

Feynman integral reduction by covariant differentiation

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We show how a large class of Feynman integrals can be efficiently reduced to master integrals by suitable covariant differentiation on the vector space dual to the one spanned by the master integrals. The connections needed in the covariant derivatives have to be built only once for a given topology and then applied to any configuration of internal propagator masses. We implement our algorithm in the *Mathematica* code method for reduction of loop integrals (MERLIN).

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I. INTRODUCTION

Any momentum integral arising in perturbation theory of relativistic quantum field theories can be written as a linear combination of so-called master integrals [1,2], with coefficients that are rational functions of Lorentz-invariants such as masses and scalar products of external momenta. The number of master integrals is finite [3–5] and they can be computed systematically using the method of differential equations [6–8]. The main task is then to find the above mentioned coefficients, which is usually done by a well-established but relatively complex algorithm [2] that has been implemented in various computer codes [9–21]. Even for quite simple cases, this algorithm needs significant computing time, and it is therefore interesting to explore alternatives [22–29].

The aim of the present paper is to develop a novel way of reducing Feynman integrals to linear combinations of master integrals. It is based on a differential operator (or covariant derivative) $d + A$, similar to the one that is needed to calculate the master integrals themselves by the method of differential equations. Even though this differential operator still needs to be found by the standard algorithm, this step can be done once and for all for any given topology of diagrams. We implement our method in the *Mathematica* code MERLIN that in this version provides an initial library of differential operator for some simple topologies of up to three loops.

An important subclass of Feynman integrals are so-called vacuum diagrams (also known as vacuum bubbles). These are diagrams that do not possess any external momenta. Their importance stems from their appearance

in the “running and matching” procedure of effective field theories (EFTs) [30–32]. Due to the fact that EFTs amount to a local derivative expansion, one needs the expansion of diagrams around vanishing momenta. The coefficients of this expansion precisely correspond to vacuum bubbles. Our method is particularly well suited for these vacuum diagrams.

This paper is organized as follows. In Sec. II we prove our general method, which is illustrated in Sec. III with some simple examples. We introduce our code MERLIN in Sec. IV and present some conclusions and outlook for future work in Sec. V.

II. REDUCTION BY COVARIANT DIFFERENTIATION

Consider a set of master integrals which we group into a vector $\mathbf{I}$ of dimension N . They are functions of the squared masses u_i , $i = 1 \dots n$ of the n internal propagators. For the time being we take the u_i to be generic, i.e., all different. If we were after this most general mass configuration we could easily obtain all the integrals K with higher powers of propagators by simple differentiation with respect to the masses of a suitable master integral I_a :

$$K(u_i) = \prod_i \frac{(-\partial_i)^{n_i}}{n_i!} I_a(u_i). \quad (1)$$

However, the calculation of the master integrals $\mathbf{I}$ is a formidable task and the resulting multiscale functions are very complicated. Most often we are however interested in a nongeneric mass configuration, sometimes with only one or two mass scales. For instance, one might be interested in the mass configuration $(u_1, u_2, u_3) = (u, u, w)$ for the case of the two-loop sunset vacuum graph in Fig. 1. Let us call this[†] configuration of interest $u_{0,i}$ and define

$$\mathbf{I}_0 \equiv \mathbf{I}(u_{0,i}). \quad (2)$$

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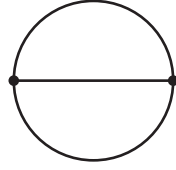


FIG. 1. Two loop vacuum diagram.

Due to the nongeneric mass configuration, a given integral may possess enhanced symmetries. For instance, for the configuration (u, u, w) for the two-loop sunset vacuum graph, the integral is symmetric under the exchange of the last two propagators. Due to these symmetry relations the vector $\mathbf{I}_0$ contains fewer than N independent entries. We will denote the vector of *independent* master integrals $\mathbf{J}$ (with dimension $M \leq N$), and introduce an $N \times M$ matrix Q_0 such that

$$\mathbf{I}_0 = Q_0 \mathbf{J}. \quad (3)$$

The $\mathbf{J}$ are precisely the master integrals one needs to compute for the mass configuration of interest. This set is not only smaller, but the integrals themselves are easier to calculate and considerably simpler functions of the masses. It would therefore be quite advantageous if one could compute integrals with higher propagator powers by differentiation of the reduced master integrals $\mathbf{I}_0$ or $\mathbf{J}$, circumventing the complicated reduction by integration by parts (IBP) identities. Of course a naive differentiation would not work, as such a derivative now acts on several propagators at once. The purpose of this paper is to show how this can nevertheless be done.

The generic set of master integrals (still with all masses different for now) satisfy a differential equation

$$\partial_i \mathbf{I} = -A_i \mathbf{I}, \quad (4)$$

where the A_i are $N \times N$ matrices whose entries are rational functions of the u_i and the spacetime dimension d [33]. For the case of vacuum diagrams this is just the usual differential system used to find the actual master integrals [6–8], for nonvacuum diagrams it is a subsystem where only the internal masses u_i are kept as variables. We write this as

$$D_i \mathbf{I} = 0, \quad (5)$$

where we defined the “covariant derivative”

$$D_i \mathbf{I} \equiv (\partial_i + A_i) \mathbf{I}. \quad (6)$$

The covariant derivative satisfy the integrability conditions $[D_i, D_j] = 0$ [34]. The matrices A_i can be calculated easily with the help of publicly available codes [9–21], and this step has to be done only once for any given graph topology. Notice that the A_i are rational functions of the u_i , as they are

the solution of a system of linear IBP relations with coefficients that are polynomial in the u_i .

Let $\{\mathbf{e}_a\}$ be a basis of unit vectors of the dual vector space (which can be naturally understood as the space of coefficients of the master integrals), then each master integral can be trivially written as

$$I_a = \mathbf{e}_a \cdot \mathbf{I}, \quad (7)$$

where $\mathbf{e}_a$ is the unit vector in the direction a . For a unique choice of n_i and $\mathbf{e}_a$, any integral K with higher powers of propagators than the ones present in $\mathbf{I}$ can be written as

$$\begin{aligned} K(u_i) &= \prod_i \frac{(-\partial_i)^{n_i}}{n_i!} I_a(u_i) \\ &= \left(\prod_i \frac{(-D_i)^{n_i}}{n_i!} \mathbf{e}_a \right) \cdot \mathbf{I}(u_i) \end{aligned} \quad (8)$$

where we used Eq. (5) and defined the contravariant derivative on the dual space

$$D\mathbf{e}_a = (\partial - A^\top) \mathbf{e}_a. \quad (9)$$

We will not distinguish in notation between the covariant and contravariant derivative, it will always be clear from the context.

Therefore, the vector

$$\mathbf{X}(u_i) \equiv \left(\prod_i \frac{(-D_i)^{n_i}}{n_i!} \mathbf{e}_a \right) \quad (10)$$

provides the coefficients for the integral K in terms of the master integrals $\mathbf{I}$. Notice that the fact that the A_i are rational functions of the u_i implies directly that $\mathbf{X}$ is rational as well.

For now, Eq. (8) appears to be just a fancy rewriting of Eq. (1). Note however that the covariant derivatives no longer act on the master integral I_a themselves but rather only on the connection A_i which are relatively simple rational functions. Naively, we could then just evaluate $\mathbf{X}$ and $\mathbf{I}$ at the nongeneric mass configuration, in particular, the components of $\mathbf{I}$ will be written in terms of the reduced set of simpler master integrals. However, the limit

$$K(u_{0,i}) = \lim_{u_i \rightarrow u_{0,i}} \mathbf{X}(u_i) \cdot \mathbf{I}(u_i) \quad (11)$$

turns out to be a little bit subtle. Even though the function $K(u_i)$ [and in particular the master integrals $\mathbf{I}(u_i)$] are regular at u_0 , the limit $\lim_{u_i \rightarrow u_{0,i}} \mathbf{X}(u_i)$ is in general singular.

In order to take the limit properly, let us choose a direction v in the mass space, and write

$$u_i = u_{0,i} + v_i t, \quad (12)$$

where t is some parameter that we want to take to zero. Then

$$K(u_{0,i}) = \lim_{t \rightarrow 0} [X(t) \cdot I(t)] \quad (13)$$

is the integral we are after. In general, this limit exists and is unique, i.e., independent of the direction v [35].

To take the limit, we may expand both X and I in t . As commented above, $X(t)$ is in general singular. More precisely, $X(t)$ is a rational function of t unless the vector v points in a direction such that some of the denominators vanish identically for any t . Therefore, we have the Laurent expansion

$$X(t) = t^{-k} X_{-k} + t^{-k+1} X_{-k+1} + \dots \quad (14)$$

for some positive integer k . This expansion can be straightforwardly done in any specific case. Next we need to expand $I(t)$ up to order t^k which can be performed as follows [36]. First, define

$$\mathcal{A}(t) \equiv v^i A_i(t), \quad (15)$$

which again is rational in t . We can use the equation

$$\left(\frac{d}{dt} + \mathcal{A}(t) \right) I(t) = 0 \quad (16)$$

in order to arrive at a recursive formula. The master integrals have a regular behavior as $t \rightarrow 0$, as the resulting integrals are simply the master integrals with some relations between the masses. We therefore make the ansatz

$$I(t) = I_0 + t I_1 + t^2 I_2 + \dots \quad (17)$$

Furthermore, we will assume that the expansion of the matrix $\mathcal{A}$ takes the form [40]

$$\mathcal{A}(t) = t^{-1} \mathcal{A}_{-1} + \mathcal{A}_0 + t \mathcal{A}_1 + \dots, \quad (18)$$

we straightforwardly get the relation (comparing coefficients of t^{n-1})

$$I_n = -\frac{1}{n + \mathcal{A}_{-1}} [\mathcal{A}_0 I_{n-1} + \dots + \mathcal{A}_{n-1} I_0] \quad n > 0. \quad (19)$$

In particular, solving for the recursion

$$I_1 = -\frac{1}{1 + \mathcal{A}_{-1}} \mathcal{A}_0 I_0, \quad (20)$$

$$I_2 = \frac{1}{2 + \mathcal{A}_{-1}} \left[\mathcal{A}_0 \frac{1}{1 + \mathcal{A}_{-1}} \mathcal{A}_0 - \mathcal{A}_1 \right] I_0, \quad (21)$$

etc. Note that the expansion coefficients are expressed entirely in terms of the reduced set of master integrals I_0 .

Moreover, making use of the relation Eq. (3) we write

$$I(t) = Q(t) J, \quad (22)$$

where $Q(t) = \sum_n t^n Q_n$, and the Q_n satisfy the same recursion relation Eq. (19) with $I_n \rightarrow Q_n$, explicitly

$$Q_1 = -\frac{1}{1 + \mathcal{A}_{-1}} \mathcal{A}_0 Q_0, \quad (23)$$

$$Q_2 = \frac{1}{2 + \mathcal{A}_{-1}} \left[\mathcal{A}_0 \frac{1}{1 + \mathcal{A}_{-1}} \mathcal{A}_0 - \mathcal{A}_1 \right] Q_0, \quad (24)$$

etc.

With the expansion coefficients Q_n and X_n at hand, we can easily obtain the limit Eq. (13),

$$K = \left(\sum_{n=0}^k X_{-n}^T \cdot Q_n \right) \cdot J. \quad (25)$$

One immediate consequence of the expansions Eqs. (17) and (18) inserted in Eq. (16) (from the term of order t^{-1}) is that

$$\mathcal{A}_{-1} I_0 = 0. \quad (26)$$

The symmetries of the reduced set of master integrals I_0 must enforce this equation. However, sometimes trivial symmetries of the diagrams can be combined with IBP identities to yield new nontrivial symmetry relations. Equation (26) can then be used to find these implicit symmetries. An example of this will be provided by the three-loop graph of Sec. III B.

III. EXAMPLES

A. Two-loop vacuum diagrams

Before introducing our code, let us illustrate the method by some examples. Consider first the two-loop vacuum diagrams

$$I_{n_1 n_2 n_3} \equiv \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \prod_{i=1}^3 \frac{1}{(k_i^2 + u_i)^{n_i}}, \quad (27)$$

where $k_1 = q_1$, $k_2 = q_2$, and $k_3 = q_1 + q_2$. These integrals contain both possible topologies, sunset (for all $n_i > 0$) and figure eight (exactly one of the $n_i = 0$ and the other positive). There are four master integrals, that may be taken as

$$I = (I_{011}, I_{101}, I_{110}, I_{111})^T. \quad (28)$$

Taking for example the mass configuration $u_1 = u_2 = u$, $u_3 = w$ the reduced set of master integrals can be taken as

$$J = (J_{011}, J_{110}, J_{111})^T, \quad (29)$$

where $J_{abc}(u, w) = I_{abc}(u, u, w)$. Taking the direction $v = (1, 0, 0)$, we get

$$\mathcal{A}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2-d}{2(u+t)} & 0 & 0 \\ 0 & 0 & \frac{2-d}{2(u+t)} & 0 \\ \frac{d-2}{q} & \frac{-(d-2)(2u-w+t)}{2(u+t)q} & \frac{-(d-2)(w+t)}{2(u+t)q} & \frac{-(d-3)(-w+t)}{q} \end{pmatrix}, \quad (30)$$

where $q = w(w - 4u) - wt + t^2$. Notice that in this particular case $\mathcal{A}_{-1} = 0$, such that the condition Eq. (26) is automatically fulfilled and we do not get any implicit symmetries.

Using Eqs. (23) and (24) one gets the immediate expansion

$$\mathcal{Q}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ \frac{d-2}{2u} & 0 & 0 \\ 0 & \frac{d-2}{2u} & 0 \\ \frac{(d-2)(2u-w)}{2u(w^2-4uw)} - \frac{d-2}{w^2-4uw} & -\frac{d-2}{2u(4u-w)} & -\frac{d-3}{w-4u} \end{pmatrix} + \dots \quad (31)$$

However, for the mass configuration (u, u, w) , neither $\mathcal{A}(t)$ nor $X(t)$ are ever singular, so one really only needs the leading term Q_0 of this expansion. A mass configuration that does give a singular behavior would for instance be provided by $(u, u, 4u)$. We will not spell out this case here.

B. Three-loop vacuum diagrams

All possible three-loop vacuum diagrams (shown in Fig. 2) can be expressed in terms of a single topology (see Fig. 3 for the labels of the propagators)

$$I_{n_1 \dots n_6} \equiv \prod_{\ell=1}^3 \int \frac{d^d q_\ell}{(2\pi)^d} \prod_{i=1}^6 \frac{1}{(k_i^2 + u_i)^{n_i}}, \quad (32)$$

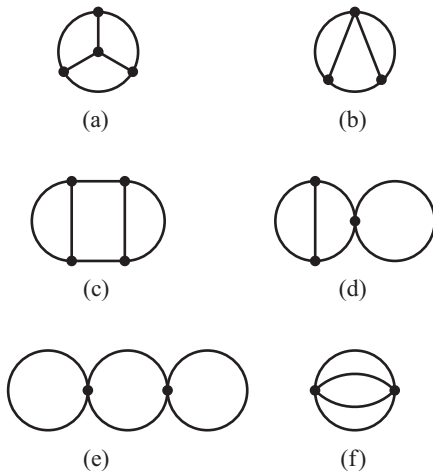


FIG. 2. Three-loop vacuum diagrams. All diagrams can be written in terms of the basic topology (a). Variants of (d) and (e) with the self-loop attached in different ways are not shown.

where $k_{1,2,3} = q_{1,2,3}$, $k_4 = q_1 - q_2$, $k_5 = q_2 - q_3$, $k_6 = q_3 - q_1$.

Any three-loop vacuum graph can be written in this way (potentially with some $n_i = 0$). [41] With all masses different, one has 47 master integrals, and consequently there are six 47×47 matrices A_i . These matrices have to be computed only once, and can be used for the reduction of any three-loop vacuum diagram of arbitrary mass configuration.

Let us say we are interested in the special mass configuration $u_i = u$. By obvious symmetries, only 6 of the 47 original master integrals are independent, one possible set is

$$\begin{pmatrix} I_{001011} \\ I_{001111} \\ I_{011101} \\ I_{011102} \\ I_{011111} \\ I_{111111} \end{pmatrix}_{u_i=u}. \quad (33)$$

However, we have the following IBP identity between the original set of master integrals:

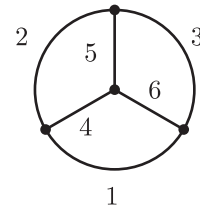


FIG. 3. Basic topology for the three-loop vacuum diagrams.

$$u_2 I_{021101} + u_3 I_{012101} + u_4 I_{011201} + u_6 I_{011102} = \frac{8-3d}{2} I_{011101}. \quad (34)$$

Not all of these integrals belong to the original set of 47 master integrals, so it cannot be used to reduce this set further. However, when all masses are equal, $u_i = u$, we have the trivial symmetry relations $I_{021101} = I_{012101} = I_{011201} = I_{011102}$ and hence

$$4u I_{011102} = \frac{8-3d}{2} I_{011101}. \quad (35)$$

Now, both integrals in Eq. (35) are in the list of master integrals Eq. (33), so we have found a new nontrivial symmetry relation between master integrals that involved the use of IBP identities. For automatization purposes, symmetry relations such as Eq. (35) are much harder to find systematically (in contrast to the trivial symmetry relations that do not involve IBP identities). The condition in Eq. (26) automatically detects these relations and hence can be very helpful to find them in practice.

A minimal set of master integrals can thus be taken as

$$J = \begin{pmatrix} I_{001011} \\ I_{001111} \\ I_{011101} \\ I_{011111} \\ I_{111111} \end{pmatrix}_{u_i=u}. \quad (36)$$

A convenient limit direction is given by $(1, -1, 0, 0, 0, 0)$, and the resulting matrix $Q(t)$ is now a 47×5 dimensional matrix that can easily be computed to any given order in t .

IV. INTRODUCING THE CODE MERLIN

A. Main functions

MERLIN is written in *Wolfram Mathematica* and is compatible with version 12.0 or later, and is available at [42]. The current version provides prebuilt connection matrices A_i for evaluating generic vacuum bubble diagrams with two and three loops, one loop diagrams with up to three external momenta, and two loop diagrams with up to two external momenta (we plan to extend this library considerably in future versions). The program uses the known master integrals of the corresponding Feynman integral families as a basis to construct the matrices A_i required for our method. These master integral datasets were generated using FIRE 6.5.2 [17,43].

The MERLIN package can be loaded with the following commands:

```
SetDirectory[NotebookDirectory[]];
Get["packages/MERLIN.wl"];
```

A minimal example of usage is

```
INITIALIZE["3-loops-vacuum"]
MASSCONFIG[u, u, w, w, w, w];
DIAGRAM[{2, 1, 1, 1, 1, 1}];
EVALUATE;
```

The program requires three prebuilt input files to operate correctly: the matrices, the master integrals, and the incidence matrix [44]. These files are already provided for a variety of simple cases of up to three-loops.

The main user-level functions are

- (i) INITIALIZE[]: Loads the initial data: matrices, master integrals, and incidence matrix that were built for a given topology (e.g., "3-loops-vacuum").
- (ii) MASSCONFIG[]: Specifies the mass configuration for the diagrams. It also reduces the list of master integrals applying the symmetries, as well as any implicit symmetries.
- (iii) DIAGRAM[]: Specifies the diagram to be evaluated. The function automatically detects multiple entries, given as lists.
- (iv) EVALUATE: Evaluates the chosen diagrams using the currently loaded data and mass configuration.

After a successful evaluation, the program saves each result in a separate .dat file, named using the structure NAME{diagram}{MASSCONFIG}, where NAME refers to the entry specified in the INITIALIZE command. The results are then automatically loaded and stored in the output variable RESULTS. It is important to note that the EVALUATE function checks whether the diagram, mass configuration, and initial data belong to the same topology and will automatically load any previously saved result [45].

B. Internal structure overview

The core evaluation routine, EVALUATE, performs all computational steps required to obtain the result for a given diagram. Internally, it constructs the coefficients of the chosen diagram, applies explicit and implicit symmetry relations, determines a valid limit direction for the expansion, and performs the corresponding series expansions of the coefficients, master integrals, and matrices.

The symmetry treatment in MERLIN is fully automated. Explicit symmetries, arising from the topology and mass configuration, as well as implicit symmetries, are handled through dedicated internal routines. This systematic reduction eliminates redundant master integrals and simplifies subsequent evaluations.

C. Integration with FIRE

As explained above, our program requires as input the connection matrices, the list of master integrals, and the incidence matrix, which in this release are provided for a variety of simple graph topologies. We plan to extend this library substantially in future versions.

However, a desirable feature would be an integration with FIRE (or similar codes) in order to provide an interface for the creation of the matrices A_i , for instance if the user would like

to use a different basis than the one provided or by going beyond the library of topologies included in the distribution. This will be added in a future version. However, already in this initial release we provide a preliminary version of a module called GRIMOIRE. It works with FIRE 6.5.2 [17,43] and builds both the connection matrices as well as the list of master integrals, along with the incidence matrix. A self-explanatory template file (GRIMOIRE_TEMPLATE) illustrating the use of GRIMOIRE is included in the above-mentioned GitHub repository.

D. Examples

1. Two-loop vacuum diagrams

Consider the two-loop example in Sec. III A. To obtain the result, one needs to use the following commands:

```
INITIALIZE["2-loops-vacuum"]
DIAGRAM[{2, 2, 1}];
MASSCONFIG[u, u, w];
EVALUATE;
```

After the evaluation is complete, the results for the chosen diagram are stored in the variable RESULTS:

```
((2-d) G[{0,1,1}]) / (2 u (4 u-w))
+ ((-2+d) G[{1,1,0}]) / (2 u (4 u-w))
- ((-3+d) G[{1,1,1}]) / (4 u-w)
```

Here, the functions $G[\{\}]$ denote the master integrals.

2. Three-loop vacuum diagrams

For the three-loop example, e.g., with all masses equal, one needs to use the following commands:

```
INITIALIZE["3-loops-vacuum"]
MASSCONFIG[u, u, u, u, u, u];
```

After executing this command, one obtains the following results for both the reduced master integral list and the implicit symmetry relations:

Reduced master integral list (MASTERRED):

```
{G[{0,0,1,0,1,1}], G[{0,0,1,1,1,1}],
G[{0,1,1,1,0,1}], G[{0,1,1,1,0,2}],
G[{0,1,1,1,1,1}], G[{1,1,1,1,1,1}]}
```

Implicit symmetries (IMPLICITRULES):

```
{G[{0,1,1,1,0,2}] -> -((-8+3 d)
G[{0,1,1,1,0,1}]) / (8 u)}
```

V. CONCLUSIONS

In this paper we have introduced a new method to reduce a large class of Feynman integrals to linear combinations of master integrals. The method proceeds by covariant differentiation on the space dual to the one spanned by the master integrals, with connections A_i that have to be computed only once for a given topology, and is then useable for

arbitrary internal mass configurations. The limit to the mass configuration of interest is then obtained by a simple power series expansion in one auxiliary variable.

We have implemented the method in the code MERLIN. Various improvements of our code are planned for future versions.

- (i) The present version contains the connections A_i for simple topologies such as the two and three-loop vacuum graphs and some simple one and two-loop nonvacuum graphs. We plan to expand this library considerably for future versions.
- (ii) Our main algorithm involves only simple algebraic operations (differentiation and matrix multiplication), these operations are very fast and the only time consuming step is the simplification of intermediate/final results. We have tried various smarter routines (for instance, by adding more intermediate simplifications). However, there seems to be no universal choice that speeds up all cases, usually one gains in some cases and loses in others. We however expect to be able to improve the performance considerably by an adaptive simplification algorithm that identifies the optimal choice for each case separately. This is also left for future work.
- (iii) Concerning the integration with FIRE mentioned in Sec. IV C, a fully tested and documented version of GRIMOIRE will be included in the future version of the code.
- (iv) The method is not yet suitable to handle general irreducible scalar products (ISPs), besides those that appear already in the master integrals themselves. This may be resolved by adding auxiliary mass variables for the ISPs.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [42].

APPENDIX A: GENERALIZATION OF THE EXPANSION FOR $I(t)$

In the main text, the expansion in Eq. (17) was derived under the assumption that Eq. (18) holds (which, in fact, was the case for all the family of Feynman graphs we have considered so far). In this appendix we outline the generalization for more singular matrices $\mathcal{A}(t)$.

Consider, for instance, the case $\mathcal{A}_k = 0$ for $k < -2$, such that the leading nonzero coefficient is $\mathcal{A}_{-2}$. Then, Eq. (16), to order t^{-2} yields $I_0 \in \ker \mathcal{A}_{-2}$, or

$$P_{\text{row}} I_0 = 0, \quad (\text{A1})$$

where P_{row} is the projector on the row space of $\mathcal{A}_{-2}$. Moreover, from order t^{-1} of the same equation we get the additional constraint

$$(1 - P_{\text{col}}) \mathcal{A}_{-1} I_0 = 0, \quad (\text{A2})$$

where P_{col} is the projector on the column space of $\mathcal{A}_{-2}$.

Next, determine I_1 in terms of I_0 . We look at $P_{\text{row}} I_1$ and $(1 - P_{\text{row}}) I_1$ separately. From terms of order t^{-1} we have

$$P_{\text{row}} I_1 = -\mathcal{A}_{-2}^+ \mathcal{A}_{-1} I_0, \quad (\text{A3})$$

where the superscript $+$ denotes the Moore-Penrose pseudoinverse, which satisfies

$$\begin{aligned} \mathcal{A}_{-2}^+ \mathcal{A}_{-2} &= P_{\text{row}} \\ \mathcal{A}_{-2} \mathcal{A}_{-2}^+ &= P_{\text{col}} \\ \mathcal{A}_{-2}^+ &= \mathcal{A}_{-2}^+ P_{\text{col}} \\ \mathcal{A}_{-2}^+ &= P_{\text{row}} \mathcal{A}_{-2}^+. \end{aligned} \quad (\text{A4})$$

The complement to Eq. (A3) can be obtained from the t^0 coefficient of Eq. (16). Projecting the latter with $(1 - P_{\text{col}})$ gives

$$\begin{aligned} (1 - P_{\text{col}})(1 + \mathcal{A}_{-1})(1 - P_{\text{row}}) I_1 \\ = -(1 - P_{\text{col}}) \mathcal{A}_0 I_0 - (1 - P_{\text{col}})(1 + \mathcal{A}_{-1}) P_{\text{row}} I_1. \end{aligned} \quad (\text{A5})$$

The right hand side is already known. As in the main text, we assume that (for $n \neq 0$), the square matrix $n + \mathcal{A}_{-1}$ has full rank. Moreover, let the ranks of $1 - P_{\text{col}}$ and $1 - P_{\text{row}}$ be M , then, barring some accidental alignment, the matrix $\mathcal{B}_n \equiv (1 - P_{\text{col}})(n + \mathcal{A}_{-1})(1 - P_{\text{row}})$ also has rank M and its column and row spaces are precisely orthogonal to those of $\mathcal{A}_{-2}$. Therefore, we can write

$$(1 - P_{\text{row}}) I_1 = -\mathcal{B}_1^+ [\mathcal{A}_0 I_0 + (1 + \mathcal{A}_{-1}) P_{\text{row}} I_1]. \quad (\text{A6})$$

The higher order terms are analogous. In particular, $P_{\text{row}} I_n$ follows from the t^{n-2} term, and $(1 - P_{\text{row}}) I_n$ follows from the t^{n-1} term. Explicitly

$$\begin{aligned} P_{\text{row}} I_n &= -\mathcal{A}_{-2}^+ [(n - 1 + \mathcal{A}_{-1}) I_{n-1} + \mathbf{K}_{n-2}] \\ (1 - P_{\text{row}}) I_n &= -\mathcal{B}_n^+ [(n + \mathcal{A}_{-1}) P_{\text{row}} I_n + \mathbf{K}_{n-1}], \end{aligned} \quad (\text{A7})$$

where

$$\mathbf{K}_n \equiv \sum_{k=0}^n A_k I_{n-k}. \quad (\text{A8})$$

The system in Eq. (A7) thus is the sought-after generalization of Eq. (19).

APPENDIX B: MERLIN INTERNAL STRUCTURE

The EVALUATE function performs all steps required to obtain the result for a given diagram. It relies on the following internal routines:

- (i) PREPAREDIFFERENTIAL: Generates the linear combination of master integrals for the chosen diagram (10).
- (ii) SYMMETRY: Automatically determines the explicit symmetries of the diagram, see Sec. II.
- (iii) IMPLICITSYM: Identifies implicit (nontopological) symmetries, see Sec. II.
- (iv) FINDLIMITDIRECTION: Determines a valid *limit direction* for expansions, see Eq. (12).
- (v) DIFFSERIESEXPANSION: Computes the series expansion of the coefficients generated by PREPAREDIFFERENTIAL, see Eq. (14).
- (vi) SERIESEXPANSION: Expands the master integrals and matrices, see Eqs. (17) and (18).

MERLIN automatically handles symmetries in vacuum diagrams, via SYMMETRY and IMPLICITSYM, from these two function ones obtain

- (a) Explicit symmetries: Evident from the topology and mass configuration, SYMMETRY.
- (b) Implicit symmetries: Subtler relations not directly apparent from topology, IMPLICITSYM.

Their results can be inspected through:

- (a) MASTERRED: The reduced list of master integrals with symmetries applied.
- (b) IMPLICITRULES: The set of implicit symmetry rules used in the final result.

MERLIN determines a suitable limit direction automatically using FINDLIMITDIRECTION. This function generates a vector of the same length as MASSCONFIG and tests whether the mass limit MASSCONFIG + t * LIMITDIRECTION introduces singularities or indeterminate expressions. If no singularities are found, the vector is stored as the LIMITDIRECTION and used in the series expansion. Note that this procedure depends only on the mass configuration and the matrices, not on the specific diagram.

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- [40] As a rational function, $\mathcal{A}$ must have an expansion $\mathcal{A} = \sum_{n=\ell}^{\infty} \mathcal{A}_n t^n$ for some integer ℓ . In all the examples that we considered, $\ell \geq -1$, which we will assume in the following for simplicity. If $\ell < -1$ the formalism can be generalized, see Appendix A.
- [41] Notice that some naively different topologies [such as the diagram in Fig. 2(c)] require the use of some algebra, i.e.,
- $$\frac{1}{k_1^2 - u_1} \frac{1}{k_1^2 - u'_1} = \frac{1}{u_1 - u'_1} \left(\frac{1}{k_1^2 - u_1} - \frac{1}{k_1^2 - u'_1} \right),$$
- etc.
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- [44] The incidence matrix B of a graph with V vertices and E edges is defined to be the $V \times E$ matrix whose entries B_{ij} are one if edge j connects to vertex i , and zero otherwise. Internal lines starting and ending at the same vertex are by convention columns with all zeroes. In MERLIN we delete the rows of B that correspond to external points of the graph. The code uses these incidence matrices to efficiently compute symmetries of the diagram.
- [45] To refresh all results, the corresponding data files must be deleted from the `results` folder.