black hole thermodynamics. All of them will be carefully redefined later in the paper, but we say now that, for example, in the Hartle-Hawking state, the particle flux is actually infinite on $\mathcal{I}_R^{+10}$ because we have a thermal spectrum in all the directions. Similarly, in the Unruh vacuum, we have a constant flux of particles at $\mathcal{I}_R^+$,¹¹ and therefore in both cases, the condition (2.24) is obviously satisfied, since the left-hand side is equal to zero. Excitations on top of these two vacua that leave the assumption (2.24) valid, correspond to “small perturbations,” in the same sense as the small perturbations we considered on the horizon. Then, taking into account this approximation, and using the fact that the covariant stress-energy tensor transforms as $\langle T_{uu} \rangle_\Psi = \kappa^2 U^2 \langle T_{UU} \rangle_\Psi$ under the change $u \rightarrow U$, Eq. (2.23) becomes

$$\frac{d^2 M}{dU^2} = \int_S \kappa \langle T_{UU} \rangle_\Psi \epsilon_S. \quad (2.25)$$

Consider then a cross section of constant parameter $U = U_0$ (with affine parameter $u_0 = \kappa^{-1} \ln U_0$) on $\mathcal{I}_R^+$, and then multiply both sides of (2.25) by $-(U - U_0)$, similarly to what we did on the horizon, so that (2.25) becomes

$$\frac{d}{dU} \left(M - (U - U_0) \frac{dM}{dU} \right) = - \int_S \kappa (U - U_0) \langle T_{UU} \rangle_\Psi \epsilon_S \quad (2.26)$$

that is similar to (2.15). The quantity

$$M_D := M - (U - U_0) \frac{dM}{dU} \quad (2.27)$$

is analogous to the dynamical entropy S_D , and we will refer to it as the *dynamical Bondi mass*. Now, if we assume that

$$M_D(\infty) = M|_{+\infty} - \kappa^{-1} \partial_u M|_{+\infty} = M|_{+\infty} \quad (2.28)$$

is finite, the last equality being true because in this case we need $\partial_u M|_{\infty} = 0$, we can define the local boost energy

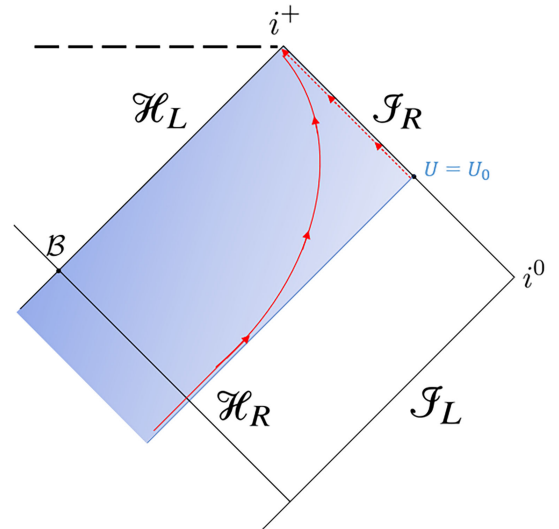


FIG. 3. The trajectory in red is generated by a vector $\bar{\chi}$ in the bulk whose vertical component is given by the asymptotic vector χ (generating the dashed red trajectory in the region $U \geq U_0$ of $\mathcal{I}_R$). It resembles a boost near the cut $U = U_0$, but it looks like a usual time translation at late times. The blue-shaded region represents the portion of spacetime where an observer traveling along the red curve can determine the physics using only the local measurements they can perform. It is delimited by the hypersurfaces $U = U_0$ and $U = +\infty$ (the event horizon).

¹⁰However, it is actually finite in the physical spacetime, since the Hartle-Hawking vacuum is a Hadamard state on the whole maximally extended Schwarzschild spacetime. The divergences at null infinity come from the fact that the falloff of the physical stress-energy tensor (which is constant near infinity) are incompatible with the usual assumptions of asymptotic flatness; see Sec. III.

¹¹In fact, in both the Hartle-Hawking and the Unruh vacuum, the Arnowitt–Deser–Misner (ADM) mass is formally infinite because of this constant flux of particle on $\mathcal{I}_R^+$. However, in the Unruh vacuum the flux on $\mathcal{I}_R^+$ is not infinite.

TABLE I. Comparing the dynamical quantities of interest and their thermodynamic counterparts on the horizon and at future null infinity. Note that we have anticipated some of the results of the next sections.

Quantity	On the future horizon	On future null infinity
Canonical charge	Black hole area $\frac{\kappa}{8\pi}A$	Bondi mass M
Improved charge	Dynamical Black hole area $\frac{\kappa}{8\pi}A_D$	Dynamical Bondi mass M_D
Monotonic flux (by imposing the NEC)	Hawking area theorem	Bondi mass loss formula
Thermodynamic interpretation	Black hole entropy (geometric)	Black hole energy (geometric)
Thermodynamic potential (quantum)	The generalized entropy	Generalized free energy
Quantum corrections	+vN entropy of QF	$-T_H \times$ vN entropy of QF

$$\begin{aligned}
M_D|_{+\infty} - M_D(U_0) &= M|_{+\infty} - M(U_0) \\
&= - \int_{\mathcal{D}_0^x} \kappa(U - U_0) \langle T_{UU} \rangle_\Psi dU \wedge \epsilon_S \\
&:= -\kappa \langle K_{U_0} \rangle_\Psi,
\end{aligned} \tag{2.29}$$

where $\mathcal{D}_0^x = (U_0, +\infty) \times S_2$. The expression (2.29) is similar to what we had on a perturbed Killing horizon. Again, we remark that K_{U_0} is written in terms of the covariant stress-energy tensor at $\mathcal{I}_R^+$. In addition, if we consider $U = U_1$ and $U = U_2 > U_1$ to be two arbitrary cross sections of $\mathcal{I}_R^+$, we have that

$$M(U_2) - M(U_1) = \kappa(\langle K_{U_2} \rangle_\Psi - \langle K_{U_1} \rangle_\Psi), \tag{2.30}$$

and it is also interesting to notice at this point that the variation of Bondi mass only requires the difference $\langle K_{U_2} \rangle_\Psi - \langle K_{U_1} \rangle_\Psi$ to be finite, which can be true even if $\langle K_{U_1} \rangle_\Psi$ and $\langle K_{U_2} \rangle_\Psi$ are not finite by themselves. In a vacuum state aiming to reproduce boundary conditions at late time compatible with a black hole collapse, we do not expect to make sense of the Bondi mass or the dynamical Bondi mass at late time, since we have a constant flux of particle at null infinity. It is the case for the example of the Unruh vacuum. However, we will still be able to make sense of the difference in (2.30) between any arbitrary cross sections by considering states lying in Hilbert spaces representations built upon a vacuum state compatible with the late-time boundary conditions.

To pursue the analogy with physics on a perturbed Killing horizon further, we will introduce the local boost field

$$\chi = \kappa(U - U_0)\partial_U, \tag{2.31}$$

which vanishes at $U = U_0$, while when $U_0 = 0$, we recover the time translation Killing field

$$\chi = \kappa U \partial_U = \partial_u, \tag{2.32}$$

that is timelike in regions I and II of the black hole background. Around $U \rightarrow +\infty$ (i.e., $u \rightarrow +\infty$), the vector field (2.31) behaves as

$$\chi_{U \rightarrow +\infty} \sim \kappa U \partial_U = \partial_u, \tag{2.33}$$

which is a usual time translation (2.32), while around $U \rightarrow U_0$ (i.e., $u \rightarrow u_0$), we have that

$$\chi_{U \rightarrow U_0} \sim \kappa(u - u_0)\partial_u \tag{2.34}$$

that is proportional to the vertical component of a usual boost field, vanishing on the cross section $u = u_0$. Therefore, an observer following a trajectory in (conformal) spacetime spanned by the local boost field χ will be an inertial observer at late time, since in this limit $\chi \sim \partial_u$, while they will start accelerating moving backward in time, so they never cross their past horizon at $U = U_0$ and remain causally disconnected from the region $U < U_0$ of $\mathcal{I}_R$. They do not have access to the interior of the black hole region and to the patch $U < U_0$ (see Fig. 3).

It is also worthy to notice that we can write (2.31) as

$$\chi = (1 - e^{-\kappa(u-u_0)})\partial_u \tag{2.35}$$

so that $\chi \sim \partial_u$ when $(u - u_0) \gg \kappa^{-1}$ is more explicit. Using (2.35), we can write (2.29) as

$$\kappa \langle K_{U_0} \rangle_\Psi = \int_{u_0}^{+\infty} \int_S (1 - e^{-\kappa(u-u_0)}) \langle T_{uu} \rangle_\Psi du \wedge \epsilon_S, \tag{2.36}$$

showing that $\langle K_{U_0} \rangle_\Psi$ is directly related to the energy flux.¹² It is also useful to consider the time

$$\bar{u} := \kappa^{-1} \ln(U - U_0) = u + \kappa^{-1} \ln(1 - e^{-\kappa(u-u_0)}) \tag{2.37}$$

that runs on $\mathbb{R} = (-\infty, +\infty)$ when the inertial time u runs from u_0 to $+\infty$. It is a nice choice of time since it covers entirely the region $U > U_0$ (i.e., $u > u_0$), and so it is a natural time used for the quantization in this region that is of most interest in the following of the paper. This concludes the study of the dynamics at $\mathcal{I}_R^+$. It is worth mentioning again that we obtained similar equations for the dynamical mass than for the dynamical entropy thanks to the hypothesis (2.24). The latter possesses a nice interpretation in terms of Markovian processes in open quantum thermodynamics, which we will come back to in the companion paper [91].

¹²Recall that the later is encoded in the covariant stress tensor written in affine null time.

3. From the horizon to null infinity

We saw in the previous two paragraphs that similar relations were displayed at the horizon and null infinity of a black hole background. An explicit comparison is summarized in Table I. There, T_H is the Hawking temperature, vN means “von Neumann” and NEC stands for “null energy conditions,” meaning that $T_{ab}n^a n^b \geq 0$ where n^a is a null normal. Finally, “QF” means “quantum fields.”

III. QUANTIZING FREE FIELDS AT NULL INFINITY

In this section, we quantize the scalar field on $\mathcal{I}_R$ in $D = 4$. We first consider general notions of time and the corresponding decompositions of the solution space into positive and negative frequency modes, from which we construct the associated one-particle Hilbert space and vacuum state. We then investigate the symmetries of these vacuum states, focusing in particular on the transformations that leave the field’s two-point function on $\mathcal{I}_R$ invariant. Finally, we introduce four physically relevant (classes of) vacuum states that will play a central role throughout this work.

Readers unfamiliar with algebraic quantization or the notion of a Weyl algebra are referred to Appendixes B and C for a brief review.

A. Positive frequency solutions

1. Generalities

We go back to the Klein-Gordon equation for the bulk scalar field Φ in Schwarzschild coordinates, which we decompose on a basis of complex spherical harmonics¹³

$$\Phi(t, r, x^A) = \frac{1}{r} f(t, r) Y_{lm}(x^A). \quad (3.1)$$

The Klein-Gordon equation for the radial part reads

$$(\partial_t^2 - \partial_{r^*}^2 + V_l(r))f(t, r) = 0, \quad (3.2)$$

$$V_l(r) = \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)$$

with r^* the tortoise coordinate, so (3.2) has the same form as (2.3). When $r \rightarrow +\infty$ (or equivalently $r^* \rightarrow +\infty$), the solutions are given by (2.7), and if we focus on $\mathcal{I}_R$, we are only interested in the right-moving modes of the space of solutions $\mathcal{S}$. In general, we need that the solutions decay fast enough so that we restrict the space of solutions $\mathcal{S}$ to the space of Schwartz functions on $\mathcal{I}_R$, $f(U, x^A) \in \mathcal{T}(\mathbb{R} \times S^2)$. Thus, the most general decomposition of the asymptotic field $\phi = \lim_{r \rightarrow +\infty} r\Phi$ is

$$\begin{aligned} \phi(U, x^A) = & \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \\ & \times (Y_m^l(x^A) a_{\Omega lm} e^{-i\Omega U} + \bar{Y}_m^l(x^A) \bar{a}_{\Omega lm} e^{i\Omega U}), \end{aligned} \quad (3.3)$$

where we used the modes $f(U, x^A) = \frac{Y_m^l(x^A)}{\sqrt{4\pi\Omega}} e^{-i\Omega U}$, $\Omega > 0$, for the decomposition. The latter are normalized with respect to the asymptotic Klein-Gordon product

$$\begin{aligned} \forall (f, g) \in \mathcal{S}^{\mathbb{C}}, \\ \Omega^{\mathbb{C}}(f, g) = i \int_{\mathcal{I}_R} (\bar{f} \partial_U g - g \partial_U \bar{f}) dU \wedge \epsilon_S, \end{aligned} \quad (3.4)$$

which is parametrization independent. Here, $\mathcal{S}^{\mathbb{C}}$ denotes the complexification of the solution space $\mathcal{S}$ of the equations of motion. Then, we promote the Fourier coefficients to quantum ladder operators $(\hat{a}_{\Omega lm}, \hat{a}_{\Omega lm}^\dagger)$ satisfying the commutation relations

$$\begin{aligned} [\hat{a}_{\Omega lm}, \hat{a}_{\Omega' l' m'}^\dagger] = & \Omega^{\mathbb{C}} \left(\frac{Y_m^l(x^A)}{\sqrt{4\pi\Omega}} e^{-i\Omega U}, \frac{Y_{m'}^{l'}(x^A)}{\sqrt{4\pi\Omega'}} e^{-i\Omega' U} \right) \\ = & \delta(\Omega - \Omega') \delta_{ll'} \delta_{mm'}, \end{aligned} \quad (3.5)$$

so, given the canonical momenta $\pi = \partial_U \phi$, we deduce that¹⁴

$$[\hat{\phi}(U_1, x_1^A), \hat{\pi}(U_2, x_2^A)] = \frac{i}{2} \delta(U - U') \delta^{(2)}(x_1^A - x_2^A). \quad (3.7)$$

On top of this, the Weyl algebra $\mathcal{A}^{\mathcal{I}}$ is obtained by smearing the momentum field π with a Schwartz function $f(U, x^A) \in \mathcal{T}(\mathbb{R} \times S^2)$,

$$\mathcal{A}^{\mathcal{I}} = \{e^{i\pi(f)} \mid \forall f \in \mathcal{T}(\mathbb{R} \times S^2)\}. \quad (3.8)$$

Note that the Weyl algebra on $\mathcal{I}_R$ is generated by the smeared momentum field π , rather than by the smeared field ϕ ; see Appendixes B and C for further details. The algebra $\mathcal{A}_0$ (respectively, $\mathcal{A}_0'$) is the subalgebra of $\mathcal{A}^{\mathcal{I}}$ spanned by functions f with support on $\mathcal{I}_R^+$ (respectively, $\mathcal{I}_R^-$) only. An other interesting Weyl algebra is

¹⁴The Dirac distribution $\delta^{(2)}(x_1^A - x_2^A)$ is normalized so that

$$\int_S \delta^{(2)}(x_1^A - x_2^A) \epsilon_S(x_2^A) = 1, \quad (3.6)$$

where S is the two-dimensional unit sphere.

¹³They satisfy among other properties $\bar{Y}_m^l = (-1)^m Y_{-m}^l$.

$$\mathcal{A}_L^T = \left\{ e^{i\pi(f)} \mid \forall f \in \mathcal{T}(\mathbb{R} \times S^2), \quad \forall g \in \mathcal{S}_{2D}, \quad \int_{\mathcal{I}_R} f(U, x^A) g(U) Y_m^l dU \wedge \epsilon_S = 0, \quad \text{if } l \geq L \right\}, \quad (3.9)$$

where $\mathcal{S}_{2D}$ is the two-dimensional space of solutions that we studied in the previous subsection. The definition (3.9) amounts to considering smearings of the field π such that in the general decomposition (3.10) we do not see the angular modes higher than the cutoff L .

The one-particle Hilbert space is built from a notion of positive frequency solutions. Simple examples are analogous to the two-dimensional case as we can choose a decomposition with respect to the null time U which covers entirely $\mathcal{I}_R$. However, in four dimensions, we have some additional freedoms. We may, for instance, consider the following decomposition of the field

$$\hat{\phi}(U, x^A) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(Y_m^l(x^A) \hat{a}_{\Omega lm} e^{-i\Omega U^{(l,m)}(U)} + \bar{Y}_m^l(x^A) \hat{a}_{\Omega lm}^\dagger e^{i\Omega U^{(l,m)}(U)} \right), \quad (3.10)$$

where $U^{(l,m)} = U^{(l,m)}(U)$ is a choice of time attached to a particular sector defined by the angular mode (l, m) , so that the time $U^{(l,m)}$ can be different for two different sectors $(l, m) \neq (l', m')$. Indeed, we can check that still

$$\Omega^{\mathbb{C}} \left(Y_m^l \frac{e^{-i\Omega U^{(l,m)}}}{\sqrt{4\pi\Omega}}, Y_{m'}^{l'} \frac{e^{-i\Omega' U^{(l',m')}}}{\sqrt{4\pi\Omega'}} \right) = \delta(\Omega - \Omega') \delta_{ll'} \delta_{mm'}, \quad \Omega, \Omega' > 0 \quad (3.11)$$

since the Klein-Gordon product of two modes with different angular momentum numbers (l, m) vanishes because of the orthogonality of the spherical harmonics. Calling $K_{\{lm\}}$ the projector into the subspaces of positive frequency solutions $\mathcal{S}_{K_{\{lm\}}}^{>0}$ spanned by the set of orthonormal modes $\left\{ Y_m^l \frac{e^{-i\Omega U^{(l,m)}}}{\sqrt{4\pi\Omega}} \right\}$, we have

$$\langle f, g \rangle_{K_{\{lm\}}} = \Omega^{\mathbb{C}}(K_{\{lm\}} f, K_{\{lm\}} g) \quad (3.12)$$

that is our positive definite inner product on the one-particle Hilbert space. The Fock space is then obtained from the one-particle Hilbert space by the usual construction; see (B9). In particular, the vacuum state $|\Omega^{\{U^{(l,m)}\}}\rangle$, where $\{U^{(l,m)}\}$ represents a sequence of times $U^{(l,m)}$ indexed by l and m , is the unique state such that

$$\forall \Omega > 0, \quad \forall (l, m), \quad \hat{a}_{\Omega lm} |\Omega^{\{U^{(l,m)}\}}\rangle = 0. \quad (3.13)$$

Anticipating a bit, we state here that $|\Omega^{\{U^{(l,m)}\}}\rangle$ can be seen as a tensor product of the vacua associated to each sector with angular momentum numbers (l, m) .

2. Symmetries of the vacua

We look at the transformations preserving $|\Omega^{\{U^{(l,m)}\}}\rangle$. First, we decompose the field operator $\hat{\phi}$ into effective two-dimensional quantum scalar fields $\hat{\phi}_{lm}$ so that

$$\begin{aligned} \hat{\phi} &= \sum_{lm} \hat{\phi}_{lm}, \\ \hat{\phi}_{lm} &= \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(Y_m^l \hat{a}_{\Omega lm} e^{-i\Omega U^{(l,m)}} + \bar{Y}_m^l \hat{a}_{\Omega lm}^\dagger e^{i\Omega U^{(l,m)}} \right) \end{aligned} \quad (3.14)$$

so that it is clear from (3.14) that an affine transformation

$$U^{(l,m)} \rightarrow \tilde{U}^{(l,m)} = a_{lm} U^{(l,m)} + b_{lm} \quad (3.15)$$

does not mix the upper and lower operators so that they leave the vacuum state $|\Omega^{\{U^{(l,m)}\}}\rangle$ invariant. However, it is not clear if (3.15) are the only symmetries allowed. To go further and get a complementary point of view, we can look at the transformations of the two-point function¹⁵ under $U^{(l,m)} \rightarrow \tilde{U}^{(l,m)}$ with $\tilde{U}^{(l,m)} = \tilde{U}^{(l,m)}(U^{(l,m)})$,¹⁶

¹⁵Since the states $|\Omega^{\{U^{(l,m)}\}}\rangle$ are Gaussian states, they are entirely determined by their two-point function.

¹⁶More rigorously, we should include the $-i0^+$ prescription:

$$\begin{aligned} &\langle \Omega^{\{U^{(l,m)}\}} | \partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) | \Omega^{\{U^{(l,m)}\}} \rangle \\ &= -\frac{1}{4\pi} \sum_{lm} Y_m^l(x_1^A) \bar{Y}_m^l(x_2^A) \frac{dU^{(l,m)}}{du}(u_1) \frac{dU^{(l,m)}}{du}(u_2) \\ &\quad \times \frac{1}{(U_1^{(l,m)} - U_2^{(l,m)} - i0^+)^2}. \end{aligned} \quad (3.16)$$

$$\begin{aligned}
\langle \Omega^{\{U^{(l,m)}\}} | \partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) | \Omega^{\{U^{(l,m)}\}} \rangle &= \sum_{lm} \langle \Omega^{\{U^{(l,m)}\}} | \partial_u \phi_{lm} \partial_u \phi_{lm} | \Omega^{\{U^{(l,m)}\}} \rangle \\
&= -\frac{1}{4\pi} \sum_{lm} Y_m^l(x_1^A) \bar{Y}_m^l(x_2^A) \frac{dU^{(l,m)}}{du}(u_1) \frac{dU^{(l,m)}}{du}(u_2) \frac{1}{(U_1^{(l,m)} - U_2^{(l,m)})^2}, \quad (3.17)
\end{aligned}$$

and then it is clear that we have invariance when changing the vacuum state $|\Omega^{\{U^{(l,m)}\}}\rangle$ into the vacuum state $|\Omega^{\{\tilde{U}^{(l,m)}\}}\rangle$ if and only if

$$\forall (l, m), \quad \frac{d\tilde{U}^{(l,m)}}{dU^{(l,m)}}(U_1^{(l,m)}) \frac{d\tilde{U}^{(l,m)}}{dU^{(l,m)}}(U_2^{(l,m)}) \frac{1}{(\tilde{U}_1^{(l,m)} - \tilde{U}_2^{(l,m)})^2} = \frac{1}{(U_1^{(l,m)} - U_2^{(l,m)})^2}, \quad (3.18)$$

which is satisfied if and only if

$$\forall (l, m): U^{(l,m)} \rightarrow \tilde{U}^{(l,m)} = \frac{a_{lm}U^{(l,m)} + b_{lm}}{c_{lm}U^{(l,m)} + d_{lm}}, \quad a_{lm}d_{lm} - b_{lm}c_{lm} \neq 0, \quad (3.19)$$

similarly to the two-dimensional case [92], but now for each sector (l, m) . Then, from the discussion on the two-dimensional case, we conclude that the vacua $|\Omega^{\{U^{(l,m)}\}}\rangle$ are invariant under local Möbius transformations for any sector (l, m) .

Note that the situation is very similar to what happens a nonexpanding horizon. In his proof of the generalized second law [28], Wall showed that the vacuum associated to the affine coordinate V on the left horizon, namely, the projection of the Hartle-Hawking vacuum, was invariant under local Möbius transformations of the affine parameter. He therefore restricted his attention to one of the (infinitely many) generators of the horizon. He also argued that the quantization of the scalar field on any complete null geodesic was equivalent to the left-moving sector of a two-dimensional conformal field theory (CFT). Indeed, because of ultralocality, we can define a vacuum state attached to each null geodesic and the total vacuum as the tensor product over all generators of all these vacua. We have something similar in this work, except that we do not attach a vacuum state to any geodesic but instead to any sector (l, m) . It is possible because we have the commutation relations

$$[\phi_{lm}(U), \pi_{l'm'}(U')] = \frac{i}{2} \delta(U - U') \delta_{ll'} \delta_{mm'} \bar{Y}_{m'}^{l'} Y_m^l, \quad (3.20)$$

where $\pi_{lm} = \partial_U \phi_{lm}$, so we can consider for each sector (l, m) an effective two-dimensional chiral conformal field theory described by the field ϕ_{lm} and a vacuum state $|\Omega^{U^{(l,m)}}\rangle$ associated to the positive frequency modes $\frac{Y_m^l}{\sqrt{4\pi\Omega}} e^{-i\Omega U^{(l,m)}}$, $\Omega > 0$ for a given time $U^{(l,m)}$ at fixed l and m . Then, our total vacuum (3.13) is obtained as a tensor product of the vacua associated to each sector (l, m) ,

$$|\Omega^{\{U^{(l,m)}\}}\rangle = \bigotimes_{lm} |\Omega^{U^{(l,m)}}\rangle. \quad (3.21)$$

Vacua of the form (3.21) are relevant for us since we are interested in the physics at null infinity in a black hole background. Because of the black hole potential, the modes emitted at the horizon undergo a backscattering, and we expect a different behavior depending on the angular momentum quantum numbers (l, m) . Therefore, vacua of defined angular momentum, like (3.21), are more relevant to our purpose than those defined on the null generators of $\mathcal{I}_R$. However, it is insightful to appreciate the analogy between (3.19) and the local Möbius transformations on the horizon

$$\begin{aligned}
V(x^A) &\rightarrow V'(x^A) = \frac{a(x^A)V(x^A) + b(x^A)}{c(x^A)V(x^A) + d(x^A)}, \\
a(x^A)d(x^A) - b(x^A)c(x^A) &\neq 0, \quad (3.22)
\end{aligned}$$

leaving the restriction of the Hartle-Hawking state to the horizon invariant, a property needed to prove the generalized second law.¹⁷

B. Examples of vacua

In this subsection, we look at different examples of vacua defined on the algebra of observables $\mathcal{A}^T$. All the vacua considered below are examples of the form (3.21). When needed, the reader is advised to look at Appendix D, where the notion of von Neumann algebra, cyclic and separating vectors are defined.

¹⁷In fact, in Ref. [28], considering only the subset of symmetries $c(x) = 0, d(x) = 1$ was enough.

1. Minkowski vacuum

For our first example, we take

$$\forall (l, m), \quad U^{(l, m)} = \begin{cases} u_+ & \text{on } \mathcal{I}_R^+ \\ u_- & \text{on } \mathcal{I}_R^- \end{cases}, \quad (3.23)$$

where we recall that u_+ (respectively, u_-) is an affine coordinate on $\mathcal{I}_R^+$ (respectively, $\mathcal{I}_R^-$). We identify the modes $\left(\frac{Y_m^l}{\sqrt{4\pi\omega_+}} e^{-i\omega_+ u_+}, \frac{Y_m^l}{\sqrt{4\pi\omega_-}} e^{-i\omega_- u_-}\right)$ with $\omega_+, \omega_- > 0$ to be the positive frequency solutions. The vacuum state $|\Omega^u\rangle := |0\rangle_M$ associated to these positive frequency solutions is the restriction of the Minkowski vacuum to $\mathcal{I}_R^+$ (and $\mathcal{I}_R^-$). It is therefore the unique state annihilated by all the operators $\hat{a}_{\omega_+ lm}$ and $\hat{a}_{\omega_- lm}$. Note that it is a pure state with respect to the algebra $\mathcal{A}_0^\mathcal{I}$. When the latter is seen as a subalgebra of $\mathcal{B}(\mathcal{H}_{\Omega^u})$, i.e., the bounded linear operators of the Gelfand-Naimark-Segal (GNS) [93,94] Hilbert space $\mathcal{H}_{\Omega^u}$ built upon $|0\rangle_M$, one finds that its double commutant is a type-von Neumann algebra; see Appendix D for related discussions and details. It will not be the case of the other representations of the algebra that we consider below, whose double commutant will be of type III.

Then, we can compute the two-point function in the state $|0\rangle_M$ for any pair of points of $\mathcal{I}_R^+$, (u_1, x_1^A) and (u_2, x_2^A) (for the rest of the paragraph, $u_+ = u$ to avoid clutter) that is, from (3.17),

$$\langle 0 |_M \partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) | 0 \rangle_M = -\frac{1}{4\pi} \frac{\delta^{(2)}(x_1^A - x_2^A)}{(u_1 - u_2)^2}, \quad (3.24)$$

where we used the completeness relation of the spherical harmonics

$$\sum_{lm} \bar{Y}_m^l(x_1^A) Y_m^l(x_2^A) = \delta^{(2)}(x_1^A - x_2^A) \quad (3.25)$$

to simplify the sum over the different sectors (l, m) . Notice that the two-point function (3.24) coincides with standard expressions appearing in Refs. [95–97]. As previously explained, we can apply the Möbius transformations on each (l, m) sector

$$u \rightarrow u' = \frac{a_{lm}u + b_{lm}}{c_{lm}u + d_{lm}}, \quad a_{lm}d_{lm} - b_{lm}c_{lm} \neq 0, \quad (3.26)$$

but since the time u is independent of the sector (l, m) , we can see from the structure of the two-point function (3.24) that the transformations

$$u \rightarrow u' = \frac{a(x)u + b(x)}{c(x)u + d(x)}, \quad ad - bc \neq 0, \quad (3.27)$$

are also symmetries of the Minkowski vacuum restricted to the algebra $\mathcal{A}_0^\mathcal{I}$. Then, the normal-ordered (with respect to $|0\rangle_M$) stress-energy tensor is obtained by subtracting the vacuum expectation value (3.24) to the momentum square

$$\begin{aligned} :T_{uu}:_{0_M} &= \lim_{(u_2, x_2) \rightarrow (u_1, x_1)} \left(\partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) \right. \\ &\quad \left. - \langle 0 |_M \partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) | 0 \rangle_M \right) \end{aligned} \quad (3.28)$$

that is a sesquilinear form for a dense set of states in $\mathcal{H}_{\Omega^u}$. In general, the normal-ordered stress-energy tensor cannot be identified as a good component of the stress-energy tensor, because such a good component had a vanishing expectation value for the Minkowski vacuum.¹⁸ However, the expectation value of (3.28) in the Minkowski vacuum vanishes by construction, and (3.28) has finite expectation value for Hadamard states at null infinity. Therefore, Eq. (3.28) is indeed a component of the renormalized stress-energy tensor that is coupled to the geometry

$$T_{uu} = :T_{uu}:_{0_M}, \quad (3.29)$$

meaning that the energy flux at $\mathcal{I}_R^+$ in the Minkowski vacuum, i.e., $\langle T_{uu} \rangle_{0_M}$, is vanishing. Therefore, $|0\rangle_M$ physically represents the absence of radiation (as it should).

2. Hartle-Hawking vacuum

Consider now the vacuum state defined using a time coordinate that is complete on $\mathcal{I}_R$, i.e., so that it runs from $-\infty$ to $+\infty$. From (2.9), we take

$$U^{(l, m)} = U = \begin{cases} e^{\kappa u_+} & \text{on } \mathcal{I}_R^+ \\ -e^{-\kappa u_-} & \text{on } \mathcal{I}_R^- \end{cases}, \quad (3.30)$$

where κ is the surface gravity. Then, we can construct a Hilbert space $\mathcal{H}_{\Omega_H}$ using the positive frequency modes associated to the time (3.30), and we denote the vacuum state $|\Omega_H\rangle$. This state is cycling and separating with respect to the representation of the algebra $\mathcal{A}_0^\mathcal{I}$ as it will turn out to be thermal. The state $|\Omega_H\rangle$ can be interpreted as the restriction of the Hartle-Hawking vacuum at null infinity. More precisely, the Hartle-Hawking state is defined using the positive frequency mode decomposition associated to the Kruskal coordinates $(\tilde{U}, \tilde{V})$ [related to the coordinates (U, V) by the relations (2.9) and (2.10)] on the horizons $\mathcal{H}_R \cup \mathcal{H}_L$, and its

¹⁸This requirement is actually one of Wald's four axioms a stress-tensor should satisfy; see Ref. [98].

restriction to an algebra of observables at null infinity can be deduced using a similar decomposition.¹⁹

The two-point function associated to the Hartle-Hawking vacuum on $\mathcal{I}_R$ is therefore given by

$$\langle \Omega_H | \partial_U \phi(U_1, x_1^A) \partial_U \phi(U_2, x_2^A) | \Omega_H \rangle = -\frac{1}{4\pi} \frac{\delta^{(2)}(x_1^A - x_2^A)}{(U_1 - U_2)^2} \quad (3.31)$$

and is similar to the result on the horizon [Eq. (1.1) of Ref. [95]]. In addition, similarly to the Minkowski vacuum, the transformations

$$U \rightarrow U' = \frac{a_{lm}U + b_{lm}}{c_{lm}U + d_{lm}}, \quad a_{lm}d_{lm} - b_{lm}c_{lm} \neq 0 \quad (3.32)$$

are not the only symmetries. From the structure of the two-point function (3.31), we can see that the local transformations

$$U \rightarrow U' = \frac{a(x)U + b(x)}{c(x)U + d(x)}, \quad a(x)d(x) - b(x)c(x) \neq 0 \quad (3.33)$$

are also symmetries of $|\Omega_H\rangle$. However, there is an important difference with what happens on the horizons: while on the horizon the Kruskal coordinate is an affine coordinate, at null infinity, the affine coordinate is given by u , not U . The “good” notion of energy flux is therefore encoded into the component T_{uu} of the renormalized stress-energy tensor. In terms of the U -coordinate, the doubly null component of the normal-ordered stress-energy tensor is

$$\begin{aligned} :T_{UU}:_{\Omega_H} &= \lim_{(U_2, x_2) \rightarrow (U_1, x_1)} \left(\partial_U \phi(U_1, x_1^A) \partial_U \phi(U_2, x_2^A) \right. \\ &\quad \left. - \langle \Omega_H | \partial_U \phi(U_1, x_1^A) \partial_U \phi(U_2, x_2^A) | \Omega_H \rangle \right), \end{aligned} \quad (3.34)$$

which is well defined as a sesquilinear form on the dense subset of $\mathcal{H}_{\Omega_H}$ spanned by $|\Omega_H\rangle$.²⁰ However, the energy flux is given by

¹⁹In fact, it is easy to show that the state $|\Omega_H\rangle$ is a Kubo-Martin-Schwinger (KMS) state with respect to the modular Hamiltonian generating the Killing flow in the subalgebra $\mathcal{A}_0^I$ and $\mathcal{A}_0^{I'}$ using the results of Appendix A and D. Therefore, the restriction of the state $|\Omega_H\rangle$ and the Hartle-Hawking state to the algebras $\mathcal{A}_0^I$ and $\mathcal{A}_0^{I'}$ are identical. The restriction of the Hartle-Hawking state to the full algebra $\mathcal{A}^I$ might be different, since the correlations might not be the same. However, since we will restrict ourselves to $\mathcal{A}_0^I$ or to one of its subalgebras in the subsequent section, we are not interested in the correlations, and we will refer for simplicity to the state $|\Omega_H\rangle$ as the Hartle-Hawking vacuum in the following.

²⁰This dense subspace comes from the GNS construction; see Appendix C.

$$\begin{aligned} \langle \Omega_H | T_{uu} | \Omega_H \rangle &= \langle \Omega_H | :T_{uu}:_{0_M} | \Omega_H \rangle \\ &= -\frac{1}{24\pi} \lim_{(U_2, x_2) \rightarrow (U_1, x_1)} S(u, U) \delta^2(x_1^A - x_2^A) \\ &= \lim_{(U_2, x_2) \rightarrow (U_1, x_1)} \frac{\kappa^2}{48\pi} \delta^2(x_1^A - x_2^A) = +\infty, \end{aligned} \quad (3.35)$$

where we used (3.29) in the first line and the general transformation law of a normal-ordered stress tensor (which includes a Schwarzian) in the second line together with the fact that $\langle :T_{UU}:_{\Omega_H} \rangle = 0$. From (3.35), we see that the restriction of the Hartle-Hawking state is singular at null infinity. The Hartle-Hawking state being a Hadamard state, the expectation value of the renormalized²¹ stress energy has finite expectation values on the physical spacetime. However, the energy density $\langle \hat{T}_{uu} \rangle_{\Omega_H}$ tends, following Stefan’s law, to a constant very far from the black hole²² [99,100]

$$\langle \hat{T}_{uu} \rangle_{\Omega_H} \underset{r \rightarrow +\infty}{\sim} T_H^4 = \left(\frac{\kappa}{2\pi} \right)^4. \quad (3.36)$$

Therefore, we see that even though the energy density is finite, its asymptotic behavior is incompatible with the falloffs required to have asymptotic flatness, which are [101]

$$r^2 \langle \hat{T}_{uu} \rangle \underset{r \rightarrow +\infty}{=} O(1). \quad (3.37)$$

This occurs because the radiation comes uniformly from all directions at infinity, rendering the black hole indistinguishable from the surrounding radiation field. Indeed, the Hartle-Hawking state describes a black hole in thermal equilibrium with incoming radiation at the Hawking temperature, so the black hole remains in equilibrium²³ since the outgoing Hawking radiation is exactly balanced by incoming radiation at the same temperature. Moreover, because the Schwarzschild potential vanishes both at the horizon and at infinity, thermal radiation propagates just as easily from past null infinity to the future horizon as outgoing radiation propagates from the past horizon to future null infinity. As a consequence, all outgoing modes are thermally populated at the Hawking temperature, for every value (l, m) of the angular momentum. In particular, although high angular-momentum modes emitted from the past horizon $\mathcal{H}_R^-$ are partially backscattered by the potential

²¹Renormalization has to be understood here from the point of view of the matter theory.

²²Note that in the Hartle-Hawking state the outgoing radiation current is exactly compensated by the ingoing radiation current, so $\langle \hat{T}_t^r \rangle_{\Omega_H} = 0$ in the bulk.

²³However, because of the negative heat capacity of the black hole, this equilibrium is in fact unstable.

barrier, their contribution is exactly compensated by high-angular-momentum modes coming from $\mathcal{I}_L^-$, which are reflected in a symmetric manner and therefore do not reach the future horizon. By contrast, a physical black hole is not in thermal equilibrium with incoming radiation and emits predominantly low angular-momentum modes, since the Hawking radiation originates only from the near-horizon region. We take this crucial physical insight into account in the next two paragraphs.

3. Hard regularization of the Hartle-Hawking state

Since restricting the Hartle-Hawking state $|\Omega_H\rangle$ to the algebra $\mathcal{A}_0^\mathcal{I}$ leads to a constant thermal flux and therefore to a divergent energy density, we are naturally led to consider alternative vacuum states, constructed in order to obtain finite energy densities at $\mathcal{I}_R^+$. It is important to emphasize that this is not merely a mathematical pathology: the singular behavior reflects the fact that the physical description itself is inappropriate. A real black hole is never immersed in a thermal bath at the Hawking temperature; rather, the radiation originates only from the black hole itself. While this radiation can be regarded as thermal near the horizon at early times,²⁴ it is certainly *not* thermal at $\mathcal{I}_R^+$. Indeed, as noted in the previous paragraph, the gravitational potential (3.2) becomes very large for modes with high angular momentum. Intuitively, this reflects the fact that the latter are not directed sufficiently “towards” the black hole and therefore cannot be efficiently absorbed—or, by time-reversal symmetry, emitted.

One possible approach is to choose a vacuum state $|\Omega_H^L\rangle$ that appears thermal [with respect to the Killing flow (2.31)] for low-angular-momentum modes $l < L$, with L fixed, while coinciding with the Minkowski vacuum for high-angular-momentum modes $l \geq L$. Hence, we choose on $\mathcal{I}_R$

$$U^{(l,m)} = \begin{cases} U & \text{if } l < L \\ u_\pm = \kappa^{-1} \ln \pm U & \text{if } l \geq L, \end{cases} \quad (3.38)$$

and we get a bunch of new vacuum states. The space of positive frequency solutions can be written as

$$\begin{aligned} \mathcal{S}_L^{>0} &:= \left\{ f \in \mathcal{S} \mid f(U, x^A) \right. \\ &= \sum_{l=0}^{L-1} \sum_{m=-l}^l \int_0^{+\infty} d\Omega a_{\Omega lm} Y_m^l e^{-i\Omega U} \\ &\quad \left. + \sum_{l=L}^{+\infty} \sum_{m=-l}^l \int_0^{+\infty} d\omega a_{\omega lm} Y_m^l e^{-i\omega u_+} \right\}, \end{aligned} \quad (3.39)$$

²⁴Or, equivalently, on the entire horizon if one follows Unruh’s construction [72].

on $\mathcal{I}_R^+$, for example. For $L = 0$, we recover the Minkowski vacuum, and for $L = +\infty$, we recover the Hartle-Hawking vacuum. The case $L = 1$ corresponds to the s -wave approximation, for which we consider only spherically symmetric radiation waves. The rationale for the choice (3.38) is that, for any finite choice of L , only a finite subset of modes is excited relative to the Minkowski vacuum, ensuring that the stress-energy density on the horizon remains finite. Moreover, the vacuum state $|\Omega_H^L\rangle$ is physically more relevant at future null infinity than the Hartle-Hawking state $|\Omega_H\rangle$, as it provides a more realistic description of an isolated black hole. We call this state the L -vacuum state.

This state is a natural interpolation between the Hartle-Hawking vacuum and the Unruh vacuum: modes with low angular momentum $l < L$ are in a thermal state, as in the Hartle-Hawking vacuum, while modes with large angular momentum remain unexcited, leading to a finite energy density on $\mathcal{I}_R^+$, as in the Unruh vacuum. Physically, the state $|\Omega_H^L\rangle$ may be interpreted (approximately) as arising from an initial condition in which an incoming thermal flux of modes with angular momentum $l < L$ at the Hawking temperature exactly compensates the outgoing Hawking radiation in those same modes, so the $l < L$ sector remains in thermal equilibrium. If L is chosen sufficiently large, the contribution of outgoing modes with $l \geq L$ to the flux at null infinity may be neglected, since such modes are strongly suppressed by the gravitational potential barrier and therefore do not efficiently escape.

Using (3.39), we can construct the Hilbert space $\mathcal{H}_{\Omega_H^L}$ for the L -vacuum state. The two-point function between two points of $\mathcal{I}_R^+$ in $|\Omega_H^L\rangle$ is given by ($u = u_+$ again)

$$\begin{aligned} &\langle \Omega_H^L | \partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) | \Omega_H^L \rangle \\ &= -\frac{1}{4\pi} \sum_{l < L, m} \bar{Y}_m^l(x_1^A) Y_m^l(x_2^A) \frac{dU}{du}(u_1) \frac{dU}{du}(u_2) \frac{1}{(U_1 - U_2)^2} \\ &\quad - \frac{1}{4\pi} \sum_{l \geq L, m} \bar{Y}_m^l(x_1^A) Y_m^l(x_2^A) \frac{1}{(u_1 - u_2)^2} \\ &= -\frac{1}{4\pi} \sum_{l < L, m} \kappa^2 \bar{Y}_m^l(x_1^A) Y_m^l(x_2^A) \frac{e^{\kappa u_1} e^{\kappa u_2}}{(e^{\kappa u_1} - e^{\kappa u_2})^2} \\ &\quad - \frac{1}{4\pi} \sum_{l \geq L, m} \bar{Y}_m^l(x_1^A) Y_m^l(x_2^A) \frac{1}{(u_1 - u_2)^2}, \end{aligned} \quad (3.40)$$

and all the symmetries given by (3.19), where $U^{(l,m)} = U$ for $l < L$ and $U^{(l,m)} = u$ for $l \geq L$, leave the L -vacuum invariant. The doubly null component of the normal-ordered affine stress-energy tensor is given by

$$\begin{aligned} :T_{uu}:_{\Omega_H^L} &= \lim_{(u_2, x_2) \rightarrow (u_1, x_1)} \left(\partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) \right. \\ &\quad \left. - \langle \Omega_H^L | \partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) | \Omega_H^L \rangle \right), \end{aligned} \quad (3.41)$$

so the doubly null component of the covariant affine stress-energy tensor is given by

$$T_{uu} = T_{uu} : \Omega_H^L + \frac{\kappa^2}{48\pi} \sum_{l < L, m} \bar{Y}_m^l(x^A) Y_m^l(x^A) < +\infty, \quad (3.42)$$

which is finite. Therefore, there is a dense subspace of Hadamard states in $\mathcal{H}_{\Omega_H^L}$ giving finite energy densities, namely, the ones of the form $A|\Omega_H^L\rangle$ for $A \in \mathcal{A}_0^L$. It is because the sum appearing in (3.42) only contained a finite number of angular momentum modes (l, m) . Note that the thermal flux at infinity in the L -vacuum is therefore effectively two dimensional, since it is a sum of thermal fluxes associated to the Hartle-Hawking vacuum of a finite number of two-dimensional scalar CFT.

4. Soft regularization of the Hartle-Hawking state

The L -vacuum studied in the previous section was used to model a finite thermal flux at future null infinity. The state was exactly thermal for the low angular-momentum modes. However, as we already said, the state to which an actual black hole relaxes is *not* thermal at all. Indeed, we have to take into account the corrections due to the graybody factors, bringing a much richer physics than a mere thermal driving. It is indeed well known that the correct vacuum state to describe the physics of a collapsing black hole from the point of view of late-time asymptotic observers is the Unruh vacuum. In this paragraph, we are not considering the Unruh vacuum directly but instead another family of vacua which will help us to appreciate the better features of the Unruh vacuum compared to the Hartle-Hawking vacuum.

To do this, we rely on the observation that different angular momentum modes are scattered in a way that depends on their angular momentum l , as we explained before. However, while for the L -vacuum we introduced a “hard” cutoff on the angular momentum modes, here we soften the regularization by introducing a series of vacuum states so that all the different angular momentum modes have their own effective temperature²⁵

$$T_l := \frac{\kappa_l}{2\pi}. \quad (3.43)$$

Physically, since the higher the angular momentum, the more backscattered the mode is, we will take $\kappa \geq \kappa_l \geq 0$, so if there is no backscattering, the effective temperature is the Hawking temperature, and $\kappa_l \rightarrow_{l \rightarrow +\infty} 0$, the latter being necessary in order to get a finite flux at $\mathcal{I}_R^+$. To construct the Hilbert space, we choose a decomposition into positive frequency modes associated to the times

$$U^{(l,m)} = \begin{cases} U^l := e^{\kappa_l u_+} & \text{on } \mathcal{I}_R^+ \\ U^l := -e^{-\kappa_l u_-} & \text{on } \mathcal{I}_R^- \end{cases} \quad (3.44)$$

so that $U^l = U$ if $\kappa_l = \kappa$ and $|U^l| = |U|^{\frac{\kappa_l}{\kappa}}$. The space of positive modes is therefore given by

$$\begin{aligned} \mathcal{S}_{\kappa_l}^{>0} &:= \left\{ f \in \mathcal{S}^{\mathbb{C}} \mid f(U, x^A) \right. \\ &= \left. \sum_{l=0}^{+\infty} \sum_{m=-l}^l \int_0^{+\infty} d\Omega a_{\Omega lm} Y_m^l e^{-i\Omega U^l} \right\}. \end{aligned} \quad (3.45)$$

The associated vacuum state $|\Omega_H^{\{\kappa_l\}}\rangle$ is called the κ_l -vacuum state, and its Hilbert space is denoted $\mathcal{H}_{\Omega_H^{\{\kappa_l\}}}$. Each sequence of positive terms $\{\kappa_l\}$ generates a different Hilbert space, and if any of the κ_l is changed, we get a representation of the algebra of observables which is unitarily inequivalent.

The two-point function on this vacuum state between two-point of $\mathcal{I}_R^+$ is given by

$$\begin{aligned} \langle \Omega_H^{\{\kappa_l\}} | \partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) | \Omega_H^{\{\kappa_l\}} \rangle \\ = -\frac{1}{4\pi} \sum_{lm} \bar{Y}_m^l(x_1^A) Y_m^l(x_2^A) \frac{dU^l}{du}(u_1) \frac{dU^l}{du}(u_2) \frac{1}{(U_1^l - U_2^l)^2} \\ = -\frac{1}{4\pi} \sum_{lm} \kappa_l^2 \bar{Y}_m^l(x_1^A) Y_m^l(x_2^A) \frac{e^{\kappa_l u_1} e^{\kappa_l u_2}}{(e^{\kappa_l u_1} - e^{\kappa_l u_2})^2}, \end{aligned} \quad (3.46)$$

and the doubly null component of the normal-ordered stress-energy tensor is given by

$$\begin{aligned} :T_{uu} :_{\Omega_H^{\{\kappa_l\}}} &= \lim_{(u_2, x_2) \rightarrow (u_1, x_1)} \left(\partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) \right. \\ &\quad \left. - \langle \Omega_H^{\{\kappa_l\}} | \partial_u \phi(u_1, x_1^A) \partial_u \phi(u_2, x_2^A) | \Omega_H^{\{\kappa_l\}} \rangle \right), \end{aligned} \quad (3.47)$$

so the doubly null component of the affine stress-energy tensor is given by

$$T_{uu} = T_{uu} :_{\Omega_H^{\{\kappa_l\}}} + \sum_{lm} \frac{\kappa_l^2}{48\pi} \bar{Y}_m^l(x^A) Y_m^l(x^A), \quad (3.48)$$

which converges if the sequence $\{\kappa_l\}$ decays sufficiently quickly.²⁶ In the latter case, there exists a dense set of Hadamard states in $\mathcal{H}_{\Omega_H^{\{\kappa_l\}}}$ for which the energy density on $\mathcal{I}_R^+$ given by (3.48) is finite. Of course, all the symmetries (3.19), where here $U^{(l,m)} = U^l$, leave the κ_l -vacuum

²⁵The precise expression for κ_l is phenomenological; see a proposal in footnote 39.

²⁶We shall see in Sec. IV that it is enough to ask $\sum_{l=0}^{+\infty} (2l+1)\kappa_l < +\infty$.

invariant. Finally, note that $|\Omega_H^{\{\kappa_l\}}\rangle$ admits the tensorial structure (3.21) with $U^{(l,m)} = U^l$.

5. Why so many vacuum states?

One may wonder why we have introduced that many different vacuum states. From a mathematical point of view, the GNS representation theorem asserts that, given an algebraic state (the vacuum state), one can always construct a Hilbert-space representation of a unital C^* -algebra like the Weyl algebra. The representation is tight to the choice of state, as two different states generically give rise to two Hilbert spaces nonrelated by a unitary transformation. Mathematics, however, do not tell how to choose the vacuum state. This depends on the physical situation at hand, what kind of process one wants to model. The study of black hole radiation and evaporation led to the construction of various vacuum states among which the Hartle-Hawking and the Unruh state. Each of them proposes a different way to model a black hole and its radiation, with more or less relevance depending on the degree of accuracy with which we want to study that process. If the Hawking radiation at late time is modeled by the Unruh vacuum, the latter cannot be written at future null infinity using a positive frequency decomposition, and its characterization under an algebra of observables at null infinity is left as an open problem. However, it is still possible to get an expression of the modular Hamiltonian starting from the thermal character of the Unruh vacuum on the past horizon and using transmission/reflexion coefficients (see the second part of this work [91]). However, as we have repeatedly emphasized, the Hartle-Hawking state is far from providing a realistic description of an evaporating black hole. This led us to propose the hard and soft regularizations of the Hartle-Hawking state at $\mathcal{I}_R$. The next section will show how the thermodynamical potential, whose monotonicity at $\mathcal{I}_R^+$ is the dual GSL, gets modified when we consider the latter vacua.

C. Beyond the scalar field

It is of main interest to extend the formalism of this section to massless field of higher spins or even to interacting field theories. We give some brief words about it in this subsection. We only focus on $\mathcal{I}_R^+$ here, for simplicity.

1. Spin-1 field

We take A_a to be a massless spin-1 free field. We can consider it to be the electromagnetic field. Since our construction for the scalar field started from the symplectic form, we have to write it in the case at hand. We can make a choice of gauge so that $A_u = 0$ on $\mathcal{I}_R^+$ [102–104]. In this case, we have that (with $a = u, A$)

$$\Omega(A_a^1, A_b^2) = -\frac{1}{e^2} \int_{\mathcal{I}_R^+} q^{AB} (E_A^1 A_B^2 - E_A^2 A_B^1) du \wedge \epsilon_S, \quad (3.49)$$

where $E_A^i = \partial_u A_A^i$ is the electric field and q_{AB} is the conformal metric (that is finite at $\mathcal{I}_R^+$), e is the coupling constant, and A_a^i , $i = 1, 2$ are solutions to the classical Maxwell equations. Therefore, the symplectic form is exactly similar to the one for the massless scalar field (3.4) at $\mathcal{I}_R^+$, and the electric field is the conjugate momentum variable. Therefore, we can define an algebra of operators using smearings of the electric field against test functions and choose a class of vacua states similar to the one that we introduced in the previous paragraphs of this section. Therefore, we do not expect to face serious difficulties when using a free spin-1 field instead of a free spin-0 field for any of the arguments in the remaining of the paper.

2. Spin-2 field

We can now treat the case of the spin-2 field. In this case, we take the example of the gravitational field and fix a gauge so that $g_{au} = 0$ on $\mathcal{I}_R^+$, $a = u, A$. Then, it is well known that in this case the symplectic form reduces to [105]

$$\begin{aligned} \Omega(C_{AB}^1, C_{A'B'}^2) \\ = -\frac{1}{16\pi} \int_{\mathcal{I}_R^+} q^{AA'} q^{BB'} (N_{AB}^1 C_{A'B'}^2 - N_{A'B'}^2 C_{AB}^1) du \wedge \epsilon_S, \end{aligned} \quad (3.50)$$

where $N_{AB} = -\partial_u C_{AB}$ is the asymptotic news and C_{AB} is the asymptotic shear. The news is the conjugate momentum of the shear which is the asymptotic configuration variable. An algebra of observables at $\mathcal{I}_R^+$ can be obtained from smearings of the news, similarly to what we did for the massless scalar field and the electromagnetic field. The symplectic structure (3.50) is of the same form as the one of the electromagnetic field (3.49) and of the scalar field (3.4), so the construction is similar to the previous cases that we have encountered. Therefore, we do not expect to face serious difficulties when dealing with the spin-2 field.

3. Interacting field theories

For interacting field theories, it is much more subtle in general. However, near $\mathcal{I}_R^+$, interacting field theories tend to behave as free field theories since the interactions spread out [102,103]. For instance, gravity is by any means an interacting field theory. However, its symplectic structure at infinity is very similar to the one of a free theory, like the massless scalar field or the electromagnetic field. Therefore, we expect to be able to treat interacting field theories on $\mathcal{I}_R^+$ as we treat free field theories and do not plan to see any additional difficulty as long as we restrict

ourselves to the algebra of observables there. However, as long as we take into account the algebra on the horizon, we should be much more careful in treating interacting field theories since an algebra on the horizon does not necessarily exist in this case [106].

IV. DUAL GENERALIZED SECOND LAW

This section constitutes the core of our work, namely, to prove the dual GSL at future null infinity. As stated in the Introduction, the GSL is proven on the horizon using the monotonicity of the relative entropy between the restriction of the Hartle-Hawking vacuum state and an arbitrary state belonging to the GNS Hilbert space representing, from that vacuum state, the algebra of observables of the horizon. The two spacelike hypersurfaces between which the GSL was proven in Ref. [28] were both starting at spatial infinity t^0 and ending at two different cuts of the future horizon. In that case, the relevant thermodynamic potential is the generalized entropy. Black hole thermodynamics has therefore been unveiled from a near-horizon perspective. We treat here a dual picture, focusing on asymptotic observers.

The dual GSL corresponds to the monotonicity of a thermodynamical potential $\mathcal{G}$ (free energy or grand potential, depending on the vacuum state) between two spacelike hypersurfaces starting from the same cut on the horizon but ending at different cuts of $\mathcal{I}_R^+$. A proof involving an effective description of the process was published by one of us in Ref. [65]. This work improves the latter, as we give rigorous algebraic foundations of all the steps. In particular, we defined precisely the algebra of observables at $\mathcal{I}_R$ and introduced three vacua, the Hartle-Hawking state and its hard and soft regularizations, from which we constructed the associated GNS Hilbert spaces in

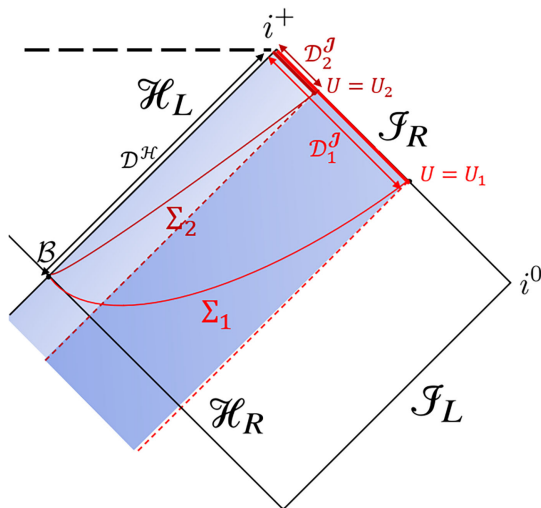


FIG. 4. Setup in which the hypersurfaces Σ_1 and Σ_2 both start at the horizon bifurcation surface $\mathcal{B}$ and end at different cuts $U = U_1$ and $U = U_2$, respectively, at $\mathcal{I}_R^+$. Also depicted are the regions $\mathcal{D}^{\mathcal{H}}$ and $\mathcal{D}_I^{\mathcal{I}}$.

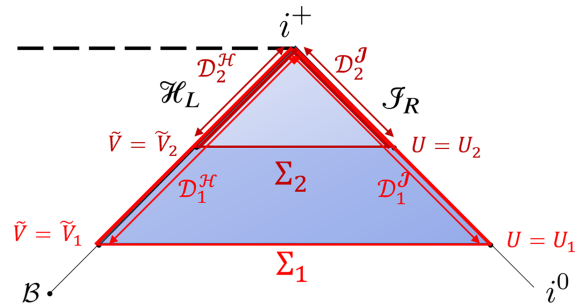


FIG. 5. The two spacelike hypersurfaces Σ_1 and Σ_l we consider in this proof as well as the regions on the horizon and null infinity on which the algebras are defined.

Sec. III. Once this is done, the proof basically amounts to applying again the monotonicity of the relative entropy. In the course of the proof appears the notion of modular Hamiltonian, which we construct in Appendix D, while in Appendix E, we relate it to the integral of some normal-ordered stress-energy tensor. Given these relations, Einstein's equations are used to make contact with the physical quantities (e.g., the Bondi mass M). Therefore, all the steps of the subsequent proof are rigorously proven from the point of view of Algebraic quantum field theory (QFT) and Modular Theory. The reader is advised to look at these Appendixes for more details.

Once the dual GSL is proven at future null infinity for the Hartle-Hawking state (Sec. IV C) and its regularizations (Secs. IV D and IV E), we consider spacelike hypersurfaces ending at different cuts on the horizon in Sec. IV F. The black hole area enters the game and imposes generalizing the definition of the thermodynamic potential $\mathcal{G}$.

A. Setup and notations

The first part of the proof focuses only on $\mathcal{I}_R^+$. Consider two spacelike hypersurfaces Σ_1 and Σ_2 starting at the bifurcation surface $\mathcal{B}$, i.e., at the cut $V = 0$ on the horizon $\mathcal{H}_L^+$, and ending at two different cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ at $\mathcal{I}_R^+$, say, at $U = U_1$ and $U = U_2$ with $U_2 > U_1 > 0$ (see Fig. 4).²⁷ Following Refs. [28,65], instead of considering directly Σ_i , we push these hypersurfaces to infinity so that they cover a certain region $\mathcal{D}_i^\beta$ of $\mathcal{H}_L^+ \cup \mathcal{I}_R^+$, namely,

$$\mathcal{D}_i^{\mathcal{B}} = \underbrace{((0, +\infty) \times S_{\mathcal{H}}^2)}_{:= \mathcal{D}^{\mathcal{H}}} \cup \underbrace{((U_i, +\infty) \times S_{\mathcal{I}}^2)}_{:= \mathcal{D}^{\mathcal{I}}}, \quad (4.1)$$

as both the horizon and null infinity have topology $\mathbb{R} \times S^2$. If we show the dual GSL between $\mathcal{D}_1^B$ and $\mathcal{D}_2^B$, then unitarity ensures that the proof extends between Σ_1 and Σ_2 . At the algebraic level, our notations are the following. We

²⁷Strictly speaking, in order to be spacelike, we should add to these regions the part between t^0 and the cut C_i . As the proof is based on differences, these regions do not matter.

denote $\mathcal{A}_i^{\mathcal{I}}$ the algebra of observables of $\mathcal{D}_i^{\mathcal{I}}$, and similarly $\mathcal{A}^{\mathcal{H}}$ is the one of $\mathcal{D}^{\mathcal{H}}$, so the total algebra of $\mathcal{D}_i^{\mathcal{B}}$ is $\mathcal{A}_i^{\mathcal{B}} = \mathcal{A}^{\mathcal{H}} \otimes \mathcal{A}_i^{\mathcal{I}}$.²⁸ In this section, we are using all the vacuum states defined and analyzed in Sec. III, at the exception of the Minkowski vacuum since we are interested in black hole backgrounds. However, the dual GSL from the Minkowski vacuum is treated in Appendix F for completeness.

B. Strategy of the proof

Let $|\Omega\rangle$ be the Hartle-Hawking state, or one of its hard or soft regularization, and $\mathcal{H}_\Omega$ be the Hilbert space built upon it via the GNS construction (see Theorem 1 of Appendix C). One can also build this Hilbert space from a set of positive frequency modes as we have done in Sec. III (see also Appendix B). This $\mathcal{H}_\Omega$ represents the algebra of observables $\mathcal{A}^{\mathcal{I}}$, which we can consider to be a von Neumann algebra (see Appendix D). Upon restriction to the subalgebra $\mathcal{A}_0^{\mathcal{I}}$ (when $U_i = 0$), the state $|\Omega\rangle$ becomes cyclic and separating (it can be seen as a straightforward extension of the Reeh-Schlieder theorem [107]). Using the symmetries we derived in the last section, it will be actually possible to show that $|\Omega\rangle$ is also cyclic and separating for any subalgebra $\mathcal{A}_i^{\mathcal{I}}$. Considering then $|\Psi\rangle \in \mathcal{H}_\Omega$ another cyclic and separating state for $\mathcal{A}_i^{\mathcal{I}}$, one can compute the relative entropy (D15) between them.

Proving the GSL amounts to use the monotonicity of the relative entropy (D16) between the algebras $\mathcal{A}_1^{\mathcal{I}}$ and $\mathcal{A}_2^{\mathcal{I}} \subset \mathcal{A}_1^{\mathcal{I}}$ (as $\mathcal{D}_2^{\mathcal{I}} \subset \mathcal{D}_1^{\mathcal{I}}$ as spacetime regions), which states that

$$S_{\mathcal{A}_1^{\mathcal{I}}}(\Psi||\Omega) \geq S_{\mathcal{A}_2^{\mathcal{I}}}(\Psi||\Omega). \quad (4.2)$$

Using the results of Appendix D especially (D22), we find that (4.2) is equivalent to

$$-S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_1^{\mathcal{I}}} + \langle K_{\Omega}^{\mathcal{A}_1^{\mathcal{I}}} \rangle_{\Psi} \geq -S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_2^{\mathcal{I}}} + \langle K_{\Omega}^{\mathcal{A}_2^{\mathcal{I}}} \rangle_{\Psi}, \quad (4.3)$$

with $S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_i^{\mathcal{I}}}$ the renormalized von Neumann entropy of $|\Psi\rangle$ with respect to $|\Omega\rangle$ [see (D23)] and with $K_{\Omega}^{\mathcal{A}_i^{\mathcal{I}}}$ the one-sided modular Hamiltonian of $|\Omega\rangle$ [see (D13)]. Consequently, we get

$$\Delta S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_i^{\mathcal{I}}} - \Delta \langle K_{\Omega}^{\mathcal{A}_i^{\mathcal{I}}} \rangle_{\Psi} \geq 0, \quad (4.4)$$

with

$$\begin{aligned} \Delta S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_i^{\mathcal{I}}} &:= S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_2^{\mathcal{I}}} - S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_1^{\mathcal{I}}} \\ \text{and } \Delta \langle K_{\Omega}^{\mathcal{A}_i^{\mathcal{I}}} \rangle_{\Psi} &:= \langle K_{\Omega}^{\mathcal{A}_2^{\mathcal{I}}} \rangle_{\Psi} - \langle K_{\Omega}^{\mathcal{A}_1^{\mathcal{I}}} \rangle_{\Psi}. \end{aligned} \quad (4.5)$$

²⁸This factorization occurs for massless fields.

An inequality analogous to (4.4) can be also written for the total algebra $\mathcal{A}_i^{\mathcal{B}} = \mathcal{A}^{\mathcal{H}} \otimes \mathcal{A}_i^{\mathcal{I}}$. One just has to take $|\Omega^{\mathcal{H}}\rangle$ an arbitrary faithful state on $\mathcal{H}_L^+$ (e.g., the restriction of the Hartle-Hawking state on the horizon) and consider the total vacuum state as

$$|\bar{\Omega}\rangle := |\Omega^{\mathcal{H}}\rangle \otimes |\Omega\rangle, \quad (4.6)$$

which is cyclic and separating for $\mathcal{A}_i^{\mathcal{B}}$. From (4.6), we construct the GNS Hilbert space $\mathcal{H}_{\bar{\Omega}}$ and consider a second cyclic and separating vector $|\bar{\Psi}\rangle \in \mathcal{H}_{\bar{\Omega}}$. Given the tensorial decomposition of the total algebra, one can repeat the steps from (4.2) to (4.4) and arrive at²⁹

$$\Delta S_{\bar{\Psi}|\bar{\Omega}}^{\text{v.N.}, \mathcal{A}_i^{\mathcal{B}}} - \Delta \langle K_{\bar{\Omega}}^{\mathcal{A}_i^{\mathcal{B}}} \rangle_{\bar{\Psi}} \geq 0, \quad (4.7)$$

which is an inequality between the total algebras $\mathcal{A}_1^{\mathcal{B}}$ and $\mathcal{A}_2^{\mathcal{B}}$, i.e., between the spacetime domain $\mathcal{D}_1^{\mathcal{B}}$ and $\mathcal{D}_2^{\mathcal{B}}$.

Equation (4.7) will be called the *fundamental inequality* and admits the same form whatever the cyclic and separating states $|\bar{\Omega}\rangle$ and $|\bar{\Psi}\rangle \in \mathcal{H}_{\bar{\Omega}}$. What changes from a vacuum state to another is the relation between the vacuum expectation value (VEV) of the one-sided modular Hamiltonian and the physical quantities of interest like the Bondi mass M (at $\mathcal{I}_R^+$) or the area of the horizon A (on $\mathcal{H}_L^+$). The link is done in two steps. First, using Appendix E, one can relate the modular Hamiltonian to the integral of the normal-ordered boost energy [see (E16)]. Second, one relates this normal-ordered quantity to the covariant one and uses the results of Sec. II C to make M appear. Finally, one can write the thermodynamic potential $\mathcal{G}_{\bar{\Omega}}$ whose monotonicity at $\mathcal{I}_R^+$ is the dual GSL.

C. Hartle-Hawking state

First, we choose $|\Omega\rangle$ to be the restriction of the Hartle-Hawking vacuum to $\mathcal{I}_R^+$, and we denote it by $|\Omega_H\rangle$. It has been constructed in Sec. III, and it is a thermal state when restricted to the algebra $\mathcal{A}_0^{\mathcal{I}}$, i.e., any pair of observables $(A, B) \in (\mathcal{A}_0^{\mathcal{I}})^2$ satisfies the KMS conditions (D14). That state is natural to consider at null infinity for the dual GSL as Wall [28] used the restriction of the Hartle-Hawking state to the black hole horizon $\mathcal{H}_L$.³⁰ It is shown in Appendix E that the one-sided modular Hamiltonian with respect to the restricted algebra $\mathcal{A}_0^{\mathcal{I}}$ is given by

²⁹Note that, due to the product state structure of (4.6), the constant terms drop out from all the inequalities, and we have that $\Delta \langle K_{\bar{\Omega}}^{\mathcal{A}_i^{\mathcal{B}}} \rangle_{\bar{\Psi}} = \Delta \langle K_{\Omega}^{\mathcal{A}_i^{\mathcal{I}}} \rangle_{\bar{\Psi}}$.

³⁰In particular, the state $|\Omega_H\rangle$ can be defined at null infinity even if the Hartle-Hawking state does not exist, as it is the case for the Kerr black hole, similarly to what had been noted on the horizon [28].

$$K_{\Omega_H}^{\mathcal{A}_0^T} = 2\pi \int_{\mathcal{I}_R^+} U : T_{UU} :_{\Omega_H} dU \wedge \epsilon_S, \quad (4.8)$$

so, formally, the restriction of $|\Omega_H\rangle$ to $\mathcal{A}_0^T$ can be written as

$$\rho_{\Omega_H}^{\mathcal{A}_0^T} = \frac{e^{-K_{\Omega_H}^{\mathcal{A}_0^T}}}{\text{Tr}\left(e^{-K_{\Omega_H}^{\mathcal{A}_0^T}}\right)}, \quad (4.9)$$

where $:T_{UU}:_{\Omega_H}$ is the normal-ordered stress-energy tensor introduced in (3.34).

Using (3.33), namely, the symmetries of the state $|\Omega_H\rangle$, we can restrict it to the future domain $\mathcal{D}_i^T$ of any cut $U = U_i$ of $\mathcal{I}_R^+$ (via the change $U \rightarrow U - U_i$) and still get a thermal state with respect to the restricted algebra of observables $\mathcal{A}_i^T$. Therefore, the one-sided modular Hamiltonian associated to $\mathcal{A}_i^T$ is given by

$$K_{\Omega_H}^{\mathcal{A}_i^T} = 2\pi \int_{\mathcal{D}_i^T} (U - U_i) : T_{UU} :_{\Omega_H} dU \wedge \epsilon_S, \quad (4.10)$$

and the formal density matrix is

$$\rho_{\Omega_H}^{\mathcal{A}_i^T} = \frac{e^{-K_{\Omega_H}^{\mathcal{A}_i^T}}}{\text{Tr}\left(e^{-K_{\Omega_H}^{\mathcal{A}_i^T}}\right)}. \quad (4.11)$$

Hence, we can apply the fundamental inequality (4.7) with (4.11). It remains to relate the latter to the Bondi mass M , using Einstein's equations (2.29). However, these equations use the covariant stress tensor T_{UU} instead of the normal-ordered version $:T_{UU}:_{\Omega_H}$. The relation between the two can be obtained from the equivalence principle, which states that

$$T_{u_+ u_+} = :T_{u_+ u_+}:_{0_M}, \quad (4.12)$$

with $u_+ = \kappa^{-1} \ln U$ the affine coordinate on $\mathcal{I}_R^+$ and $|0\rangle_M$ the Minkowski vacuum. Under the change $u_+ \rightarrow U$ the l.h.s. of (4.12) transforms covariantly, while the r.h.s. transforms with the Schwarzian derivative (3.35). The latter physically corresponds to the zero point energy, i.e., the energy density of the Hartle-Hawking vacuum $|\Omega_H\rangle$ compared to the one of the Minkowski vacuum $|0\rangle_M$ on $\mathcal{I}_R^+$. This quantity, as we said precisely in (3.35), is infinite, and that was why regularizations of the Hartle-Hawking state were necessary to introduce. However, an important proof of concepts can be performed if we allow ourselves for a moment to manipulate this infinite Schwarzian. Formally, then, the relation between the covariant and normal-ordered stress-energy tensor admits the form

$$T_{UU} = :T_{UU}:_{\Omega_H} + \text{Schwarzian term}, \quad (4.13)$$

so from Einstein's equations (2.29), we get [the state $|\bar{\Psi}\rangle \in \mathcal{H}_{\bar{\Omega}_H}$, with $|\bar{\Omega}_H\rangle = |\Omega_H^L\rangle \otimes |\Omega_H\rangle$, has to be chosen such that (2.24) holds]

$$\Delta\langle K_{\Omega_H}^{\mathcal{A}_i^T} \rangle_{\bar{\Psi}} = \Delta M - \Delta M_{\Omega_H}, \quad (4.14)$$

where ΔM_{Ω_H} is the variation of Bondi mass in the Hartle-Hawking state coming from the Schwarzian. It is infinite.³¹ From the fundamental inequality (4.7), we get

$$\Delta M - \Delta M_{\Omega_H} - T_H \Delta S_{\bar{\Psi}|\bar{\Omega}_H}^{\text{v.N.}, \mathcal{A}_i^B} \leq 0. \quad (4.15)$$

The term ΔM_{Ω_H} can actually be related to a notion of entropy using the following reasoning. The outgoing radiation in the state $|\Omega_H\rangle$ not only carries energy but also entropy, and it is clear that $S_{\Omega_H|\bar{\Omega}_H}^{\text{v.N.}, \mathcal{A}_i^B} = 0$, so we also need to add the entropy variation of the reference state ΔS_{Ω_H} . However, the Hartle-Hawking state is a thermal state. On the slices Σ_i of constant volume,³² the entropy variation can only come from the energy variation between Σ_1 and Σ_2 , given by ΔM_{Ω_H} . Using the equilibrium relation

$$\left. \frac{\partial S}{\partial E} \right|_V = \frac{1}{T_H}, \quad (4.16)$$

we conclude³³ that

$$\Delta S_{\Omega_H} = \frac{\Delta M_{\Omega_H}}{T_H}. \quad (4.17)$$

Therefore, by adding (4.17) to (4.15), we get

$$\Delta M - T_H \Delta S \leq 0, \quad (4.18)$$

where

$$\Delta S = \Delta S_{\bar{\Psi}|\bar{\Omega}_H}^{\text{v.N.}, \mathcal{A}_i^B} + \Delta S_{\Omega_H} \quad (4.19)$$

would be the total entropy variation. The thermodynamic potential of interest in the Hartle-Hawking state is therefore the free energy

³¹Notice that, even if there is no net energy flux in the bulk in the Hartle-Hawking vacuum, there is an outgoing energy flux at $\mathcal{I}_R^+$ since at future null infinity the outgoing radiation is not compensated by the incoming one.

³²The volume of the two spacelike slices Σ_1 and Σ_2 is infinite but fixed.

³³Equivalently, we can use the Clausius relation $\Delta S = \frac{Q}{T_H}$, valid at first order in perturbation around an equilibrium state. Since the radiation in the Hartle-Hawking state at $\mathcal{I}_R^+$ is exactly thermal, there is no work that can be extracted by an asymptotic observer, so $W = 0$, and we have $\Delta S = \frac{\Delta E}{T_H}$ using the first law of thermodynamics.

$$\mathcal{G}_{\Omega_H} = M - T_H S := \mathcal{F}. \quad (4.20)$$

It seems that we have proven in (4.18) the decreasing of the free energy (4.20) in the Hartle-Hawking state. A similar relation has been argued to hold in Sec. 3 of Ref. [65]. Notice that the variation $\Delta\mathcal{F}$ is finite and well defined and vanishes if $|\bar{\Psi}\rangle = |\bar{\Omega}_H\rangle$. However, both quantities appearing in (4.17) are *stricto sensu* infinite in that state. It is because the radiation does not only come from the black hole but also from every direction in space, so the energy density of the physical stress-energy tensor is constant and does not decay at infinity. In particular, the black hole is in thermal equilibrium with the surrounding radiation and cannot be distinguished from the latter. Still, a real black hole is not in equilibrium with some incoming radiation and only emits low angular momenta modes, because the Hawking radiation comes *only* from the neighborhood of the black hole horizon. Therefore, in order to interpret (4.18), we need to regulate the high angular momentum modes. This is what we do now by considering the hard and soft regularizations we introduced in Sec. III.

D. Hard regularization: L -vacuum

We consider in this section the L -vacuum $|\Omega_H^L\rangle$ constructed from the notion of time (3.38) and the space of positive solutions (3.39). Before proceeding with the expression of the modular Hamiltonian, we prove a general result.

1. Useful lemma

Assume that on any section S of $\mathcal{I}_R^+$ there exist a constant C_Ω (which depends on the vacuum state $|\Omega\rangle$) such that the relation between the integrated stress tensor (on S) and its integrated normal-ordered version is of the form

$$\int_S T_{UU} dU \wedge \epsilon_S = \int_S :T_{UU}:_\Omega dU \wedge \epsilon_S + \frac{C_\Omega}{U^2} dU. \quad (4.21)$$

Then, it follows that

$$\begin{aligned} \int_{\mathcal{D}_i^\Lambda} (U - U_i) T_{UU} dU \wedge \epsilon_S &= \int_{\mathcal{D}_i^\Lambda} (U - U_i) :T_{UU}:_\Omega dU \wedge \epsilon_S \\ &+ \int_{u_i}^\lambda C_\Omega (1 - e^{-\kappa(u-u_i)}) \kappa du, \end{aligned} \quad (4.22)$$

where $\mathcal{D}_i^\Lambda = (U_i, \Lambda) \times S^2$ with $\lambda = \kappa^{-1} \ln \Lambda$, and $U = \Lambda$ (i.e., $u = \lambda$) is a cross section of $\mathcal{I}_R^+$ at (almost) arbitrary late time, as we assume that $\lambda - u_i \gg \kappa^{-1}$. Then, in the limit $\lambda \rightarrow +\infty$ (equivalently $\Lambda \rightarrow +\infty$), the second term in the right-hand side of (4.22) diverges, while

$$\int_{\mathcal{D}_i^\Lambda} (U - U_i) :T_{UU}:_\Omega dU \wedge \epsilon_S \xrightarrow{\Lambda \rightarrow +\infty} \frac{1}{2\pi} K_\Omega^{\mathcal{A}_i^T} \quad (4.23)$$

converges to the one-sided modular Hamiltonian. Therefore, we conclude that the left-hand side of (4.22) necessarily diverges. However, computations related to the dual GSL only involve differences like, e.g.,

$$\begin{aligned} \Delta K_{U_i}^\Lambda &:= \int_{\mathcal{D}_2^\Lambda} (U - U_i) T_{UU} dU \wedge \epsilon_S \\ &- \int_{\mathcal{D}_1^\Lambda} (U - U_i) T_{UU} dU \wedge \epsilon_S, \end{aligned} \quad (4.24)$$

which is well defined for $\Lambda > U_2$ and converges when $\Lambda \rightarrow +\infty$. Indeed,

$$\Delta K_{U_i}^{+\infty} = \frac{1}{2\pi} \Delta K_\Omega^{\mathcal{A}_i^T} - (u_2 - u_1) \kappa C_\Omega, \quad (4.25)$$

so for any $|\Psi\rangle \in \mathcal{H}_\Omega$, we have that

$$\Delta \langle K_{U_i}^{+\infty} \rangle_\Psi = \frac{1}{2\pi} \Delta \langle K_\Omega^{\mathcal{A}_i^T} \rangle_\Psi - (u_2 - u_1) \kappa C_\Omega \quad (4.26)$$

that is finite. Recalling (2.30), i.e.,

$$\Delta \langle K_{U_i}^{+\infty} \rangle_\Psi := \Delta \langle K_{U_i} \rangle_\Psi = \frac{\Delta M}{\kappa}, \quad (4.27)$$

we find that now the Bondi mass variation is finite. Finally, when (4.21) holds, the relation between the variation of the one-sided modular Hamiltonian and the variation of Bondi mass reads

$$\Delta M = T_H \Delta \langle K_\Omega^{\mathcal{A}_i^T} \rangle_\Psi - (u_2 - u_1) \kappa^2 C_\Omega, \quad (4.28)$$

and we define, by analogy with (4.14),

$$\Delta M_\Omega = -(u_2 - u_1) \kappa^2 C_\Omega, \quad (4.29)$$

the variation of Bondi mass in the vacuum state $|\Omega\rangle$. The main message of this section is that having a nonvanishing Schwarzian (i.e., an energy density in the vacuum state $|\Omega\rangle$) is fine as long as the integrated stress-energy tensor on a cross section S can be written in a form (4.21). Under such an hypothesis, the variation of Bondi mass will be finite between two arbitrary cross sections. Note that (4.26)–(4.28) remain valid upon replacing $|\Psi\rangle$ by $|\bar{\Psi}\rangle \in \mathcal{H}_{\bar{\Omega}}$ with $|\bar{\Omega}\rangle$ defined in (4.6).

2. Modular Hamiltonian of the L -vacuum

To make contact with the last lemma and to avoid rendering the notations even more heavy, we shall consider in this paragraph the integrated version of the stress-energy tensor over the spatial cross sections of $\mathcal{I}_R^+$. When this is the case, we replace the canonical capital T by a lowercase t , e.g.,

$$:t_{UU}:\Omega = \int_S :T_{UU}:\Omega \epsilon_S. \quad (4.30)$$

From the general results of Appendix E and using the symmetries (3.19), it can be shown that the one-sided modular Hamiltonian of $|\Omega_H^L\rangle$ restricted to the algebra $\mathcal{A}_i^{\mathcal{T}}$ is

$$\begin{aligned} K_{\Omega_H^L}^{\mathcal{A}_i^{\mathcal{T}}} &:= \sum_{l=0}^{+\infty} \int_{\mathcal{D}_i^{\mathcal{T}}} k_l \\ &= 2\pi \sum_{l=0}^{L-1} \int_{U_i}^{+\infty} (U - U_i) :t_{UU}^l:\Omega_H^L dU \\ &\quad + 2\pi \sum_{l=L}^{+\infty} \int_{u_i}^{+\infty} (u - u_i) :t_{uu}^l:\Omega_H^L du, \end{aligned} \quad (4.31)$$

where (we omit the hat, as it is clear that we are dealing with quantum fields)

$$\begin{aligned} \phi &= \sum_{l=0}^{+\infty} \phi_l, \\ \phi_l &= \sum_{m=-l}^{m=+l} \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(a_{\Omega lm} Y_m^l e^{-i\Omega U} + \bar{Y}_m^l a_{\Omega lm}^\dagger e^{i\Omega U} \right), \end{aligned} \quad (4.32)$$

so

$$:t_{uu}^l:\Omega_H^L = \int_{S^2} (\partial_u \phi_l \partial_u \phi_l - \langle \partial_u \phi_l \partial_u \phi_l \rangle_{\Omega_H^L}) \epsilon_S. \quad (4.33)$$

The covariant stress-energy tensor, integrated on the spatial sections of $\mathcal{I}_R^+$, is

$$t_{uu} := \int_{S^2} T_{uu} \epsilon_S = \sum_{l=0}^{+\infty} t_{uu}^l, \quad (4.34)$$

where we defined the effective two-dimensional stress-energy tensor associated to the chiral CFTs with the same quantum number l to be

$$t_{uu}^l := :t_{uu}^l:_{0_M} = \int_{S^2} (\partial_u \phi_l \partial_u \phi_l - \langle \partial_u \phi_l \partial_u \phi_l \rangle_{0_M}) \epsilon_S. \quad (4.35)$$

Since for the modes $l \geq L$ the L -vacuum is indistinguishable from the Minkowski vacuum, we have that

$$l \geq L \Rightarrow :t_{uu}^l:\Omega_H^L = :t_{uu}^l:_{0_M} = t_{uu}^l. \quad (4.36)$$

Therefore, in order to relate the one-sided modular Hamiltonian (4.31) to the real stress-energy distribution (integrated on a cross section S), we change coordinate and deal with the Schwarzian derivative only when $l < L$. At the end, we get

$$\begin{aligned} 2\pi(U - U_i) t_{UU}^l dU &= \frac{2\pi}{\kappa} (1 - e^{-\kappa(u-u_i)}) t_{uu}^l du \\ &= \int_{S^2} k_l + \mathcal{R}_l, \end{aligned} \quad (4.37)$$

where

$$\begin{cases} \mathcal{R}_l = \frac{2\pi}{\kappa} (1 - e^{-\kappa(u-u_i)}) \frac{(2l+1)\kappa^2}{48\pi} du & \text{if } l < L \\ \mathcal{R}_l = \left[-2\pi(u - u_i) + \frac{2\pi}{\kappa} (1 - e^{-\kappa(u-u_i)}) \right] t_{uu}^l du & \text{if } l \geq L. \end{cases} \quad (4.38)$$

It is clear that the first term in (4.38) diverges if $L \rightarrow +\infty$ while the second one goes to zero (if we consider a state $|\Psi\rangle \in \mathcal{H}_{\Omega_H^L}$ that have finite energy density). Then, at fixed L , we consider the states $|\Psi\rangle \in \mathcal{H}_{\Omega_H^L}$ so that

$$\sum_{l=0}^{L-1} \langle \mathcal{R}_l \rangle_\Psi = \frac{2\pi}{\kappa} (1 - e^{-\kappa(u-u_i)}) \frac{L^2 \kappa^2}{48\pi} du \gg \sum_{l=L}^{+\infty} \langle \mathcal{R}_l \rangle_\Psi \quad (4.39)$$

that is a condition always satisfied for $|\Omega_H^L\rangle$ since $\sum_{l=L}^{+\infty} \langle \mathcal{R}_l \rangle_{\Omega_H^L} = 0$ and that is true for most of the states if we take $L \gg 1$ [and true in a sense for any state if $L = +\infty$ since the left-hand side of (4.39) diverges].³⁴ Then, since

$$2\pi(U - U_i) t_{UU} dU = \sum_{l=0}^{+\infty} 2\pi(U - U_i) t_{UU}^l dU, \quad (4.40)$$

we have that

$$\begin{aligned} \frac{2\pi}{\kappa} \int_{U_i}^\Lambda \kappa(U - U_i) \langle t_{UU} \rangle_\Psi dU \\ = \sum_{l=0}^{+\infty} \int_{\mathcal{D}_i^\Lambda} \langle k_l \rangle_\Psi + \frac{2\pi}{\kappa} \int_{u_i}^\Lambda (1 - e^{-\kappa(u-u_i)}) \frac{L^2 \kappa^2}{48\pi} du, \end{aligned} \quad (4.41)$$

which is of the form (4.21) with $C_L = \frac{L^2}{48\pi}$. Therefore, in virtue of the lemma, we get in the limit $\Lambda \rightarrow +\infty$

$$\frac{2\pi}{\kappa} \Delta M = \Delta \langle K_{\Omega_H^L}^{\mathcal{A}_i} \rangle_\Psi + \frac{2\pi}{\kappa} \Delta M_{\Omega_H^L}, \quad (4.42)$$

where

$$\Delta M_{\Omega_H^L} := -(u_2 - u_1) \frac{L^2 \kappa^2}{48\pi} \quad (4.43)$$

³⁴This condition is immediately satisfied for the states $|\Psi\rangle \in \{\mathcal{A}_i^L |\Omega_H^L\rangle\}$ where the algebra $\mathcal{A}_i^L$ is given by (3.9) [$\mathcal{A}_i^L$ is the subalgebra of (3.9) restricted to the region $\mathcal{D}_i$].

is the stress-energy density on $\mathcal{I}_R^+$ in the state $|\Omega_H^L\rangle$ and is therefore the zero-point energy density. This is true as long as (2.24) and (4.39) are valid for $|\Psi\rangle$ (it is all the more true when L is large).

3. Second law from the L -vacuum

If we fix $L \in \mathbb{N}^*$ [assuming that it is “large enough” for (4.39) to hold] and write the fundamental inequality (4.7),

$$\Delta S_{\bar{\Psi}|\bar{\Omega}_H^L}^{\text{v.N.}, \mathcal{A}_i^B} - \Delta \langle K_{\Omega_H^L}^{\mathcal{A}_i^T} \rangle_{\bar{\Psi}} \geq 0, \quad (4.44)$$

where $|\bar{\Omega}^L\rangle = |\Omega^{\mathcal{H}}\rangle \otimes |\Omega_H^L\rangle$ and $|\bar{\Psi}\rangle \in \mathcal{H}_{\bar{\Omega}^L}$. All the steps between (4.38) and (4.42) remain valid as long as $|\bar{\Psi}\rangle$ also satisfies (4.39). We see with (4.42) that the one-sided modular Hamiltonian is related to the difference between the Bondi mass variation in the state $|\bar{\Psi}\rangle$ and the one in the L -vacuum. To make the latter quantity disappear, we use the Clausius relation, as we discussed in Sec. IV C. Indeed, the radiation in the L -vacuum carries not only energy but also entropy. Exactly as we argued before, the radiation is thermal, so we cannot extract work from it; hence, the energy dissipated should be counted as a heat flux. Then, since in the state $|\Omega_H^L\rangle$ the total amount of energy is strictly speaking infinite³⁵ and the radiation is in thermal equilibrium, we can apply the Clausius relation to obtain the variation of entropy in the L -vacuum (the zero point entropy variation)

$$\Delta S_{\Omega_H^L} = \frac{2\pi}{\kappa} \Delta M_{\Omega_H^L}. \quad (4.45)$$

Notice that, strictly speaking, the vacuum state $|\Omega_H^L\rangle$ is thermal with respect to the flow generated by the boost field $(U - U_i)\partial_U$ only if we restrict ourselves to the algebra $\mathcal{A}_i^L$ defined in (3.9).³⁶ However, in the state $|\Omega_H^L\rangle$, the modes $l \geq L$ are not excited with respect to the Minkowski vacuum $|0\rangle_M$, so they do not carry any energy and entropy, and therefore the thermal reservoir justifying the use of the Clausius relation (4.45) is only made of the modes with angular momentum $l \leq L$. Then, by combining (4.45), (4.42), and (4.44), we get

³⁵In the L -vacuum, the total amount of energy is infinite, as it is the case in the Hartle-Hawking vacuum state or in the more physical Unruh vacuum state, even if the local energy density at $\mathcal{I}_R^+$ is not. It means that if we calculate the ADM mass in all these states we would get infinity. It is not a problem if we only look at the local observables and local backreactions, as we are doing here. For instance, the black hole luminosity at $\mathcal{I}_R^+$, computed in the Unruh vacuum state, is a local quantity that has physical significance, even if the ADM mass of the Unruh vacuum is infinite.

³⁶This means that the KMS conditions in the vacuum state $|\Omega_H^L\rangle$ are satisfied only for any pair of observables $(A, B) \in \mathcal{A}_i^L$ if we take the modular Hamiltonian to be conjugated to the “time” generator $\xi = (U - U_i)\partial_U$.

$$\Delta M - T_H \Delta S \leq 0, \quad (4.46)$$

where

$$\Delta S := \Delta S_{\bar{\Psi}|\bar{\Omega}_H^L}^{\text{v.N.}, \mathcal{A}_i^B} + \Delta S_{\Omega_H^L} \quad (4.47)$$

and we obtain that the relevant thermodynamic potential in the L -vacuum is again the free energy

$$\mathcal{G}_{\Omega_H^L} = M - T_H S := \mathcal{F}. \quad (4.48)$$

Notice that if we take $L = +\infty$ the relation (4.46) still makes sense and is perfectly well defined, but ΔM and ΔS become infinite; we recover the Hartle-Hawking case (4.18). As we argued in Sec. III, a nice notion of black hole at thermal equilibrium at the Hawking temperature would be to take L arbitrarily large but finite. Then, the only outgoing energy flux that is not compensated by an incoming one would be coming from the modes $l \geq L$, which are actually exponentially suppressed when l is big, so if we take L to be large enough, we can neglect the contribution of this flux for arbitrarily large time intervals.³⁷

E. Soft regularization: The κ_l -vacuum

We consider finally the κ_l -vacuum state $|\Omega_H^{\{\kappa_l\}}\rangle$ defined using the notion of time (3.44) and the positive frequency modes (3.45). This state admits the tensor product decomposition (3.21) with $U^{(l,m)} = U^l = e^{\kappa_l u}$ on $\mathcal{I}_R^+$. Then, we decompose the field operator ϕ in a sum of scalar fields ϕ_l corresponding to the different sectors l ,

$$\begin{aligned} \phi &= \sum_l \phi_l \Rightarrow \phi_l \\ &= \sum_{m=-l}^l \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \left(Y_m^l \hat{a}_{\Omega l m} e^{-i\Omega U^{(l,m)}} + \bar{Y}_m^l \hat{a}_{\Omega l m}^\dagger e^{i\Omega U^{(l,m)}} \right), \end{aligned} \quad (4.49)$$

so that the l -stress-energy tensor [renormalized with respect to the vacuum $|\Omega_H^{U^l}\rangle$ attached to each sector l ; see (3.21)] reads

$$:T_{U^l U^l}^l:_{\Omega^{U^l}} = \partial_{U^l} \phi_l \partial_{U^l} \phi_l - \langle \partial_{U^l} \phi_l \partial_{U^l} \phi_l \rangle_{\Omega^{U^l}}. \quad (4.50)$$

Like in the previous paragraph on the L -vacuum, we denote with t_{uu} the component of the stress-energy tensor integrated over a spatial cross section of $\mathcal{I}_R$, i.e., for instance,

$$:t_{U^l U^l}^l:_{\Omega_H^{U^l}} = \int_S :T_{U^l U^l}^l:_{\Omega_H^{U^l}} \epsilon_S. \quad (4.51)$$

³⁷Notice that this equilibrium would nevertheless be very instable; see footnote 23.

1. Modular Hamiltonian of the κ_l -vacuum

From the general results of Appendix E, we can compute the one-sided modular Hamiltonian of the state $|\Omega_H^{U^l}\rangle$ with respect to the algebra $\mathcal{A}_i^T$,

$$K_{\Omega_H^{U^l}}^{\mathcal{A}_i^T} = 2\pi \int_{\mathcal{D}_i^T} (U^l - U_i^l) : T_{U^l U^l}^l :_{\Omega_H^{U^l}} dU^l \wedge \epsilon_S. \quad (4.52)$$

The relation between the covariant and normal-ordered stress-energy tensor, integrated over S , reads

$$t_{U^l U^l}^l = : t_{U^l U^l}^l :_{\Omega_H^{U^l}} + \frac{1}{(\kappa_l U_l)^2} \frac{(2l+1)\kappa_l^2}{48\pi}, \quad (4.53)$$

that is in the form (4.21) with $C_l = \frac{(2l+1)}{48\pi}$. Therefore, we deduce that

$$\begin{aligned} \frac{2\pi}{\kappa_l} \kappa_l (U^l - U_i^l) t_{U^l U^l}^l dU^l &= 2\pi (U^l - U_i^l) : t_{U^l U^l}^l :_{\Omega_H^{U^l}} dU^l \\ &+ \frac{2\pi}{\kappa_l} (1 - e^{-\kappa_l(u-u_i)}) \frac{(2l+1)\kappa_l^2}{48\pi} du. \end{aligned} \quad (4.54)$$

The integration of (4.54) over the whole spacetime domain $\mathcal{D}_i^\Lambda$ (it remains to perform the integral over the null time direction) gives

$$K_{\Omega_H^{U^l}}^{\mathcal{A}_i^T} = \frac{2\pi}{\kappa_l} \left(E^l + \int_{u_i}^\lambda (e^{-\kappa_l(u-u_i)} - 1) \frac{(2l+1)\kappa_l^2}{48\pi} du \right), \quad (4.55)$$

that is well defined although the two terms in the r.h.s. of (4.55) are not, so we need to introduce an infrared regulator $\Lambda := \kappa^{-1} \ln \lambda$ to manipulate them individually. We also have introduced E^l , the *energy of mode l* ,

$$\begin{aligned} E^l &:= \int_{\mathcal{D}_i^T} \kappa_l (U^l - U_i^l) T_{U^l U^l}^l dU^l \wedge \epsilon_S \\ &= \int_{\mathcal{D}_i^T} (1 - e^{-\kappa_l(u-u_i)}) T_{uu}^l du \wedge \epsilon_S, \end{aligned} \quad (4.56)$$

that is also regularized with an infrared cutoff Λ as in the proof of the lemma (4.21). Since in the end one will only look at differences of quantities like (4.56), the presence of such cutoffs is only temporary, and the final results will be of course regularization independent. Therefore, for the sake of clarity, the presence of the infrared cutoff Λ in (4.56) is kept implicit. At this stage, it is interesting to introduce a new quantity $\tilde{\mu}_l$ in order to relate our analysis to usual thermodynamics. We set

$$\tilde{\mu}_l := 1 - \frac{\kappa}{\kappa_l}, \quad (4.57)$$

and we notice that, since $\kappa \geq \kappa_l$, $\tilde{\mu}_l$ is negative and vanishes if we have a perfect transmission, i.e., $\tilde{\mu}_l = 0$ when $\kappa_l = \kappa$. As we shall show latter in the proof, $\tilde{\mu}_l$ can be interpreted as a *chemical potential* for the two-dimensional effective CFT underpinning the field at angular momentum mode l . Then, we can rewrite (4.55) as

$$K_{\Omega_H^{U^l}}^{\mathcal{A}_i^T} = \frac{2\pi}{\kappa} (1 - \tilde{\mu}_l) \left(E^l + \int_{u_i}^\lambda (e^{-\kappa_l(u-u_i)} - 1) \frac{(2l+1)\kappa_l^2}{48\pi} du \right), \quad (4.58)$$

so the one-sided modular Hamiltonian of the κ_l -vacuum is obtained only via a sum over the one-sided modular Hamiltonians associated to each sector l ,

$$K_{\Omega_H^{\{\kappa_l\}}}^{\mathcal{A}_i^T} := \sum_l K_{\Omega_H^{U^l}}^{\mathcal{A}_i^T} = \sum_l \left[\frac{2\pi}{\kappa} (1 - \tilde{\mu}_l) \left(E^l + \int_{u_i}^\lambda (e^{-\kappa_l(u-u_i)} - 1) \frac{(2l+1)\kappa_l^2}{48\pi} du \right) \right]. \quad (4.59)$$

Of course, we choose the sequence $\{\kappa_l\}_{l \geq 0}$ so that³⁸

$$\sum_{l \geq 0} (2l+1)\kappa_l < +\infty \quad (4.60)$$

in order to get finite quantities.³⁹

2. Monotonicity of relative entropy

Now, we look at the relative entropy between a state $|\tilde{\Psi}\rangle \in \mathcal{H}_{\Omega_H^{\{\kappa_l\}}}$ and the total vacuum $|\bar{\Omega}_H^{\{\kappa_l\}}\rangle = |\Omega^{\mathcal{H}}\rangle \otimes |\Omega_H^{\{\kappa_l\}}\rangle$ with respect to the algebra $\mathcal{A}_i^B$. Then, from the fundamental inequality (4.7), we can write

$$\Delta S_{\tilde{\Psi}|\bar{\Omega}_H^{\{\kappa_l\}}}^{\text{v.N.}, \mathcal{A}_i^B} - \Delta \langle K_{\Omega_H^{\{\kappa_l\}}}^{\mathcal{A}_i^T} \rangle_{\tilde{\Psi}} \geq 0. \quad (4.61)$$

Given (4.59), we have

$$\begin{aligned} \Delta \langle K_{\Omega_H^{\{\kappa_l\}}}^{\mathcal{A}_i^T} \rangle_{\tilde{\Psi}} &= \frac{2\pi}{\kappa} \sum_l (\Delta E^l - \Delta E_{\Omega_H^{U^l}}^l) \\ &- \frac{2\pi}{\kappa} \sum_l \tilde{\mu}_l (\Delta E^l - \Delta E_{\Omega_H^{U^l}}^l), \end{aligned} \quad (4.62)$$

³⁸Notice that $\sum_{l \geq 0} (2l+1)\kappa_l < +\infty$ also implies that $\sum_{l \geq 0} (2l+1)\kappa_l^2 < +\infty$.

³⁹Since the potential barrier has a height that is of order $l(l+1)$, it would be natural to choose $\kappa_l \sim e^{-\alpha l(l+1)}$ with $\alpha > 0$ a constant, to try to model a real black hole. Such a choice of κ_l ensures convergence.

where $\Delta E_{\Omega_H^{U^l}}^l$ is the energy variation of the mode l in the vacuum $|\Omega_H^{U^l}\rangle$ attached to the angular mode l . Following (4.29), the latter reads

$$\Delta E_{\Omega_H^{U^l}}^l = -(u_2 - u_1) \frac{(2l+1)\kappa_l^2}{48\pi}, \quad (4.63)$$

where all the quantities defined above are finite due to the lemma (4.21).⁴⁰ Now, we use an assumption similar to (2.24) on the state $|\bar{\Psi}\rangle$,

$$\forall l, \quad \kappa_l^{-1} \partial_u \langle T_{uu}^l \rangle_{\bar{\Psi}} \ll \langle T_{uu}^l \rangle_{\bar{\Psi}}, \quad (4.64)$$

so we can neglect the exponential term in (4.56) in front of the constant one,⁴¹ so we can relate the variation of total energy in the state $|\bar{\Psi}\rangle$, i.e., $\Delta E = \sum_l \Delta E^l$ to the Bondi mass via (2.30)

$$\Delta M = \sum_l \Delta \langle E^l \rangle_{\bar{\Psi}}. \quad (4.67)$$

Then, as we did in the previous subsections, since the energy variations in the vacuum state appears in (4.62), we can use the Clausius relation to relate it to the entropy variation of the radiation in the state. Note that in the state $|\Omega_H^{U^l}\rangle$ the modes with angular momentum l are in thermal equilibrium at temperature $T_l = \frac{\kappa_l}{2\pi}$. Modes of the quantum field with different angular momentum l are therefore emitted from distinct reservoirs with local temperature

⁴⁰In particular, the dependence on the cutoff Λ has been removed at this stage.

⁴¹Indeed, notice that by integrating by part the second term in the r.h.s. of (4.58) we have that

$$\begin{aligned} & \int_{u_i}^{\lambda} \int_S (1 - e^{-\kappa_l(u-u_i)}) \langle T_{uu}^l \rangle_{\bar{\Psi}} du \wedge \epsilon_S \\ &= \int_{u_i}^{\lambda} \int_S \langle T_{uu}^l \rangle_{\bar{\Psi}} du \wedge \epsilon_S + \int_{u_i}^{\lambda} \int_S \kappa_l^{-1} \partial_u \langle T_{uu}^l \rangle_{\bar{\Psi}} du \wedge \epsilon_S \\ & \quad - \kappa_l^{-1} \int_S \langle T_{uu}^l \rangle_{\bar{\Psi}}|_{\lambda} + O\left(\int_{u_i}^{\lambda} \int_S \kappa_l^{-1} \partial_u \langle T_{uu}^l \rangle_{\bar{\Psi}} du \wedge \epsilon_S\right), \end{aligned} \quad (4.65)$$

if we take $\lambda - u_i \gg \kappa_l^{-1}$, which we can always do by taking λ sufficiently large. Then, by subtracting the above result for the two regions $\mathcal{D}_1^\lambda$ and $\mathcal{D}_2^\lambda$, we get

$$\begin{aligned} & \Delta \left(\int_{\mathcal{D}_i^\lambda} (1 - e^{-\kappa_l(u-u_i)}) \langle T_{uu}^l \rangle_{\bar{\Psi}} du \wedge \epsilon_S \right) \\ &= \Delta \left(\int_{\mathcal{D}_i^\lambda} (\langle T_{uu}^l \rangle_{\bar{\Psi}} + \kappa_l^{-1} \partial_u \langle T_{uu}^l \rangle_{\bar{\Psi}} + O(\kappa_l^{-1} \partial_u \langle T_{uu}^l \rangle_{\bar{\Psi}})) du \wedge \epsilon_S \right) \end{aligned} \quad (4.66)$$

and taking the limit $\lambda \rightarrow +\infty$, the l.h.s. of (4.66) becomes ΔE_l if we assume (4.64).

$T_l = \frac{\kappa_l}{2\pi}$. The Clausius relation applied on each reservoir l tells us that

$$\frac{\Delta E_{\Omega_H^{U^l}}^l}{T_l} = \frac{\Delta E_{\Omega_H^{U^l}}^l}{T_H} - \frac{\tilde{\mu}_l}{T_H} \Delta E_{\Omega_H^{U^l}}^l = \Delta S_{\Omega_H^{U^l}}^l, \quad (4.68)$$

and since the vacuum state $|\Omega_H^{\{\kappa_l\}}\rangle$ is in a factorized state of the form (3.21), one sums over l to get

$$\frac{\Delta M_{\Omega_H^{\{\kappa_l\}}}}{T_H} - \sum_l \frac{\tilde{\mu}_l}{T_H} \Delta E_{\Omega_H^{U^l}}^l = \sum_l \Delta S_{\Omega_H^{U^l}}^l = \Delta S_{\Omega_H^{\{\kappa_l\}}}, \quad (4.69)$$

so (4.62) becomes

$$\Delta \langle K_{\Omega_H^{\{\kappa_l\}}}^{A_i^T} \rangle_{\bar{\Psi}} = \frac{\Delta M}{T_H} - \sum_{l=0}^{+\infty} \frac{\tilde{\mu}_l}{T_H} \langle \Delta E^l \rangle_{\bar{\Psi}} - \Delta S_{\Omega_H^{\{\kappa_l\}}}. \quad (4.70)$$

Equation (4.61) can therefore be written as

$$\frac{\Delta M}{T_H} - \sum_{l=0}^{+\infty} \frac{\tilde{\mu}_l}{T_H} \langle \Delta E^l \rangle_{\bar{\Psi}} - \Delta S_{\bar{\Psi}} \leq 0 \quad (4.71)$$

with

$$\Delta S_{\bar{\Psi}} = \Delta S_{\bar{\Psi}|\bar{\Omega}_H^{\{\kappa_l\}}}^{\text{v.N.}\dots A_i^B} + \Delta S_{\Omega_H^{\{\kappa_l\}}} \quad (4.72)$$

being the total entropy radiated away. Consequently, within the soft regularization, the free energy is no longer the adequate thermodynamic potential. Instead, there emerges the grand potential $\mathcal{G}_{\Omega_H^{\{\kappa_l\}}}$

$$\Delta \mathcal{G}_{\Omega_H^{\{\kappa_l\}}} := \Delta M - \sum_{l=0}^{+\infty} \tilde{\mu}_l \langle \Delta E^l \rangle_{\bar{\Psi}} - T_H \Delta S_{\bar{\Psi}} \leq 0, \quad (4.73)$$

which gives us information on the spontaneous evolution of our physical system. The change of thermodynamic potential, with respect to the hard regularization, is tied to the fact that the vacuum state $|\Omega_H^{\{\kappa_l\}}\rangle$ is not thermal anymore with respect to the usual Hamiltonian. Indeed, we have now an infinite number of reservoirs at different temperatures. Having a bunch of reservoirs at hand allows in principle for work extraction from the quantum fields in the vacuum state $|\Omega_H^{\{\kappa_l\}}\rangle$. These considerations are left for the second part of this work [91].

3. Enlightening rewriting

Before coming to the last part of the proof in which the black hole area enters the game, we want to draw the reader's attention to the fact that we can rewrite

$$E_l - \langle E_l \rangle_{\Omega_H^{\{\kappa_l\}}} = \sum_{m=-l}^{m=+l} \sum_{\omega>0} \omega (N_{\omega lm} - \langle N_{\omega lm} \rangle_{\Omega_H^{\{\kappa_l\}}}) \quad (4.74)$$

since it corresponds to the contribution of the l -modes to the one-sided boost energy on top the vacuum distribution. Therefore, since any vector $|\tilde{\Psi}\rangle \in \mathcal{H}_{\Omega_H^{\{\kappa_l\}}}$ can be obtained through a finite excitation on top of the κ_l -vacuum, the operator in (4.74) is well defined on $\mathcal{H}_{\Omega_H^{\{\kappa_l\}}}$ (although unbounded). Alternatively, we can write (4.74) as

$$E^l - \langle E^l \rangle_{\Omega_H^{\{\kappa_l\}}} = \sum_{m=-l}^{m=+l} \int_0^{+\infty} \omega \bar{n}_{\omega lm} d\omega, \quad (4.75)$$

$$\bar{n}_{\omega lm} = \lim_{\Delta\omega \rightarrow 0} \sum_{\omega, \omega+\Delta\omega} \frac{N_{\omega lm} - \langle N_{\omega lm} \rangle_{\Omega_H^{\{\kappa_l\}}}}{\Delta\omega},$$

where $\bar{n}_{\omega lm}$ is the excess density of modes compared to the vacuum $|\Omega_H^{\{\kappa_l\}}\rangle$. Of course, we should not forget that ω is here the frequency conjugated to the time (2.37) associated to the region $u \geq u_0$. Likewise, we can decompose Eq. (4.63) so that

$$\Delta E_{\Omega_H^{\{\kappa_l\}}}^l = -(u_2 - u_1)(2l+1) \int_0^{+\infty} \frac{1}{2\pi} \frac{\omega}{e^{\frac{2\pi\omega}{\kappa_l}} - 1} d\omega$$

$$= -(u_2 - u_1) \sum_{m=-l}^{m=+l} \int_0^{+\infty} \omega \langle N_{\omega lm} \rangle_{\Omega_H^{\{\kappa_l\}}} d\omega, \quad (4.76)$$

where of course $\langle N_{\omega lm} \rangle_{\Omega_H^{\{\kappa_l\}}}$ is the density of particles in the state $|\Omega_H^{\{\kappa_l\}}\rangle$ on $\mathcal{I}_R^+$, given by a Bose-Einstein distribution at the temperature $T_l = \frac{\kappa_l}{2\pi}$. Now, we can set

$$\mu_{\omega l} := \omega \tilde{\mu}_l = \omega \left(1 - \frac{\kappa}{\kappa_l} \right) \quad (4.77)$$

so that (4.71) can now be written

$$\frac{\Delta M}{T_H} - \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} \int_0^{+\infty} \frac{\mu_{\omega l}}{T_H} \langle \Delta n_{\omega lm} \rangle_{\tilde{\Psi}} d\omega - \Delta S_{\tilde{\Psi}} \leq 0, \quad (4.78)$$

where

$$\Delta n_{\omega lm} := \Delta \bar{n}_{\omega lm} - (u_2 - u_1) \langle N_{\omega lm} \rangle_{\Omega_H^{\{\kappa_l\}}} \quad (4.79)$$

is the spectral density operator. By analogy to ordinary laws of thermodynamics, the quantity $\mu_{\omega l}$ is indeed a *chemical potential* and was introduced first in Ref. [65]. In addition, we recover exactly the same relation as in Ref. [65] between the effective temperature and the chemical potential, namely, Eq. (4.77).

F. Complete proof

Consider now that the two spacelike hypersurfaces Σ_1 and Σ_2 start at different cuts S_1 and S_2 on the horizon $\mathcal{H}_L^+$, say, at $\tilde{V} = \tilde{V}_1$ and $\tilde{V} = \tilde{V}_2$ with $\tilde{V}_2 > \tilde{V}_1 > 0$, and end at two different cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ at $\mathcal{I}_R^+$, say, at $U = U_1$ and $U = U_2$ with $U_2 > U_1 > 0$ (see Fig. 5).⁴² Pushing the Σ_i to infinity, we observe that they correspond to the domains

$$\mathcal{D}_i^{\text{tot}} = \underbrace{((\tilde{V}_i, +\infty) \times S_{\mathcal{H}}^2)}_{:=\mathcal{D}_i^{\mathcal{H}}} \cup \underbrace{((U_i, +\infty) \times S_{\mathcal{I}}^2)}_{:=\mathcal{D}_i^{\mathcal{I}}}, \quad (4.80)$$

to which we attach the algebra of observables $\mathcal{A}_i^{\text{tot}} = \mathcal{A}_i^{\mathcal{H}} \otimes \mathcal{A}_i^{\mathcal{I}}$. We give explicit details on how to complete our proof in this more general setup only in the case of the L -vacuum. We consider as a reference state the tensor product state between the Hartle-Hawking state on the horizon⁴³ $|\Omega_H^{\mathcal{H}_L^+}\rangle$ and the L -vacuum at future null infinity

$$|\Omega_L^{\text{tot}}\rangle = |\Omega_H^{\mathcal{H}_L^+}\rangle \otimes |\Omega_H^L\rangle. \quad (4.81)$$

Consequently, the total one-sided modular Hamiltonian is the sum of contributions from the horizon and null infinity

$$K_{\Omega_L^{\text{tot}}}^{\text{tot}, \mathcal{A}_i^{\text{tot}}} := K_{\Omega_H^{\mathcal{H}_L^+}}^{\mathcal{A}_i^{\mathcal{H}}} + K_{\Omega_H^L}^{\mathcal{A}_i^{\mathcal{I}}} \quad (4.82)$$

with

$$K_{\Omega_H^{\mathcal{H}_L^+}}^{\mathcal{A}_i^{\mathcal{H}}} = 2\pi \int_{\mathcal{D}_i^{\mathcal{H}}} (\tilde{V} - \tilde{V}_i) : T_{\tilde{V}\tilde{V}} :_{\Omega_H^{\mathcal{H}_L^+}} d\tilde{V} \wedge \epsilon_S$$

$$= 2\pi \int_{\mathcal{D}_i^{\mathcal{H}}} (\tilde{V} - \tilde{V}_i) T_{\tilde{V}\tilde{V}} d\tilde{V} \wedge \epsilon_S. \quad (4.83)$$

From Eq. (2.18), i.e., from the dynamics of the horizon, we relate the variation of area between $\tilde{V}_1$ and $\tilde{V}_2$ to the change of VEV of modular Hamiltonian via

$$\Delta \left(\frac{A}{4G} \right) = \Delta \langle K_{\Omega_H^{\mathcal{H}_L^+}}^{\mathcal{A}_i^{\mathcal{H}}} \rangle_{\tilde{\Psi}}. \quad (4.84)$$

The arbitrary state $|\tilde{\Psi}\rangle \in \mathcal{H}_{\Omega_L^{\text{tot}}}$ is taken to be such that hypotheses (2.24) and (4.39) are both satisfied, which requires to take L to be large enough, but finite. Finally, the monotonicity of the relative entropy between such a state

⁴²Strictly speaking, in order to be spacelike, we should add to these regions the part between the bifurcation surface $\mathcal{B}$ and the cut S_i together with the part between ι^0 and the cut $\mathcal{C}_i$; see footnote 27.

⁴³Recall that the radiation emitted by a black hole is *exactly* thermal on the horizon.

$|\bar{\Psi}\rangle$ and our reference state (4.81) straightforwardly leads to the decreasing of the *generalized free energy*

$$\Delta \mathcal{F}_{\text{gen}} \leq 0 \quad (4.85)$$

with

$$\mathcal{F}_{\text{gen}} = M - T_H \left(S_i + \frac{A}{4G} \right), \quad (4.86)$$

where

$$S_i = S_{\bar{\Psi}|\Omega_L^{\text{tot}}}^{\text{v.N.}\dots A_i^{\text{tot}}} + S_{\Omega_H^L} \quad (4.87)$$

is the total entropy variation between the two open slices Σ_i .⁴⁴ We thus recover the generalized entropy

$$S_{i,\text{gen}} = S_i + \frac{A}{4G}. \quad (4.88)$$

First established between the two regions $\mathcal{D}_i^{\text{tot}}$, we propagate the result (4.85) to the arbitrary hypersurfaces Σ_i by unitarity as usual. We have therefore shown that in an eternal Schwarzschild black hole background the quantity which decreases between two arbitrary hypersurfaces during any physical process is, in the hard regularized Hartle-Hawking state, the generalized free energy (4.86). Within the soft regularization scheme, we would have gotten the generalized version of the grand potential (4.73) with the chemical potentials $\mu_{\omega l}$.

V. CONCLUSIONS

We showed that from the point of view of asymptotic observers the spontaneous evolution of quantum fields in a perturbed Schwarzschild black hole background was equivalent to the monotonicity of a thermodynamic potential built upon the relevant geometric quantity at future null infinity, namely, the Bondi mass, and eventually additional non geometric terms to which we provided a thermodynamic interpretation. Depending on the manner the Hartle-Hawking state, unsuitable for such a study due to its divergent net energy flux at $\mathcal{I}_R^+$, is regularized, the dual GSL states the decrease of either the free energy or the grand potential. Nongeometric corrections are encoded in a set of chemical potentials modeling the backreaction process due to the potential barrier. When the hypersurfaces are opened both at infinity and on the horizon, geometric quantities associated to both semiclassical dynamics (area

and mass) enters the, now generalized, thermodynamic potential. The monotonicity of these thermodynamic potentials has been proven in this setting within the semiclassical regimes for a large class of states representing local excitations of the field on top of the Hartle-Hawking vacuum state and/or its regularizations.

The first natural extension of our work is to consider the Unruh state, which is the adapted state to consider in order to model accurately the state of a black hole after a collapse. While the different classes of vacuum states defined on the algebra of observables at null infinity were modeling some features of the Unruh vacuum, such as the fact that modes with different angular momentum are backscattered differently by the black hole background, none of them describes accurately the late-time Hawking radiation. Indeed, the dependence in the frequency of the transmission coefficients prevents us from describing the Unruh vacuum as the natural vacuum state associated to a positive frequency decomposition attached with a time parameter at maximally extended null infinity. Instead, as proposed in Ref. [65], one should study a transmission process between the white hole horizon where the radiation is exactly thermal at Hawking temperature and future null infinity. Because of the potential barrier, not all the angular modes will be observable by an asymptotic observer, and hence chemical potentials are expected to appear in the relevant thermodynamic potential, exactly as what happened for late-time boundary conditions compatible with the κ_l -vacua [see (4.78)]. In Ref. [65], preliminary computations showed that the latter depend precisely on the transmission coefficients. The more algebraic setup presented in this work can be used to put this proof on more solid grounds. Additionally, treating the Unruh state allows to deal with rotating black holes and therefore to study the dual GSL also in that context.

A second extension is based on several observations we have made along this work. On the one hand, as stated in the Introduction, finding the free energy or the grand potential in the dual GSL shows that, from the point of view of usual thermodynamics, there exists a deeper connection between quantization of a field on a nonexpanding null hypersurface and the theory of open quantum systems. One further hint toward that direction was the assumption (2.24), comparing a dynamical time $\tau_D = \frac{\langle T_{uu} \rangle_\Psi}{\partial_u \langle T_{uu} \rangle_\Psi}$ related to the characteristic variation of the stress-energy tensor to what can be interpreted as a correlation time $\tau_B = \kappa^{-1} \sim \frac{\hbar}{T_H}$. In the theory of open quantum systems weakly interacting with a thermal bath, the typical scale at which one can rely on Markovian dynamics is tied to the inverse temperature [48].⁴⁵ This suggests a close interplay

⁴⁴In the Hartle-Hawking vacuum (regularized here to the L -vacuum with large L), there is no entropy flux on the horizon while there is an entropy flux at $\mathcal{I}_R^+$. The discussion is similar to the energy flux: the increase in entropy due to the outgoing radiation is exactly compensated by the entropy of the ingoing radiation falling into the black hole.

⁴⁵To be precise, the Markovian approximation also depends on additional timescales set by the bath spectral density, including possible ultraviolet cutoffs. However, for thermal equilibrium baths, the inverse temperature defines a universal thermal correlation timescale.

between the relaxation of quantum fields toward a vacuum state at null infinity (e.g., the Hartle-Hawking state) and the equilibration of a small system, regarded as a subpart of a larger bath, toward a (generalized) Gibbs state. It would be interesting to unveil this analogy between black hole physics and Markovian systems, explaining in particular why, while being a very strong hypothesis, Markovianity is immediately achieved once we place ourself on a null hypersurface. Lastly, the correlation with thermodynamics of open quantum systems can be pushed further by going back to the expression (4.78). The nongeometric terms depending on the chemical potential are related to work terms. This seems to tell us that, once correctly described by an appropriate quantum state at infinity,⁴⁶ work could in principle be extracted from the nonthermal radiation emitted by a nonrotating black hole.

We leave the treatment of the Unruh state, the connection with the thermodynamics of open systems, and the construction of a machine which can extract work from the radiation emitted by a nonrotating black hole for the companion paper [91].

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

⁴⁶By this, we mean not the Hartle-Hawking state.

APPENDIX A: UNRUH EFFECT ON A NONEXPANDING NULL HYPERSURFACE

In this appendix, we present a comprehensive proof of the Unruh effect from a direct quantization of a free massless field on a nonexpanding null hypersurface, completing and making explicit some details of standard proofs, e.g., Refs. [72,95,98,108–110]. This effect was the key point in Sec. IV to ensure that the vacuum state $|\Omega\rangle$, defined through a positive frequency decomposition induced by a choice of time, was indeed thermal when restricted to the subalgebras of observables we considered.

Let $\mathcal{N}$ be a nonexpanding null hypersurface parametrized by a coordinate $U \in (-\infty, +\infty)$ (e.g., $\mathcal{I}_R$). Its normal vector is therefore ∂_U . On this background, we quantize a massless scalar field.⁴⁷ The name of the game being to restrict the projected bulk vacuum state to a subregion of $\mathcal{N}$, we start by analyzing a restriction to $U > 0$. Our proof is divided into two parts. First, we show that such a restriction generates a Bose-Einstein distribution⁴⁸ of particles for the average value of the number operator. Then, we explicitly show that the projected vacuum state on $\mathcal{N}$ can be written as a thermal entanglement between the eigenmodes of the field in the two regions, above and below the cut $U = 0$. Note that in general, especially when using the Unruh effect in proofs of the GSL the vacuum is invariant (in particular) under translations on $\mathcal{N}$, see Sec. III, making the precise locus of the cut in U irrelevant. Therefore, we take $U = 0$ for simplicity.

We give all the details of the proof in the two-dimensional case while we only display the end result for the four-dimensional extension.

1. Global vacuum state

Consider a Klein-Gordon field $\hat{\phi}(U)$ on $\mathcal{N}$, which we canonically quantize using the (correctly normalized)⁴⁹ modes $\frac{e^{-i\Omega U}}{\sqrt{4\pi\Omega}}$ with $\Omega > 0$. This gives

$$\hat{\phi}(U) = \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} (\hat{a}_\Omega e^{-i\Omega U} + \hat{a}_\Omega^\dagger e^{+i\Omega U}), \quad (\text{A1})$$

which, together with its canonical momenta $\hat{\pi}(U) \equiv \partial_U \hat{\phi}$, satisfies the canonical equal-time commutation relations

$$[\hat{\phi}(U), \hat{\pi}(U')] = \frac{i}{2} \delta(U - U'), \quad (\text{A2})$$

given that

⁴⁷Our proof extends immediately to fermionic fields, by simply changing the commutation relations of the ladder operators into their fermionic counterpart.

⁴⁸Or Fermi-Dirac for fermionic fields.

⁴⁹With respect to the scalar product (3.4).

$$[\hat{a}_\Omega, \hat{a}_{\Omega'}^\dagger] = \delta(\Omega - \Omega'). \quad (\text{A3})$$

From this QFT construction, we get the one-particle Hilbert space associated with the coordinate U whose the vacuum state $|\Omega^U\rangle$ is the state such that for all frequencies $\Omega > 0$ we have $\hat{a}_\Omega|\Omega^U\rangle = 0$.

2. Quantization in the region $U > 0$

We restrict our attention to the upper half of $\mathcal{N}$, namely, the region $U > 0$ that we shall denote region I. There, U is no longer a good coordinate for quantization as it does not range in the whole set of real numbers. We consider instead the usual change of coordinates

$$u = \ln U, \quad (\text{A4})$$

which defines $u \in (-\infty, +\infty)$ as we go along region I. Our objective is now to decompose the restriction of $\hat{\phi}$ in region I, denoted $\hat{\phi}_+$, but into positive-frequency modes for the coordinate u , i.e., in the basis $\frac{e^{-i\omega u}}{\sqrt{4\pi\omega}}$. This leads to

$$\hat{\phi}_+(u) = \int_0^{+\infty} \frac{d\omega}{\sqrt{4\pi\omega}} (\hat{a}_{\omega,I} e^{-i\omega u} + \hat{a}_{\omega,I}^\dagger e^{+i\omega u}). \quad (\text{A5})$$

This field should coincide with the restriction of (A1) to region I; therefore, the positive modes Ω should be decomposed in the basis of modes ω . Hence, as nothing prevents negative ω modes from appearing, we need the following relations to hold when $U > 0$,

$$\hat{A}_I = \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \hat{a}_\Omega e^{-i\Omega U} \equiv \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{4\pi|\omega|}} \tilde{a}_{\omega,I} e^{-i\omega u}, \quad (\text{A6})$$

$$\hat{A}_I^\dagger = \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \hat{a}_\Omega^\dagger e^{i\Omega U} \equiv \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{4\pi|\omega|}} \tilde{a}_{\omega,I}^\dagger e^{i\omega u}, \quad (\text{A7})$$

so that of course $\hat{\phi}_+ = \hat{A} + \hat{A}^\dagger$ and the $\tilde{a}_\omega$ are implicitly defined by

$$\tilde{a}_{\omega,I} = \left(\hat{A}, \frac{e^{-i\omega u}}{\sqrt{4\pi|\omega|}} \right)_{\text{KG}} \quad \text{and} \quad \tilde{a}_{\omega,I}^\dagger = - \left(\hat{A}^\dagger, \frac{e^{i\omega u}}{\sqrt{4\pi|\omega|}} \right)_{\text{KG}}, \quad (\text{A8})$$

where we recall for completeness the Klein-Gordon product in region I,

$$(f, g)_{\text{KG}} = i \int_{-\infty}^{+\infty} (\tilde{f}(u) \partial_u g(u) - g(u) \partial_u \tilde{f}(u)) du, \quad (\text{A9})$$

where the bar denotes complex conjugation. In (A6) and (A7), due to the integral over all real frequencies, the operators $\tilde{a}_{\omega,I}$ are not the ladder operators of the field in coordinate u . They are just convenient variables for our purposes, and the actual ladder operators will turn out to be linear combinations of the $\hat{a}_\Omega$ and $\hat{a}_\Omega^\dagger$. We start by considering $\hat{A}_I$; from (A6) and (A8), we get

$$\hat{A}_I(u) = \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{d\omega}{\sqrt{4\pi|\omega|}} \frac{d\Omega}{2\pi} \hat{a}_\Omega \sqrt{\frac{\omega}{\Omega}} F(\Omega, \omega) e^{-i\omega u}, \quad (\text{A10})$$

where the function $F(\Omega, \omega)$ reads for positive Ω

$$F(\Omega, \omega) = \int_{-\infty}^{+\infty} e^{i\Omega U} e^{-i\omega u} du = \int_0^{+\infty} e^{i\Omega U} U^{-i\omega-1} dU. \quad (\text{A11})$$

Similarly, we get for the Hermitian conjugate $\hat{A}^\dagger$

$$\hat{A}_I^\dagger(u) = \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{d\omega}{\sqrt{4\pi|\omega|}} \frac{d\Omega}{2\pi} \hat{a}_\Omega^\dagger \sqrt{\frac{\omega}{\Omega}} \bar{F}(\Omega, \omega) e^{i\omega u}, \quad (\text{A12})$$

where $\bar{F}$, the complex conjugate of F , appears. It turns out that the following relation holds:

$$F(\Omega, \omega) = F(-\Omega, \omega) e^{\pi\omega}. \quad (\text{A13})$$

To show (A13), we can complexify the coordinate U and integrate on the upper-half complex plane where $\Im \mathbf{m}(U) > 0$ between the half-circles of radius $R \gg 1$ and $r \ll 1$. In particular, if we set $U' = -U$, we have

$$\begin{aligned} F(\Omega, \omega) &= \lim_{r \rightarrow 0; R \rightarrow +\infty} \int_r^R dU e^{i\Omega U} U^{-i\omega-1} \\ &= e^{\pi\omega} \lim_{r \rightarrow 0; R \rightarrow +\infty} \int_{-r}^{-R} dU e^{-i\Omega U} U^{-i\omega-1}, \end{aligned} \quad (\text{A14})$$

where we have used $-1 = e^{+i\pi}$.⁵⁰ The integral on the circle of radius $R \rightarrow +\infty$ vanishes because of the real exponential and the condition $\Omega > 0$, while the integral on the circle $r \rightarrow 0$ vanishes since the contribution in the integrand is proportional to $r^{i\omega}$ that vanishes in that limit.⁵¹ Therefore, we deduce (A13) from (A14). Similarly, for the complex conjugate, we get

⁵⁰We have to write $-1 = e^{+i\pi}$ because we have to take the upper-half circle of radius $r \ll 1$ in order to have no residue. Therefore, the cut of the logarithm must be chosen in the upper half-plane, and so we have $-1 = e^{+i\pi}$ and not $-1 = e^{-i\pi}$.

⁵¹When $r \rightarrow 0$, we can take $\omega' = \omega' - i\epsilon$, $\epsilon > 0$, and then taking the limit $\epsilon \rightarrow 0$, we get the advertised result.

$$\bar{F}(\Omega, \omega) := F(-\Omega, -\omega) = F(\Omega, -\omega)e^{\pi\omega}. \quad (\text{A15})$$

Now, starting from the decomposition $\hat{\phi}_+ = \hat{A} + \hat{A}^\dagger$, using the expressions (A10) and (A12) and the relations (A13) and (A15), we can decompose the field only in terms of positive frequencies in ω ,

$$\hat{\phi}_+(u) = \int_{\omega=0}^{+\infty} \frac{d\omega}{\sqrt{4\pi\omega}} (\hat{a}_{\omega,I} e^{-i\omega u} + \hat{a}_{\omega,I}^\dagger e^{i\omega u}), \quad (\text{A16})$$

where the “true” ladder operators for $\omega > 0$ read

$$\hat{a}_{\omega,I} \equiv \frac{1}{2\pi} \int_{\Omega=0}^{+\infty} \sqrt{\frac{\omega}{\Omega}} d\Omega F(\Omega, \omega) (\hat{a}_\Omega + e^{-\pi\omega} \hat{a}_\Omega^\dagger) \quad (\text{A17})$$

$$\hat{a}_{\omega,I}^\dagger \equiv \frac{1}{2\pi} \int_{\Omega=0}^{+\infty} \sqrt{\frac{\omega}{\Omega}} d\Omega \bar{F}(\Omega, \omega) (\hat{a}_\Omega^\dagger + e^{-\pi\omega} \hat{a}_\Omega). \quad (\text{A18})$$

This concludes our objective of expressing our field in positive frequencies of the accelerated observer at coordinate u .⁵²

3. Commutation relations and number operator

Being genuine ladder operators, the commutators between (A17) and (A18) match the canonical ones. We indeed find

$$[\hat{a}_{\omega,I}, \hat{a}_{\omega',I}] = [\hat{a}_{\omega,I}^\dagger, \hat{a}_{\omega',I}^\dagger] = 0, \quad (\text{A19})$$

together with

$$[\hat{a}_{\omega,I}, \hat{a}_{\omega',I}^\dagger] = \delta(\omega - \omega'). \quad (\text{A20})$$

Using the definitions (A17) and (A18) together with the commutators (A3), we get

$$\begin{aligned} [\hat{a}_{\omega,I}, \hat{a}_{\omega',I}^\dagger] &= \frac{1}{4\pi^2} \int_0^{+\infty} d\Omega \int_0^{+\infty} d\Omega' \sqrt{\frac{\omega\omega'}{\Omega\Omega'}} F(\Omega, \omega) \bar{F}(\Omega, \omega) \\ &\quad \times \left([\hat{a}_\Omega, \hat{a}_{\Omega'}^\dagger] - e^{-\pi(\omega-\omega')} [\hat{a}_{\Omega'}, \hat{a}_\Omega^\dagger] \right) \\ &= \frac{\sqrt{\omega\omega'}}{4\pi^2} \int_0^{+\infty} \frac{d\Omega}{\Omega} F(\Omega, \omega) \bar{F}(\Omega, \omega') (1 - e^{-\pi(\omega+\omega')}) \\ &= \frac{\sqrt{\omega\omega'}}{4\pi^2} (1 - e^{-\pi(\omega+\omega')}) \int_0^{+\infty} \frac{d\Omega}{\Omega} \\ &\quad \times \left[\int_0^{+\infty} dU e^{i\Omega U} U^{-i\omega-1} \right] \\ &\quad \times \left[\int_0^{+\infty} dV e^{-i\Omega V} V^{i\omega-1} \right], \end{aligned} \quad (\text{A21})$$

⁵²Note *en passant* that the vacuum state in region I is the state $|\Omega^u\rangle$ annihilated by all positive modes $\hat{a}_\omega$ for $\omega > 0$.

where to get the second line we used the definition of F in (A11) and of its complex conjugate. Applying successively the changes of variables $(U, V) = (e^u, e^v)$ and $(u, v) = (x + y, x - y)$, we find

$$\begin{aligned} [\hat{a}_{\omega,I}, \hat{a}_{\omega',I}^\dagger] &= \frac{\sqrt{\omega\omega'}}{4\pi^2} (1 - e^{-\pi(\omega+\omega')}) \int_0^{+\infty} \frac{d\Omega}{\Omega} \int_{-\infty}^{+\infty} dx \\ &\quad \times \int_{-\infty}^{+\infty} dy e^{-i\Omega e^x (e^y - e^{-y})} e^{i(\omega-\omega')x} e^{i(\omega+\omega')y}, \end{aligned} \quad (\text{A22})$$

and we observe that the integral

$$I(x, \omega, \omega') := \int_0^{+\infty} \frac{d\Omega}{\Omega} \int_{-\infty}^{+\infty} dy e^{-i\Omega e^x (e^y - e^{-y})} e^{i(\omega+\omega')y} \quad (\text{A23})$$

is independent of x .⁵³ Therefore, it follows that

$$\begin{aligned} [\hat{a}_{\omega,I}, \hat{a}_{\omega',I}^\dagger] &= \frac{\sqrt{\omega\omega'}}{4\pi^2} (1 - e^{-\pi(\omega+\omega')}) I(\omega, \omega') \\ &\quad \times \int_{-\infty}^{+\infty} dx e^{i(\omega-\omega')x} \\ &= \frac{\omega}{2\pi} (1 - e^{-\pi(\omega+\omega')}) I(\omega, \omega) \delta(\omega - \omega'). \end{aligned} \quad (\text{A24})$$

Ultralocality in ω is therefore expected with the appearance of the delta function. Using (A24) in a previous step of the computation, namely, the first line of (A21), gives

$$\begin{aligned} [\hat{a}_{\omega,I}, \hat{a}_{\omega',I}^\dagger] &= \frac{\omega}{4\pi^2} (1 - e^{-2\pi\omega}) \int_0^{+\infty} \frac{d\Omega}{\Omega} |F(\Omega, \omega)|^2 \delta(\omega - \omega') \\ &= \delta(\omega - \omega'), \end{aligned} \quad (\text{A25})$$

in virtue of (A20). Therefore, we find the relation

$$\frac{1}{4\pi^2} \int_0^{+\infty} \frac{d\Omega}{\Omega} \omega |F(\Omega, \omega)|^2 = \frac{1}{1 - e^{-2\pi\omega}}. \quad (\text{A26})$$

We emphasize that this result crucially depends on the $\frac{1}{2}$ factor appearing in (A2), therefore on the fact that we are dealing with a null hypersurface. We are now equipped to compute the number of particles detected by an accelerated observer (with coordinate u) at frequency ω , while the vacuum state was described by the coordinate U and modes Ω . The VEV of the number operator $\hat{N}_{\omega,I}$ in the vacuum $|\Omega^U\rangle$ is

⁵³One has to set $\tilde{\Omega} = \Omega e^x$.

$$\begin{aligned}
N_{\omega,I} &= \langle \Omega^U | \hat{N}_{\omega,I} | \Omega^U \rangle = \langle \Omega^U | \hat{a}_{\omega,I}^\dagger \hat{a}_{\omega,I} | \Omega^U \rangle \\
&= \frac{1}{4\pi^2} \int_0^{+\infty} \frac{d\Omega}{\Omega} \omega |F(\Omega, \omega)|^2 e^{-2\pi\omega} \\
&= \frac{1}{e^{2\pi\omega} - 1},
\end{aligned} \tag{A27}$$

which gives, as expected, a Bose-Einstein distribution. We recovered the first step toward the Unruh effect but in the more general setup of an arbitrary null hypersurface. Note that the assumption of nonexpansion is useless here, as a nonvanishing expansion only changes the volume form of the space of generators, which does not exist when the bulk is two dimensional.

However, it is not enough to compute the average number of particles at a given frequency to find the exact expression of the vacuum. The latter will be thermal if one can show a thermal entanglement between the region $U > 0$ and the region $U < 0$. We therefore turn our attention to this last point.

4. Thermal entanglement of the vacuum state

The quintessence of the Unruh effect comes from the fact that we cut the spacetime in two regions and observe the vacuum state from only one of the two, by tracing out the degrees of freedom of the other region. To get a thermal state upon tracing, the density matrix of the vacuum state $|\Omega^U\rangle$ should be written as a sum of entangled states between the two regions, with exponential coefficients.⁵⁴ This is what we shall show in this section.

The region $U > 0$ has been taken into account in the last section, so we should now focus on the complementary region $U < 0$. Quantities related to the region $U < 0$ will be denoted with a subscript II. Again, to get a good coordinate for quantization, we can choose

$$u_- = \ln(-U). \tag{A28}$$

Even though it ranges in $(-\infty, +\infty)$ as it should, one should remark that the cut $U = 0$ is located at $u_- = -\infty$, and therefore u_- is past directed, and so the positive frequency modes for u_- (of frequency $\tilde{\omega} > 0$) are written as $\frac{e^{i\tilde{\omega}u_-}}{\sqrt{4\pi\tilde{\omega}}}$ with a plus sign. The counterpart of the relations (A6) and (A7) are

$$\hat{A}_{II} = \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \hat{a}_\Omega e^{-i\Omega U} \equiv \int_{-\infty}^{+\infty} \frac{d\tilde{\omega}}{\sqrt{4\pi|\tilde{\omega}|}} \tilde{a}_{\tilde{\omega},II} e^{i\tilde{\omega}u_-}, \tag{A29}$$

$$\hat{A}_{II}^\dagger = \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \hat{a}_\Omega^\dagger e^{i\Omega U} \equiv \int_{-\infty}^{+\infty} \frac{d\tilde{\omega}}{\sqrt{4\pi|\tilde{\omega}|}} \tilde{a}_{\tilde{\omega},II}^\dagger e^{-i\tilde{\omega}u_-}, \tag{A30}$$

and this defines the “convenient” operators we use in region II,

$$\begin{aligned}
\tilde{a}_{\tilde{\omega},II} &= - \left(\hat{A}_{II}, \frac{e^{i\tilde{\omega}u_-}}{\sqrt{4\pi|\tilde{\omega}|}} \right)_{\text{KG}} \\
\text{and } \tilde{a}_{\tilde{\omega},II}^\dagger &= \left(\hat{A}_{II}^\dagger, \frac{e^{-i\tilde{\omega}u_-}}{\sqrt{4\pi|\tilde{\omega}|}} \right)_{\text{KG}}.
\end{aligned} \tag{A31}$$

The equivalent of F in (A10) is given here by G such that

$$\begin{aligned}
G(\Omega, \tilde{\omega}) &= \int_{-\infty}^{+\infty} e^{i\Omega U} e^{i\tilde{\omega}u_-} du_- \\
&= \int_0^{+\infty} e^{-i\Omega U} U^{i\tilde{\omega}-1} dU = F(-\Omega, -\tilde{\omega}) \\
&= \bar{F}(\Omega, \tilde{\omega}),
\end{aligned} \tag{A32}$$

where we have used (A11) and (A15). We also have the following relation on the complex conjugate of G ,

$$\bar{G}(\Omega, \omega) = G(-\Omega, -\omega) = F(\Omega, \omega) \tag{A33}$$

using (A32). From (A32) and (A33), we can deduce the expression of the genuine ladder operators for the coordinate u_- . Calling them $\hat{a}_{\omega,II}$ and $\hat{a}_{\omega,II}^\dagger$, we find

$$\hat{a}_{\omega,II} = \frac{1}{2\pi} \int_0^{+\infty} \sqrt{\frac{\omega}{\Omega}} d\Omega \bar{F}(\Omega, \omega) (\hat{a}_\Omega + e^{-\pi\omega} \hat{a}_\Omega^\dagger) \tag{A34}$$

$$\hat{a}_{\omega,II}^\dagger = \frac{1}{2\pi} \int_0^{+\infty} \sqrt{\frac{\omega}{\Omega}} d\Omega F(\Omega, \omega) (\hat{a}_\Omega^\dagger + e^{-\pi\omega} \hat{a}_\Omega). \tag{A35}$$

These operators also satisfy the canonical commutation relations (A19) and (A20), but we should emphasize the fact that the couple $(\hat{a}_{\omega,II}, \hat{a}_{\omega,II}^\dagger)$ acts on the region $U < 0$ while $(\hat{a}_{\omega,I}, \hat{a}_{\omega,I}^\dagger)$ was acting on the region $U > 0$. The two couples of operators do not act on the same Hilbert space, and therefore they necessarily commute with each other.

Recalling that the global vacuum state on $\mathcal{N}$ is by definition annihilated by all $\hat{a}_\Omega$ for positive Ω , we deduce from (A17), (A18), (A34), and (A35) that the following combination of the restricted operators also annihilate $|\Omega^U\rangle$:

$$(\hat{a}_{\omega,II} - e^{-\pi\omega} \hat{a}_{\omega,I}^\dagger) |\Omega^U\rangle = 0 \tag{A36}$$

⁵⁴This is equivalent in the well-known Rindler framework to a decomposition of the vacuum density matrix in the Rindler basis.

$$(\hat{a}_{\omega,I} - e^{-\pi\omega}\hat{a}_{\omega,II}^\dagger)|\Omega^U\rangle = 0. \quad (\text{A37})$$

These relations imply that the number operators $\hat{N}_{\omega,I}$ and $\hat{N}_{\omega,II}$ in each of the two regions are necessarily identical since one can deduce from (A36) and (A37)

$$\hat{a}_{\omega,II}^\dagger \hat{a}_{\omega,II} |\Omega^U\rangle = \hat{a}_{\omega,I}^\dagger \hat{a}_{\omega,I} |\Omega^U\rangle, \quad (\text{A38})$$

meaning that one can choose a basis of the total Hilbert space $\mathcal{H}_\Omega$ made of vectors representing N -particle states in the two regions, i.e., states of the form $|N_{\omega,I}, N_{\omega,II}\rangle = |N_{\omega,I}\rangle \otimes |N_{\omega,II}\rangle$, where as we said $N_{\omega,I} = N_{\omega,II}$. In this basis, the vacuum state can be written as

$$|\Omega^U\rangle = \prod_{\omega=0}^{+\infty} Z_\omega \sum_{N_{\omega,I}=N_{\omega,II}} A_{N_\omega}(\omega) |N_{\omega,I}\rangle \otimes |N_{\omega,II}\rangle \quad (\text{A39})$$

with $A_{N_\omega}(\omega)$ coefficients yet to be determined and Z_ω a (ω -dependent) normalization factor. The thermal character (or not) of the restricted vacuum state is all contained in these coefficients. Acting on (A39) with a particular frequency ω , we get a recursion relation for the A_{N_ω} ,

$$A_{N_\omega} = e^{-\pi\omega} A_{N_\omega-1}, \quad (\text{A40})$$

from which we deduce $A_{N_\omega} = e^{-\pi N_\omega \omega} A_0$ by iteration. Knowing A_{N_ω} , we get from the requirement that $\langle 0|0\rangle_U = 1$ the expression of the normalization factor ($A_0 = 1$ necessarily)

$$Z_\omega = \sqrt{1 - e^{-2\pi\omega}}. \quad (\text{A41})$$

Hence, the density matrix of the $\mathcal{N}$ vacuum reads

$$|\Omega^U\rangle\langle\Omega^U| = \prod_{\omega=0}^{+\infty} (1 - e^{-2\pi\omega}) \sum_{N_{\omega,I}=N_{\omega,II}} e^{-\pi\omega N_\omega} |N_{\omega,I}\rangle \otimes |N_{\omega,II}\rangle \langle N_{\omega,I}| \otimes \langle N_{\omega,II}|, \quad (\text{A42})$$

displaying a thermal entanglement between the two subregions. Any partial tracing over one of the two Hilbert spaces, i.e., any restriction of the vacuum state to one of the two subregions, will lead to a thermal state. This is the Unruh effect in two dimensions.

5. Four-dimensional proof

We complement the coordinate U by two angular coordinates collectively denoted by x^A with $A = 1, 2$ and expand the field as

$$\begin{aligned} \hat{\phi}(U, x^A) = & \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \int_0^{+\infty} \frac{d\Omega}{\sqrt{4\pi\Omega}} \\ & \times \left(\hat{a}_{\Omega lm} Y_m^l(x^A) e^{-i\Omega U} + \hat{a}_{\Omega lm}^\dagger \bar{Y}_m^l(x^A) e^{+i\Omega U} \right), \end{aligned} \quad (\text{A43})$$

where the modes are correctly normalized as indeed

$$\left(Y_m^l(x^A) \frac{e^{-i\Omega U}}{\sqrt{4\pi\Omega}}, Y_{m'}^{l'}(x^A) \frac{e^{-i\Omega' U}}{\sqrt{4\pi\Omega'}} \right)_{\text{KG}} = \delta_{ll'} \delta_{mm'} \delta(\Omega - \Omega'), \quad (\text{A44})$$

with the Klein-Gordon scalar product, which reads now

$$(f(U, x^A), g(U, x^A))_{\text{KG}} = i \int_{-\infty}^{+\infty} \int_{S^2} dU \wedge \epsilon_S (\partial_U \bar{f} g - \partial_U g \bar{f}). \quad (\text{A45})$$

We see that there appears the volume form of the two-dimensional spatial sections of $\mathcal{N}$, ϵ_S , which we take to be metrically a unit 2-sphere S^2 .⁵⁵ The latter is meant to change when the null hypersurface at hand possesses a nonzero expansion. Consequently, nothing guarantees the normalization condition (A44) to be valid everywhere on $\mathcal{N}$. As our proof requires the mode decomposition (A43) to be valid everywhere, we are led to assume that $\mathcal{N}$ is a nonexpanding null hypersurface.

After restricting the field to region I and to region II, we get the combinations of operators of region I and of region II that annihilate the vacuum

$$(\hat{a}_{\omega lm, II} - (-1)^m e^{-\pi\omega} \hat{a}_{\omega l-m, I}^\dagger) |\Omega^U\rangle = 0 \quad (\text{A46})$$

$$(\hat{a}_{\omega lm, I} - (-1)^m e^{-\pi\omega} \hat{a}_{\omega l-m, II}^\dagger) |\Omega^U\rangle = 0. \quad (\text{A47})$$

These two equations lead, using similar techniques as in the two-dimensional case, to an expression of the $\mathcal{N}$ -vacuum density matrix

$$|\Omega^U\rangle\langle\Omega^U| = \prod_{\omega lm} (1 - e^{-2\pi\omega}) \sum_{N_{\omega lm, I}=N_{\omega l-m, II}} e^{-\pi\omega N_\omega} |N_{\omega lm, I}\rangle \otimes |N_{\omega l-m, II}\rangle \langle N_{\omega lm, I}| \otimes \langle N_{\omega l-m, II}|, \quad (\text{A48})$$

⁵⁵This choice is convenient as it allows to decompose the field in the well-known basis of spherical harmonics. Actually, it is only necessary to ask the space of generators to be topologically a 2-sphere. Indeed, if the space of generators is not a metric 2-sphere, one just shifts $Y_m^l \rightarrow \frac{Y_m^l}{\sqrt{q}}$, where q is the conformal factor so that $\epsilon_S = q \epsilon_S^0$ with ϵ_S^0 being the volume form of a unit metric 2-sphere. Now, new modes are normalized, and (A44) is satisfied. This small modification allows one to quantize on a Kerr black hole horizon or at $\mathcal{I}^+$ in a non-Bondi frame using the framework presented here.

betraying the appearance of a thermal state upon partial tracing. All these computations remain valid upon replacing U by $U^{(l,m)}$ defined in (3.44). This concludes our proof of the Unruh effect on any arbitrary nonexpanding null hypersurface. We shed the light once again on the importance of both assumptions, while being on a null hypersurface is used to prove the Bose-Einstein distribution of the number of particles in both subregions, the nonexpanding character of $\mathcal{N}$ is essential in four dimensions to be able to decompose the field in a basis of (everywhere) correctly normalized modes.

APPENDIX B: SPACETIME APPROACH TO QFT IN CURVED SPACETIMES

1. Generalities

In Minkowski spacetime, Klein-Gordon fields form an infinite collection of harmonic oscillators, some with positive frequency and others with negative frequency. The classical field, i.e., the solution of the Klein-Gordon equation, is expanded in a basis of positive frequency modes (e.g., $e^{-i\omega t}$, $\omega > 0$ with t the Poincaré global time), and the Fourier coefficients a_ω, a_ω^* are promoted to quantum operators satisfying the canonical commutation relations. This gives the quantum version of the field from which the notion of vacuum state, one-particle Hilbert space, and Fock space are built as usual. Hence, in a nutshell, quantizing a classical field amounts first to defining a space of positive frequency modes from which the one-particle Hilbert space arises and then to construct the associated Fock space. What renders Minkowski spacetime peculiar with respect to other spacetimes (in particular curved ones) is the existence of a global notion of time t . The field is then basically quantized on a $t = \text{cst}$ hypersurface.

In curved spacetimes, there does not exist generically⁵⁶ a unique notion of time, even locally. One then selects one time, among others, and starts by considering the space of initial data of a configuration variable (the field) and its conjugate momenta (namely the derivative of the field with respect to the selected notion of time). This “initial” surface can be either spacelike or null, and this leads to the quantization of fields on spacelike or null hypersurfaces. It is actually more convenient to step back and look at a more covariant manner to quantize the theory, without specifying any particular initial surface data nor any notion of equal time. The basic idea is to focus on the solutions of the equations of motion themselves. The fundamental commutation relations are then encoded by the symplectic form of the theory, defined on the solution space. The symplectic form plays therefore a fundamental role in the quantization of fields in curved

spacetimes. Once an invertible symplectic form is defined on the phase space,⁵⁷ the latter is decomposed in positive and negative frequency modes with respect to the symplectic inner product, and finally the Fock space of the theory is built upon the one-particle Hilbert space coming from the positive frequency modes.

Given our knowledge on the spacelike case, we present in this appendix the quantization of fields on a nonexpanding null-hypersurface, as a complement to Sec. III. Care has to be taken when translating the spacelike steps to this situation as, e.g., the naive definition of the symplectic form yields a (undesirable) degenerate quantity. After imposing therefore a slight prescription, we obtain the one-particle Hilbert space and the Fock space. As a genuine feature of QFT where infinitely many degrees of freedom are considered, the constructed Hilbert spaces crucially depend on the choice of positive frequency modes.⁵⁸ We can thus construct nonunitarily related Hilbert spaces. This observation is at the heart of the Unruh effect proved in Appendix A. Following Wald [98], we explain how the unavoidable UV and IR divergences of QFT are a guide toward the reduction of the indeterminacy in the choice of Hilbert space.

Hereafter, we use the abstract index notation when necessary. Most of the material presented here is a translation of well-known material to the case of null hypersurfaces. The interested reader can look at reviews like, e.g., Refs. [98,112].

2. Quantization on a nonexpanding null hypersurface

Consider ϕ a massless free scalar field and a nonexpanding null hypersurface. The assumption of nonexpansion is important both for applications of the present formalism at $\mathcal{I}_R$ and for reasons related to the change of spatial volume form; see (B3) below.

The process of quantization requires a nondegenerate symplectic structure. The latter is naturally defined on the theory’s phase space and then extended to the whole set of solutions of the equations of motion. In the spacelike case, the phase space Γ of the Klein-Gordon theory is defined to be the space of initial data (ϕ, π) with π the conjugate momenta of the field ϕ , which are smooth and of compact support on the spacelike Cauchy surface Σ , i.e., $\phi, \pi \in \mathcal{C}_0^\infty(\Sigma)$. While, at least locally, a unique family of spacelike Cauchy surfaces is enough in that case, when dealing with null initial value surface, two families of hypersurfaces are needed. Indeed, the Klein-Gordon equation generically separates between the left- and right-

⁵⁷Which, as we just said, is in one to one correspondence with the space of solutions of the equations of motion.

⁵⁸The general statement, tamed in the Stone-von Neumann theorem [111], is that for a finite number of degrees of freedom there is only one unitary equivalent class of Hilbert-spaces one can construct with the canonical procedure. Therefore, physics is undistinguishable between two constructions.

⁵⁶Especially for nonstationary spacetimes.

moving modes. The spacelike initial value problem becomes a null characteristic initial value problem where initial data live, e.g., at future null infinity and on the future light cone emanating from the origin (see, e.g., the general discussion of Ref. [113] and the explicit case of the scalar field recently developed in Ref. [114]).⁵⁹ In our work, however, we stick onto one of the two null hypersurfaces, namely, either only the horizon or only right maximally extended null infinity $\mathcal{I}_R$; therefore, we describe only half of the modes.

The bulk manifold $\mathcal{M}$ is charted by the coordinates (U, V, x^A) , and $\mathcal{I}_R$ is a nonexpanding null hypersurface located at $V = +\infty$. There lives only the U -modes of the Klein-Gordon equation, namely, fields of the form $\phi = \phi(U, x^A)$ with conjugate momenta $\pi = \pi(U, x^A) := \partial_U \phi$. The description hereafter can be repeated to treat the V -modes on the complementary null hypersurface at $U = +\infty$. The space of solutions of the equations of motion, $\mathcal{S}$, is therefore made of any function independent of the coordinate V , i.e., and can be restricted to the space of Schwartz functions $f(U, x^A) \in \mathcal{T}(\mathbb{R} \times S^2)$. This reinterpretation of the solution space in terms of test functions leads on the one hand for a smooth transition toward the algebraic approach described in Appendix C, while on the other hand, it allows for a rigorous definition of the quantum field operator as an operator-valued distribution. Consider all possible smearing of the scalar field $\phi \in \mathcal{S}$ by elements $f \in \mathcal{T}(\mathbb{R} \times S^2)$,

$$\phi(f) = \int_{\mathcal{I}_R} f \phi dU \wedge \epsilon_S, \quad (\text{B1})$$

where the integral is over the support of f and with ϵ_S the volume form of the spatial cross sections S of $\mathcal{I}_R$. We get from the classical theory the following symplectic product:

$$\forall \phi \in \mathcal{S}, \forall f \in \mathcal{T}(\mathbb{R} \times S^2);$$

$$\Omega(f, \phi) = \int_{\mathcal{I}_R} (f \partial_U \phi - \phi \partial_U f) dU \wedge \epsilon_S. \quad (\text{B2})$$

Contrarily to the spacelike case, the null symplectic product (B2) is generically degenerate due to the zero modes in U of ϕ . Hence, one can quantize the theory by using the operators

$$\Omega(f, \phi) := 2\pi(f) \quad (\text{B3})$$

⁵⁹This observation comes from the fact that a Cauchy surface is either a one-sheeted spacelike hypersurface or a two-sheeted null hypersurface. A relation can be made between the characteristic initial value problem and the equivalent Cauchy problem in a neighborhood of the codimension-2 surface in which the two null hypersurfaces intersect; see, e.g., Ref. [113].

instead of (B1).⁶⁰ From (B3), we deduce the commutation relations

$$\forall (f, g) \in \mathcal{T}(\mathbb{R} \times S^2), [\hat{\pi}(f), \hat{\pi}(g)] = -\frac{i}{4} \Omega(f, g), \quad (\text{B4})$$

which indeed leads to the usual formula

$$[\hat{\phi}(U, x_1^A), \hat{\pi}(U', x_2^A)] = \frac{i}{2} \delta(U - U') \delta^{(2)}(x_1^A - x_2^A). \quad (\text{B5})$$

Note that, as we stated in the main content of the article, that one-half factor in (B5), or equivalently the one-fourth factor in (B4), depicts the fact that our construction discards half of the initial conditions, namely, the ones living on the complementary null hypersurface (here, $\mathcal{I}_R = \mathcal{I}_I^+ \cup \mathcal{I}_I^-$).

To define the space of positive frequency solutions, we consider the complexified solution space $\mathcal{S}_0^{\mathbb{C}}$ of Schwartz functions on $\mathcal{I}_R$ and define on it the complex symplectic form which is called the *Klein-Gordon product*,

$$\forall (f, g) \in (\mathcal{S}_0^{\mathbb{C}})^2,$$

$$\Omega^{\mathbb{C}}(f, g) = i \int_{\mathcal{I}_R} (\bar{f} \partial_U g - g \partial_U \bar{f}) dU \wedge \epsilon_S$$

$$= 2i \int_{\mathcal{I}_R} \bar{f} \partial_U g dU \wedge \epsilon_S. \quad (\text{B6})$$

It is antilinear in the first argument and linear in the second one. Also, it satisfies $\Omega^{\mathbb{C}}(f, g) = \overline{\Omega^{\mathbb{C}}(g, f)}$; therefore, it is almost an inner product. To make it a genuine inner product, we consider a projector K , which splits the solution space into the direct sum of positive and negative frequency modes $\mathcal{S}_0^{\mathbb{C}} = \mathcal{S}_K^{>0} \oplus \mathcal{S}_K^{<0}$, with $\mathcal{S}_K^{<0} = \overline{\mathcal{S}_K^{>0}}$. In that case, we have that $\forall (f, g) \in \mathcal{S}_0^{\mathbb{C}}$,

$$\Omega^{\mathbb{C}}(Kf, Kf) > 0, \quad \Omega^{\mathbb{C}}(\bar{K}f, \bar{K}f) < 0, \quad \Omega^{\mathbb{C}}(Kf, \bar{K}g) = 0, \quad (\text{B7})$$

where $\bar{K}$ the projector onto $\mathcal{S}_K^{<0}$. The one-particle Hilbert space $\mathcal{H}_1^K$ is then the Cauchy completion of $\mathcal{S}_K^{>0}$. The vacuum state $|\Omega^K\rangle$ is the unique state annihilated by all operators

$$a(Kf) = \Omega^{\mathbb{C}}(Kf, \phi) = \langle f, \phi \rangle_K, \quad f \in \mathcal{S}_K^{>0}, \quad (\text{B8})$$

and the (symmetrized) Fock space reads

⁶⁰Note that the restriction to operators (B3) prevents us from representing the soft sector of the theory, namely, the difference between the value of the field at infinite past and late time. As in this work we only focus on local observables, this issue does not matter.

$$\mathcal{F}_\phi^K := \mathcal{F}_s(\mathcal{H}_1^K) = \bigoplus_{n=0}^{\infty} \left(\text{sym} \bigotimes_{k=0}^n \mathcal{H}_K^1 \right), \quad (\text{B9})$$

where⁶¹ “sym” denotes the symmetrization.⁶² Given the Fock space, quantum states and observables of the theory can be defined and computations of outcomes of physical measurements can be performed. One has therefore achieved giving a quantum version of the classical Klein-Gordon theory on a null hypersurface.

A fundamental remark is that many choices of decomposition of $\mathcal{S}_0^{\mathbb{C}}$, i.e., choices of projector K are possible. For instance, on $\mathcal{I}_R^+$, we can use the positive frequencies Ω , i.e., the orthonormal modes $\frac{e^{-i\Omega U}}{\sqrt{4\pi\Omega}}$, or the space spanned by the positive frequencies ω , i.e., the orthonormal modes $\frac{e^{-i\omega u}}{\sqrt{4\pi\omega}}$ using another notion of time $u = u(U) = \ln U$ in the mode decomposition. The two vacua and Fock spaces will generically be different and be related to different boundary conditions at late time. To lift the huge ambiguity, one can use the UV and IR behavior of the field [98]. In the UV, one can impose the *Hadamard condition*,⁶³ which allows one to define a stress-energy tensor and therefore to use the semiclassical Einstein’s equations, as we did in Sec. II. In the IR, one imposes boundary conditions (which on $\mathcal{I}_R$ are late-time conditions) on the field.⁶⁴ This provides a prescription for the final state to which the quantized field will relax. For example, a field can be asked to evolve toward a thermal state which, in a Schwarzschild black hole background, selects the class of Hilbert spaces built from the Hartle-Hawking vacuum state.

⁶¹When $n = 0$ and $k = 0$, the sum reduces to $\mathbb{C}$.

⁶²The symmetrized version of the Fock space has been chosen in agreement with the spin statistic theorem, which would have required to take the antisymmetrized version in the case of the Dirac field.

⁶³This condition imposes the field’s two-point function in the state $|\Psi\rangle$ to be expressed as

$$\langle \phi(x)\phi(x') \rangle_\Psi = \frac{\alpha(x, x')}{\sigma} + \beta(x', x) \ln \sigma + H_\Psi(x, x'), \quad (\text{B10})$$

where σ is the squared geodesic distance between x and x' with $x = (U, V, x^A)$ collectively. In a neighborhood of a nonexpanding null hypersurface, one can always write

$$\frac{1}{\sigma^2} = \frac{1}{(U - U')(V - V') + (x^A - x'^A)^2}, \quad (\text{B11})$$

where (U, V) are null coordinates. The coefficients $\alpha(x', x)$ and $\beta(x', x)$ only depend on the spacetime geometry, while $H_\Psi(x, x')$ is state dependent and can be chosen to vanish in the Minkowski vacuum.

⁶⁴A standard result [98, 115] states that in a closed Universe (namely, a globally hyperbolic spacetime which possesses a compact Cauchy surface) there basically exist only one class of Hilbert spaces which satisfy the Hadamard condition.

APPENDIX C: ALGEBRAIC APPROACH TO QFT IN CURVED SPACETIMES

The spacetime approach to QFT led to nonunitary related notions of Hilbert-spaces, undermining our desire for universality.⁶⁵ Even though the UV and IR behavior of the field can help choosing between them, we are forced to admit that the very notion of a Hilbert-space is not fundamental. The canonical commutation relations of the field with its momentum,⁶⁶ like, e.g., in (B4), are, however, the same whatever the Hilbert-space representative and therefore fundamentally characterize the theory we try to quantize. The Hilbert-space formalism for QFT emphasizes the notion of states, while one could like to focus more on the observables of the theory and the algebraic relations (like the commutation relations) between them. This route is the one followed by algebraic quantum field theory.

To any dynamical classical theory, one can associate a symplectic space, namely, the solution space $\mathcal{S}$, with symplectic structure Ω . When suitably smeared against test functions $f \in \mathcal{S}$, the field π induces,⁶⁷ by exponentiation, a set of bounded operators,

$$\forall f \in \mathcal{S}, \quad W(f) := e^{i\pi(f)}. \quad (\text{C1})$$

As a consequence of the canonical commutation relations (B4) expressed in their algebraic form, the $W(f)$ satisfy the relations

$$W(f)W(g) = e^{\frac{i}{2}\Omega(f, g)}W(f+g) \quad \text{and} \quad W(f)^* = W(-f), \quad (\text{C2})$$

where the first one is proven thanks to the Baker-Campbell-Hausdorff formula, while the second one is immediate.⁶⁸

⁶⁵The appearance of nonunitarily related Hilbert-spaces is not tight to QFT in curved spacetimes; the van Hove model [116] (reviewed in Ref. [117]) is an example in flat space where nonunitarity arises because of a coupling between a scalar field and a distributional external source.

⁶⁶Or equivalently, the canonical commutation relations of the ladder operators in second quantization.

⁶⁷Here, we treat the null case in which first the solution space and the space of test functions coincide and second the use of the momentum was necessary to render the symplectic structure (B2) nondegenerate.

⁶⁸The usual Weyl algebra relations are often written

$$W(f)W(g) = e^{\frac{i}{2}\Omega(f, g)}W(f+g) \quad \text{and} \quad W(f)^* = W(-f) \quad (\text{C3})$$

instead of (C2), but it is because the Weyl unitaries $W(f)$ are usually defined as

$$W(f) := e^{i\Omega(f, \phi)}, \quad \frac{\Omega(f, \phi)}{2} = \pi(f) \quad (\text{C4})$$

instead of (C1). However, the two formulations are exactly equivalent.

The set of $W(f)$ for $f \in \mathcal{S}$ defines the *Weyl algebra*.⁶⁹ The latter is naturally endowed with the structure of a $\mathbb{C}^*$ -algebra, namely, a normed algebra over $\mathbb{C}$, complete for its norm topology and which satisfies the $*$ -property.⁷⁰ These algebras admits Hilbert-space representations via the GNS construction theorem (see below). Therefore, the classical theory naturally induces an algebra from which one can build an Hilbert space.

In the Hilbert-space formulation of quantum physics, pure states are seen as rays of a Hilbert-space, while mixed states are density matrices, i.e., positive and trace-class operators of the Hilbert space normalized to 1. Observables are self-adjoint operators on the Hilbert space, which preexists; there is a hierarchy in which states precede observables. Algebraic Quantum Field Theory, on the other hand, reverses this hierarchy. It starts from observables, which are defined as the self-adjoint elements of a unital $\mathbb{C}^*$ -algebra⁷¹ $\mathcal{A}$ (see the definition below), which characterizes the physical system at hand.⁷² This subset of $\mathcal{A}$ is dubbed the *algebra of observables*. *Algebraic states* are maps $\omega: \mathcal{A} \rightarrow \mathbb{C}$ which are positive,⁷³

$$\forall A \in \mathcal{A}, \quad \omega(A^*A) \geq 0, \quad (\text{C6})$$

and normalized,

$$\omega(1_{\mathcal{A}}) = 1. \quad (\text{C7})$$

A state will be *mixed* if it is the combination of other states, i.e., $\omega = \lambda\omega_1 + \mu\omega_2$ with $\lambda + \mu = 1$; otherwise, the state is *pure*.

The main examples of $\mathbb{C}^*$ -algebras are the algebra of bounded operators of a Hilbert space $\mathcal{B}(\mathcal{H})$ and the Weyl algebra (C2).⁷⁴ The Weyl algebra in particular, as it

⁶⁹The theory of operators and $\mathbb{C}^*$ -algebra proves that to any symplectic vector is associated a unique Weyl algebra, up to a $*$ -isomorphism. Therefore, such algebras really constitute the algebraic counterpart of the physical canonical commutation relations.

⁷⁰Namely,

$$\forall A \in \mathcal{A}, \quad \|A^*A\| = \|A\|\|A^*\| = \|A\|^2. \quad (\text{C5})$$

⁷¹The $\mathbb{C}^*$ structure is not strictly necessary, though; having at hand a unital $*$ -algebra is enough to define the notion of observables.

⁷²Hence, the reason why we are interested in the Weyl algebra.

⁷³The observables form the subspace of $\mathcal{A}$ which are self-adjoint. However, a general element $A \in \mathcal{A}$ has no reason *a priori* to be self-adjoint itself.

⁷⁴In the special case of $\mathcal{B}(\mathcal{H})$, one can explicitly show the relation between a state $|\Psi\rangle \in \mathcal{H}$ and an algebraic state $\omega_{\Psi}: \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ via

$$\forall A \in \mathcal{B}(\mathcal{H}), \quad \omega_{\Psi}(A) = \langle \Psi | A | \Psi \rangle. \quad (\text{C8})$$

The same can be done for mixed states ρ via $\omega_{\rho}(A) = \text{Tr} \rho A$. In (C8), A is directly an operator on a Hilbert space, but for a general $\mathbb{C}^*$ -algebra like the Weyl algebra, there is no notion of Hilbert space *a priori*, hence the need for the subsequent representation theorem.

emanates from the symplectic structure, contains all the information about the theory we want to quantize. We define the process of quantizing a theory as the construction of a Hilbert space in which the quantum states and the observables lie. From the algebraic viewpoint, we seek a representation of the $\mathbb{C}^*$ -algebra $\mathcal{A}$ in terms of a Hilbert space. This can be achieved using the GNS construction [93,94].⁷⁵ From a state ω , we get a triple $(\mathcal{H}, \mathcal{D}, \pi)_{\omega}$ (which depends on ω) and a unit vector $|\Omega_{\omega}\rangle \in \mathcal{D}$ such that $\mathcal{D} = \{\pi(A)|\Omega_{\omega}\rangle | A \in \mathcal{A}\}$ and $\forall A \in \mathcal{A}, \omega(A) = \langle \Omega_{\omega} | A | \Omega_{\omega} \rangle$. The representation $(\mathcal{H}, \mathcal{D}, \pi)_{\omega}$ is unique up to unitary equivalence. The state ω is pure if and only if the representation is irreducible. Via the map π , the abstract algebra $\mathcal{A}$ can now be seen as a subalgebra of the linear operators of $\mathcal{H}_{\omega}$, i.e., $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H}_{\omega})$. However, in the following and through the main body of the paper, the notation for the algebra elements $A \in \mathcal{A}$ is also used for their representatives $\pi(A) \in \mathcal{B}(\mathcal{H}_{\omega})$ in order to lighten the writing. Finally, due to point 1. one says that $|\Omega_{\omega}\rangle$ is *cyclic*, i.e., $\mathcal{H} = \overline{\{\pi(A)|\Omega_{\omega}\rangle, A \in \mathcal{A}\}}$.

The state ω is usually referred as the *vacuum state*. In many practical purposes, and for all the vacuum states considered in the main body of this paper, the algebraic state ω is taken to be a quasifree (i.e., Gaussian) state, defined so that the expectations values of the Weyl unitary elements are given by⁷⁶

$$\omega(e^{i\pi(f)}) = e^{-\frac{1}{8}\langle f, f \rangle_K} \quad (\text{C10})$$

and where a choice of positive definite inner product $\langle \cdot, \cdot \rangle_K$ had to be made. It can be shown that if $\langle \cdot, \cdot \rangle_K$ is a positive definite inner product constructed from the Klein-Gordon inner product (B6) then the positivity condition (C6) is satisfied for any element A of the Weyl algebra. To make the connection between the definition of a quasifree state (C10), given by a choice of inner product, and the definition of the vacuum state $|\Omega^K\rangle = |\Omega_{\omega}\rangle$ via a positive-frequency decomposition—namely, as the state annihilated by the operators $a(Kf)$ —more explicit, it suffices to compute

$$\begin{aligned} W(\pi(f)) &= \langle \Omega_{\omega} | e^{i\pi(f)} | \Omega_{\omega} \rangle \\ &= \langle \Omega_{\omega} | e^{i\frac{a^{\dagger}(Kf)}{2}} e^{i\frac{a(Kf)}{2}} | \Omega_{\omega} \rangle e^{\frac{1}{8}[ia(Kf), ia^{\dagger}(Kf)]} \\ &= e^{-\frac{1}{8}\langle f, f \rangle_K} \end{aligned} \quad (\text{C11})$$

⁷⁵A proof of the GNS construction can be found in Ref. [117].

⁷⁶Usually, the quasifree state is defined via the expectation value

$$\omega(e^{i\Omega(f, \phi)}) = e^{-\frac{1}{2}\langle f, f \rangle_K}, \quad (\text{C9})$$

but here the unitary elements of the Weyl algebra are defined by $W(f) = e^{i\pi(f)} = e^{i\frac{\Omega(f, \phi)}{2}}$, that is equivalent to rescale $f \rightarrow \frac{f}{2}$, hence the factor $\frac{1}{8}$ instead of $\frac{1}{2}$ in (C10); see also footnote 68.

given that

$$[a(Kf), a^\dagger(Kg)] = \Omega^\mathbb{C}(Kf, Kg) := \langle f, g \rangle_K. \quad (\text{C12})$$

We note that for a given vacuum state ω we get a whole class of Hilbert spaces all related by unitary transformations. However, from two different vacuum states ω_1 and ω_2 , we generally obtain nonunitarily related representations. To illustrate this point, we give the following example in Minkowski spacetime. Consider the Hilbert space $\mathcal{H}_M$ built from the Minkowski vacuum ω_M on $\mathcal{I}_R^+$, by using the modes $\frac{e^{-i\omega u_+}}{\sqrt{4\pi\omega}}$, $\omega > 0$, as positive frequency modes. One could have also considered the Hilbert space $\mathcal{H}_R$ coming from the state ω_R above a cut $u_+ = u_i$, which is the vacuum state for the modes $\frac{e^{-i\omega_R u_R}}{\sqrt{4\pi\omega_R}}$ [where $u_R = \kappa \ln(u_+ - u_i)$ and $\omega_R > 0$]. Because of the Unruh effect, $|0\rangle_M = |\Omega^{u_+}\rangle$ corresponds to an infinite number of particles, while $|0\rangle_R = |\Omega^{u_R}\rangle$ is a true vacuum (i.e., a pure state) with no particles. No unitary transformations can create an infinite amount of particles from zero, and therefore the two GNS constructions are nonequivalent.

Note finally that the link between the algebraic and spacetime approaches is made via the support of the test functions we use to define the $W(f)$. When the latter lies in a spacetime region $\mathcal{O}$, we construct the algebra of observables of that region, $\mathcal{A}(\mathcal{O})$, which is a subalgebra of the algebra of the total space $\mathcal{M}$. When $\mathcal{M}$ is globally hyperbolic, the net of local algebras $\{\mathcal{A}(\mathcal{O}) | \mathcal{O} \subset_{\text{open}} \mathcal{M}\}$ satisfies the axioms of isotony [$\mathcal{O} \subset \mathcal{O}' \Rightarrow \mathcal{A}(\mathcal{O}) \subset_{\text{subalgebra}} \mathcal{A}(\mathcal{O}')$] and causality [if $\mathcal{O}$ and $\mathcal{O}'$ are causally disjoint, then $[\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}')] = 0$]. Restriction of an algebra of observables to one of its subalgebra translates in spacetime as restricting our attention to a subregion, and this amounts to consider test functions whose support lies in the latter instead of the full space. Many results can be derived from a restriction to a subalgebra of observables, like the Unruh effect derived in Appendix A.

APPENDIX D: MODULAR THEORY AND ARAKI'S RELATIVE ENTROPY

In this appendix, we present basics on the theory of von Neumann algebras and on Tomita-Takesaki's modular theory [118, 119]. This mathematical framework allows us to rigorously define the notions of modular Hamiltonian and relative entropy, which we widely use in the main content. We assume the reader to be familiar with $*$ -algebras and the GNS representation theorem, both reviewed in Appendix C. We do not pretend to be comprehensive, just to introduce the tools underpinning the proof of the dual GSL. More details can be found, e.g., in the reviews [120–123].

Along the Appendix, we shall exemplify the notions using the algebra of observables $\mathcal{A}$ of a given region of spacetime which we assume to be decomposed into two

complementary regions I and II. $\mathcal{A}_I$ will be the algebra of region I, and $\mathcal{A}_{II}$ will be the one of region II. The total GNS Hilbert space which represents an arbitrary algebra $\mathcal{A}$ with vacuum state $|\Omega\rangle$ is denoted as usual $\mathcal{H}_\Omega$, and $\mathcal{A}$ can be seen as a subalgebra of⁷⁷ $\mathcal{B}(\mathcal{H}_\Omega)$ via the representation map π of the GNS theorem. As said in Appendix C, the state $|\Omega\rangle$ is cyclic for $\mathcal{A}$. If the vacuum state $|\Omega\rangle$ is defined through a positive frequency decomposition using a global notion of time U covering $\mathcal{I}_R$ [or times $\{U^{(l,m)}\}$ attached to each sector (l, m)], a simple extension of the Reeh-Schlieder theorem [107] states that for a subalgebra $\mathcal{A}_I^\mathcal{I} \subset \mathcal{A}^\mathcal{I}$ of operators of an open region $\mathcal{D}_I^\mathcal{I} = (U_i, +\infty) \times S^2 \subset \mathcal{I}_R$ the vacuum state $|\Omega\rangle$ is also cyclic, i.e.,⁷⁸

$$\overline{\pi(\mathcal{A}_I^\mathcal{I})|\Omega\rangle} = \overline{\pi(\mathcal{A}^\mathcal{I})|\Omega\rangle} = \mathcal{H}_\Omega, \quad (\text{D1})$$

so we get the same Hilbert spaces from the two GNS constructions. Recall that we use the same notation for the abstract algebra $\mathcal{A}$ and for its representation, and similarly we denote A instead of $\pi(A)$ an arbitrary element of $\mathcal{A}$ acting on a vector of $\mathcal{H}_\Omega$. Assuming a GNS construction has been performed, we restrict our attention to subalgebras $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ for a given Hilbert space $\mathcal{H}$.

1. Von Neumann algebras

We start by recalling some basics on von Neumann algebras. Such algebras are characterized by the fact of being equal to their double commutant. Recall that the *commutant* $\mathcal{A}'$ of an algebra $\mathcal{A}$ is the set $\mathcal{A}' = \{B \in \mathcal{B}(\mathcal{H}) | \forall A \in \mathcal{A}, [A, B] = 0\}$. A von Neumann algebra therefore satisfies $\mathcal{A}'' = \mathcal{A}$. From a physical system, one gets a von Neumann algebra by taking the double commutant of the GNS representation of the Weyl algebra $\pi(\mathcal{A}_W)''$. Note that the commutant of an algebra of observables of a region $\mathcal{O}$ (not necessarily a von Neumann algebra, though) can be related to the algebra of observables of the complementary region $\mathcal{O}'$: we have $\mathcal{A}'_{\mathcal{O}} \subset \mathcal{A}_{\mathcal{O}'}$. Instances for which we have the equality $\mathcal{A}'_{\mathcal{O}} = \mathcal{A}_{\mathcal{O}'}$ are said to satisfy *Haag's duality* [124]. We assume for our examples that $\mathcal{A}'_I = \mathcal{A}_{II}$. As also $\mathcal{A}'_{II} = \mathcal{A}_I$, both of

⁷⁷Recall that observables are built by (C1), so we get bounded operators.

⁷⁸In the two-dimensional case, if U is charting $\mathcal{I}_R$, then the cyclic vacuum state $|\Omega\rangle$ obtained from the positive frequency decomposition associated to the time U is invariant under boosts and translations $U \rightarrow aU + b$ and the generator of the time translations has a positive spectrum. This is basically the set of hypotheses needed to prove the Reeh-Schlieder theorem. In the four-dimensional case, the vacuum state $|\Omega\rangle$ considered can be expressed as a tensor product of two-dimensional vacua on each sector (l, m) , see (3.21), all satisfying the positivity conditions and invariant under $U^{(l,m)} \rightarrow a_{lm}U^{(l,m)} + b_{lm}$, so the two-dimensional argument can immediately be extended to the case at hand.

them are von Neumann algebras.⁷⁹ When considering $\mathcal{I}_R$ and cutting it at $U = 0$, we also have that $\mathcal{A}_0^{\mathcal{I}}$ and its commutant $\mathcal{A}_0^{\mathcal{I}'}$ are von Neumann algebras. If the cut is at an arbitrary $U = U_i$, then, of course, the commutant of the algebra $\mathcal{A}_i^{\mathcal{I}}$ is the algebra of observables of the region $U < U_i$, and both algebras are von Neumann.

Von Neumann algebras can be classified in several types depending on their factors. We will not enter the details here (we refer to, e.g., Refs. [11,122]), but for physical applications, one can remember that *type I* von Neumann algebras have pure states and density matrices, *type II* have only density matrices, and *type III* have neither of them, as notions such as traces no longer exist. For example, when we restrict an algebra of observables $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_\Omega)$ to one of its subalgebra $\mathcal{A}_I$, the only difference is that $\mathcal{A}_I'$ is a type III von Neumann algebra ($|\Omega\rangle$ is not pure), while $\mathcal{A}''$ is a type I von Neumann algebra (as $|\Omega\rangle$ is pure with respect to the global algebra).

2. Cyclic and separating state

The second ingredient of modular theory is the notion of cyclic and separating vector in a Hilbert space. From now on, we consider the Hilbert space at hand to be the GNS space representing $\mathcal{A}$ with vacuum state $|\Omega\rangle$, i.e., $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_\Omega)$. A *cyclic* vector $|\Psi\rangle \in \mathcal{H}_\Omega$ is such that the space $\{A|\Psi\rangle, A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\Omega$. It is in particular the case for $|\Omega\rangle$. A *separating* vector $|\Psi\rangle \in \mathcal{H}_\Omega$ is such that⁸⁰ $A|\Psi\rangle = 0 \Rightarrow A = 0$. If we assume that $\mathcal{A}$ is a von Neumann algebra, then a vector $|\Psi\rangle \in \mathcal{H}_\Omega$ is cyclic for $\mathcal{A}$ if and only if it is separating for its commutant $\mathcal{A}'$.

Consider $|\Omega\rangle \in \mathcal{H}_\Omega$ a vacuum state and the algebra $\mathcal{A}$. Then, $|\Omega\rangle$ is cyclic and separating for the subalgebra⁸¹ $\mathcal{A}_I$, and so it is for the commutant $\mathcal{A}_I' = \mathcal{A}_{II}$. For algebras admitting trace operators and density matrices, we can give an enlightening decomposition of such a state in a basis of the total Hilbert space $\mathcal{H}_\Omega = \mathcal{H}_I \otimes \mathcal{H}_{II}$ as

$$|\Omega\rangle = \sum_{i \in J; i' \in J'} c_{ii'} |i\rangle_I \otimes |i'\rangle_{II} \quad (\text{D2})$$

with $c_{ii'} \neq 0$ and $\{|i\rangle\}_I$ (respectively, $\{|i'\rangle\}_{II}$) is an orthonormal basis of $\mathcal{H}_I$ (respectively, $\mathcal{H}_{II}$) indexed by a set J (respectively, J') which is at most countably infinite. It is a pure state from the point of view of the algebra $\mathcal{A}$. Its restriction to the subalgebra $\mathcal{A}_I$ is obtained via partial tracing

⁷⁹Note that, even though the restricted algebras are von Neumann algebras, this is not always the case for the total algebra of observables $\mathcal{A}$.

⁸⁰The restriction of the Hartle-Hawking state $|\Omega_H\rangle$ on the future horizon $\mathcal{H}_L^+$ is an example of cyclic and separating state.

⁸¹Again, it is a consequence of the Reeh-Schlieder theorem, and it implies the Unruh effect; see next paragraph.

$$\hat{\rho}_\Omega^{A_I} = \text{Tr}_{\mathcal{A}_{II}} |\Omega\rangle\langle\Omega|, \quad (\text{D3})$$

where the use of density matrices is mandatory as we deal now with a mixed state. One says that $|\Omega\rangle$ is the *purification* of $\hat{\rho}_\Omega^{A_I}$ in $\mathcal{H}_\Omega$. The paragon of a cyclic and separating state is the vacuum state (A48), which is pure from the point of view of the algebra on the complete null hypersurface $\mathcal{N}$ but becomes thermal upon restriction to region I or region II. It is therefore cyclic with respect to the full algebra $\mathcal{A}^\mathcal{N}$ but becomes cyclic *and* separating for $\mathcal{A}_I^\mathcal{N}$.

3. Modular operator

Consider $\mathcal{A}$ a von Neumann algebra on $\mathcal{H}_\Omega$ and a state $|\Psi\rangle \in \mathcal{H}_\Omega$ which is cyclic and separating for $\mathcal{A}$.⁸² Modular theory starts by the introduction of the *Tomita operator* S_Ψ on $\mathcal{H}_\Omega$, which is antilinear and satisfies

$$\forall A \in \mathcal{A}, \quad S_\Psi(A|\Psi\rangle) = A^\dagger|\Psi\rangle. \quad (\text{D4})$$

The requirement of cyclicity is essential for $A|\Psi\rangle$ to belong to $\mathcal{H}_\Omega$ while asking the vector to be separating ensures (D4) to be consistent.⁸³ The Tomita operator depends on the choice of cyclic and separating vector $|\Psi\rangle$. In general, $S_\Psi \notin \mathcal{B}(\mathcal{H}_\Omega)$, so in particular, it does not belong to $\mathcal{A}$. Under the action of the Tomita operator, $|\Psi\rangle$ is invariant ($S_\Psi|\Psi\rangle = |\Psi\rangle$), and we also have that $S_\Psi^2 = \mathbb{1}_{\mathcal{H}_\Omega}$, the identity operator on $\mathcal{H}_\Omega$. One can define the Hermitian conjugate $S_\Psi^\dagger$ and observe that it is just the Tomita operator of the commutant algebra $\mathcal{A}'$.

Given a Tomita operator and its Hermitian conjugate, we introduce the *modular operator* of $|\Psi\rangle$ with respect to $\mathcal{A}$ as

$$\Delta_\Psi := S_\Psi^\dagger S_\Psi. \quad (\text{D5})$$

This operator is positive definite on $\mathcal{H}_\Omega$ because $|\Psi\rangle$ is cyclic and separating. Going back to the case where trace and density matrices are defined, we consider a state $|\Psi\rangle$ of the form (D2), cyclic and separating for $\mathcal{A}_I$ (it is so also for $\mathcal{A}_{II}$). The mixed state $\hat{\rho}_\Psi^{A_I} \in \mathcal{B}(\mathcal{H}_I)$ obtained via (D3) can be written as

$$\hat{\rho}_\Psi^{A_I} = \sum_{i \in J} c_{ii}^2 |i\rangle\langle i|, \quad c_{ii} \neq 0, \quad (\text{D6})$$

and a similar expression holds for $\hat{\rho}_\Psi^{A_{II}}$. One can easily show that

⁸²To fix ideas, one can take $|\Psi\rangle = |\Omega\rangle$ and $\mathcal{A} = \mathcal{A}_I$ the restricted algebra. However, there is no reason why the vacuum should be the only cyclic and separating state.

⁸³If $|\Psi\rangle$ was not separating, a non-zero operator A_0 would exist such that $A_0|\Psi\rangle = 0$ and then $S_\Psi A_0|\Psi\rangle = S_\Psi(0) = 0$ by antilinearity on the one hand while $S_\Psi A_0|\Psi\rangle = A_0^\dagger|\Psi\rangle \neq 0$ on the other hand.

$$\Delta_\Psi = \hat{\rho}_\Psi^{A_I} \otimes (\hat{\rho}_\Psi^{A_{II}})^{-1} \quad (\text{D7})$$

with $(\hat{\rho}_\Psi^{A_{II}})^{-1}$ the inverse of $\hat{\rho}_\Psi^{A_{II}}$.

Given any two cyclic and separating vectors $|\Omega\rangle$ and $|\Psi\rangle$ of $\mathcal{H}_\Omega$, one can also consider the *relative Tomita operator*, denoted $S_{\Psi|\Omega}$, which satisfies

$$\forall A \in \mathcal{A}, \quad S_{\Psi|\Omega}(A|\Psi\rangle) = A^\dagger|\Omega\rangle. \quad (\text{D8})$$

Of course, $S_{\Omega|\Omega} = S_\Omega$.⁸⁴ Given $S_{\Psi|\Omega}$, we can immediately introduce a *relative modular operator*,

$$\Delta_{\Psi|\Omega} := S_{\Psi|\Omega}^\dagger S_{\Psi|\Omega}. \quad (\text{D9})$$

When density matrices exist, consider again $\mathcal{H}_\Omega = \mathcal{H}_I \otimes \mathcal{H}_{II}$ and now two (possibly mixed) states $\hat{\rho}_\Omega^{A_I} \in \mathcal{H}_I$ and $\hat{\rho}_\Omega^{A_{II}} \in \mathcal{H}_{II}$ whose purifications in $\mathcal{H}_\Omega$ are given by the cyclic and separating states

$$\begin{aligned} |\Omega\rangle &= \sum_{i \in I; i' \in I'} c_{ii'} |i\rangle \otimes |i'\rangle \quad \text{and} \quad |\Psi\rangle \\ &= \sum_{j \in J; j' \in J'} c_{jj'} |j\rangle \otimes |j'\rangle \end{aligned} \quad (\text{D10})$$

for some bases $(|i\rangle)_{i \in I}$ and $(|j\rangle)_{j \in J}$, which may be different. Therefore, we have $\hat{\rho}_\Omega^{A_I} = \text{Tr}_{\mathcal{H}_{II}}(|\Omega\rangle\langle\Omega|)$ and $\hat{\rho}_\Omega^{A_{II}} = \text{Tr}_{\mathcal{H}_I}(|\Psi\rangle\langle\Psi|)$. Then, we get

$$\Delta_{\Psi|\Omega} = \hat{\rho}_\Omega^{A_I} \otimes (\hat{\rho}_\Omega^{A_{II}})^{-1}. \quad (\text{D11})$$

4. Modular Hamiltonian

We still consider a cyclic and separating vector $|\Psi\rangle \in \mathcal{H}_\Omega$. The modular operator (D5) being positive definite, one can write it as

$$\Delta_\Psi = e^{-K_\Psi} \Leftrightarrow K_\Psi = -\ln \Delta_\Psi, \quad (\text{D12})$$

where K_Ω is the *modular Hamiltonian* of $|\Psi\rangle$. Generically, $K_\Psi \notin \mathcal{B}(\mathcal{H}_\Omega)$. When we restrict to the subalgebra $\mathcal{A}_I$, using (D7), we can generically write the total modular Hamiltonian as the combination of two terms, one coming from the algebra $\mathcal{A}_I$ (and acting on $\mathcal{H}_I$) and one coming from the complementary algebra $\mathcal{A}'_I = \mathcal{A}_{II}$ (and acting on $\mathcal{H}_{II}$)

$$K_\Psi = K_\Psi^{A_I} - K_\Psi^{A_{II}}, \quad (\text{D13})$$

where operators appearing in this sum are called *one-sided modular Hamiltonians*. The right-sided modular Hamiltonian $K_\Psi^{A_{II}}$ commutes with the algebra $\mathcal{A}_I$, while the left-sided modular Hamiltonian $K_\Psi^{A_I}$ commutes with its commutant algebra $\mathcal{A}_{II}$. In the cases of interest, the modular Hamiltonian will be expressed as the integral of local operators on $\mathcal{I}_R$, so we take the one-sided modular Hamiltonian $K_\Psi^{A_I}$ by restricting the integral to the region $U > 0$ (times cross sections); see Appendix E where we give the explicit expressions of the one-sided modular hamiltonians in terms of the stress-energy tensor of the theory at hand.

5. KMS conditions

A fundamental use of the modular operator and Hamiltonian, Eqs. (D5) and (D12), is in the *Kubo-Martin-Schwinger (KMS) conditions* [53,54]. A state $|\Psi\rangle$ satisfies the KMS conditions with respect to a von Neumann algebra $\mathcal{A}$ if

$$\begin{aligned} \forall (A, B) \in \mathcal{A}^2, \\ \langle \Psi | \Delta_\Psi^{-1} A \Delta_\Psi B | \Psi \rangle := \langle \Psi | e^{K_\Psi} A e^{-K_\Psi} B | \Psi \rangle = \langle \Psi | B A | \Psi \rangle. \end{aligned} \quad (\text{D14})$$

When (D14) holds, one says that $|\Psi\rangle$ is at thermal equilibrium, with temperature $T = 1$, with respect to the modular Hamiltonian K_Ψ ; see (D12).⁸⁵ Actually, the cyclic and separating vector $|\Psi\rangle \in \mathcal{H}_\Omega$ satisfies the KMS conditions with respect to $\mathcal{A}$.

Two important remarks are in order here. First, the fact that $|\Psi\rangle$ satisfies the KMS conditions is reminiscent of the cyclicity of the trace for von Neumann algebras in which this operation does not always exist. Second, observe that any cyclic and separating vector field of a von Neumann algebra is a KMS state. We must be careful not to overinterpret the satisfaction of the KMS condition, saying, e.g., that any quantum state is a thermal state. The thermal behavior is with respect to a peculiar operator, namely, the modular Hamiltonian K_Ψ , which has no reasons *a priori* to be related to a well-defined notion of energy in the physical system at hand.⁸⁶ However, if $|\Psi\rangle$ is, for instance, the Minkowski vacuum, the modular Hamiltonian turns out to be the boost Hamiltonian, and so the previous statement gives us a straightforward proof of the Unruh effect. The exact same reasoning applies to the restriction of the Hartle-Hawking state on the future horizon $\mathcal{H}_L^+$ and future null infinity $\mathcal{I}_R^+$.

⁸⁴ Although $|\Omega\rangle$ must be cyclic and separating for consistency, $|\Psi\rangle$ can actually be any state. However, if $|\Psi\rangle$ is not cyclic and separating, the relative entropy we shall define later in (D15) can be infinite.

⁸⁵ Be careful not to overstate this observation; see next paragraph.

⁸⁶ That is to say that the modular Hamiltonian needs not be the Noether charge associated to “time” translations.

To end this section on basics of modular theory, it is important to state Tomita's theorem (see, e.g., Ref. [123] for a nice discussion on the proof), namely, that the modular flow preserves the von Neumann algebra $\mathcal{A}$, i.e., $\forall t \in \mathbb{R}, \Delta_{\Psi}^{it} \mathcal{A} \Delta_{\Psi}^{-it} \subset \mathcal{A}$. Therefore, we understand why, geometrically, the modular Hamiltonian is a “boost.” It is because it needs to preserve the algebra of observables $\mathcal{A}(\mathcal{O})$ of some spacetime region $\mathcal{O}$, so it must preserve $\mathcal{O}$. In the main content, we considered subregions of $\mathcal{I}_R$ above a certain cut $U = U_i$ and observed that indeed the modular Hamiltonian of our vacuum states were preserving that region, as they were associated to the boost vector field $\xi = \kappa(U - U_i)\partial_U$.

We have now introduced all the tools required to define the relative entropy.

6. Relative entropy

Through the notion of relative entropy, the relative modular operator (D9) allows us to build a quantity one can use to answer the question about *how close two quantum states are* (with respect to some algebra of observables). Given two cyclic and separating states $|\Psi\rangle$ and $|\Omega\rangle$ for a von Neumann algebra $\mathcal{A}$, Araki's definition [33] of *relative entropy* is

$$S(\Psi||\Omega) := \langle \Psi | (-\ln \Delta_{\Psi|\Omega}) | \Psi \rangle. \quad (\text{D15})$$

Often, $|\Omega\rangle$ is the vacuum state. Note that if one of the states is not separating then the relative modular operator may contain some zero eigenvalues, rendering the above definition divergent. Several immediate properties are that $S(\Omega||\Omega) = 0$, $S(\Psi||\Omega) \geq 0$ and when there exists a unitary element of the commutant $A' \in \mathcal{A}'$ (i.e., $A'^{\dagger}A' = 1_{\mathcal{A}'}$) such that $|\Psi\rangle = A'|\Omega\rangle$ then $S(\Omega||A'\Omega) = 0$.

The fundamental property for this work though is the following. If $\mathcal{B} \subset \mathcal{A}$ is a subalgebra of $\mathcal{A}$, then we have for any two states $|\Psi\rangle$ and $|\Omega\rangle$ that

$$S_{\mathcal{A}}(\Psi||\Omega) \geq S_{\mathcal{B}}(\Psi||\Omega). \quad (\text{D16})$$

This is called the *monotonicity of the relative entropy*, and it formalizes the fact that if we have at our disposal a smaller set of observables (the subalgebra $\mathcal{B}$) then it will be harder to distinguish between the two states $|\Psi\rangle$ and $|\Omega\rangle$. For systems in which one can deal with density matrices, the relative entropy is decreasing under the action of a complete positive trace preserving map Λ ,

$$S(\hat{\rho}_{\Psi}^{\mathcal{A}}||\hat{\rho}_{\Omega}^{\mathcal{A}}) \geq S(\Lambda(\hat{\rho}_{\Psi}^{\mathcal{A}})||\Lambda(\hat{\rho}_{\Omega}^{\mathcal{A}})), \quad (\text{D17})$$

and can be interpreted as the fact that the two states $\hat{\rho}_{\Psi}^{\mathcal{A}}$ and $\hat{\rho}_{\Omega}^{\mathcal{A}}$ are less distinguishable after the application of the channel Λ .

Finally, we should be sure that the definition (D15) gives back the usual expression in the case where traces are well

defined. We use again the states $|\Omega\rangle$ and $|\Psi\rangle$ which are cyclic and separating for both restricted algebras $\mathcal{A}_I$ and $\mathcal{A}_{II}$. Recall then that $\hat{\rho}_{\Omega}^{\mathcal{A}_I} = \text{Tr}_{\mathcal{A}_{II}}(|\Omega\rangle\langle\Omega|)$ and $\hat{\rho}_{\Psi}^{\mathcal{A}_{II}} = \text{Tr}_{\mathcal{A}_I}(|\Psi\rangle\langle\Psi|)$ so that for any observable A we have $\langle A \rangle_{\Psi} = \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_{II}} A)$ when $A \in \mathcal{A}_{II}$ while $\langle A \rangle_{\Psi} = \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_I} A)$ when $A \in \mathcal{A}_I$. In that case, the modular operator takes the form (D11), and so

$$\begin{aligned} S(\Psi||\Omega) &= \langle \Psi | (-\ln \hat{\rho}_{\Omega}^{\mathcal{A}_I} + \ln \hat{\rho}_{\Psi}^{\mathcal{A}_{II}}) | \Psi \rangle \\ &= \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_{II}} \ln \hat{\rho}_{\Psi}^{\mathcal{A}_{II}}) - \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_I} \ln \hat{\rho}_{\Omega}^{\mathcal{A}_I}) \\ &= \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_I} \ln \hat{\rho}_{\Psi}^{\mathcal{A}_I}) - \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_I} \ln \hat{\rho}_{\Omega}^{\mathcal{A}_I}), \end{aligned} \quad (\text{D18})$$

where we recognize the usual quantum relative entropy.⁸⁷ Therefore, Eq. (D15) is the natural extension of the relative entropy to cases where traces and density matrices do not exist.

7. Decomposition of the relative entropy

Let $|\Omega\rangle$ and $|\Psi\rangle$ in $\mathcal{H}_{\Omega}$ be two cyclic and separating states for the restricted algebra $\mathcal{A}_I$. In this section, we relate the relative entropy between them to the one-sided modular Hamiltonian of $|\Omega\rangle$.

Consider first that both states $|\Omega\rangle$ and $|\Psi\rangle$ can be written as density matrices $\hat{\rho}_{\Omega}^{\mathcal{A}_I}$ and $\hat{\rho}_{\Psi}^{\mathcal{A}_I}$. We also assume that traces exists. The state $|\Omega\rangle$ satisfies the KMS conditions (D14), so it can be written as a generalized Gibbs state of the form

$$\hat{\rho}_{\Omega}^{\mathcal{A}_I} = \frac{e^{-K_{\Omega}^{\mathcal{A}_I}}}{\text{Tr}(e^{-K_{\Omega}^{\mathcal{A}_I}})} \quad (\text{D19})$$

with $K_{\Omega}^{\mathcal{A}_I}$ the one-sided modular Hamiltonian. On the one hand, we can write the relative entropy as in (D18),

$$\begin{aligned} S(\hat{\rho}_{\Psi}^{\mathcal{A}_I}||\hat{\rho}_{\Omega}^{\mathcal{A}_I}) &= \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_I} \ln \hat{\rho}_{\Psi}^{\mathcal{A}_I}) - \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_I} \ln \hat{\rho}_{\Omega}^{\mathcal{A}_I}) \\ &= \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_I} \ln \hat{\rho}_{\Psi}^{\mathcal{A}_I}) + \ln [\text{Tr}(e^{-K_{\Omega}^{\mathcal{A}_I}})] + \langle K_{\Omega}^{\mathcal{A}_I} \rangle_{\Psi}, \end{aligned} \quad (\text{D20})$$

where $\langle \cdot \rangle_{\Psi}$ denotes the VEV in the state $|\Psi\rangle$ and we have used that $\text{Tr} \hat{\rho}_{\Psi}^{\mathcal{A}_I} = 1$. On the other hand, we have

$$\begin{aligned} \text{Tr}(\hat{\rho}_{\Omega}^{\mathcal{A}_I} \ln \hat{\rho}_{\Omega}^{\mathcal{A}_I}) &= -\langle K_{\Omega}^{\mathcal{A}_I} \rangle_{\Omega} - \ln [\text{Tr}(e^{-K_{\Omega}^{\mathcal{A}_I}})] \\ &= -\ln [\text{Tr}(e^{-K_{\Omega}^{\mathcal{A}_I}})] \end{aligned} \quad (\text{D21})$$

as $\langle K_{\Omega}^{\mathcal{A}_I} \rangle_{\Omega} = 0$ in virtue of $\Delta_{\Omega}|\Omega\rangle = |\Omega\rangle$. Combining (D20) with (D21), we finally get

⁸⁷We have of course from our construction $\text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_{II}} \ln \hat{\rho}_{\Psi}^{\mathcal{A}_{II}}) = \text{Tr}(\hat{\rho}_{\Psi}^{\mathcal{A}_I} \ln \hat{\rho}_{\Psi}^{\mathcal{A}_I})$.

$$\begin{aligned} S(\hat{\rho}_\Psi^{\mathcal{A}_I} || \hat{\rho}_\Omega^{\mathcal{A}_I}) &= \text{Tr}(\hat{\rho}_\Psi^{\mathcal{A}_I} \ln \hat{\rho}_\Psi^{\mathcal{A}_I}) - \text{Tr}(\hat{\rho}_\Omega^{\mathcal{A}_I} \ln \hat{\rho}_\Omega^{\mathcal{A}_I}) + \langle K_\Omega^{\mathcal{A}_I} \rangle_\Psi \\ &= -S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_I} + \langle K_\Omega^{\mathcal{A}_I} \rangle_\Psi, \end{aligned} \quad (\text{D22})$$

where we see appearing the *renormalized von Neumann entropy* of $|\Psi\rangle$ with respect to $|\Omega\rangle$,

$$S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_I} = \text{Tr}(\hat{\rho}_\Omega^{\mathcal{A}_I} \ln \hat{\rho}_\Omega^{\mathcal{A}_I}) - \text{Tr}(\hat{\rho}_\Psi^{\mathcal{A}_I} \ln \hat{\rho}_\Psi^{\mathcal{A}_I}). \quad (\text{D23})$$

The renormalization of the von Neumann entropy is needed in quantum field theory because of universal ultraviolet and infrared divergences. However, in some Hilbert space $\mathcal{H}_\Omega$, we expect that there exists a dense subspace of states sharing the same ultraviolet and infrared behaviors, so that the *difference* of von Neumann entropies makes sense, even if for any state it is itself divergent. Therefore, Eq. (D23) might be well defined even if both terms on the r.h.s. are themselves divergent. In (D20)–(D23), the traces are computed with respect to the complementary algebra to $\mathcal{A}_I$, namely, $\mathcal{A}_{II}$.

Even if we cannot write the states $|\Psi\rangle$ and $|\Omega\rangle$ as density matrices with respect to the algebra $\mathcal{A}_I$, e.g., when dealing with $\mathcal{I}_R$, the relative entropy (D15) is always well defined, so we can evaluate it. The same is true for the one-sided modular Hamiltonian (D13), as long as $|\Psi\rangle$ has a finite normal-ordered boost energy. Then, we take a state $|\Psi\rangle \in \mathcal{H}_\Omega$ in the domain of $K_\Omega^{\mathcal{A}_I^T}$ and *define* a quantity $S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_I^T}$ by

$$S_{\Psi|\Omega}^{\text{v.N.}, \mathcal{A}_I^T} := S_{\mathcal{A}_I^T}(\Psi || \Omega) - \langle K_\Omega^{\mathcal{A}_I^T} \rangle_\Psi \quad (\text{D24})$$

and interpret it as a (generalized) renormalized von Neumann entropy. Since $\langle K_\Omega^{\mathcal{A}_I^T} \rangle_\Omega = 0$, Eq. (D24) vanishes for $|\Psi\rangle = |\Omega\rangle$. By doing this, we have decomposed Araki's relative entropy into two finite components as long as $|\Psi\rangle$ belongs to the domain of the one-sided modular Hamiltonian $K_\Omega^{\mathcal{A}_I^T}$. Therefore, Eq. (D22) is the general decomposition of the relative entropy in terms of one-sided modular Hamiltonian and renormalized von Neumann entropy, valid for whatever the type of von Neumann algebras we deal with. It remains to relate the one-sided modular Hamiltonian to a physical quantity of interest, namely, the (normal-ordered) stress-energy tensor of the theory; this is to be done in Appendix E.

APPENDIX E: MODULAR HAMILTONIAN AND BOOST ENERGY

In this appendix, we prove that the modular Hamiltonian for a vacuum state $|\Omega\rangle$ with respect to a von Neumann algebra of observables $\mathcal{A}$ [introduced in (D12)] can be written as an integral, over the spacetime region whose observables lies in $\mathcal{A}$, of the normal-ordered boost energy.

Consider that $\mathcal{A}$ is the algebra of observables at $\mathcal{I}_R$ and that we decompose the latter in two regions: region I for $U > U_0$ and region II for $U < U_0$. Like in Appendix D, we consider $\mathcal{H}_\Omega$ the GNS Hilbert space representing $\mathcal{A}^T$, and we have that $\mathcal{A}^T$ decomposes into $\mathcal{A}_I^T$ for region I and $\mathcal{A}_{II}^T = (\mathcal{A}_I^T)'$ for region II. The state $|\Omega\rangle$ is cyclic and separating for $\mathcal{A}_I^T$ and $\mathcal{A}_{II}^T$. The Tomita operator S_Ω^I acts on operators of $\mathcal{A}_I^T$, so in particular, it acts on the annihilation operator $\hat{a}_{\omega,I}$ ⁸⁸ [defined in Eq. (A17)] via

$$S_\Omega^I(\hat{a}_{\omega,I}|\Omega\rangle) = \hat{a}_{\omega,I}^\dagger|\Omega\rangle, \quad (\text{E1})$$

and similarly for its Hermitian conjugate $(S_\Omega^I)^\dagger$ which is the Tomita operator for the algebra $\mathcal{A}_{II}^T$ in region II. Therefore,

$$(S_\Omega^I)^\dagger(\hat{a}_{\omega,II}|\Omega\rangle) = \hat{a}_{\omega,II}^\dagger|\Omega\rangle. \quad (\text{E2})$$

Recall also the relations (A36) and (A37), which we display again here for convenience:

$$(\hat{a}_{\omega,II} - e^{-\pi\omega}\hat{a}_{\omega,I}^\dagger)|\Omega\rangle = 0 \quad (\text{E3})$$

$$(\hat{a}_{\omega,I} - e^{-\pi\omega}\hat{a}_{\omega,II}^\dagger)|\Omega\rangle = 0. \quad (\text{E4})$$

They give a relation between the ladder operators of region I and those of region II once they act on the total vacuum state $|\Omega\rangle$. Using (E1)–(E4), we can show the following relation for the modular operator (D5),

$$\begin{aligned} \Delta_\Omega(a_{\omega,I}^\dagger|\Omega\rangle) &= (S_\Omega^I)^\dagger S_\Omega^I(a_{\omega,I}^\dagger|\Omega\rangle) = (S_\Omega^I)^\dagger(a_{\omega,I}|\Omega\rangle) \\ &= (S_\Omega^I)^\dagger(e^{-\pi\omega}a_{\omega,II}^\dagger|\Omega\rangle) = e^{-\pi\omega}(S_\Omega^I)^\dagger(a_{\omega,II}^\dagger|\Omega\rangle) \\ &= e^{-\pi\omega}S_\Omega^{II}(a_{\omega,II}^\dagger|\Omega\rangle) \\ &= e^{-\pi\omega}a_{\omega,II}|\Omega\rangle = e^{-2\pi\omega}a_{\omega,I}^\dagger|\Omega\rangle; \end{aligned} \quad (\text{E5})$$

i.e., the state $\hat{a}_{\omega,I}^\dagger|\Omega\rangle$ is an eigenstate of the modular operator with eigenvalue $e^{-2\pi\omega}$. This works exactly the same in region II for the state $\hat{a}_{\omega,II}^\dagger|\Omega\rangle$. By induction, we have that

$$\Delta_\Omega(\hat{a}_{\omega,I}^\dagger)^N|\Omega\rangle = e^{-2\pi N\omega}(\hat{a}_{\omega,I}^\dagger)^N|\Omega\rangle \quad (\text{E6})$$

$$\Delta_\Omega(\hat{a}_{\omega,II}^\dagger)^N|\Omega\rangle = e^{-2\pi N\omega}(\hat{a}_{\omega,II}^\dagger)^N|\Omega\rangle, \quad (\text{E7})$$

and we have a similar relation for the states $\hat{a}_{\omega,I}^N|\Omega\rangle$ and $\hat{a}_{\omega,II}^N|\Omega\rangle$, which are eigenvectors of Δ_Ω with eigenvalues $e^{2\pi N\omega}$, so the modular Hamiltonian K_Ω can be expressed as

⁸⁸Note that $\hat{a}_{\omega,I}$ is not in the Weyl algebra, since it is unbounded. However, we can obtain it in a particular limit of bounded operator.

$$K_\Omega = 2\pi \sum_\omega \omega (\hat{N}_{\omega,I} - \hat{N}_{\omega,II}). \quad (\text{E8})$$

From that, we define the one-sided modular Hamiltonian of region I as

$$K_\Omega^{\mathcal{A}_I^\tau} = 2\pi \sum_\omega \omega (\hat{N}_{\omega,I} - \langle \hat{N}_{\omega,I} \rangle_\Omega), \quad (\text{E9})$$

where we have subtracted the VEV in the global vacuum $|\Omega\rangle$ to get a finite quantity. A similar expression to (E9) can be written for $K_\Omega^{\mathcal{A}_{II}^\tau}$ using $\hat{N}_{\omega,II}$.

Now, consider the affine stress-energy tensor of region I, and take its doubly null time component T_{uu} , where $u = \ln(U - U_0)$. In terms of ladder operators, it reads

$$\begin{aligned} T_{uu} = \partial_u \phi \partial_u \phi = & - \int_0^{+\infty} \int_0^{+\infty} \frac{\sqrt{\omega\omega'}}{4\pi} (\hat{a}_{\omega,I} \hat{a}_{\omega',I} e^{-i(\omega+\omega')u} + \hat{a}_{\omega,I}^\dagger \hat{a}_{\omega',I}^\dagger e^{i(\omega+\omega')u}) d\omega d\omega' \\ & + \int_0^{+\infty} \int_0^{+\infty} \frac{\sqrt{\omega\omega'}}{4\pi} (\hat{a}_{\omega,I}^\dagger \hat{a}_{\omega',I} e^{i(\omega-\omega')u} + \hat{a}_{\omega,I} \hat{a}_{\omega',I}^\dagger e^{i(\omega-\omega')u}) d\omega d\omega'. \end{aligned} \quad (\text{E10})$$

The one-sided integral of the normal-ordered stress tensor is defined by

$$\hat{K}^I = \int_{-\infty}^{+\infty} :T_{uu}:\Omega du = \int_{U_0}^{+\infty} (U - U_0) :T_{UU}:\Omega dU, \quad (\text{E11})$$

which, in terms of ladder operators gives, from (E10)

$$\hat{K}^I = \int_0^{+\infty} \omega (\hat{a}_{\omega,I}^\dagger \hat{a}_{\omega,I} - \langle \hat{a}_{\omega,I}^\dagger \hat{a}_{\omega,I} \rangle_\Omega) d\omega = \frac{K_\Omega^{\mathcal{A}_I^\tau}}{2\pi}. \quad (\text{E12})$$

In (E12), we see the number operator of region I appearing, namely, $\hat{N}_{\omega,I} = \hat{a}_{\omega,I}^\dagger \hat{a}_{\omega,I}$. Similarly, one could have led an exactly similar analysis in region II, i.e., $U < U_0$, using the coordinate $u_- = -\ln(U_0 - U)$ and obtain

$$\hat{K}^{II} = - \int_{-\infty}^{U_0} (U_0 - U) :T_{UU}:\Omega dU, \quad (\text{E13})$$

so in terms of ladder operators of region II, we get an expression of the form

$$\hat{K}^{II} = \int_0^{+\infty} \omega (\hat{a}_{\omega,II}^\dagger \hat{a}_{\omega,II} - \langle \hat{a}_{\omega,II}^\dagger \hat{a}_{\omega,II} \rangle_\Omega) d\omega = - \frac{K_\Omega^{\mathcal{A}_{II}^\tau}}{2\pi}. \quad (\text{E14})$$

Finally, we get the subsequent fundamental result:

$$\begin{aligned} & 2\pi \int_{-\infty}^{+\infty} (U - U_0) :T_{UU}:\Omega dU \\ & = 2\pi \int_0^{+\infty} \omega (\hat{N}_{\omega,I} - \hat{N}_{\omega,II}) d\omega. \end{aligned} \quad (\text{E15})$$

Combining the latter equation with (E8), we find the advertised relation between the modular Hamiltonian and the normal-ordered stress-energy tensor, namely,

$$K_\Omega^{\mathcal{A}_I^\tau} = 2\pi \int_{-\infty}^{+\infty} (U - U_0) :T_{UU}:\Omega dU. \quad (\text{E16})$$

The one-sided modular Hamiltonian (E9) is obtained by restricting the time integral of (E16) to the spacetime domain corresponding to $\mathcal{A}_I^\tau$, i.e., in the case at hand the region $U > U_0$; see (E12).

This proof can be immediately extended to the four-dimensional case and to any notion of time. In particular, if we start with the κ_l -vacuum states defined from the choice of times (3.44), one can apply the previous analysis to each sector l and immediately find the expression (4.59).

APPENDIX F: DUAL GSL FROM THE MINKOWSKI VACUUM

If there is no black hole, $\mathcal{I}_R^+$ is a complete characteristic data surface for free massless fields (for half of the modes of courses as explained in Sec. II). As emphasized in Sec. III, the natural state to consider for massless fields is the Minkowski vacuum ω_M attached to the inertial (affine) time $u := u_+$. Further details on its properties—most notably its symmetry structure—and on the Hilbert space $\mathcal{H}_{0_M}$ obtained as the GNS representation induced by ω_M are provided in Sec. III B 1. This is the framework adopted, for example, in Refs. [73,74,125]. To obtain the generalized second law in this setup, one first has to restrict the vacuum state $|0_M\rangle$ to a subalgebra $\mathcal{A}_i^\tau \subset \mathcal{A}_0^\tau$ attached to the region $\mathcal{D}_i^\tau = (u_i, +\infty) \times S^2$. The one-sided modular Hamiltonian of the Minkowski vacuum associated to the algebra $\mathcal{A}_i^\tau$ is given by

$$\begin{aligned} K_{0_M}^{\mathcal{A}_i^\tau} &= 2\pi \int_{\mathcal{D}_i^\tau} (u - u_i) :T_{uu}:\omega_M du \wedge \epsilon_S \\ &= 2\pi \int_{\mathcal{D}_i^\tau} (u - u_i) T_{uu} du \wedge \epsilon_S \end{aligned} \quad (\text{F1})$$

that is the one-sided boost energy. It can be shown that the integral boost energy is related to the variation of the renormalized area $\bar{A}$ [73,126]. To check that it is true, we consider a null Rindler horizon at $v = v_0$. Its spacelike cross sections are orthogonal to the normal $\mathbf{n} = \partial_v$, which points toward a fixed angular direction x^A on the celestial sphere. Taking the limit $v_0 \rightarrow \infty$, the Rindler plane approaches, in the conformal compactification, a single null generator of $\mathcal{I}_R^+$ associated with the chosen angle x^A .⁸⁹ In this limit, all outgoing radiation reaching $\mathcal{I}_R^+$ at the geometric angle x^A eventually crosses the corresponding null Rindler horizon. The evolution of its cross sectional area is therefore governed by the local null Raychaudhuri equation, which can be used to relate the accumulated energy flux to the variation of the renormalized area

$$\partial_u^2 \bar{a}(x^A) = -8\pi T_{uu}(x^A) \epsilon_S, \quad \bar{a}(x^A) = a(x^A) - a_0(x^A) \quad (\text{F2})$$

at first order, where a is the area of cross section of the plane and a_0 corresponds to the (infinite) area of the Rindler plane when there is no radiation, so $\bar{a}$ can be interpreted as a renormalized area. The application of the second law to null Rindler planes necessarily requires such a renormalization, since the physical area of these planes is,

⁸⁹Therefore, one studies the radiative data on the celestial sphere geodesic by geodesic, as in Ref. [127].

strictly speaking, infinite. By considering the family of Rindler planes normal to all solid angles x^A on the celestial sphere and summing the corresponding contributions, we then obtain that

$$\partial_u^2 \bar{A} = -8\pi \int_S T_{uu} \epsilon_S \bar{A} = - \int_S \bar{a}(x^A), \quad (\text{F3})$$

where S is a cross section $\mathcal{I}_R^+$. Multiplying both sides of the equation (F3) by the factor $-(u - u_i)$, we get

$$\frac{d}{du} \left[\frac{1}{4} \left(\bar{A} - (u - u_i) \frac{d\bar{A}}{du} \right) \right] = -2\pi \int_S (u - u_i) T_{uu} \epsilon_S \quad (\text{F4})$$

that is exactly similar to (2.15), but with u the affine coordinate. We have related the boost Hamiltonian (F1) to the variation of renormalized area. Then, by considering a cyclic and separating vector $|\Psi\rangle \in \mathcal{H}_{0_M}$, we can run the derivation of the second law similarly to what we did in Sec. IV and find that

$$\Delta \left(\frac{\bar{A}}{4G} + S \right) \geq 0 \quad (\text{F5})$$

since the one-sided modular Hamiltonian of the Minkowski vacuum is given by (F1). We recover therefore the result of Ref. [73]. Note that (F5) is not surprising since it can be obtained directly from the standard proof of the generalized second law on Rindler horizons, as in Ref. [27].

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