

Hamiltonian renormalization. VIII. $P(\Phi)_2$ quantum field theory

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In previous works in this series we focused on Hamiltonian renormalization of free field theories in all spacetime dimensions. In this paper, we address the Hamiltonian renormalization of the self-interacting scalar field in two spacetime dimensions with polynomial potential, called $P(\Phi)_2$. We consider only the finite volume case. The $P(\Phi)_2$ theory is one of the few interacting quantum field theories that can be rigorously constructed nonperturbatively. We find that our Hamiltonian renormalization flow finds this theory indeed as a fixed point.

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I. INTRODUCTION

Constructing interacting quantum field theories (QFTs) rigorously in four and higher spacetime dimensions remains one of the most difficult challenges in theoretical and mathematical physics [1]. The difficulties come from the fact that quantum fields are operator-valued distributions, which means that products thereof as they appear typically in Hamiltonians are *a priori* ill defined, being plagued by both short distance (UV) and large distance (IR) divergences. In the constructive QFT (CQFT) approach [2], one tames both types of divergences by introducing both UV (M) and IR cutoffs (R) to the effect that only a finite number of degrees survive at finite M , R . For instance, R could be a compactification radius and M a lattice spacing. Then, at finite M , R one is in the safe realm of quantum mechanics. The problem is then how to remove the cutoffs. Usually one removes first M (continuum limit) and then R (thermodynamic limit). In this process, the parameters (coupling constants) are taken to be cutoff dependent and they are tuned or renormalized in such a way that the limiting theory is well defined when possible.

Nonperturbative renormalization in CQFT (not to be confused with renormalization in the perturbative approach to QFT) has a long tradition [3] and comes in both the functional integral language and the Hamiltonian language (see, e.g., [4] and references therein). Focusing on UV cutoff removal, we consider quantum mechanical systems labeled by the cutoff M . If these quantum mechanical systems all descend from a well-defined continuum theory,

then in the functional integral approach one obtains the theory at resolution M by integrating out all degrees of freedom referring to higher resolution, while in the Hamiltonian approach one projects those out. This, in particular, implies that if one takes the quantum mechanical theory at resolution M' and integrates or projects out the degrees of freedom at resolutions between $M < M'$ and M' one obtains the quantum mechanical theory at resolution M . Vice versa, when this necessary set of “consistency conditions” is met, this typically also is sufficient to define a continuum theory.

The family of theories that one starts with are constructed making various choices such as representations, factor orderings, discretization errors, etc. and the aforementioned consistency conditions are generically violated. However, one can define a sequence of such quantum mechanical theory families by defining a new theory at resolution M by integrating/projecting out the degrees of freedom between M and $M'(M)$ of the old theory at resolution $M'(M) > M$ where $M': M \mapsto M'(M)$ is a fixed function on the set of resolution scales. Such a process is called a block spin transformation or coarse graining operation, which typically leads to a renormalization of the coupling constants. At a fixed point of this renormalization flow of theories, the consistency condition is enforced by construction and therefore fixed points qualify as continuum theories.

In previous parts of this series, we have considered a Hamiltonian projection scheme [4,5] that is motivated by the functional integral approach via Osterwalder-Schrader reconstruction. It was then applied to free QFT in Minkowski space [6–10] in any dimension and parametrized QFT [11] in two dimensions, which shares some features with the free bosonic string. In all those cases, the fixed point of the flow could be computed and was shown to coincide with the known continuum theory. In the present paper, for the first time we expose the formalism to interacting QFT, specifically to the self-interacting scalar quantum field theory in two spacetime dimensions with

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polynomial potential, called $P(\Phi)_2$ theory [12]. We focus on the UV part of the renormalization and thus consider the cylinder spacetime $\mathbb{R} \times [0, R)$ with periodic boundary conditions where the circumference $2\pi R$ of the circle is fixed. Once the theory is constructed at finite R the thermodynamic limit $R \rightarrow \infty$ is taken by the methods described in [12] and references therein should one be interested in the spacetime $\mathbb{R}^2$.

This work is organized as follows:

In Sec. II, we introduce the classical and quantum $P(\Phi)_2$ model on the cylinder. Its quantum theory is well defined in the Fock representation selected by the free part of its Hamiltonian no matter what the polynomial degree of its potential is, even if it is not bounded from below, as long as the potential is normal ordered with respect to the same Fock structure [13,14]. An elementary proof of this astonishing fact unique to two spacetime dimensions is provided in Appendix A.

In Sec. III, we introduce the Hamiltonian renormalization of this model. We pick the Dirichlet kernel [15] to define the block spin transformation mentioned above. An account of Dirichlet kernel renormalization techniques, which are closely related to wavelet theory [16], may be found in Appendix B. The Dirichlet kernel is a compromise between position locality and momentum decay properties; in particular, it is smooth and therefore has advantages over the often chosen Schwarz kernel. The Schwarz kernel has better position locality properties but it has distributional derivatives that are potentially problematic with respect to treating the necessarily occurring field derivatives.

In Sec. IV, we derive the flow of the model and show that its fixed point at finite resolution coincides with blocking from the known continuum QFT reviewed in Sec. II.

In Sec. V, we summarize and conclude.

II. CLASSICAL AND QUANTUM $P(\Phi)_2$ ON THE CYLINDER

In the first subsection, we introduce the classical $P(\Phi)_2$ theory and, in the second, we show that its normal ordered self-interacting Hamiltonian is a densely defined, symmetric operator in the Fock representation selected by its free part.

A. Classical $P(\Phi)_2$ on the cylinder

We consider the cylinder spacetime manifold $\mathbb{R} \times [0, R)$ where R is any finite, fixed circumference of the circle. For convenience, we pass from dimensionful coordinates (cs, y) on that spacetime to dimensionless coordinates $X = (t = \frac{cs}{R}, x = \frac{y}{R})$ so that $x \in [0, 1)$ with end points identified. Then the classical action can be rewritten as

$$\begin{aligned} S[\Phi] &= \int_{\mathbb{R}} dt \int_0^1 dx \left\{ \frac{1}{2} [\dot{\Phi}]^2 - [\Phi']^2 - p^2 \Phi^2 - P(\Phi) \right\} (X) \\ &=: \int_{\mathbb{R} \times [0, 1)} d^2 L(\Phi(x), \Phi'(x), \dot{\Phi}(X)), \end{aligned} \quad (2.1)$$

where p is the mass, $(\dot{}) = \partial_t()$, $(\cdot)' = \partial_x(\cdot)$, and

$$P(\Phi) = \sum_{k=0}^N g_k \Phi^k \quad (2.2)$$

is any finite polynomial in the fields, i.e., a finite linear combination of powers Φ^k , $k = 0, \dots, N$ with real-valued, dimensionless coefficients g_k called coupling constants. We assume that whatever the value of g_2 is, we have $p^2 = [MR]^2 > 0$ where M is the dimensional mass. This can always be achieved by redefining g_2 and is done in order to avoid the special treatment of zero modes. We take Φ dimension free and have dropped a constant prefactor from the action that has the dimension of an action. The spacetime field Φ is subject to periodic boundary conditions $\Phi(t, 0) = \Phi(t, 1)$ for all $t \in \mathbb{R}$. Therefore, in all that follows we never have to worry about boundary terms when integrating by parts with respect to x .

The Hamiltonian formulation is straightforward: The real-valued time zero field $\phi(x) := \Phi(0, x)$ has the real-valued conjugate momentum $\pi(x) = [\partial_t \Phi](0, x)$ which obeys the canonical Poisson brackets

$$\begin{aligned} \{\phi(x), \phi(y)\} &= \{\pi(x), \pi(y)\} = 0, \\ \{\pi(x), \phi(y)\} &= \delta(x, y), \end{aligned} \quad (2.3)$$

where $\delta(x, y)$ is the periodic delta distribution (see, e.g., Appendix B) and the Hamiltonian is obtained by the Legendre transformation

$$\begin{aligned} h[\phi, \pi] &= \text{extr}_u \left\{ \int_0^1 dx [\pi(x)u(x) - L(\phi(x), \phi'(x), u(x))] \right\} \\ &= \int_0^1 dx \left\{ \frac{1}{2} [\pi^2 + [\phi']^2 + p^2 \phi^2] + P(\phi) \right\} (x). \end{aligned} \quad (2.4)$$

B. Quantum $P(\Phi)_2$ on the cylinder

The structure of the Hamiltonian (2.4) suggests the natural split

$$\begin{aligned} h &= h_0 + v, \quad h_0 = \int_0^1 dx \left\{ \frac{1}{2} [\pi^2 + \phi \omega^2 \cdot \phi] \right\} (x), \\ v &= \int_0^1 dx P(\phi(x)), \end{aligned} \quad (2.5)$$

where

$$\omega^2 := p^2 - \Delta, \quad \Delta = \partial_x^2 \quad (2.6)$$

is a self-adjoint operator on the one-particle Hilbert space

$$L = L_2([0, 1), dx), \quad (2.7)$$

with pure point spectrum $\hat{\omega}_n^2 = p^2 + [2\pi in]^2$, $n \in \mathbb{Z}$. The corresponding eigenfunctions are $e_n(x) = e^{2\pi inx}$, which define an orthonormal basis of L . For obvious reasons, h_0, v are called the free part and interacting part, respectively.

We define the unital Weyl $*$ algebra $\mathfrak{A}$ in the usual way via its generating Weyl elements $w[f] = \exp(i\langle f, \phi \rangle_L)$, $w[g] = \exp(i\langle g, \pi \rangle_L)$ for real-valued $f, g \in L$ that are subject to the Weyl relations

$$\begin{aligned} w[g]w[f]w[-g] &= e^{-i\langle g, f \rangle_L} w[f], & w[f]w[f'] &= w[f + f'], \\ w[g]w[g'] &= w[g + g'], & w[f]^* &= w[-f], \\ w[g]^* &= w[-g]. \end{aligned} \quad (2.8)$$

We define a cyclic Fock representation $(\rho, \mathcal{H}, \Omega)$ of $\mathfrak{A}$ from the Fock state¹ ω_F ,

$$\omega_F(w[f]w[g]) := e^{\frac{i}{2}\langle f, g \rangle_L} e^{-\frac{i}{4}(\langle f, \omega^{-1} \cdot f \rangle_L + \langle g, \omega \cdot g \rangle_L)} \quad (2.9)$$

via the Gel'fand-Naimark-Segal (GNS) construction. It is not difficult to show that (2.9) is unitarily equivalent to defining annihilation operator-valued distributions corresponding to the algebra element

$$a(x) := \frac{1}{\sqrt{2}} [\omega^{1/2} \cdot \phi - i\omega^{-1/2} \cdot \pi](x), \quad (2.10)$$

so that $\rho(a(x))\Omega = 0$ and everything else follows from the commutation relations of which the nonvanishing ones are

$$[a(x), a^*(y)] = \delta(x, y). \quad (2.11)$$

The linear span $\mathcal{D}$ of the Fock vector states $\psi_{f_1, \dots, f_n} := \rho(\langle f_1, a \rangle^* \dots \rho(\langle f_n, a \rangle^*)\Omega$ for $f_1, \dots, f_n \in L$ together with Ω is dense in the Fock representation space $\mathcal{H}$. The number n is called the particle number of the Fock vector where we assign $n = 0$ to the vacuum Ω void of particles. We will, in fact, consider $\mathcal{D}$ corresponding to $f_1, \dots, f_n \in L_0$ where L_0 is the span of the functions e_n . This $\mathcal{D}$ is still dense and has the advantage that finite products of functions in L_0 are still in L_0 .

We use capital letters in order to denote the operator representatives of algebra elements, e.g., $A(x) = \rho(a(x))$, $A^\dagger(x) = \rho(a^*(x))$. We use this Fock structure in order to define both H_0 and V by their normal ordered symbols

¹Note that the symbol ω is reserved for both the covariance operator and the Fock state introduced in Eq. (2.9); to distinguish the latter, it is denoted with a subscript F . In contrast, the letters w and W are used exclusively to denote Weyl elements.

$$\begin{aligned} H_0 &:= \rho(h_0) = \int_0^1 dx A^\dagger(x) [\omega \cdot A](x), & V &= \sum_{k=0}^N g_k V_k, \\ V_k &:= \rho(v_k) = 2^{-k/2} \sum_{l=0}^k \binom{k}{l} V_k(l), \\ V_k(l) &:= \int_0^1 dx \{ [\omega^{-1/2} \cdot A^\dagger](x) \}^{k-l} [\omega^{-1/2} \cdot A](x) \}^l. \end{aligned} \quad (2.12)$$

By construction, $\mathcal{D}$ is an invariant, dense domain of H_0 . The astonishing fact, unique to two spacetime dimensions, when $m > 0$ and only when the spacetime is spatially compact, is that $\mathcal{D}$ is also a dense domain for V , albeit no longer an invariant one when $g_k \neq 0$ for at least one of $k = 2, \dots, N$. We give an elementary proof in Appendix A.

III. HAMILTONIAN RENORMALIZATION OF $P(\Phi)_2$

An account of the version of Hamiltonian renormalization used below and employing the Dirichlet kernel can be found in Appendix B; see [4,5] for its motivation. Furthermore, in the following sections we adopt the notation introduced in Appendix B: capital letters such as F, G denote functions in L ; capital letters with a subscript M , such as F_M, G_M , refer to functions in L_M ; and lowercase letters with a subscript M , such as f_M, g_M , denote functions in l_M . Elements of the abstract algebra are still written in lowercase, while their representatives (after GNS construction) are denoted by capital letters.

The starting point of the renormalization scheme is to provide a family of theories $(\rho_M^{(0)}, \mathcal{H}_M^{(0)}, \Omega_M^{(0)}, H_M^{(0)})$ consisting of a Hilbert space $\mathcal{H}_M^{(0)}$ with cyclic vector $\Omega_M^{(0)}$ that carries a representation $\rho_M^{(0)}$ of some $*$ algebra $\mathfrak{A}_M$ and a Hamiltonian operator $H_M^{(0)}$ densely defined on some subspace $\mathcal{D}_M^{(0)}$ of $\mathcal{H}_M^{(0)}$. Equivalently, we consider a state $\omega_{M,F}^{(0)}$ on $\mathfrak{A}_M$ for which $(\rho_M^{(0)}, \mathcal{H}_M^{(0)}, \Omega_M^{(0)})$ are its GNS data. The objects $\mathfrak{A}_M^{(0)}$ and $H_M^{(0)}$ are to be thought of as discretized versions of the continuum objects $\mathfrak{A}$ and H . The label M is, in general, taken from a partially ordered and directed index set $\mathbb{O}$ that describes the (location, momentum, energy, etc.) resolution at which we probe the theory. In the present case, the central tool is the Dirichlet kernel

$$\begin{aligned} P_M(x, y) &= \sum_{n \in \mathbb{Z}_M} e_n(x - y), \\ \mathbb{Z}_M &= \left\{ n \in \mathbb{Z}; |n| \leq \frac{M-1}{2} \right\}, \end{aligned} \quad (3.1)$$

which can be considered as a tamed version of the δ distribution on the circle. The index set is taken to be the set of

odd naturals with ordering relation $M < M'$ if and only if $M'/M \in \mathbb{O}$. The Dirichlet kernel can be considered as an orthogonal projection P_M in L with image L_M . We define the Weyl algebra $\mathfrak{A}_M$ in analogy to (2.8) as the abstract $*$ algebra generated by the Weyl elements $w_M[F_M], w_M[G_M]$ with real-valued $F_M, G_M \in L_M$ subject to the relations

$$\begin{aligned} w_M[G_M]w_M[F_M]w_M[-G_M] &= e^{-i\langle G_M, F_M \rangle_{L_M}} w_M[F_M], \\ w_M[F_M]w_M[F'_M] &= w_M[F_M + F'_M], \\ w_M[G_M]w_M[G'_M] &= w_M[G_M + G'_M], \\ w_M[F_M]^* &= w_M[-F_M], \\ w_M[G_M]^* &= w_M[-G_M], \end{aligned} \quad (3.2)$$

where the scalar product on the finite resolution one-particle Hilbert space L_M coincides with the one on L . The state $\omega_{M,F}^{(0)}$ is chosen to be in analogy to (2.9) as the Fock state

$$\begin{aligned} \omega_{F,M}^{(0)}(w_M[F_M]w_M[G_M]) \\ = e^{i\langle F_M, G_M \rangle_{L_M}} e^{-\frac{1}{4}[\langle F_M, [\omega_M^{(0)}]^{-1} \cdot F_M \rangle_{L_M} + \langle G_M, [\omega_M^{(0)}] \cdot G_M \rangle_{L_M}]}, \end{aligned} \quad (3.3)$$

which requires as a central input the definition of the kernel $\omega_M^{(0)}$ on L_M . Note that the unfortunate doubling of symbols ω for both an algebraic state and a kernel on the one-particle Hilbert space is resolved by attaching an extra index “F” to the state to indicate “Fock”. To motivate the choice of $\omega_M^{(0)}$, we follow the general prescription of Appendix B and define h_M as a quantization of

$$\begin{aligned} h_M[\phi_M, \pi_M] &:= h[\phi_M, \pi_M], \quad \phi_M = P_M \cdot \phi, \\ \pi_M &= P_M \cdot \pi, \end{aligned} \quad (3.4)$$

where h is the classical continuum Hamiltonian (2.5). For its free part, we find

$$\begin{aligned} h_{0,M} &= \frac{1}{2} \int_0^1 dx [\pi_M(x)^2 + [\partial_x \phi_M(x)]^2 + p^2 \phi_M(x)^2] \\ &= \frac{1}{2} [\langle \pi_M, \pi_M \rangle_{L_M} + \langle \partial \phi_M, \partial \phi_M \rangle_{L_M} + p^2 \langle \phi_M, \phi_M \rangle_{L_M}]. \end{aligned} \quad (3.5)$$

Integrating the derivative term by parts (note that P_M preserves the space of periodic L_2 functions), we obtain

$$h_{0,M} = \frac{1}{2} [\langle \pi_M, \pi_M \rangle_{L_M} - \langle \phi_M, \Delta_M \cdot \phi_M \rangle_{L_M} + p^2 \langle \phi_M, \phi_M \rangle_{L_M}], \quad (3.6)$$

with the Laplacian

$$\Delta_M := \partial_M^2, \quad \partial_M = P_M \cdot \partial \cdot P_M. \quad (3.7)$$

We have used the projection property $P_M \cdot P_M = P_M$ of the Dirichlet kernel to obtain (3.7). The specific form of $h_{0,M}$ suggests a Fock quantization with annihilators

$$\begin{aligned} a_M(x) &= 2^{-1/2} [\omega_M]^{1/2} \cdot \phi_M - i[\omega_M]^{-1/2} \cdot \pi_M(x), \\ [\omega_M]^2 &= p^2 1_M - \Delta_M, \end{aligned} \quad (3.8)$$

which obey nontrivial commutation relations

$$[a_M(x), a_M(y)^*] = P_M(x, y), \quad (3.9)$$

if one defines the Poisson brackets between ϕ_M, π_M as the result of considering these as functions on the continuum phase space and using the continuum Poisson bracket

$$\begin{aligned} \{\pi_M(x), \phi_M(y)\} \\ := \int du \int dv P_M(x, u) P_M(y, v) \{\pi(u), \phi(v)\} \\ = P_M(x, y). \end{aligned} \quad (3.10)$$

Then we see that (3.3) and (3.8) match, provided that we interpret $w_M[F_M] = e^{i\langle F_M, \phi_M \rangle_{L_M}}, w_M[G_M] = e^{i\langle G_M, \pi_M \rangle_{L_M}}$.

We can finish the initialization of the Hamiltonian renormalization flow by defining

$$\begin{aligned} H_M^{(0)} &= H_{0,M}^{(0)} + V_M^{(0)}, \\ H_{0,M}^{(0)} &= \int_0^1 dx A_M^\dagger(x) [\omega_M^{(0)} \cdot A_M](x), \\ V_M &= \int_0^1 dx: V([2\omega_M^{(0)}]^{-1/2} \cdot [A_M + A_M^\dagger])(x):_M, \end{aligned} \quad (3.11)$$

where A_M is the operator representative of a_M and $\vdots_M$ denotes normal ordering of the $A_M, A_M^\dagger$.

IV. HAMILTONIAN RENORMALIZATION FLOW OF $P(\Phi)_2$

In the first subsection, we block the theory from the continuum in order to determine which fixed point family the renormalization flow should find. In the second subsection, we use projector maps P_M —as defined in Appendix B—to project from L to the L_M subspaces and then discretize the fields, work in the l_M spaces of square summable sequences, and compute the renormalization flow. In the two frameworks, we show that the renormalization flow of the initial family defined in the previous section is already at its fixed point. The reason for displaying the strictly equivalent flows in terms of projections and discretizations, respectively, is that the former is in the spirit of renormalization schemes outside a lattice context, while the latter emphasizes the traditional real space block spin interpretation of the renormalization flow.

A. Blocking from the continuum

Blocking from the continuum means to define a state $\omega_{F,M}$ and a Hamiltonian H_M out of the continuum state ω_F and Hamiltonian H defined in Sec. II via the formulas

$$\begin{aligned} \omega_{F,M}(w_M[F_M]w_M[G_M]) &:= \omega_F(w[F_M]w[G_M]), \\ \langle W_M[F_M]\Omega_M, H_M W_M[G_M] \rangle_{\mathcal{H}_M} &:= \langle W[F_M]\Omega, HW[G_M]\Omega \rangle_{\mathcal{H}}, \end{aligned} \quad (4.1)$$

where $(\rho_M, \mathcal{H}_M, \Omega_M)$ and $(\rho, \mathcal{H}, \Omega)$ are the GNS data of $\omega_{F,M}$ and ω_F , respectively, and we have denoted the operator representatives by capital letters, i.e., $W_M[F_M] = \rho_M(w_M[F_M])$ and $W[F] = \rho(w[F])$. On the right-hand side of (4.1), the elements $F_M, G_M \in L_M$ are to be considered as elements of the continuum one-particle Hilbert space L by trivial embedding $F_M \mapsto F$.

We start with the first line of (4.1) and use (2.9),

$$\omega_F(w[F_M]w[G_M]) := e^{\frac{i}{2}\langle F_M, G_M \rangle_L} e^{-\frac{i}{4}[(F_M, \omega^{-1} \cdot F_M)_L + (G_M, \omega \cdot G_M)_L]}, \quad (4.2)$$

We have explicitly $\hat{\omega}_n = \sqrt{p^2 + [2\pi n]^2}$ using a resolution of identity with respect to the orthonormal basis (ONB) e_n of L ,

$$\begin{aligned} \langle F_M, \omega^{-1} \cdot F'_M \rangle_L &= \sum_{n \in \mathbb{Z}} \langle F_M, \omega^{-1} e_n \rangle_L \langle e_n, F'_M \rangle_L \\ &= \sum_{n \in \mathbb{Z}} \hat{\omega}_n^{-1} \langle F_M, e_n \rangle_L \langle e_n, F'_M \rangle_L \\ &= \sum_{n \in \mathbb{Z}_M} \hat{\omega}_n^{-1} \langle F_M, e_n \rangle_{L_M} \langle e_n, F'_M \rangle_{L_M} \\ &=: \langle F_M, \omega_M^{-1} \cdot F'_M \rangle_{L_M}, \end{aligned} \quad (4.3)$$

where we used that F_M, F'_M are orthogonal to the $e_n, n \notin \mathbb{Z}_M$. To interpret the operator ω_M on L_M , we explicitly compute

$$\begin{aligned} \partial_M \cdot F_M &= \sum_{n \in \mathbb{Z}_M} e_n \langle e_n, \partial_M \cdot F_M \rangle_{L_M} = \sum_{n \in \mathbb{Z}_M} e_n \langle e_n, \partial \cdot F_M \rangle_{L_M} \\ &= - \sum_{n \in \mathbb{Z}_M} e_n \langle \partial \cdot e_n, F_M \rangle_{L_M} \\ &= \sum_{n \in \mathbb{Z}_M} [2\pi i n] e_n \langle e_n, F_M \rangle_{L_M}, \end{aligned} \quad (4.4)$$

hence

$$\begin{aligned} \langle W[F_M]\Omega, V_k(l)W[F'_M]\Omega \rangle_{\mathcal{H}} &= i^{l-(k-l)} \int dx ([\omega^{-1/2} \cdot F_M](x))^{k-l} ([\omega^{-1/2} \cdot F'_M](x))^l \langle W[F_M]\Omega, W[F'_M]\Omega \rangle_{\mathcal{H}} \\ &= i^{l-(k-l)} \int dx ([\omega_M^{-1/2} \cdot F_M](x))^{k-l} ([\omega_M^{-1/2} \cdot F'_M](x))^l \langle W_M[F_M]\Omega_M, W_M[F'_M]\Omega_M \rangle_{\mathcal{H}_M} \\ &= \langle W_M[F_M]\Omega_M, [V_k(l)]_M W_M[F'_M]\Omega_M \rangle_{\mathcal{H}_M}, \end{aligned} \quad (4.9)$$

where

$$[p^2 - \Delta_M] \cdot F_M = \sum_{n \in \mathbb{Z}_M} \hat{\omega}_n^2 e_n \langle e_n, F_M \rangle_{L_M}. \quad (4.5)$$

It follows that $\omega_M = \sqrt{p^2 - \Delta_M}$, which is, in fact, identical to the natural choice $\omega_M^{(0)}$ of Sec. III used to define the initial family of states. Concluding, we find that $\omega_{F,M}$ coincides with $\omega_{F,M}^{(0)}$ defined in (3.3).

Considering the second line of (4.1), we use the elementary identity

$$\begin{aligned} A(x)W[F] &= W[F]W[F]^{-1}A(x)W[F] \\ &= W[F](A(x) - i([\langle 2\omega \rangle^{-1/2} \cdot F, A + A^\dagger]_L, A(x))) \\ &= W[F](A(x) + i([\langle 2\omega \rangle^{-1/2} \cdot F](x)1_{\mathcal{H}_M})) \end{aligned} \quad (4.6)$$

to find for the free part

$$\begin{aligned} \langle W[F_M]\Omega, H_0 W[F'_M]\Omega \rangle_{\mathcal{H}} &= \int dx \int dy \omega(x, y) \langle A(x)W[F_M]\Omega, A(y)W[F'_M]\Omega \rangle_{\mathcal{H}} \\ &= \int dx \int dy \omega(x, y) ([\langle 2\omega \rangle^{-1/2} \cdot F_M](x) ([\langle 2\omega \rangle^{-1/2} \cdot F'_M](y) \langle W[F_M]\Omega, W[F'_M]\Omega \rangle_{\mathcal{H}} \\ &= \frac{1}{2} \langle \omega^{-1/2} \cdot F_M, \omega \cdot \omega^{-1/2} \cdot F'_M \rangle_L \langle W_M[F_M]\Omega_M, W_M[F'_M]\Omega_M \rangle_{\mathcal{H}_M} \\ &= \frac{1}{2} \langle F_M, F'_M \rangle_{L_M} \langle W_M[F_M]\Omega_M, W_M[F'_M]\Omega_M \rangle_{\mathcal{H}_M}, \end{aligned} \quad (4.7)$$

where $\omega(x, y) = \sum_{n \in \mathbb{Z}} \hat{\omega}_n e_n(x - y)$ is the integral kernel of ω and in the step before the last one we used that we already know that the GNS data of $\omega_{F,M}$ are those of $\omega_{F,M}^{(0)}$.

Accordingly, $H_M = H_M^{(0)}$ is defined in (3.11). For the interacting part, it suffices to consider a field monomial of order k ,

$$\begin{aligned} V_k &= \int dx : \{ ([\langle 2\omega \rangle^{-1/2} \cdot [A + A^\dagger]](x)) \}^k \\ &:= \sum_{l=0}^k \binom{k}{l} 2^{-k/2} V_k(l), \\ V_k(l) &= \int dx ([\omega^{-1/2} \cdot A]^\dagger(x))^{k-l} ([\omega^{-1/2} \cdot A](x))^l. \end{aligned} \quad (4.8)$$

Using (4.6), it follows that

$$[V_k(l)]_M = \int dx ([\omega_M^{-1/2} \cdot A_M]^\dagger(x))^{k-l} (\omega_M^{-1/2} \cdot A_M)(x))^l. \quad (4.10)$$

It follows that the initial family defined in Sec. III coincides with the family blocked from the continuum. The reason for this is simply the identity

$$\omega \cdot P_M = P_M \cdot \omega_M, \quad (4.11)$$

which is because the derivative operator ∂ preserves the subspace L_M . This particular feature of the Dirichlet kernel is not shared by most other kernels, such as the Schwarz kernel, and in that case the initial family of Sec. III does not coincide with the family blocked from the continuum of Sec. IV A even for free theories; see [5–8].

B. Renormalization flow in terms of the projected fields

By definition, for $M < M'$ we consider the trivial embedding $L_M \rightarrow L_{M'}$; $F_M \mapsto F_{M'}$ which is possible because the L_M are nested subspaces of L . Then the analog of (4.1) is given by

$$\begin{aligned} & \omega_{F,M}^{(n+1)} (w_M[F_M] w_M[G_M]) \\ & := \omega_{F,M'}^{(n)} (w_{M'}[F_M] w_{M'}[G_M]), \\ & \langle w_M[F_M] \Omega_M^{(n+1)}, H_M^{(n+1)} w_M[F_M] \Omega_M^{(n+1)} \rangle_{\mathcal{H}_M^{(n+1)}} \\ & := \langle w_{M'}[F_M] \Omega_{M'}^{(n)}, H_{M'}^{(n)} w_{M'}[F_M] \Omega_{M'}^{(n)} \rangle_{\mathcal{H}_{M'}^{(n)}}, \end{aligned} \quad (4.12)$$

where $(\rho_M^{(n+1)}, \mathcal{H}_M^{(n+1)}, \Omega_M^{(n+1)})$ and $(\rho_{M'}^{(n)}, \mathcal{H}_{M'}^{(n)}, \Omega_{M'}^{(n)})$ are the GNS data of $\omega_{F,M}^{(n+1)}$ and $\omega_{F,M'}^{(n)}$, respectively, and we have denoted the operator representatives by capital letters, i.e., $W_M^{(n+1)}[F_M] = \rho_M^{(n+1)}(w_M[F_M])$ and $W_{M'}[F_M] = \rho_{M'}^{(n)}(w_{M'}[F_M])$. On the right-hand side of (4.12), the elements F_M, G_M, F'_M are to be considered as elements of the continuum one-particle Hilbert space $L_{M'}$ by trivial embedding $F_M \mapsto F_{M'}$.

Going through literally the same steps as in Sec. IV A, we find that the flow is trivial: $\omega_{F,M}^{(n)} = \omega_{F,M}^{(0)} = \omega_{F,M}^* = \omega_{F,M}$ and $H_M^{(n)} = H_M^{(0)} = H_M^* = H_M$; that is, every sequence element of the family coincides with the initial element, which is also the element blocked from the continuum and thus the fixed point element. This is again due to (for $M' > M$)

$$\omega_{M'} \cdot P_M = P_M \cdot \omega_M. \quad (4.13)$$

C. Renormalization flow in terms of discretized fields

In order to test the version of Hamiltonian renormalization method described in this series of papers [4–11] less trivially for the solvable and self-interacting $P(\Phi)_2$ theory,

one therefore should use an initial family that deviates from the natural choice of Sec. III. For example, instead of the position nonlocal momentum ONB e_n of L_M , $n \in \mathbb{Z}_M$ one may use the position quasilocal ONB χ_m^M , $m \in \mathbb{N}_M = \{0, 1, \dots, M-1\}$ of L_M constructed in Appendix B; see (B7) and (B8). Indeed, we can alternatively write (4.10) as

$$[V_k(l)]_M = M^{-k} \sum_{m_1, \dots, m_k \in \mathbb{N}_M} g_{M;m_1, m_2, \dots, m_k} \prod_{s=1}^{k-l} [\omega_M^{-1} \cdot A_M](m_s) \cdot A_M]^*(m_s), \quad (4.14)$$

where the “coupling constant” is given by

$$\begin{aligned} & g_{M;m_1, m_2, \dots, m_k} \\ & = \int_0^1 dx \prod_{s=1}^k \chi_{m_s}^M(x) \\ & = \sum_{|n_1|, \dots, |n_k| \leq (M-1)/2} \delta_{n_1 + \dots + n_k = 0} \prod_{s=1}^k e^{2\pi i n_k x_{m_s}^M} \\ & = \sum_{|n_1|, \dots, |n_{k-1}|, |n_1 + \dots + n_{k-1}| \leq (M-1)/2} \prod_{s=1}^{k-1} e^{2\pi i n_k (x_{m_s}^M - x_{m_k}^M)} \end{aligned} \quad (4.15)$$

with $x_m^M = m/M$. If it was not for the constraint $|n_1 + \dots + n_{k-1}| \leq (M-1)/2$, this would collapse to

$$\prod_{s=1}^{k-1} \chi_{m_s}^M(x_{m_k}^M) = M^{k-1} \prod_{s=1}^{k-1} \delta_{m_s, m_k} \quad (4.16)$$

and we have

$$[\omega_M^{-1/2} \cdot A_M](m) = \langle \chi_m^M, \omega_M^{-1/2} \cdot A_M \rangle_{L_M} = \langle \chi_m^M, \omega^{-1/2} A \rangle_{L_M}, \quad (4.17)$$

because $\chi_m^M \in L_M$ and due to (4.11). Then (4.14) could be considered as the k -fold Riemann sum of the coupling constant times the displayed polynomial of discretized annihilation and creation operators as $\epsilon_M = x_{m+1}^M - x_m^M = M^{-1}$.

However, due to the constraint, the coupling constant (4.15) is only quasilocal and it is for (4.15) that the flow is at its fixed point and not for (4.16). However, one could start from the naive guess (4.16)

$$g_{M;m_1, m_2, \dots, m_k}^{(0)} = M^{k-1} \prod_{s=1}^{k-1} \delta_{m_s, m_k}, \quad (4.18)$$

and then run the renormalization flow. Already after the first iterations step one finds that the form (4.18) is not preserved and that $g_{M;m_1, m_2, \dots, m_k}^{(1)}$ takes a more general form. Starting from this more general form, one sees that (4.15) is a fixed point of the corresponding renormalization flow equation.

In more detail, we make the general ansatz

$$[V_k^{(r)}(l)]_M = M^{-k} \sum_{m_1, \dots, m_k \in \mathbb{N}_M} g_{M;m_1, m_2, \dots, m_k}^{(r)} \prod_{s=1}^{k-l} [\omega_M^{-1} \cdot A_M]^*(m_s) \prod_{s=k-l+1}^k [\omega_M^{-1} \cdot A_M](m_s), \quad (4.19)$$

for the r th renormalization step with initial condition (4.18). Then, due to $\omega_{3M}^{-1} I_{M3M} = I_{M3M} \omega_{3M}^{-1}$ sandwiching $[V_k^{(r)}(l)]_{3M}$ between states of the form $w_{3M}[I_{M3M} f_M] \Omega_{3M}$ and requiring this to be $[V_k^{(r+1)}(l)]_M$, we see that we obtain

$$[V_k^{(r+1)}(l)]_M = (3M)^{-k} \sum_{m'_1, \dots, m'_k \in \mathbb{N}_{3M}} g_{3M;m'_1, m'_2, \dots, m'_k}^{(r)} \prod_{s=1}^{k-l} [I_{M3M} \omega_M^{-1} \cdot A_M]^*(m'_s) \prod_{s=k-l+1}^k [I_{M3M} \omega_M^{-1} \cdot A_M](m'_s), \quad (4.20)$$

which produces the flow equation

$$g_{M;m_1, \dots, m_k}^{(r+1)} = 3^{-k} \sum_{m'_1, \dots, m'_k} \left[\prod_{s=1}^k I_{M3M}(m'_s, m_s) \right] g_{3M;m'_1, \dots, m'_k}^{(r)}. \quad (4.21)$$

It is easy to see that (4.21) has (4.15) as a fixed point because

$$\begin{aligned} \sum_{m'} I_{M3M}(m', m) \chi_{m'}^{3M}(x) &= M^{-1} \sum_{m'} \langle \chi_{m'}^{3M}, \chi_m^M \rangle \chi_{m'}^{3M}(x) \\ &= 3(P_{3M} \chi_m^M)(x) = 3\chi_m^M(x), \end{aligned} \quad (4.22)$$

where completeness $\sum_m \chi_m^M(x) \chi_m^M(y) = M P_M(x, y)$ and $\chi_m^M \in L_M \subset L_{3M}$ was used.

We now use (4.16) as the initial condition for the flow defined by (4.21). We need for $m' \in \mathbb{N}_{3M}, m \in \mathbb{N}_M, n' \in \mathbb{Z}_{3M}, n \in \mathbb{Z}_M$,

$$\begin{aligned} \langle \chi_{m'}^{3M}, \chi_m^M \rangle &= \sum_{n, n'} e^{2\pi i [n' x_{m'}^{3M} - n x_m^M]} < e^{2\pi i n'}, e^{2\pi i n} \\ &= \sum_n e^{2\pi i n [x_{m'}^{3M} - n x_m^M]} \end{aligned} \quad (4.23)$$

since $\mathbb{Z}_M \subset \mathbb{Z}_{M'}$. Thus, explicitly

$$\begin{aligned} g_{M;m_1, \dots, m_k}^{(1)} &= \frac{1}{(3M)^k} \sum_{m' \in \mathbb{N}_{3M}^k} \prod_{s=1}^k \langle \chi_{m'_s}^{3M}, \chi_{m_s}^M \rangle g_{3M;m'_1, \dots, m'_k}^{(0)} \\ &= \frac{1}{3M} \sum_{m' \in \mathbb{N}_{3M}} \prod_{s=1}^k \langle \chi_{m'_s}^{3M}, \chi_{m_s}^M \rangle \\ &= \frac{1}{3M} \sum_{n \in \mathbb{Z}_M^k} \sum_{m' \in \mathbb{N}_{3M}} \prod_{s=1}^k e^{2\pi i n_s [x_{m'_s}^{3M} - x_{m_s}^M]} \\ &= \sum_{n \in \mathbb{Z}_M^k} \left[\prod_{s=1}^k e^{-2\pi i n_s x_{m_s}^M} \right] \delta_{n_1 + \dots + n_k, 0 \pmod{3M}}, \end{aligned} \quad (4.24)$$

where in the last step we took care of the fact that summing over $m' \in \mathbb{N}_{3M}$ produces $3M$ when $n_1 + \dots + n_k$ is an integer multiple of $3M$ and zero else. Comparing with (4.15), we see that the difference between the two expressions consists in the modulo $3M$ support of the Kronecker symbol because we can relabel $n_s \rightarrow -n_s$ as both sums are over the reflection invariant domain $\mathbb{Z}_M$.

At the next iteration step, we find

$$\begin{aligned} g_{M;m_1, \dots, m_k}^{(2)} &= \frac{1}{(3M)^k} \sum_{m' \in \mathbb{N}_{3M}^k} \prod_{s=1}^k \langle \chi_{m'_s}^{3M}, \chi_{m_s}^M \rangle g_{3M;m'_1, \dots, m'_k}^{(1)} \\ &= \frac{1}{(3M)^k} \sum_{n' \in \mathbb{Z}_{3M}^k} \sum_{m' \in \mathbb{N}_{3M}^k} \left[\prod_{s=1}^k \langle \chi_{m'_s}^{3M}, \chi_{m_s}^M \rangle e^{-2\pi i n'_s x_{m'_s}^{3M}} \right] \delta_{n'_1 + \dots + n'_k, 0 \pmod{3(3M)}} \\ &= \frac{1}{(3M)^k} \sum_{n' \in \mathbb{Z}_{3M}^k} \sum_{n \in \mathbb{Z}_M^k} \sum_{m' \in \mathbb{N}_{3M}^k} \left[\prod_{s=1}^k e^{-2\pi i n_s x_{m'_s}^M} e^{2\pi i (n_s - n'_s) x_{m'_s}^{3M}} \right] \delta_{n'_1 + \dots + n'_k, 0 \pmod{3^2 M}} \\ &= \sum_{n' \in \mathbb{Z}_{3M}^k} \sum_{n \in \mathbb{Z}_M^k} \left[\prod_{s=1}^k e^{-2\pi i n_s x_{m'_s}^M} \delta_{n_s, n'_s} \right] \delta_{n'_1 + \dots + n'_k, 0 \pmod{3^2 M}} \\ &= \sum_{n \in \mathbb{Z}_M^k} \left[\prod_{s=1}^k e^{-2\pi i n_s x_{m_s}^M} \right] \delta_{n_1 + \dots + n_k, 0 \pmod{3^2 M}}, \end{aligned} \quad (4.25)$$

where we used that $\delta_{n'-n,0(\bmod 3M)} = \delta_{n'-n,0}$ as $|n' - n| \leq \frac{3M-1+M-1}{2} < 2M < 3M$, i.e., the modulo operation can be dropped. We see that in the second iteration step the only change that happened compared to the first is that the modulo operation is now with respect to 3^2M rather than 3^1M .

Iterating (formally one proceeds by induction) we find explicitly for $r \geq 1$,

$$g_{M;m_1,\dots,m_k}^{(r)} = \sum_{n \in \mathbb{Z}_M^k} \left[\prod_{s=1}^k e^{-2\pi i n_s x_s^M} \right] \delta_{n_1+\dots+n_k,0 \pmod{3^r M}}. \quad (4.26)$$

Now $|n_1 + \dots + n_k| \leq k(M-1)/2$. For fixed resolution M , the number of steps r required such that the modulo operation can be dropped is when that estimate is lower than the smallest possible nonzero integer multiple of $3^r M$, i.e., $k(M-1)/2 < 3^r M$, i.e., $3^r > k \frac{M-1}{2M}$. Thus, independent of M for a polynomial potential of degree k , we need at most $r_k := 1 + \lceil \frac{\ln(k/2)}{\ln(3)} \rceil$ renormalization steps until $g_{M;m_1,\dots,m_k}^{(r)} = g_{M;m_1,\dots,m_k}$ for all $r \geq r_k$, where $\lceil \cdot \rceil$ denotes the Gauss bracket. Note that this quick convergence of the flow would not occur had we not defined $\omega_M^2 = p^2 - \Delta_M$,

$\Delta_M = \partial_M^2$, $\partial_M = I_M^\dagger \partial I_M$ but instead had replaced ∂_M by some of the more common discrete lattice derivatives {e.g., forward derivative $(\partial_M^+ f_M)(m) = M[f_M(m+1) - f_M(m)]$ } because then the intertwining property $I_{M3M} \cdot \partial_M = \partial_{3M} \cdot I_{M3M}$ would not hold.

We end this section by illustrating graphically the degree of nonlocality of the fixed point coupling (4.15) for the case $k = 3$. Explicitly, we consider

$$f_M(y_1, y_2) := M^{-2} \sum_{|n_1|, |n_2|, |n_1+n_2| < (M-1)/2} e^{2i(n_1 y_1 + n_2 y_2)}, \quad (4.27)$$

which yields

$$f(y_1 = \pi[x_{m_1}^M - x_{m_3}^M], y_2 = \pi[x_{m_2}^M - x_{m_3}^M]) = M^{-2} g_{M;m_1,m_2,m_3}, \quad (4.28)$$

which we expect to be a quasilocal version of $\delta_{m_1,m_3} \delta_{m_2,m_3}$. The sums in (4.27) can be computed in closed form using the geometric series summation formula $\sum_{n=0}^{N-1} z^n = (z^N - 1)/(z - 1)$. We subdivide the summation domain into the three sets

$$\begin{aligned} & \{n_1 = \left\{1, \dots, \frac{M-1}{2}\right\}, n_2 = \left\{-\frac{M-1}{2}, \dots, \frac{M-1}{2} - n_1\right\}\} \\ & \cup \{n_1 = \left\{-1, \dots, -\frac{M-1}{2}\right\}, n_2 = \left\{-\frac{M-1}{2} - n_1, \dots, \frac{M-1}{2}\right\}\} \\ & \cup \{n_1 = \{0\}, n_2 = \left\{-\frac{M-1}{2}, \dots, \frac{M-1}{2}\right\}\}. \end{aligned} \quad (4.29)$$

One finds

$$f(y_1, y_2) = \frac{1}{2M^2} \left\{ \frac{\cos(M(y_1 - y_2))}{\sin(y_1) \sin(y_2)} + \frac{\cos(My_2)}{\sin(y_1) \sin(y_1 - y_2)} - \frac{\cos(My_1)}{\sin(y_2) \sin(y_1 - y_2)} \right\}. \quad (4.30)$$

Despite its appearance, this function is everywhere smooth and bounded. Using $z_j = x_k - x_l$, $\epsilon_{jkl} = 1$ with $y_1 = x_1 - x_3$, $y_2 = x_2 - x_3$, one can write this in the manifestly permutation invariant form

$$f(z_1, z_2, z_3) = -\frac{1}{2M^2} \left\{ \frac{\cos(Mz_1)}{\sin(z_2) \sin(z_3)} + \frac{\cos(Mz_2)}{\sin(z_3) \sin(z_1)} + \frac{\cos(Mz_3)}{\sin(z_1) \sin(z_2)} \right\}, \quad (4.31)$$

noticing the identity $z_1 + z_2 + z_3 = 0$ that displays the totally symmetric, smooth, and bounded form (4.15) of this

function. For any $z_j = 0 \Rightarrow z_k = -z_l = z$, $\epsilon_{jkl} = 1$, it reduces to

$$\tilde{f}(z) = \frac{1}{2M^2 \sin^2(z)} \{M \sin(Mz) \sin(z) + 1 - \cos(Mz) \cos(z)\}, \quad (4.32)$$

as one can confirm by using de l'Hôpital's theorem. Using l'Hôpital's theorem one more time when taking $z \rightarrow 0$, one can explicitly check that the maximum of this function is given not by unity but by $f(0,0) = \tilde{f}(0) = \frac{3}{4} + \frac{1}{4M^2}$, corresponding to the fact that the constraints $|n_1 + n_2| \leq (M-1)/2$ on n_1, n_2 roughly delete one quarter of the unconstrained M^2 points. Away from the maximum, the function is of order M^{-2} at a generic lattice point, while it is of order M^{-1} on one of the coordinate axes and on the diagonal $y_1 = y_2$ (these are precisely the points $z_j = 0$, $j = 1, 2, 3$), but well away from the origin. This can be seen analytically noticing that $\cos(Mz_j) = (-1)^{m_k - m_l}$, $\epsilon_{jkl} = 1$, while $\sin(Mz_j) = 0$. Accordingly, if all $m_j - m_k$ are of the

order $(M-1)/2$, the sin functions in the denominator of f are close to unity. If, on the other hand, say, $m_2 = m_3$ corresponding to $z_1 = z_2 = 0, -z := z_3 = -z_2 = \pi(m_1 - m_2)/M =: m/M$, one finds $\tilde{f}(z) = \frac{1 - (-1)^m \cos(m\pi/M)}{2M^2 \sin^2(m\pi/M)}$ which for m odd exceeds $\frac{1}{2M^2 \sin^2(m\pi/M)}$ and for large M decreases quasicontinuously from approximately $[2\pi^2]^{-1}$ to $[2M^2]^{-1}$ between $m = 1$ and $m = (M-1)/2$. It is of order $\pi/(2M)$ for $m \approx \sqrt{M/\pi}$, which is well away from the vicinity of the origin but for large M is outside of approximately the fraction $\sqrt{M}/M = 1/\sqrt{M}$ of the available points.

In Figs. 1–3, we illustrate this quasilocality graphically for a few fixed resolutions $M = 11, 21, 51, 71, 111$, where we pick $m_3 = 5, 10, 25, 35, 55$, respectively, while $m_1, m_2 \in \mathbb{N}_M$ take full range. We computed the quasilocal coupling at the exact lattice points and then let *Mathematica* interpolate between those values. One sees the effect that the maximum ≈ 0.75 becomes more pronounced and concentrated as the resolution increases, while the function continues to display nontrivial oscillations in the vicinity of the maximum and along the coordinate axes and the diagonal.

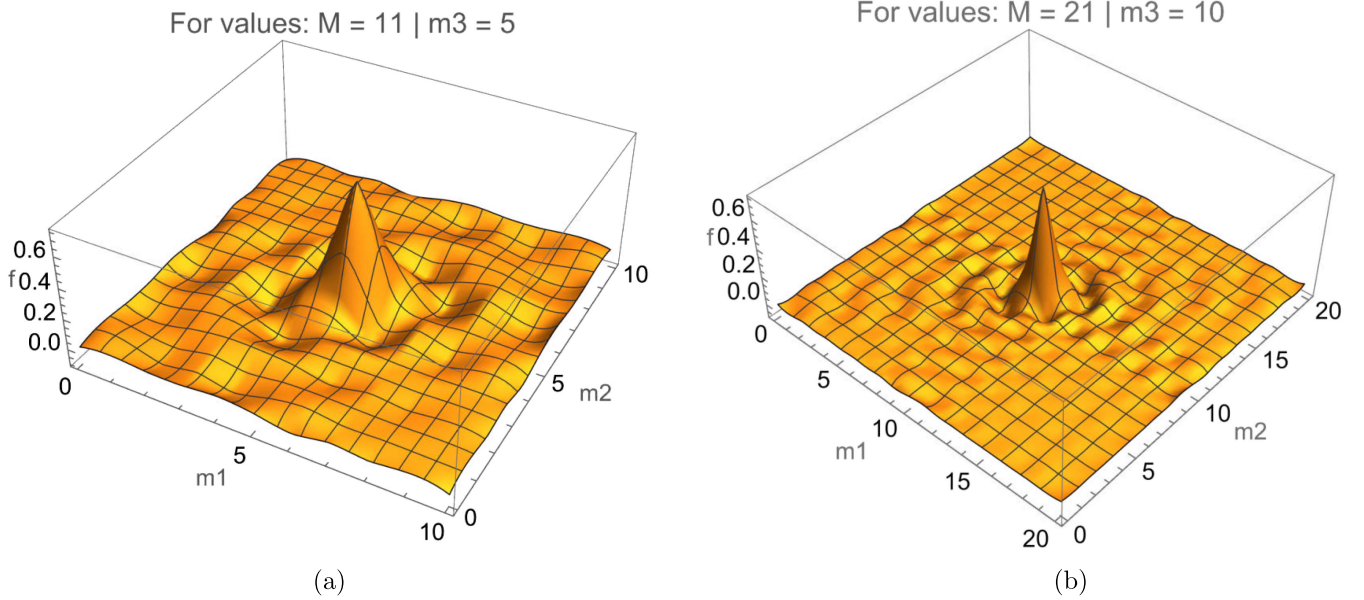


FIG. 1. (a) Coupling for $M = 11$ and $m_3 = 5$. (b) Coupling for $M = 21$ and $m_3 = 10$.

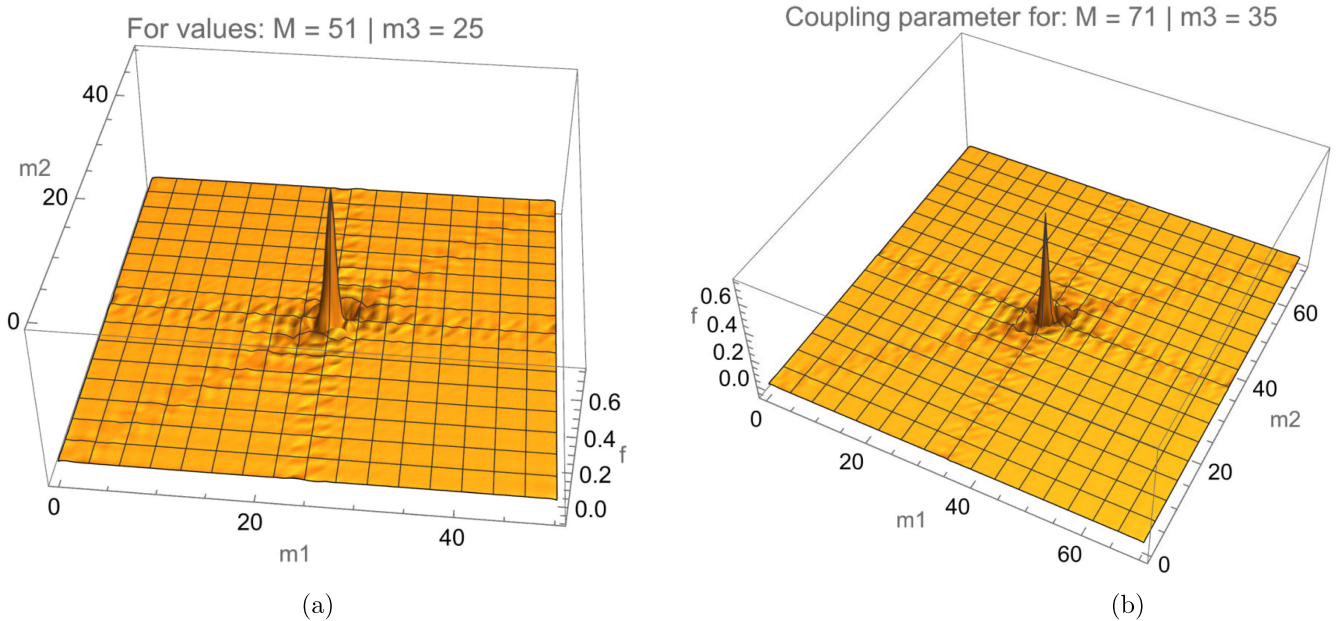
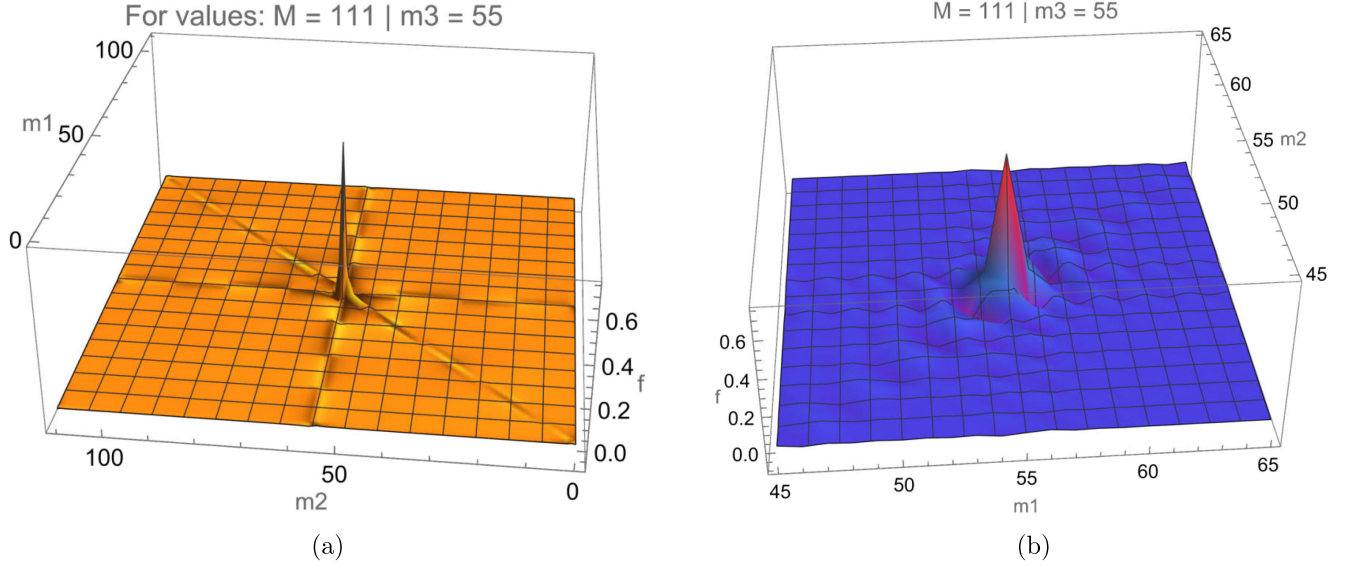
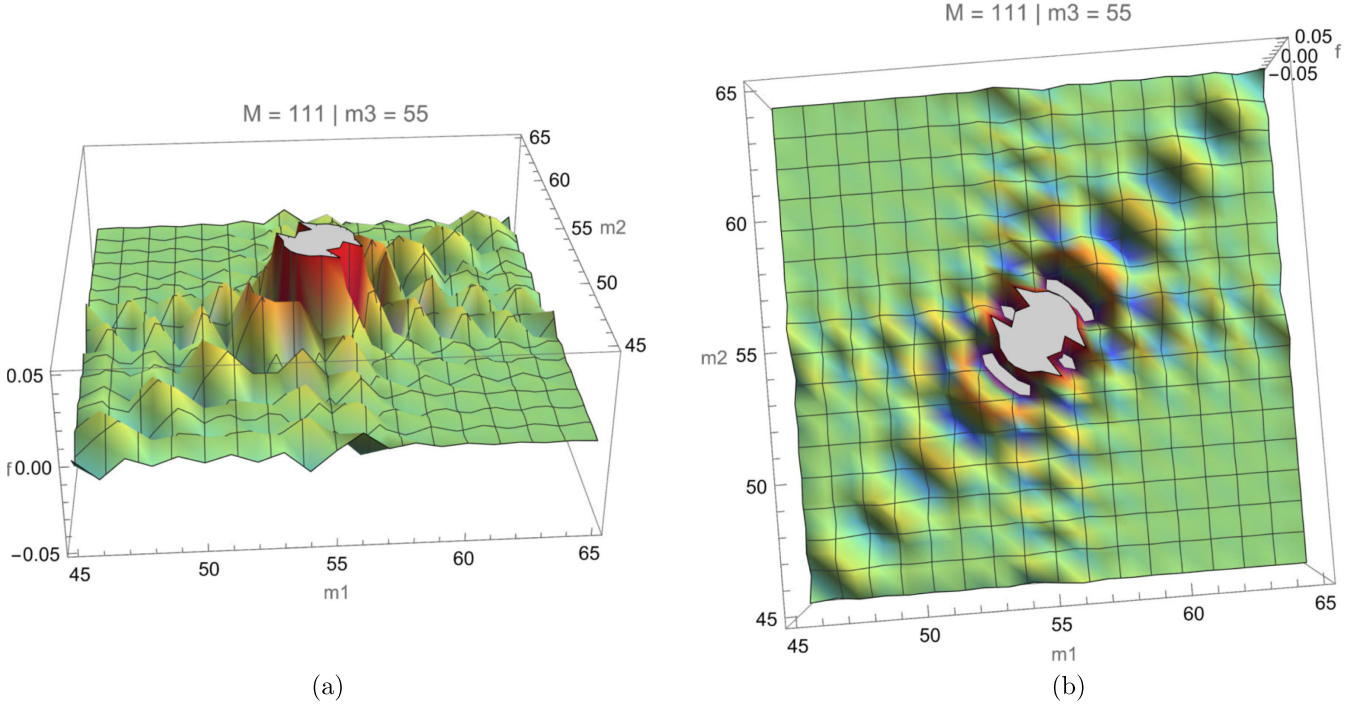


FIG. 2. (a) Coupling for $M = 51$ and $m_3 = 25$. (b) Coupling for $M = 71$ and $m_3 = 35$.

FIG. 3. Coupling for $M = 111$ and $m_3 = 55$. (a) Full range. (b) Vicinity of maximum.FIG. 4. High resolution in the vicinity of the maximum for coupling for $M = 111$ and $m_3 = 55$. (a) Side view. (b) Top view.

In Fig. 4, we focus on the $\sqrt{M}$ vicinity of the maximum at $m_1 = m_2 = m_3 = 55$ for $M = 111$. We cut off the maximum peak at a convenient value in order not to suppress the values of the coupling in its vicinity.

V. CONCLUSIONS

We have successfully applied the version of Hamiltonian renormalization developed in the current sequence of papers

for the first time to an interacting QFT in finite volume in $1 + 1$ spacetime dimensions and showed that the renormalization flow finds the known rigorous fixed point solution that is available in this case. The flow, in fact, stabilizes rather quickly if one uses as coarse graining tools not the familiar position localized block spin transformations but rather smooth versions thereof, which compromise between position and momentum locality and have better smoothness properties. This demonstrates that the tools developed in the

course of this series, which so far has focused on free QFT, are also successfully applicable in the interacting case and does find the correct continuum Hamiltonian operator.

Extrapolating to interacting QFT in higher, specifically, four spacetime dimensions, the Hamiltonian $H = H_0 + V$ when quantized in the Fock representation adapted to H_0 is known to be no longer an operator but merely a quadratic form. Following the steps that we have carried out for the present model to any interacting QFT with Hamiltonian $H = H_0 + V$ on the $D + 1$ manifold $\mathbb{R} \times T^D$ shows that the corresponding flow would indeed find that quadratic form as a fixed point in the Fock representation adapted to H_0 . However, ideally, renormalization is designed to actually construct a continuum theory with H as an operator rather than merely a quadratic form which by Haag's theorem [1] requires one to leave the realm of Fock representations adapted to H_0 . To the best of our knowledge, the single known example where this actually worked in higher than $1 + 1$ dimensions is Φ^4 theory in $2 + 1$ dimensions [17]. In addition to making the coupling constants g_k to depend on M (which in the present model was not necessary), a new and nontrivial step successfully performed in that seminal work was to invent invertible “dressing transformations” T_M at resolution M on the cutoff Fock space $\mathcal{H}_M$ with modes confined to $n \in \mathbb{Z}_M$ such that in $H'_M := T_M^{-1} H_M T_M$ those terms are removed that prevent H_M from being an operator rather than a quadratic form as $M \rightarrow \infty$ (these are normal ordered monomials in field operator-valued distributions with more than one creation operator). If that is the case, then $\mathcal{D}$ is an invariant domain for H'_M , where $\mathcal{D}$ is the span of Fock states in the free Fock space $\mathcal{H}$. Then H_M , defined on the new domain $\mathcal{D}'_M = T_M \mathcal{D}$, preserves $\mathcal{D}'_M$. The price to pay is that the norm of the vectors in $\mathcal{D}'_M$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ diverges as $M \rightarrow \infty$. In [17], it was shown that one can define a new inner product $\langle \cdot, \cdot \rangle'_M := \frac{\langle \cdot, \cdot \rangle}{\langle T_M \Omega, T_M \rangle}$ on $\mathcal{D}'_M$, where Ω is the Fock vacuum of H_0 , such that one can take $M \rightarrow \infty$ and $H'_M \rightarrow H'$ becomes an operator densely defined on $\mathcal{D}'_M \rightarrow \mathcal{D}'$ whose Hilbert space completion with respect to $\langle \cdot, \cdot \rangle'_M \rightarrow \langle \cdot, \cdot \rangle'$ results in a new Hilbert space $\mathcal{H}'$ for the interacting theory. In order for this to work, the norms of vectors $T_M \psi$, $\psi \in \mathcal{D}$ with respect to $\langle \cdot, \cdot \rangle$ must all diverge at the same rate. We expect that similar mechanisms must be invoked in the present renormalization scheme as well.

In a forthcoming work [18], we will apply the version of Hamiltonian renormalization developed in this series of papers to the self-interacting $U(1)^3$ model of the weak Newton constant realm of Euclidian signature vacuum quantum gravity in four spacetime dimensions [19]. This model has two known solutions, one in terms of operators [20] using a Narnhofer-Thirring type of representation [21] and one in terms of quadratic forms [22] in Fock representations [22] in the sense that the full algebra of quantum constraints (Gauss, spatial diffeomorphism, and Hamiltonian) closes without anomaly. We will investigate the Hamiltonian

renormalization flow for both types of representations with the aim to gain new insights into how to renormalize full quantum gravity, e.g., in the loop quantum gravity representation [23] in order to remove or reduce present quantization ambiguities.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: FINITENESS OF $P(\Phi)_2$ QFT IN FOCK REPRESENTATIONS

As motivated in Sec. II, we pick the Fock representation selected by the free part of the Hamiltonian H_0 . We want to show that the interaction part V is densely defined on $\mathcal{D}$, the linear span of Fock vectors with smearing functions in L_0 , and the L_2 functions with compact momentum support. Below we present an elementary proof that requires just first year's calculus knowledge.

To show this, consider a general Fock vector ψ . Then we have the elementary estimate

$$\|V\psi\|_{\mathcal{H}} \leq \frac{|g_k|}{2^{k/2}} \sum_{l=0}^k \binom{k}{l} \|V_k(l)\psi\|_{\mathcal{H}}, \quad (\text{A1})$$

and writing $\psi = \sum_{n=0}^N \psi_n$, where ψ_n is a Fock vector of particle number n , we have further the estimate

$$\|V_k(l)\psi\|_{\mathcal{H}} \leq \sum_{n=l}^N \|V_k(l)\psi_n\|_{\mathcal{H}}. \quad (\text{A2})$$

Here we used that $V_k(l)$ annihilates l particles and thus the rhs of (A2) vanishes for $N < l$. Let then $n \geq l$ and $\psi_0 = c\Omega$, $c = \text{const}$ or $\psi_n = \prod_{r=1}^n [\langle f_r, A \rangle]^\dagger \Omega$, $n > 0$. Then for $l \geq 1$ we have

$$V_k(l)\psi_n = \sum_{I \in S_l} \int_0^1 dx [\omega^{-1/2} \cdot A^*](x)^{k-l} \prod_{i \in I} [\omega^{-1/2} \cdot f_i](x) \prod_{j \notin I} \langle f_j, A \rangle^* \Omega, \quad (\text{A3})$$

where S_l is the set of subsets of $\{1, \dots, n\}$ with l elements and for $l = 0$ we have

$$V_k(0)\psi_n = \int_0^1 dx [\omega^{-1/2} \cdot A^*](x)^k \psi_n. \quad (\text{A4})$$

Using further elementary estimates, it follows that V is densely defined on $\mathcal{D}$ if and only if objects of the form

$$\int_0^1 dx F_l(x) [[\omega^{-1/2} \cdot A^*](x)]^{k-l} \psi_{n-l} \quad (\text{A5})$$

are normalizable vectors in the Fock space where F_l is either the constant function equal to unity for $l = 0$ or a product of l functions in L_0 for $l > 0$. It is understood that $k, n \geq l$ and $k = 0, \dots, N$. For $k = 0$ and thus $l = 0$, there is nothing to show as (A5) just equals $\langle 1, 1 \rangle \psi_n$, which is finite as $1 \in L$ when space is compact. For $k = 1$, (A5) equals $\langle 1, A \rangle^* \psi_n$ and $\langle 1, F_1 \rangle \psi_{n-1}$, respectively, which are both Fock vectors. The interesting terms therefore come from $k \geq 2$.

The norm squared of (A5) is

$$\int_0^1 dx \int_0^1 dy F_l^*(x) F_l(y) < \psi_{n-l}, \\ [[\omega^{-1/2} \cdot A](x)]^{k-l} [[\omega^{-1/2} \cdot A^*](y)]^{k-l} \psi_{n-l}. \quad (\text{A6})$$

When moving the annihilators to the right, we get a sum of terms with $0 \leq r \leq k - l$ factors of

$$[(\omega^{-1/2} \cdot A)(x), (\omega^{-1/2} \cdot A^*)(y)] \\ = \omega^{-1}(x, y) = \sum_{n \in \mathbb{Z}} \hat{\omega}_n^{-1} e_n(x - y), \quad (\text{A7})$$

which is the integral kernel of the operator ω^{-1} and another $k - l - r$ factors of functions in L_0 (for $k - l - r < 0$ that term vanishes). Altogether we see that V is densely defined on $\mathcal{D}$ if and only if integrals of the form

$$\int_0^1 dx \int_0^1 dy F_{k-r}^*(x) G_{k-r}(y) [\omega^{-1}(x, y)]^r \quad (\text{A8})$$

converge with $2 \leq k \leq N$ and $0 \leq r \leq k$ with $F_{k-r}, G_{k-r} \in L_0$ when $k - r > 0$ and $F_0 = G_0 = 1$. Let $\hat{F}_n = \langle e_n, F \rangle$ be the Fourier transform of $F \in L$, then (A8)

is either

$$\sum_{K \in \mathbb{Z}} \hat{F}_{k-r}^*(K) \hat{G}_{k-r}(K) \sum_{n_1, \dots, n_r \in \mathbb{Z}} \delta_{n_1 + \dots + n_r, K} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \quad (\text{A9})$$

for $k - r > 0$ or

$$\sum_{n_1, \dots, n_r \in \mathbb{Z}} \delta_{n_1 + \dots + n_r, 0} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \quad (\text{A10})$$

for $k - r = 0$, where, again, compactness of space was important as the kernel is translation invariant. As functions in L_0 have compact momentum support, the sum over K in (A9) is finite and it is sufficient to show that the sums

$$\sum_{n_1, \dots, n_r \in \mathbb{Z}} \delta_{n_1 + \dots + n_r, K} \prod_{s=1}^r \hat{\omega}_{n_s} \quad (\text{A11})$$

converge for every $2 \leq r \leq N$ and any finite $K \in \mathbb{Z}$.

The intuitive reason why this is, in fact, the case is as follows: If it was not for the Kronecker symbol, the sums would decouple into a product of r sums each of which behaves as the $M \rightarrow \infty$ limit of $\zeta_M(1)$, which diverges as $\ln(M)$ and where $\zeta_M(z) = \sum_{n=1}^M n^{-z}$ is the M cutoff of the Riemann ζ function. However, due to the Kronecker symbol, we only get $r - 1$ sums over $n_1, \dots, n_{r-1}$ but still r factors, one depending on $|n_s|, s = 1, \dots, n_{r-1}$ and one depending on $|n_1 + \dots + n_{r-1} - K|$, which roughly adds an additional factor of $\hat{\omega}_{n_s}^{1/(r-1)}$. Thus, each of the $r - 1$ sums behave as $\zeta_M(1 + \frac{1}{r-1})$, which converges for every $r > 1$. Of course, this heuristic argument is not rigorous.

We therefore rigorously estimate the multiple sum in (A11) as follows: First, we simplify

$$\begin{aligned} \sum_{n_1, \dots, n_r \in \mathbb{Z}} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \delta_{n_1 + \dots + n_r, K} &\leq \sum_{n_1, \dots, n_{r-1} \in \mathbb{Z} \cup \{0\}} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \delta_{n_1 + \dots + n_r, K} \\ &= \sum_{\sigma_1, \dots, \sigma_r = \pm 1} \sum_{n_1, \dots, n_r \in \mathbb{N}_0} \prod_{s=1}^r \hat{\omega}_{\sigma_s n_s}^{-1} \delta_{\sigma_1 n_1 + \dots + \sigma_r n_r, K} \\ &= \sum_{\sigma_1, \dots, \sigma_r = \pm 1} \sum_{n_1, \dots, n_r \in \mathbb{N}_0} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \delta_{\sigma_1 n_1 + \dots + \sigma_r n_r, K} \\ &= \sum_{t=0}^r \binom{r}{t} \sum_{n_1, \dots, n_r \in \mathbb{N}_0} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \\ &\quad \times \{ \theta(K - 1) \delta_{[n_1 + \dots + n_t] - [n_{t+1} + \dots + n_r + |K|], 0} + \theta(-K - 1) \delta_{[n_1 + \dots + n_t + |K|] - [n_{t+1} + \dots + n_r], 0} + \delta_{K, 0} \delta_{[n_1 + \dots + n_t] - [n_{t+1} + \dots + n_r], 0} \}, \quad (\text{A12}) \end{aligned}$$

where we made use of reflection invariance $\hat{\omega}_n = \hat{\omega}_{-n}$ and that the product of the inverse eigenvalues is invariant under permutations of r summation variables $n_1, \dots, n_r$ so that the sum over $\sigma_1, \dots, \sigma_r$ can be reduced to a summation over the number $t = 0, 1, \dots, r$ of those $\sigma_1, \dots, \sigma_r$ which take the value $+$ which can be taken to be the $\sigma_1, \dots, \sigma_t$.

The first term vanishes when $t = 0$ and for $t = r$ the sum is constrained by $n_1, \dots, n_r \leq |K|$ as all summation variables are non-negative. Likewise, the second term vanishes for $t = r$, and for $t = 0$ the sum is constrained by $n_1, \dots, n_r \leq |K|$. Finally, the third term collapses to $n_1 = \dots = n_r = 0$ for $t = 0, t = r$. Thus, the terms with $t = 0, r$ are trivially finite and it is sufficient to consider only the terms $0 < t < r$. For those, in the second term we relabel $t \rightarrow r - t$ and switch the two groups $n_1, \dots, n_t$ and $n_{t+1}, \dots, n_r$. As $\theta(K - 1) + \theta(-K - 1) + \delta_{K,0} = 1$, we end up with the relevant contribution

$$\begin{aligned} S_r(K) &:= \sum_{t=1}^{r-1} \binom{r}{t} \sum_{n_1, \dots, n_r \in \mathbb{N}_0} \prod_{s=1}^r \hat{w}_{n_s}^{-1} \delta_{[n_1 + \dots + n_t] - [n_{t+1} + \dots + n_r + |K|], 0} \\ &= m^r \sum_{t=1}^{r-1} \binom{r}{t} \sum_{L=|K|}^{\infty} \left[\sum_{n_1, \dots, n_t \in \mathbb{N}_0} \prod_{s=1}^t w_{n_s}^{-1} \delta_{n_1 + \dots + n_t, L} \right] \left[\sum_{n_{t+1}, \dots, n_r \in \mathbb{N}_0} \prod_{s=t+1}^r w_{n_s}^{-1} \delta_{n_{t+1} + \dots + n_r, L - |K|} \right], \end{aligned} \quad (\text{A13})$$

where $\kappa := \frac{2\pi}{m}$ and $w_n = \sqrt{1 + [\kappa n]^2}$ and m is the mass parameter. Using the abbreviation for $1 \leq t \leq r - 1, M \geq 0$,

$$s_t(M) = \sum_{n_1, \dots, n_t \in \mathbb{N}_0} \prod_{s=1}^t w_{n_s}^{-1} \delta_{n_1 + \dots + n_t, M}, \quad (\text{A14})$$

which for $t = 1$ just equals w_M^{-1} , we have

$$S_r(K) = m^r \sum_{t=1}^{r-1} \binom{r}{t} \sum_{L=|K|}^{\infty} s_t(L) s_{r-t}(L - |K|). \quad (\text{A15})$$

The idea is now to show that $s_t(M)$ decays at least as $|M|^{-p}$ for some $p > 1/2$ no matter the value of t , for then the sum over L converges and the sum over t is anyway finite. For

$t = 1$ this is trivial; hence, in what follows, we consider $t \geq 2$.

To this end, we use the basic estimate

$$1 + x^2 \geq c(1 + |x|)^2 \Leftrightarrow \left[|x| - \frac{c}{1-c} m \right]^2 + \frac{1-2c}{(1-c)^2} m^2 \geq 0, \quad (\text{A16})$$

which holds for $0 < c \leq \frac{1}{2}$. We pick $c = \frac{1}{4}$ and thus

$$w_n^{-1} \leq \frac{2}{1 + \kappa n}. \quad (\text{A17})$$

It follows:

$$\begin{aligned} s_t(M) &\leq 2^t \sum_{n_1=0}^M [1 + \kappa n_1]^{-1} \sum_{n_2=0}^{M-n_1} [1 + \kappa n_2]^{-1} \dots \\ &\quad \sum_{n_{t-2}=0}^{M-[n_1 + \dots + n_{t-3}]} [1 + \kappa n_{t-2}]^{-1} \sum_{n_{t-1}=0}^{M-[n_1 + \dots + n_{t-2}]} [1 + \kappa n_{t-1}]^{-1} [1 + \kappa(M - [n_1 + \dots + n_{t-1}])]^{-1}. \end{aligned} \quad (\text{A18})$$

To estimate the sum over n_{t-1} we use the abbreviation $l = M - [n_1 + \dots + n_{t-2}]$ (which is defined to equal M when $t = 2$),

$$\begin{aligned} \sum_{n=0}^l [1 + \kappa n]^{-1} [1 + \kappa(l - n)]^{-1} &= [1 + \kappa l]^{-1} \sum_{n=0}^l ([1 + \kappa n]^{-1} + [1 + \kappa(l - n)]^{-1}) = \frac{2}{1 + \kappa l} \sum_{n=0}^l [1 + \kappa n]^{-1} \\ &= \frac{2}{1 + \kappa l} \left[1 + \sum_{n=1}^l [1 + \kappa n]^{-1} \right] = \frac{2}{1 + \kappa l} \left[1 + \sum_{n=1}^l \int_{n-1}^n dk [1 + \kappa n]^{-1} \right] \\ &\leq \frac{2}{1 + \kappa l} \left[1 + \sum_{n=1}^l \int_{n-1}^n dk [1 + \kappa k]^{-1} \right] \\ &= \frac{2}{1 + \kappa l} \left[1 + \int_0^l dk [1 + \kappa k]^{-1} \right] = \frac{2}{1 + \kappa l} [1 + \kappa^{-1} \ln(1 + \kappa l)]. \end{aligned} \quad (\text{A19})$$

Consider now any $0 < p < 1$ and note that $z := 1 + \kappa l \geq 1$, then

$$\frac{\ln(z)}{z} \leq z^{-p} (1 - p)^{-1}, \quad z^{-1} \leq z^{-p}. \quad (\text{A20})$$

Thus, there exists a constant c_1 (depending on p_1, κ) such that (A19) is bounded from above by $c_1[1 + \kappa l]^{-p_1}$ for any $0 < p_1 < 1$. It follows that

$$s_t(M) \leq 2^t c_1 \sum_{n_1=0}^M [1 + \kappa n_1]^{-1} \sum_{n_2=0}^{M-n_1} [1 + \kappa n_2]^{-1} \dots \sum_{n_{t-3}=0}^{M-[n_1+\dots+n_{t-4}]} [1 + \kappa n_{t-3}]^{-1} \sum_{n_{t-2}=0}^{M-[n_1+\dots+n_{t-3}]} [1 + \kappa n_{t-2}]^{-1} [1 + \kappa(M - [n_1 + \dots + n_{t-2}])]^{-p_1}. \quad (\text{A21})$$

To estimate the sum over n_{t-2} we use the abbreviation $l = M - [n_1 + \dots + n_{t-3}]$ (which is defined to equal M when $t = 3$) and use again $z^{-1} < z^{-p_1}$ for $z \geq 1$ to find

$$\sum_{n=0}^l [1 + \kappa n]^{-1} [1 + \kappa(l - n)]^{-p_1} \leq \sum_{n=0}^l [1 + \kappa n]^{-p_1} [1 + \kappa(l - n)]^{-p_1} \leq [1 + \kappa l]^{-p_1} \sum_{n=0}^l [[1 + \kappa n]^{-1} + [1 + \kappa(l - n)]^{-1}]^{p_1}. \quad (\text{A22})$$

For any two numbers $a, b > 0$ and $p > 0$, we have

$$[a + b]^p = \begin{pmatrix} a^p [1 + b/a]^p & a \geq b \\ b^p [1 + a/b]^p & b \geq a \end{pmatrix} \leq \begin{pmatrix} (2a)^p & a \geq b \\ (2b)^p & b \geq a \end{pmatrix} \leq 2^p [a^p + b^p]. \quad (\text{A23})$$

Therefore, (A22) can be further estimated by

$$\begin{aligned} 2^{p_1} [1 + \kappa l]^{-p_1} \sum_{n=0}^l \{ [1 + \kappa n]^{-p_1} + [1 + \kappa(l - n)]^{-p_1} \} &= 2^{1+p_1} [1 + \kappa l]^{-p_1} \sum_{n=0}^l [1 + \kappa n]^{-p_1} \\ &= \frac{2^{1+p_1}}{[1 + \kappa l]^{p_1}} \left[1 + \sum_{n=1}^l [1 + \kappa n]^{-p_1} \right] = \frac{2^{1+p_1}}{[1 + \kappa l]^{p_1}} \left[1 + \sum_{n=1}^l \int_{n-1}^n dk [1 + \kappa n]^{-p_1} \right] \\ &\leq \frac{2^{1+p_1}}{[1 + \kappa l]^{p_1}} \left[1 + \sum_{n=1}^l \int_{n-1}^n dk [1 + \kappa k]^{-p_1} \right] \\ &= \frac{2^{1+p_1}}{[1 + \kappa l]^{p_1}} \left[1 + \int_0^l dk [1 + \kappa k]^{-p_1} \right] = \frac{2^{1+p_1}}{[1 + \kappa l]^{p_1}} [1 + \kappa^{-1} (1 - p_1)^{-1} ([1 + \kappa l]^{1-p_1} - 1)] \\ &\leq \frac{2^{1+p_1}}{[1 + \kappa l]^{p_1}} \left[\left| 1 - \frac{1}{\kappa(1 - p_1)} \right| + \kappa^{-1} (1 - p_1)^{-1} [1 + \kappa l]^{1-p_1} \right]. \end{aligned} \quad (\text{A24})$$

We choose p_1 so close to 1 that $p_2 := 2p_1 - 1$ still obeys $0 < p_2 < 1$, which is the case for $1/2 < p_1 < 1$. Then $p_2 < p_1$ so that also $[1 + \kappa l]^{-p_1} < [1 + \kappa l]^{-p_2}$. Thus, there exists a constant c_2 (depending on p_1, κ) such that (A24) is bounded from above by $c_2[1 + \kappa l]^{-p_2}$ for any $1/2 < p_1 < 1$ with $p_2 = 2p_1 - 1$. It follows that

$$s_t(M) \leq 2^t c_1 c_2 \sum_{n_1=0}^M [1 + \kappa n_1]^{-1} \sum_{n_2=0}^{M-n_1} [1 + \kappa n_2]^{-1} \dots \sum_{n_{t-4}=0}^{M-[n_1+\dots+n_{t-5}]} [1 + \kappa n_{t-4}]^{-1} \sum_{n_{t-3}=0}^{M-[n_1+\dots+n_{t-4}]} [1 + \kappa n_{t-3}]^{-1} [1 + \kappa(M - [n_1 + \dots + n_{t-3}])]^{-p_2}. \quad (\text{A25})$$

Comparing (A21) and (A25) we see that we can now iterate and define a sequence $p_u = 2p_{u-1} - 1, u = 1, \dots, t-1$ of powers, provided we can keep them in the range $0 < p_u < 1$. The iteration is easily solved by

$$p_u = 2^{u-1} p_1 - (2^{u-1} - 1). \quad (\text{A26})$$

Pick $p_1 = 1 - 2^{-T}$ then

$$p_u = 1 - 2^{u-1-T}, \quad (\text{A27})$$

which satisfies $0 < p_u < 1$ for $u = 1, \dots, t-1$ if $T \geq t$. As we have to perform $t-1$ summations in $s_t(M)$, for the choice $T = t$ we have $p_{t-1} = 3/4$ and find that for some constant c_t we have

$$s_t(M) \leq c_t [1 + \kappa M]^{-3/4}. \quad (\text{A28})$$

Correspondingly,

$$S_r(K) \leq m^r \left[\sum_{t=1}^{r-1} \binom{r}{t} c_t c_{r-t} \right] \times \sum_{L=|K|}^{\infty} ([1 + \kappa L][1 + \kappa(L - |K|)])^{-3/4}. \quad (\text{A29})$$

Let $\kappa' = \min(1, \kappa)$ so that $1 + \kappa M \geq \kappa'(1 + M)$, then for some constant $C_{r,m}$,

$$S_r(K) \leq C_{r,m} \sum_{L=|K|}^{\infty} ([1 + L][1 + L - |K|])^{-3/4} \leq C_{r,m} \zeta\left(\frac{3}{2}\right), \quad (\text{A30})$$

where we introduced $L - |K| = 0, 1, 2, \dots$ as a new summation variable and estimated $1 + L + |K| \geq 1 + L$ so that the Riemann zeta $\zeta(z)$ appears, which converges for $\Re(z) > 1$ (this fact can, of course, also be proved elementarily using the above technique to estimate the sum by an integral). Note that the bound (A30) is, in fact, independent of K and can certainly be optimized if needed.

APPENDIX B: RENORMALIZATION TOOLS

More details on this appendix, in particular, the relation to wavelet theory [16], can be found in [15].

We work on spacetimes diffeomorphic to $\mathbb{R} \times \sigma$. In a first step, the spatial D manifold σ is compactified to T^D . Therefore, all constructions that follow have to be done directionwise for each copy of S^1 . On $X := S^1$, understood as $[0, 1)$ with end points identified, we consider the Hilbert space $L = L_2([0, 1), dx)$ with orthonormal basis

$$e_n(x) := e^{2\pi i n x}, n \in \mathbb{Z} \quad (\text{B1})$$

with respect to the inner product

$$\langle F, G \rangle_L := \int_0^1 dx \overline{F(x)} G(x). \quad (\text{B2})$$

Let $\mathbb{O} \subset \mathbb{N}$ be the set of positive odd integers. We equip $\mathbb{O}$ with a partial order, namely,

$$M < M' \Leftrightarrow \frac{M'}{M} \in \mathbb{N}. \quad (\text{B3})$$

Note that this is not a linear order, i.e., not all elements of $\mathbb{O}$ are in relation, but $\mathbb{O}$ is directed; that is, for each $M, M' \in \mathbb{O}$ we find $M'' \in \mathbb{O}$ such that $M, M' < M''$, e.g., $M'' = MM'$. For each $M \in \mathbb{O}$, called a resolution scale, we introduce the subsets $\mathbb{N}_M \subset \mathbb{N}_0, \mathbb{Z}_M \subset \mathbb{Z}, X_M \subset X$ of respective cardinality M defined by

$$\begin{aligned} \mathbb{N}_M &= \{0, 1, \dots, M-1\}, \\ \mathbb{Z}_M &= \left\{ -\frac{M-1}{2}, -\frac{M-1}{2} + 1, \dots, \frac{M-1}{2} \right\}, \\ X_M &= \left\{ x_m^M := \frac{m}{M}, m \in \mathbb{N}_M \right\}. \end{aligned} \quad (\text{B4})$$

It is easy to check that we have the lattice relation

$$X_M \subset X_{M'} \Leftrightarrow M < M'. \quad (\text{B5})$$

The subspace $L_M \subset L$ is defined by

$$L_M := \text{span}(\{e_n, n \in \mathbb{Z}_M\}). \quad (\text{B6})$$

On L_M we use the same inner product as on L , hence the $e_n, n \in \mathbb{Z}_M$ provide an ONB for L_M . An alternative basis for L_M is defined by the functions

$$\chi_m^M(x) := \sum_{n \in \mathbb{Z}_M} e_n(x - x_m^M). \quad (\text{B7})$$

The motivation to introduce these functions is that, in contrast to the plane waves e_n , they are (1) spatially concentrated at $x = x_m^M$ and (2) real valued. This makes them useful for renormalization purposes. In addition, in contrast to characteristic functions that have better spatial location properties, they are smooth. This is a crucial feature because quantum field theory involves products of derivatives of the fields and derivatives of characteristic functions yield δ distributions. More in general, renormalization tools must make a compromise between localization and smoothness.

The functions χ_m^M are still orthogonal, but not orthonormal,

$$\langle \chi_m^M, \chi_{\hat{m}}^M \rangle_{L_M} = M \delta_{m, \hat{m}}. \quad (\text{B8})$$

We choose not to normalize them in order to minimize the notational clutter in what follows. Let l_M be the space of square summable sequences $f_M = (f_{M,m})_{m \in \mathbb{N}_M}$ with M members and inner product given by

$$\langle f_M, g_M \rangle_{l_M} := \frac{1}{M} \sum_{m \in \mathbb{N}_M} \overline{f_{M,m}} g_{M,m}. \quad (\text{B9})$$

If we interpret $f_{M,m} = F(x_m^M)$, then (B9) is a lattice approximant of $\langle F, G \rangle_L$. We define

$$\begin{aligned} I_M: l_M &\rightarrow L_M; (I_M \cdot f_M)(x) := \langle \chi_m^M(x), f_M \rangle_{l_M} \\ &= \frac{1}{M} \sum_{m \in \mathbb{N}_M} f_{M,m} \chi_m^M(x). \end{aligned} \quad (\text{B10})$$

Its adjoint is defined by the requirement that

$$\langle I_M^* \cdot F_M, g_M \rangle_{l_M} = \langle F_M, I_M \cdot g_M \rangle_{L_M}, \quad (\text{B11})$$

which demonstrates

$$I_M^*: L_M \rightarrow l_M, (I_M^* \cdot F_M)_m = \langle \chi_m^M, F_M \rangle_{L_M}. \quad (\text{B12})$$

One easily checks, using (B8), that

$$I_M^* \cdot I_M = 1_{l_M}, \quad \langle I_M^* \cdot, I_M \cdot \rangle_{L_M} = \langle \cdot, \cdot \rangle_{l_M}, \quad (\text{B13})$$

which shows that L_M, l_M are in 1-1 correspondence and that I_M is an isometry. Likewise,

$$P_M := I_M \cdot I_M^* = 1_{L_M}. \quad (\text{B14})$$

We can consider I_M also as a map $I_M: l_M \rightarrow L$ with image L_M and then $\langle I_M^* \cdot, \cdot \rangle_{l_M} = \langle \cdot, I_M \rangle_L$ shows that $I_M^*: L \rightarrow l_M$ is given by the same formula (B12) with $F \in L$, but now $P_M: L \rightarrow L_M$ is an orthogonal projection

$$P_M \cdot P_M = P_M, \quad P_M^* = P_M. \quad (\text{B15})$$

We have, explicitly,

$$\begin{aligned} (P_M \cdot F)(x) &= \int_0^1 dy P_M(x, y) F(y), \quad P_M(x, y) \\ &= \sum_{n \in \mathbb{Z}_M} e_n(x - y), \end{aligned} \quad (\text{B16})$$

i.e., $P_M(x, y)$ is the M cutoff of the δ distribution on X , i.e., modes $|n| > \frac{M-1}{2}$ are discarded.

Given a continuum function $F \in L$, we call $f_M = I_M^* \cdot F \in l_M$ or $F_M = P_M \cdot F \in L_M$ the discretization of F at resolution M . In particular, if we have a Hamiltonian field theory on X with conjugate pair of fields (Φ, Π) , i.e., the nonvanishing Poisson brackets are

$$\{\Pi(x), \Phi(y)\} = \delta_X(x, y) = \sum_{n \in \mathbb{Z}} e_n(x - y), \quad (\text{B17})$$

then their discretizations obey

$$\begin{aligned} \{\pi_{M,m}, \phi_{M,\tilde{m}}\} &= M \delta_{m,\tilde{m}}, \quad \{\Pi_M(x), \Phi_M(y)\} \\ &= P_M(x, y) = \frac{\sin(M\pi(x - y))}{\sin(\pi(x - y))}. \end{aligned} \quad (\text{B18})$$

The latter formula is known as the Dirichlet kernel.

Given a functional $H[\Pi, \Phi]$ of the continuum fields, we define its discretization by

$$\begin{aligned} h_M[\pi_M, \phi_M] &:= H_M[\Pi_M, \Phi_M] = H[\Pi_M, \Phi_M] \\ &= (I_M^* H)[\pi_M, \phi_M], \end{aligned} \quad (\text{B19})$$

where I_M^* denotes the pullback by I_M . That is, in the continuum formula for H one substitutes $\Pi \rightarrow \Pi_M, \Phi \rightarrow \Phi_M$ in the formula for H upon which H is restricted to Π_M, Φ_M , i.e., H_M is that restriction, and then uses the identity $\Pi_M = I_M \cdot \pi_M, \Phi_M = I_M \cdot \phi_M$. In order for this to be well defined, it is important that I_M is sufficiently smooth as H typically depends on derivatives of Π, Φ . This is granted by our choice of I_M . In particular, as the derivative $\partial = \frac{\partial}{\partial x}$ preserves each of the spaces L_M , we have a canonical discretization of the derivative defined by

$$\partial_M := I_M^* \cdot \partial \cdot I_M, \quad (\text{B20})$$

which obeys $\partial_M^n = I_M^* \cdot \partial^n \cdot I_M$ because $I_M \cdot I_M^* = P_M$ and $[\partial, P_M] = 0$.

Concerning quantization, in the continuum we define the Weyl algebra $\mathfrak{A}$ generated by the Weyl elements

$$W[F] = e^{-i\langle F, \Phi \rangle_L}, \quad W[G] = e^{-i\langle G, \Pi \rangle_L} \quad (\text{B21})$$

for real valued $F, G \in L$ (or a dense subspace thereof with additional properties such as smoothness and rapid momentum decrease of its Fourier modes $\langle e_n, F \rangle_L, \langle e_n, G \rangle_L$). That is, the nontrivial Weyl relations are

$$\begin{aligned} W[G]W[F]W[-G] &= e^{-i\langle G, F \rangle_L} W[F], \\ W[F]W[F'] &= W[F + F'], \\ W[G]W[G'] &= W[G + G'] \\ W[0] &= 1_{\mathfrak{A}}, \\ W[F]^* &= W[-F], \\ W[G]^* &= W[-G]. \end{aligned} \quad (\text{B22})$$

Cyclic representations $(\rho, \mathcal{H}, \Omega)$ of $\mathfrak{A}$ with $\Omega \in \mathcal{H}$ a cyclic vector [i.e., $\mathcal{D} := \rho(\mathfrak{A})\mathcal{H}$ is dense] are generated from states (positive, linear, normalized functionals) ω on $\mathfrak{A}$ via the GNS construction [24]. The correspondence is given by

$$\omega(A) = \langle \Omega, \rho(A)\Omega \rangle_{\mathcal{H}}. \quad (\text{B23})$$

We may proceed analogously with the discretized objects. For each M we define the Weyl algebra $\mathfrak{A}_M$ generated by the Weyl elements

$$\begin{aligned} W_M[F_M] &= e^{-i\langle F_M, \Phi_M \rangle} = w_M[f_M] = e^{-i\langle f_M, \phi_M \rangle_{l_M}}, \\ W_M[G_M] &= e^{-i\langle G_M, \Pi_M \rangle_{L_M}} = w_M[g_M] = e^{-i\langle g_M, \pi_M \rangle_{l_M}}, \end{aligned} \quad (\text{B24})$$

where $F_M = I_M \cdot f_M, G_M = I_M \cdot g_M$ are real valued. Accordingly,

$$\begin{aligned} W_M[G_M]W_M[F_M]W_M[-G_M] &= e^{-i\langle G_M, F_M \rangle_{L_M}} W_M[F_M], \\ W_M[F_M]W_M[F'_M] &= W_M[F_M + F'_M], \\ W_M[G_M]W_M[G'_M] &= W_M[G_M + G'_M], \\ W_M[0] &= 1_{\mathfrak{A}_M}, \\ W_M[F_M]^* &= W_M[-F_M], \\ W_M[G_M]^* &= W_M[-G_M] \end{aligned} \quad (\text{B25})$$

and completely analogous for ϕ_M, π_M if we substitute lower case letters for capital letters in (B25). For each M , we define a state ω_M on $\mathfrak{A}_M$ that gives rise to GNS data $(\rho_M, \mathcal{H}_M, \Omega_M)$ and the dense subspace $\mathcal{D}_M = \mathfrak{A}_M \Omega_M$. Note that $\mathfrak{A}_M$ is a subalgebra of $\mathfrak{A}_{M'}$ for $M < M'$ and that $\mathfrak{A}_M$ is a subalgebra of $\mathfrak{A}$. This follows from the identities

$$W_{M'}[F_M] = W_M[F_M], \quad W_M[F_M] = W[F_M] \quad (\text{B26})$$

due to $P_{M'} \cdot P_M = P_M$, since $L_M \subset L_{M'}$ and $P_M \cdot P_M = P_M$, respectively.

The sole reason for discretization is as follows: While finding states on $\mathfrak{A}$ is not difficult (e.g., Fock states), it is tremendously difficult to find such states that allow one to define nonlinear functionals of Π, Φ such as Hamiltonians densely on $\mathcal{D}$ due to UV singularities arising from the fact that Π, Φ are promoted to operator-valued distributions whose product is *a priori* ill defined. In the presence of the UV cutoff M , this problem can be solved because, e.g., $\Phi_M(x)^2$ is perfectly well defined (Φ is smeared with the smooth kernel P_M). Suppose then that h_M or, equivalently, H_M are somehow quantized on $\mathcal{D}_M$. We denote these quantizations by $\rho_M(h_M, c_M)$ or $\rho_M(H_M, c_M)$, respectively, to emphasize that these operators are (1) densely defined on

$\rho_M(\mathfrak{A}_M)\Omega_M$, (2) correspond to the classical symbol h_M of H_M , respectively, and (3) depend on a set of choices c_M for each M such as factor or normal ordering, etc. It is therefore not at all clear whether the theories defined for each M , in fact, descend from a continuum theory. By “descendance” we mean that ω_M is the restriction of ω to $\mathfrak{A}_M$ and that $\rho_M(H_M, c_M)$ is the restriction of $\rho(H, c)$ to $\mathcal{D}_M$ as a quadratic form (i.e., in the sense of matrix elements). In formulas, this means

$$\begin{aligned} \omega_M(A_M) &= \omega(A_M), \\ \langle \rho_M(A_M)\Omega_M, \rho_M(H_M, c_M)\rho_M(B_M)\Omega_M \rangle_{\mathcal{H}_M} \\ &= \langle \rho(A_M)\Omega, \rho(H, c)\rho(B_M)\Omega \rangle_{\mathcal{H}}, \end{aligned} \quad (\text{B27})$$

for all $M \in \mathbb{O}$ and all $A_M, B_M \in \mathfrak{A}_M$. If they did, then we obtain the following identities for $M < M'$:

$$\begin{aligned} \omega_{M'}(A_M) &= \omega_M(A_M), \\ \langle \rho_M(A_M)\Omega_M, \rho_M(H_M, c_M)\rho_M(B_M)\Omega_M \rangle_{\mathcal{H}_M} \\ &= \langle \rho_{M'}(A_M)\Omega_{M'}, \rho_{M'}(H_{M'}, c_{M'})\rho_{M'}(B_M)\Omega_{M'} \rangle_{\mathcal{H}_{M'}}, \end{aligned} \quad (\text{B28})$$

called consistency conditions. This follows from the fact that $A_{M'} := A_M, B_{M'} := B_M$ can be considered as elements of $\mathfrak{A}_M$ and then using (B27). With some additional work [4], one can show that (B28) are necessary and sufficient for $\omega, \rho(H)$ to exist (at least as a quadratic form).

In CQFT [2], one proceeds as follows. One starts with an ansatz of a family of discretized theories $(\omega_M^{(0)}, \rho_M^{(0)}(H_M, c_M^{(0)}))_{M \in \mathbb{O}}$. That ansatz generically violates (B28). We now define a renormalization flow of states and quantizations by defining the sequence $(\omega_M^{(k)}, \rho_M^{(k)}(H_M, c_M^{(k)}))_{M \in \mathbb{O}}$ for $k \in \mathbb{N}_0$ via

$$\begin{aligned} \omega_M^{(k+1)}(A_M) &:= \omega_{M'(M)}^{(n)}(A_M), \\ \langle \rho_M^{(k+1)}(A_M)\Omega_M^{(k+1)}, \rho_M^{(k+1)}(H_M, c_M^{(k+1)})\rho_M^{(k+1)}(B_M)\Omega_M^{(k+1)} \rangle_{\mathcal{H}_M^{(k+1)}} \\ &= \langle \rho_{M'}^{(k)}(A_M)\Omega_{M'}^{(k)}, \rho_{M'}^{(k)}(H_{M'}, c_{M'}^{(k)})\rho_{M'}^{(k)}(B_M)\Omega_{M'}^{(k)} \rangle_{\mathcal{H}_{M'}^{(k)}}, \end{aligned} \quad (\text{B29})$$

where $M': \mathbb{O} \rightarrow \mathbb{O}$ is a fixed map with the property that $M'(M) > M, M'(M) \neq M$. The first relation defines a new state at the coarser resolution M as the restriction of the old state at the finer resolution $M'(M)$. This then defines also new GNS data $(\rho_M^{(k+1)}, \mathcal{H}_M^{(k+1)}, \Omega_M^{(k+1)})$ via the GNS construction. The second relation defines the matrix elements of an operator or quadratic form in that new representation and with new quantization choices to be made at coarser resolution in terms of the restriction of the matrix elements of the old operator or quadratic form with old quantization choices in the old representation at finer resolution. A fixed point family $(\omega_M^*, \rho_M^*(H_M, c_M^*))_{M \in \mathbb{O}}$ of the flow (B29)

solves (B28) at least for $M' = M'(M)$ and all M and thus all $[M']^n(M), n \in \mathbb{N}_0$ and all M . This typically implies that (B28) holds for all $M' < M$. In practice, we will work with $M'(M) := 3M$.

Note that for a general operator or quadratic form O defined densely on $\mathcal{D}$ it is not true that we find an element $a \in \mathfrak{A}$ such that $\rho(a) = O$ (e.g., unbounded operators), which is why the above statements cannot be made just in terms of the states ω . If one tried, one would need to use sequences or nets $a_n \in \mathfrak{A}$ whose limits lie outside of $\mathfrak{A}$. On the other hand, if one prefers to work with the Weyl elements $W_M[F_M]$ one may relate the spaces l_M, L_M via the

identities $w_M[f_M] = W_M[F_M]$, $f_M = I_M^* \cdot F_M$. The $w_M[f_M]$, $w_{M'}[f'_M]$ at resolution M , M' , respectively, can be related via the “coarse graining” map $I_{MM'} := I_{M'}^* \cdot I_M$; $l_M \rightarrow l_{M'}$ such that $w_{M'}[I_{MM'} \cdot f_M] = w_M[f_M]$. This map obeys $I_{M_2 M_3} \cdot I_{M_1 M_2} = I_{M_1 M_3}$ for $M_1 < M_2 < M_3$ because the image of I_M is L_M which is a subspace of $L_{M'}$; thus, $I_{M_2 M_3} \cdot I_{M_1 M_2} = I_{M_3}^* \cdot P_{M_2} \cdot I_{M_1} = I_{M_3}^* \cdot I_{M_1}$. Then $W_{M'}[F_M] = w_{M'}[I_{M'}^* F_M] = w_{M'}[I_{M'}^* \cdot I_M \cdot f_M] = w_{M'}[I_{MM'} \cdot f_M]$ indeed. For the same reason $W_{M'}[F_M] = W_M[F_M]$ as L_M is embedded in $L_{M'}$ by the identity map. The renormalization flow in terms of Weyl elements $w_M[f_M]$ and the coarse graining map $I_{M,M'}$ takes the form

$$\omega_M^{(k+1)}(w_M[f_M]) := \omega_{M'}^{(k)}(w_{M'}[I_{M,M'} f'_M])$$

$$\langle w_M[f'_M] \Omega_M^{(k+1)}, H_M^{(k+1)} w_M[f_M] \Omega_M^{(k+1)} \rangle_{\mathcal{H}_M^{(k+1)}} := \langle w_{M'}[I_{M,M'} f'_M] \Omega_{M'}^{(k)}, H_{M'}^{(k)} w_{M'}[I_{M,M'} f'_M] \Omega_{M'}^{(k)} \rangle_{\mathcal{H}_{M'}^{(k)}}. \quad (\text{B30})$$

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