

$g_{tt}g_{rr} = -1$ black hole thermodynamics in general quasitopological gravityJohanna Borissova^{*}*Abdus Salam Centre for Theoretical Physics, Imperial College London, London SW7 2AZ, United Kingdom*

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We present a unified framework for the discussion of black hole thermodynamics of d -dimensional static black holes with spherical or compact hyperbolic horizon topology satisfying $g_{tt}g_{rr} = -1$ in Schwarzschild gauge. To that end, we consider any such black hole as a solution to an integrable two-dimensional effective dilaton theory and thereby as a vacuum solution to a generalized notion of d -dimensional quasitopological gravity. We show that the generating function determining $f(r) = -g_{tt}$ in the integrated equation of motion provides the thermodynamic mass in a generalized first law with entropy computed as the Wald entropy. The framework presented here can be applied to singular and regular black holes with flat or anti-de Sitter asymptotics.

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The Bekenstein-Hawking area law for the entropy of a black hole in general relativity (GR) [1,2] is expected to receive corrections for beyond-GR black holes. These corrections can be accessed through Wald's Noether charge formalism [3,4]—provided a given beyond-GR black hole arises as a known solution to a generally covariant gravitational theory. This is often not the case for phenomenological black hole models inspired by quantum gravity.¹ In this case dealing with the thermodynamics of black holes typically requires imposing the first law by hand. The purpose of this paper is to establish a unified framework for the discussion of black hole thermodynamics of $(d \geq 4)$ -dimensional static black holes satisfying $g_{tt}g_{rr} = -1$ in the Schwarzschild gauge as gravitational vacuum solutions to a generalized notion of d -dimensional *quasitopological gravity* (QTG), which allows us to ground the discussion on integrable two-dimensional generalized dilaton theories.²

The precise albeit *implicit* definition [34] of a *general* QTG adopted here is detailed further below and can be summarized as follows. A general QTG is *any* d -dimensional gravitational theory whose reduction on spherically symmetric backgrounds—or backgrounds with compact hyperbolic or toroidal symmetry—yields a two-dimensional Horndeski theory satisfying an integrability condition that is necessary and sufficient for vacuum solutions to be static and to satisfy $g_{tt}g_{rr} = -1$ in Schwarzschild gauge, whereby the first-order equation determining $g_{tt} = -f(r)$ can be integrated into an algebraic equation involving a theory-dependent generating function $\Omega(r, f)$.³ We will henceforth refer to the reduced theory as a two-dimensional *integrable* Horndeski theory. Conversely, lifting such a two-dimensional integrable Horndeski theory to a $(d \geq 4)$ -dimensional gravity [34–36] by this definition produces a general QTG. This definition is operational in what concerns the derivation of spherically symmetric vacuum solutions, and does not require or assume knowledge about the precise form of the d -dimensional gravitational action.

We will carefully distinguish this generalised notion of a QTG from the following notions of QTGs so far appearing in the literature. The first one are QTGs as an *explicit* set of theories representing polynomial higher-curvature extensions of general relativity [37–45] whose reduction on spherically symmetric backgrounds produces a subset of 2-dimensional integrable Horndeski theories [34,45–48]. We will refer to such theories here explicitly as *polynomial curvature* QTGs. These are contained in the larger set of QTGs built from arbitrary but not necessarily analytic

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¹See, e.g., [5–16] for a nonexhaustive set of quantum gravity inspired examples related to regular black hole models such as those proposed originally in [17–19]. Here we are not discussing the case when such models are produced by coupling GR to exotic matter violating energy conditions, such as nonlinear electrodynamic sources [20–30].

²See [31,32] for the early stages of development of two-dimensional dilaton theory and, e.g., [33] for a review.

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³The exceptions to this statement are theories for which $\partial_f \Omega(r, f) = 0$ on-shell [34].

functions of curvature invariants without covariant derivatives [34,35,49]. We will refer here to these latter theories as *nonpolynomial curvature* QTGs. General QTGs contain polynomial and nonpolynomial curvature QTGs as a subclass, but represent an extended class of theories satisfying the above defining properties with an action containing in general arbitrary functions of curvature and curvature-derivative invariants [34]. This is the relevant class of theories on which our discussion is grounded.

The notion of a general QTG is not to be confused with the notion of a *generalized* QTG (GQTG) used in previous works [41–44] to refer to gravitational theories representing polynomial higher-curvature extensions of general relativity that admit static spherically symmetric vacuum solutions satisfying $g_{tt}g_{rr} = -1$, whereby the equation determining $g_{tt} = -f(r)$ is algebraic *only* if the GQTG is a polynomial curvature QTG. The reduction on spherically symmetric backgrounds of a GQTG that is not a polynomial curvature QTG thus cannot be a two-dimensional integrable Horndeski theory.

II. GENERAL QUASITOPOLOGICAL GRAVITIES

Consider a d -dimensional static spacetime

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{d-2}^2, \quad (1)$$

where $d\Sigma_{d-2}^2$ is the surface element of a $(d-2)$ -dimensional compact space of constant curvature $k = \pm 1$ corresponding to spherical or compact hyperbolic symmetry. Any such spacetime depending on a constant dimensionful parameter can be constructed as a vacuum solution to a general QTG [34].⁴

By a general QTG we mean here a generally covariant theory⁵

$$S[g] = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \mathcal{L}_g, \quad (2)$$

$$\mathcal{L}_g = -2\Lambda + R + \Delta \mathcal{L}_g, \quad (3)$$

⁴Extending this statement in full generality to spacetimes (1) with $k = 0$ would require finding covariant densities that separate the variables φ and χ for a general warped-product metric (4) as done for $k \neq 0$ in [34]. It is not clear whether such densities exist. The thermodynamic relations derived here nevertheless apply to spacetimes (1) with $k = 0$ if the generating function $\Omega(r, f)$ determined by the inversion of $f(r)$ with respect to the dimensionful parameter—as discussed later—takes the particular form $\Omega(r, f) \propto r^{d-1} h(\frac{f}{r})$, where h is an arbitrary function of its argument. These spacetimes can be constructed as vacuum solutions to a general QTG [34,35].

⁵We work in units where $c = \hbar = k_B = 1$ and Newton's constant has mass dimension $[G_N] = 2 - d$.

with second-order equations on warped-product backgrounds

$$g_{\mu\nu}(x)dx^\mu dx^\nu = q_{ab}(y)dy^a dy^b + \varphi(y)^2 d\Sigma_{d-2}^2, \quad (4)$$

$$d\Sigma_{d-2}^2 = \gamma_{ij}(\theta)d\theta^i d\theta^j, \quad (5)$$

for which the two-dimensional reduced theory

$$S_{2D}[q, \varphi] = \frac{(d-2)\Sigma_{d-2}}{16\pi G_N} \int d^2 y \sqrt{-q} \mathcal{L}_{q,\varphi}, \quad (6)$$

$$\mathcal{L}_{q,\varphi} = \frac{1}{d-2} \varphi^{d-2} \mathcal{L}_g \Big|_{(4)}, \quad (7)$$

satisfies an integrability condition that is necessary and sufficient for the existence of solutions (1) and the equation of motion for f to be integrable into an algebraic equation. The generating function determining f for a black hole can then be interpreted as the thermodynamic mass, as we will discuss here.

Since the reduction of a general QTG on (4), by definition, is a two-dimensional Horndeski theory for q_{ab} and φ , the Lagrangian density of the reduced action (6) takes the form [50,51]

$$\mathcal{L}_{q,\varphi} = h_2(\varphi, \chi) - h_3(\varphi, \chi) \square \varphi + h_4(\varphi, \chi) \mathcal{R} + 2\partial_\chi h_4(\varphi, \chi) [(\square \varphi)^2 - \nabla_a \nabla_b \varphi \nabla^a \nabla^b \varphi], \quad (8)$$

where ∇_a is the covariant derivative associated with q_{ab} , $\mathcal{R}$ is its Ricci scalar, and h_i for $i = 2, 3, 4$ are theory-dependent functions of the scalar field φ and its kinetic term $\chi = \nabla_a \varphi \nabla^a \varphi$. The former is considered as a field with mass dimension $[\varphi] = -1$ in the reduced action (6).

The integrability condition satisfied by the reduced action (6) of a general QTG is [34,35]

$$\partial_\chi \alpha(\varphi, \chi) - \partial_\varphi \beta(\varphi, \chi) = 0, \quad (9)$$

where

$$\alpha(\varphi, \chi) = h_2 + \chi \partial_\varphi (h_3 - 2\partial_\varphi h_4), \quad (10)$$

$$\beta(\varphi, \chi) = \chi \partial_\chi (h_3 - 2\partial_\varphi h_4) - \partial_\varphi h_4 \quad (11)$$

are two functions with mass dimensions $[\alpha] = 4 - d$ and $[\beta] = 3 - d$ which determine the equations of motion for q_{ab} and φ .

The implications of the integrability condition (9) can be seen as follows. According to the principle of symmetric criticality that holds for compact symmetry groups [52–56], the field equations derived from the generally covariant action (2) and evaluated on the ansatz (4) are equivalent to the field equations derived from the reduced action (6). The corresponding variations are related by

$$\begin{aligned} \mathcal{E}_{\mu\nu} \Big|_{(4)} &:= \frac{16\pi G_N}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \Big|_{(4)} \\ &= (d-2)\varphi^{2-d}\mathcal{E}_{ab}\delta_\mu^a\delta_\nu^b - \frac{1}{2}\varphi^{5-d}\mathcal{E}_\varphi\gamma_{ij}\delta_\mu^i\delta_\nu^j, \end{aligned} \quad (12)$$

where

$$\mathcal{E}_{ab} := \frac{16\pi G_N}{(d-2)\Sigma_{d-2}} \frac{1}{\sqrt{-q}} \frac{\delta S_{2D}}{\delta q^{ab}}, \quad (13)$$

$$\mathcal{E}_\varphi := \frac{16\pi G_N}{(d-2)\Sigma_{d-2}} \frac{1}{\sqrt{-q}} \frac{\delta S_{2D}}{\delta \varphi}. \quad (14)$$

When the integrability condition (9) is satisfied, the equation of motion $\mathcal{E}_{ab} = 0$ for q_{ab} is explicitly [57,58]

$$\frac{1}{2}(\alpha + 2\beta\Box\varphi)q_{ab} - \beta\nabla_a\nabla_b\varphi = 0, \quad (15)$$

whereas the equation of motion $\mathcal{E}_\varphi = 0$ for φ is redundant on-shell on the equation of motion for q_{ab} due to the off-shell Bianchi identity

$$\nabla^a\mathcal{E}_{ab} + \frac{1}{2}\mathcal{E}_\varphi\nabla_b\varphi = 0 \quad (16)$$

resulting from two-dimensional general covariance of (6) provided the scalar field is nonconstant. This is equivalent to the statement that the angular components of the equation of motion $\mathcal{E}_{\mu\nu} = 0$ for $g_{\mu\nu}$ in (4) are redundant due to the off-shell Bianchi identity $\nabla^\mu\mathcal{E}_{\mu\nu} = 0$ resulting from the d -dimensional general covariance of (2). For an ansatz

$$q_{ab}(y)dy^a dy^b = -n(t,r)^2 f(t,r)dt^2 + \frac{dr^2}{f(t,r)}, \quad (17)$$

$$\varphi(y) = r, \quad (18)$$

i.e., $\chi = f$, the independent components of the equation of motion (15) yield the three equations

$$\partial_t f = 0, \quad \partial_r n = 0, \quad \alpha + \beta\partial_r f = 0, \quad (19)$$

where $\beta \neq 0$ is assumed. A general QTG thus satisfies the following Birkhoff theorem [34]: For generic solutions of the form

$$ds^2 = -n(t,r)^2 f(t,r)dt^2 + \frac{dr^2}{f(t,r)} + r^2 d\Sigma_{d-2}^2 \quad (20)$$

in vacuum, the metric function f is time-independent, and one may set $n = 1$ after a redefinition of the coordinate t such that the solutions are static and satisfy $g_{tt}g_{rr} = -1$. The integrability condition (9) moreover implies that the

remaining equation determining $f = f(r)$ in (19) can be written in terms of a generating function Ω satisfying the defining off-shell relations [57,58]

$$\alpha(\varphi, \chi) = \frac{16\pi G_N}{(d-2)\Sigma_{d-2}} \partial_\varphi \Omega(\varphi, \chi), \quad (21)$$

$$\beta(\varphi, \chi) = \frac{16\pi G_N}{(d-2)\Sigma_{d-2}} \partial_\chi \Omega(\varphi, \chi), \quad (22)$$

thus having mass dimension $[\Omega] = 1$, as

$$\frac{d}{dr}\Omega(r, f) = \partial_r \Omega(r, f) + \partial_f \Omega(r, f)f' = 0, \quad (23)$$

and can be integrated into an algebraic equation for f ,

$$\Omega(r, f) = M, \quad (24)$$

where M is an integration constant. We will see that this identification provides the thermodynamic mass in a first law for d -dimensional static black holes (1).

General QTGs include Lovelock theories [59–61] and polynomial curvature QTGs [37–44] satisfying an analog version of the above Birkhoff theorem [45]. For these theories the generating function Ω takes the form [62–64]

$$\Omega(r, f) = \frac{(d-2)\Sigma_{d-2}}{16\pi G_N} r^{d-1} h\left(\frac{k-f}{r^2}\right), \quad (25)$$

where h is a power series in its argument. The reduction of GR with a negative cosmological constant is described by the function

$$h\left(\frac{k-f}{r^2}\right) = -\frac{2\Lambda}{(d-1)(d-2)} + \frac{k-f}{r^2} \quad (26)$$

$$=: \frac{1}{\ell^2} + \frac{k-f}{r^2}, \quad (27)$$

where ℓ is the anti-de Sitter radius. The integrated equation of motion (24) in this case yields

$$f(r) = k + \frac{r^2}{\ell^2} - \frac{2m}{r^{d-3}}, \quad (28)$$

where the parameter m is related to the integration constant M by

$$M = \frac{(d-2)\Sigma_{d-2}}{16\pi G_N} 2m. \quad (29)$$

Thus, M is identical to the Arnowitt-Deser-Misner mass [65,66] of the spacetime.

Polynomial curvature QTGs beyond GR exist only in $d \geq 5$ dimensions [41,44], whereas nonpolynomial

curvature QTGs can be constructed also in $d = 4$ dimensions [34,35,49,67], and in this case Ω can have a more general dependence on r and f than in (25) [34,35].⁶ More generally, the equations of motion of generic two-dimensional Horndeski theories can be obtained as the reduced equations of motion on (4) of purely gravitational theories (2) involving nonpolynomial functions of curvature and curvature-derivative invariants in $d \geq 4$ dimensions [34–36]. Assuming a given two-dimensional Horndeski theory satisfies the integrability condition (9); i.e., dealing with the warped-product sector (4) of a general QTG, the generating function Ω can have an arbitrary dependence on r and f as independent variables [34].

From a reversed perspective, the previous discussion implies the following remarkably general statement. Consider a static black hole (1) asymptotically of the form (28) with an invertible dependence of f on the mass M . It follows that any such black hole can be reconstructed as a vacuum solution to a d -dimensional generally covariant gravitational theory [34]. To that end, one may identify the black hole mass M with the integration constant in the integrated equation of motion (24) of an associated two-dimensional Horndeski theory satisfying the integrability condition (9). This identification determines the generating function Ω off-shell after performing the replacement $r \mapsto \varphi$ and $f \mapsto \chi$. Therefrom the functions h_i can be reconstructed, which make the configuration

$$q_{ab}(y)dy^a dy^b = -f(r)dt^2 + \frac{dr^2}{f(r)}, \quad (30)$$

$$\varphi(y) = r, \quad (31)$$

a solution to the equations of motion for q_{ab} and φ derived from (8) [57,58]. One may then interpret this Horndeski theory as the effective two-dimensional theory obtained from the reduction on (4) of a general d -dimensional QTG (2) [34].⁷

A concrete realization of the functions h_i in terms of the generating function Ω is provided by [87]

⁶For applications of $d \geq 5$ polynomial and $d = 4$ nonpolynomial curvature QTGs in the context of black holes and cosmology, see, e.g., [37–46,48,62–64,68–78] and [34,35,49,67,79–86].

⁷Such a reconstruction procedure has been developed originally in the context of a unified extension of the Einstein equations on the space of warped-product spacetimes (4) based on a second-order covariantly conserved rank-2 tensor constructed from the variations of the two-dimensional Horndeski action [58] (cf. also [57]). For a generic two-dimensional metric q_{ab} and scalar field φ , the tensor constructed in [58] can be generated as the variation of a d -dimensional covariant action for $g_{\mu\nu}$ evaluated on (4), concretely Eq. (12) [34]. Thus, the extended Einstein equations [58] in vacuum are in fact equivalent to the reduced field equations of a d -dimensional covariant theory for $g_{\mu\nu}$ [34].

$$h_2(\varphi, \chi) = \frac{16\pi G_N}{(d-2)\Sigma_{d-2}} \partial_\varphi \Omega(\varphi, \chi), \quad (32)$$

$$h_3(\varphi, \chi) = -\frac{32\pi G_N}{(d-2)\Sigma_{d-2}} \partial_\chi \Omega(\varphi, \chi), \quad (33)$$

$$h_4(\varphi, \chi) = -\frac{16\pi G_N}{(d-2)\Sigma_{d-2}} \int d\varphi \partial_\chi \Omega(\varphi, \chi). \quad (34)$$

The goal here is to use the above results for a unified discussion of black hole thermodynamics based on the effective two-dimensional reduced theory (6) without knowledge of its d -dimensional generally covariant form (2).

III. THERMODYNAMICS OF $g_{tt}g_{rr} = -1$ BLACK HOLES

Throughout we will assume that f has a most outer zero at $r = r_+$ corresponding to the event horizon of the black hole. From now on we will only deal with functions $\mathcal{F}(r, f)$ evaluated at $r = r_+$ and hence $f = 0$, and for this purpose we adopt the shorthand notion $\mathcal{F}(r_+) := \mathcal{F}(r_+, 0)$. The integrated equation of motion (24) implies that an on-shell configuration $f = f(r)$ corresponds to a level curve of Ω , and therefore the mass M can be written as

$$M = \Omega(r_+). \quad (35)$$

We proceed by expressing the Hawking temperature T in terms of Ω . The former can be derived by an application of Euclidean path integral methods [88,89]. The Euclidean metric obtained after a Wick rotation of the time coordinate $t \rightarrow i\tau$ expanded at $r = r_+$ has a conical singularity unless τ is periodically identified with period $\kappa/2\pi$, where $\kappa = f'(r_+)/2 > 0$ is the surface gravity computed from the timelike Killing vector field $\xi^\mu = \partial_t^\mu$ according to $\xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu$. The imaginary time period fixes the inverse of the Hawking temperature

$$T = \frac{f'(r_+)}{4\pi}. \quad (36)$$

For a black hole solution to (24) with generating function Ω , we can use (23) to obtain

$$T = -\frac{1}{4\pi} \frac{\partial_{r_+} \Omega(r_+)}{\partial_f \Omega(r_+)}. \quad (37)$$

Now we will evaluate the black hole entropy by an application of Wald's Noether charge formalism.

IV. WALD ENTROPY

The defining formula for the Wald entropy S of a black hole derived from a generally covariant action (2) is [3,4]

$$S = -2\pi \oint_{\mathcal{H}} d^{d-2}x \sqrt{h} \left[\frac{P^{\mu\nu\rho\sigma}}{16\pi G_N} \right] \epsilon_{\mu\nu} \epsilon_{\rho\sigma}, \quad (38)$$

where h is the induced metric on the horizon $\mathcal{H}$ and $\epsilon_{\mu\nu}$ the binormal to $\mathcal{H}$ normalized to $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$. The entropy tensor P is defined as the variational derivative of the Lagrangian with respect to the Riemann tensor,

$$P^{\mu\nu\rho\sigma} = \frac{\delta \mathcal{L}_g}{\delta R_{\mu\nu\rho\sigma}}. \quad (39)$$

Its contraction with the binormals in (38) must be evaluated onshell. For a static black hole (1) the horizon binormal is given in terms of the unit spacelike and timelike normal vectors r^μ and t^ν as $\epsilon_{\mu\nu} = 2r_{[\mu}t_{\nu]}$ and has the non-zero components $\epsilon_{01} = -\epsilon_{10} = 1$. The contraction of the entropy tensor with the binormals in this case is constant on the horizon, and the Wald formula (38) simplifies to

$$S = -\frac{\Sigma_{d-2}}{8G_N} r_+^{d-2} P^{abcd} \epsilon_{ab} \epsilon_{cd} \Big|_{r=r_+}. \quad (40)$$

The key step now is to notice that realizing such a black hole as a solution to a general QTG, the remaining tensorial contraction can be computed from the two-dimensional reduced Lagrangian density

$$\mathcal{L}_g \Big|_{(4)} = (d-2) \varphi^{2-d} \mathcal{L}_{q,\varphi} \quad (41)$$

defined as in (8) without knowledge of its d -dimensional generally covariant form.⁸ The former depends only linearly on

$$R_{abcd} \Big|_{(4)} = \mathcal{R} q_{a[c} q_{d]b} \quad (42)$$

via the two-dimensional Ricci scalar in (8), and in particular does not depend on covariant derivatives of the curvature. Thus, the variational derivative in (40) reduces to a partial derivative. The relevant part of the reduced Lagrangian density is

$$\mathcal{L}_g \Big|_{(4)} \ni (d-2) \varphi^{2-d} h_4(\varphi, \chi) \mathcal{R}. \quad (43)$$

⁸Analog techniques for computing the Wald entropy in polynomial and nonpolynomial curvature QTGs and more general gravitational theories admitting solutions of the form (1) have been used, e.g., in [36,42,43,62–64].

Using $q^{a[c} q^{d]b} \epsilon_{ab} \epsilon_{cd} = 2q^{-1}$ where the determinant of q_{ab} is $q = -1$ for the ansatz (30), the Wald entropy evaluates to⁹

$$S = \frac{\Sigma_{d-2}}{4G_N} r_+^{d-2} [\partial_{\mathcal{R}} \mathcal{L}_g]_{(4)} \Big|_{(30)-(31)} \Big|_{r=r_+} = \frac{(d-2)\Sigma_{d-2}}{4G_N} h_4(r_+). \quad (44)$$

The function $h_4(r_+)$ in (44) still needs to be expressed on-shell for a solution. When the black hole is realized through the representation of the function h_4 as in (34), we can write¹⁰

$$S = -4\pi \int dr_+ \partial_f \Omega(r_+). \quad (46)$$

Using the expressions for the mass M in (35) and temperature T in (37), this result can be written as

$$S = \int dr_+ \frac{\partial_{r_+} M}{T}, \quad (47)$$

wherefrom the first law of thermodynamics

$$\delta M = T \delta S \quad (48)$$

follows. In particular the entropy formula (46) combined with the integrated equation of motion (24) provide all ingredients required for a consistent discussion of the thermodynamics of d -dimensional static black holes (1).

V. GENERALIZED FIRST LAW

The above derivation assumes the thermodynamic mass to be a function of r_+ only [cf. Eq. (35)]. More generally, when the black hole mass is used to determine the generating function Ω through (24)—the key step to make this spacetime a solution— Ω evaluated at $r = r_+$, and

⁹When the two-dimensional reduced theory (6) features a linear nonminimal coupling $h_4(\varphi, \chi) \propto \varphi$, such as in Jackiw-Teitelboim gravity [90,91], Eq. (44) reduces to the well-known statement that the entropy is proportional to the value of the dilaton field φ on the horizon. See, e.g., [92] for a review.

¹⁰One may verify that this expression reproduces the entropy of asymptotically flat or anti-de Sitter black holes in $d \geq 5$ polynomial curvature QTGs [62–64]. Changing variables $(r, f) \mapsto (r, \psi)$, where $\psi(r, f) = (k - f)/r^2$ with $k = \pm 1$, and inserting (25) into (46), results in

$$S = -\frac{(d-2)\Sigma_{d-2}k^{d/2-1}}{8G_N} \int d\psi_+ \frac{h'(\psi_+)}{\psi_+^{d/2}}, \quad (45)$$

where $\psi_+ = \psi(r_+) = k/r_+^2$.

hence M will be a function

$$M = M(r_+, \Lambda, \alpha_i) \quad (49)$$

with a dependence on all dimensionful scales involved in f , such as Λ and other mass scales α_i . In this case one may consider a generalization of the first law (48) in the form

$$\delta M = T\delta S + V\delta P + \sum_i \mu_{\alpha_i} \delta \alpha_i, \quad (50)$$

where

$$P = -\frac{\Lambda}{8\pi G_N} \quad (51)$$

is a thermodynamic pressure associated with the cosmological constant [93–95] with conjugate variable V given by

$$V = \left(\frac{\partial M}{\partial P} \right)_{S, \alpha_i} \quad (52)$$

and interpreted as the thermodynamic volume of the black hole [96–101], and μ_{α_i} are generalized thermodynamic potentials conjugate to the couplings α_i . Interpreting their contribution in (50) as work terms, the black hole mass M must be identified with the chemical enthalpy $H = E + PV$ rather than the internal energy E as in (48)—an observation that has critically sparked the field of black hole chemistry [102–104].¹¹ The generalized first law (50) in the absence of work terms associated with α_i was discovered originally as a way to construct a generalization of the Smarr mass formula [112] for an anti-de Sitter Schwarzschild black hole (28) [96,98],

$$(d-3)M = (d-2)TS - 2VP, \quad (53)$$

for which the additional $V\delta P$ term in (50) compared to (48) is essential.¹² One way to see this is to observe that the expression for the Wald entropy (46) is invariant under a shift of the generating function Ω by a function of r ,

$$\Omega(r, f) \rightarrow \Omega(r, f) + \omega(r), \quad (54)$$

¹¹A holographic perspective on black hole chemistry and associated interpretation of P as a central charge can be found in [105–109]. A perspective challenging the view of Λ as a thermodynamic variable is provided in [110]. See also [111] for a proposal of a holographic definition of thermodynamic pressure and volume for black holes.

¹²Generalizations of the Smarr formula (53) for higher-curvature gravitational theories with dimensionful couplings have been constructed via the introduction of chemical potentials for Lovelock theories, e.g., in [113–115] and for more general polynomial curvature QTGs, e.g., in [63,64,116].

whereas the mass M in (35) is not. Such a shift modifies the solution to the equation of motion (24). The cosmological constant term in (26) is an example of such an r -dependent shift of Ω .¹³

$$\Omega_\Lambda(r) = \frac{(d-2)\Sigma_{d-2}}{16\pi G_N} \left[\frac{-2\Lambda}{(d-1)(d-2)} \right] r^{d-1}. \quad (55)$$

Upon performing the identification (24), the thermodynamic volume V in (52) is therefrom computed as

$$V = \frac{\Sigma_{d-2}}{d-1} r_+^{d-1}. \quad (56)$$

A generalization of the Smarr formula (53) to take into account the new dimensionful parameters α_i can be derived by an application of Euler's theorem for a homogenous function

$$M(\lambda r_+, \lambda^p \Lambda, \lambda^{q_i} \alpha_i) = \lambda^s M(r_+, \Lambda, \alpha_i), \quad (57)$$

which yields the scaling relation

$$sM(r_+, \Lambda, \alpha_i) = \left(\frac{\partial M}{\partial r_+} \right)_{\Lambda, \alpha_i} r_+ + p \left(\frac{\partial M}{\partial \Lambda} \right)_{r_+, \alpha_i} \Lambda + \sum_i q_i \left(\frac{\partial M}{\partial \alpha_i} \right)_{r_+, \Lambda, \alpha_{j \neq i}} \alpha_i. \quad (58)$$

The assumption of homogeneity of M appears restrictive from the point of view of a unified picture of d -dimensional black hole thermodynamics on the space of general QTGs considered here, but can be achieved under quite generic circumstances. For example, let us assume that the function Ω specifying the integrated equation of motion (24) of the two-dimensional reduced theory can be expanded as a series of terms beyond GR, in the form

$$\Omega(r, f) = \frac{(d-2)\Sigma_{d-2}}{16\pi G_N} r^{d-1} h(r, f), \quad (59)$$

$$h(r, f) = \frac{1}{\ell^2} + \frac{k-f}{r^2} + \sum_i \alpha_i \eta_i(r, f), \quad (60)$$

¹³More generally, the addition of a scalar potential $V(\varphi)$ encoded in the function h_2 in (8) amounts to such an r -dependent shift of Ω ; cf. the defining relations of Ω in (21) and (22) based on the functions α and β in (10) and (11). This suggests that one might also deal with black holes exhibiting more general nonflat asymptotics than anti-de Sitter, which would require an extended notion of black hole volume.

where h can have a general dependence on r and f through the functions η_i .¹⁴ These functions have mass dimensions $[\eta_i] = 2 - [\alpha_i]$ and are assumed to not depend on additional dimensionful variables beyond r . Hence, they must be of the form

$$\eta_i(r, f) = \hat{\eta}_i(f) r^{[\alpha_i]-2}. \quad (61)$$

The mass M in (35) can thus be expressed as

$$M(r_+, \Lambda, \alpha_i) = \frac{(d-2)\Sigma_{d-2}}{16\pi G_N} r_+^{d-1} \left[\frac{1}{\ell^2(\Lambda)} + \frac{k}{r_+^2} + \sum_i \alpha_i \hat{\eta}_i(0) r_+^{[\alpha_i]-2} \right] \quad (62)$$

and is a homogenous function with scaling dimension $s = d - 3$, whereas the scaling dimensions of Λ and α_i are $p = -2$ and $q_i = -[\alpha_i]$. In this case the Euler relation (58) yields

$$(d-3)M = \left(\frac{\partial M}{\partial r_+} \right)_{\Lambda, \alpha_i} r_+ - 2VP - \sum_i [\alpha_i] \left(\frac{\partial M}{\partial \alpha_i} \right)_{r_+, \Lambda, \alpha_{j \neq i}} \alpha_i. \quad (63)$$

The entropy can be obtained according to (46) with Ω given in (59) as

$$S = \frac{A}{4G_N} \left[1 - (d-2) \sum_i \frac{\alpha_i \hat{\eta}'_i(0)}{d-2 + [\alpha_i]} r_+^{[\alpha_i]} \right], \quad (64)$$

where $A = \Sigma_{d-2} r_+^{d-2}$ is the $(d-2)$ -dimensional area of the horizon. Using this result to compute

$$\left(\frac{\partial M}{\partial r_+} \right)_{\Lambda, \alpha_i} r_+ = T \left(\frac{\partial S}{\partial r_+} \right)_{\Lambda, \alpha_i} r_+, \quad (65)$$

one may construct a generalized Smarr formula

$$(d-3)M = (d-2)TS - 2VP - \sum_i [\alpha_i] \mu_{\alpha_i} \alpha_i \quad (66)$$

for suitably defined theory-dependent potentials μ_{α_i} .

¹⁴For polynomial curvature QTGs [cf. Eq. (25)], h is a power series in the variable $\psi = (k-f)/r^2$, i.e., $\eta_i \propto \psi^i$ for $i \geq 2$. This variable represents one of the scalar invariants characterizing the Riemann tensor of the spacetimes (1). Such a scalar structure combining r and f can be separated so as to achieve a dependence of h on r and f as independent variables by allowing covariant derivatives apart from curvatures in the action (2) [34].

VI. GLOBAL CHARGES

An electric charge Q of the black hole originating from the coupling to an electromagnetic field that recovers Maxwell theory for $r \rightarrow \infty$ is manifested in an asymptotic r -dependent falloff term $\propto r^{2(3-d)}$ in f , which is subleading compared to the mass term. Upon performing the identification (24) with M taken to be the black hole mass, such a global charge contribution would amount to a universal shift of Ω of the form (49) in the limit $r \rightarrow \infty$ but would require a model-dependent discussion of the conjugate potential Φ when considered as arising from the coupling of (2) to an electromagnetic source. Here we have generated the black hole as an effective vacuum solution through (24) such that a global charge parameter Q should be viewed as just another mass scale similar to Λ and α_i . A proper way to treat charged black holes in this framework is to first generate the corresponding black hole with $Q = 0$ as a vacuum solution through (24) and subsequently supplement the action (2) by the coupling to an electromagnetic source. The derivation of the Wald entropy (46) is insensitive to a minimal coupling, whereas the solution to the equation of motion (24) is not. This would then be compensated by a work term $\Phi \delta Q$ in (50). A generalization of the framework is presented here to include an electromagnetic field and is an apparent future extension.¹⁵

Example: Let us finish with an exemplary step-by-step application of the previous discussion. Consider the asymptotic anti-de Sitter regular Bardeen black hole in $d = 4$ [17],

$$f(r) = -1 + \frac{r^2}{\ell^2} - \frac{2mr^2}{[r^2 + \gamma^2]^{\frac{3}{2}}}, \quad (67)$$

where the parameter m is related to the black hole mass M by

$$M = \frac{\Sigma_2}{4\pi G_N} m, \quad (68)$$

as can be read off by taking the limit $r \rightarrow \infty$ of (67) and comparing with (28). Moreover, γ is a regularization length parameter whose value we assume to be such that f has a real zero, and hence, the metric describes a black hole. This metric belongs to a simple class of models for which f is linear in M . Invoking the integrated equation of motion (24), i.e., inverting (67) for M , the generating function Ω is identified as

$$\Omega(r, f) = \frac{\Sigma_2}{8\pi G_N \ell^2} \frac{1}{r^2} [r^2 + \gamma^2]^{\frac{3}{2}} [r^2 - (1+f)\ell^2]. \quad (69)$$

¹⁵This could potentially broaden the perspective on near-extremal charged black holes beyond Einstein-Maxwell theory widely discussed based on two-dimensional Jackiw-Teitelboim gravity (see, e.g., [117–120]).

The thermodynamic mass M in (35), temperature T in (37), and entropy S in (46) are then computed as

$$M = \frac{\Sigma_2}{8\pi G_N \ell^2} \frac{1}{r_+^2} [r_+^2 + \gamma^2]^{\frac{3}{2}} [r_+^2 - \ell^2], \quad (70)$$

$$T = \frac{1}{4\pi \ell^2} \frac{1}{r_+} \frac{1}{r_+^2 + \gamma^2} [3r_+^4 - \ell^2 r_+^2 + 2\gamma^2 \ell^2], \quad (71)$$

$$S = \frac{\Sigma_2}{4G_N r_+} \left[[r_+^2 - 2\gamma^2] \sqrt{r_+^2 + \gamma^2} + 3\gamma^2 r_+ \operatorname{arctanh} \left[\frac{r_+}{\sqrt{r_+^2 + \gamma^2}} \right] \right], \quad (72)$$

wherefrom the first law (48) can be verified. In the limit $\gamma \rightarrow 0$ the entropy reduces to the Schwarzschild-anti-de Sitter area law $A/4G_N$ where $A = \Sigma_2 r_+^2$.

VII. CONCLUSIONS

The entropy formula (46) establishes explicitly the corrections to the Bekenstein-Hawking area law for d -dimensional static and asymptotically flat or anti-de Sitter beyond-GR black holes satisfying $g_{tt}g_{rr} = -1$ in Schwarzschild gauge when regarded as vacuum solutions to a general QTG. It will be insightful to supplement the

derivation of the Wald entropy by a computation of the entropy based on Euclidean path integral methods [88,89] to see whether these are equivalent or not [121]. Even though the boundary terms of the d -dimensional theory are generally unknown, one could consider an analysis at the level of the two-dimensional reduced theory using the boundary terms for Horndeski theory [122]. For static black holes these would be a generalized Gibbons-Hawking-York term involving the nonminimal coupling h_4 and supplemented by appropriate counterterms depending on the background asymptotics. It is overall fascinating to anticipate the depth of possibilities of addressing open questions in black hole physics [123–125] based on *generic* two-dimensional dilaton theory as reduced d -dimensional gravity.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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Correction: A typographical error in the title has been fixed.