

Perturbative hyperboloidal extraction of gravitational waves in 3 + 1 numerical relativity

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We present a framework to propagate to null infinity gravitational waves computed at timelike world tubes in the interior of 3 + 1 (Cauchy) numerical relativity simulations. In our method, numerical relativity data are used as the inner inflowing boundary of a perturbative time-domain Regge-Wheeler-Zerilli simulation in hyperboloidal coordinates that reaches null infinity. We showcase waveforms from (3 + 1)D simulations of gravitational collapse of rotating neutron stars, binary black hole mergers and scattering, and binary neutron star mergers and compare them to other extrapolation methods. Our perturbative hyperboloidal extraction provides a simple yet efficient procedure to compute gravitational waves with data quality comparable to the Cauchy-characteristic extraction for several practical applications. Nonlinear effects in the wave propagation are not captured by our method, but the present work is a stepping stone toward more sophisticated hyperboloidal schemes for gravitational-wave extraction.

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I. INTRODUCTION

Gravitational-wave astronomy relies on numerical relativity (NR) computations for accurate predictions of gravitational waves from strongly gravitating and fast moving sources. Examples of such sources are stars undergoing gravitational collapse and compact binary mergers. Formalisms and algorithms for wave extraction play a key role for the computation of NR waveforms; the main target is to compute unambiguous data at future null infinity ($\mathcal{I}^+$).

In the context of 3 + 1 NR with Cauchy-type foliations, waveforms can be extracted at coordinate spheres at finite radii on spacelike hypersurfaces. Waveform extraction algorithms are built on either on the Regge-Wheeler-Zerilli-Moncrief (metric) formalism for perturbations of spherical spacetimes [1–7] or the Newman-Penrose (curvature) formalism for perturbations of Kerr spacetimes (in particular, using Weyl’s ψ_4 scalar) [8]. Independent of the specific technique, these waveforms are affected by two main systematic errors: (i) the “artificial” timelike boundary conditions of the main evolution system and (ii) the finite radius of extraction. A Cauchy perturbative matching extraction algorithm, in which the Cauchy evolution of the

strong-field spacetime region is matched to a perturbative evolution that provides both waveforms and boundary condition, was proposed to alleviate these issues [2,3]. However, the method still suffers from finite extraction error and it has been abandoned in modern simulations.

A commonly employed technique for the computation of waveforms at $\mathcal{I}^+$ is to extrapolate in radius and (approximate) retarded time, e.g., [9–12]. Alternatively, a more sophisticated approach is the Cauchy-characteristic extraction (CCE). In CCE, data from a world tube in the interior of a Cauchy domain are propagated to null infinity using a null-cones formulation of Einstein equations, i.e., a characteristic evolution [13–19]. CCE allows, in principle, a more rigorous extraction of waveforms at $\mathcal{I}^+$. However, it retains the boundary condition issue and it is, in practice, sensitive to the choice of the world tube [18–20]. The development of CCE was originally motivated as a preparatory step for Cauchy-characteristic matching [13,21,22]. In the latter approach, the Cauchy evolution of the strong-field spacetime region is matched to a characteristic evolution that provides both waveforms at $\mathcal{I}^+$ and boundary conditions, see [23] for a review. Significant progress has recently been made with this technique, e.g., [17,19,22,24,25], which can deliver robust waveforms for modeling purposes. Note, however, that the characteristic formulation of the Einstein field equations is weakly hyperbolic with potential implications on the convergence of numerical data [26].

The use of hyperboloidal foliations in applications to gravitational-wave astronomy and waveform extraction is

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less developed. Hyperboloidal foliations provide a natural framework for the initial-boundary value problem in general relativity and has been extensively studied in mathematical relativity, e.g., [27–32] (and [33–36] for reviews). However, the hyperboloidal approach is not yet fully established for astrophysical solutions to the full Einstein equations in $(3+1)$ dimensions $[(3+1)\text{D}]$. Recent efforts included a full solution of the perturbation problem in hyperboloidal coordinates [37–40], advances in the understanding of the quasinormal mode spectrum [41–45], spherically symmetric black hole evolutions with $\mathcal{I}^+$ -fixing compactified hyperboloidal coordinates [46–49], dual foliation approach with asymptotic regularization [50–52], and the first simulation of gravitational waves from past to future null infinity with conformal field equations [53].

This work puts forward a simple and yet effective hyperboloidal method to propagate to null infinity the gravitational waveforms computed in a $3+1$ NR simulation. The method builds on the CCE idea of propagating data from a world tube in the interior of a Cauchy domain to null infinity. However, here the propagation is performed perturbatively using compactified hyperboloidal slices of Schwarzschild spacetime. This perturbative hyperboloidal extraction (PHE) is described in Sec. II.

The PHE method is validated in Sec. III, where PHE waveforms at $\mathcal{I}^+$ are presented from $(3+1)\text{D}$ simulations of gravitational collapse, binary black holes, and binary neutron star mergers. PHE waveforms are systematically compared to waveforms extracted at finite radius, waveform extrapolated in radius, and CCE waveforms computed from the same simulations.

The paper ends with conclusions that discuss limitations and future development toward nonlinear evolutions with hyperboloidal foliations.

Throughout this paper, we use geometric units $G=c=1$ and express masses and lengths in solar masses ($M_\odot$). Partial derivatives are indicated with a suffix, e.g., $\Psi_x \equiv \partial_x \Psi$. Complex waveform modes are decomposed into amplitude and phase according to $\Psi_{\ell m} = A_{\ell m} e^{-i\phi_{\ell m}}$.

II. METHOD

Numerical relativity simulations in the $3+1$ Cauchy approach provide waveforms on a timelike world tube in the interior of the Cauchy domain by stacking data in time at a coordinate Schwarzschild radius $\bar{r}$. The (approximation to the) strain at finite radius is given by the spin-weighted ($s = -2$) mode expansion

$$h_+ - ih_\times = \sum_{\ell \geq 2} \sum_{m=-\ell}^{\ell} \frac{N_\ell}{D_L} \left(\Psi_{\ell m}^{(e)}(u) + i\Psi_{\ell m}^{(o)}(u) \right)^{-2} Y_{\ell m}(\theta, \phi), \quad (1)$$

where D_L is the luminosity distance, $\Psi_{\ell m}^{(e/o)}(u)$ are the modes of the Regge-Wheeler-Zerilli (RWZ) master

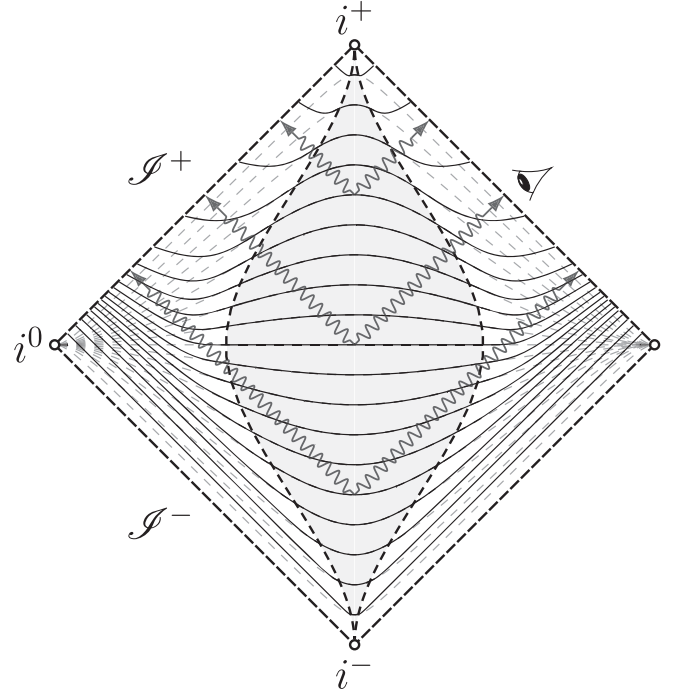


FIG. 1. Hyperboloidal layer foliation in a representative Penrose diagram of an asymptotically simple spacetime. The shaded region contains a truncated Cauchy foliation. Solid lines show the Schwarzschild foliation employed in our work [40]; the two dashed lines connecting past (i^-) and future (i^+) timelike infinity indicate the transition layer between Cauchy and hyperboloidal coordinates ($r_* = \rho = r_*^L$). PHE gets inner boundary data at $r_* \lesssim r_*^L$ from a $3+1$ NR simulation and propagates them to $\mathcal{I}^+$.

functions of even and odd parity as a function of the retarded time u and $N_\ell = \sqrt{(\ell+2)!/(\ell-2)!}$. In perturbation theory of spherical spacetimes and far from the source, these functions satisfy the $1+1$ RWZ equation,

$$\Psi_{tt} - \Psi_{r_* r_*} + V\Psi = 0, \quad (2)$$

which is written above in tortoise coordinates (t, r_*) and where V is the ℓ -dependent potential of the background metric [superscripts (e/o) and multipolar labels are dropped in the notation for Ψ and V].

To perform PHE, we inject the RWZ mode $\Psi^{\text{NR}}(t, r)$ from a $3+1$ NR simulation as ingoing boundary data into a $1+1$ RWZ evolution code using hyperboloidal slices, which propagates the waveform to future null infinity ($\mathcal{I}^+$). Specifically, here we employ a hyperboloidal layer [39,40], which is illustrated in Fig. 1. The algorithm and implementation details are described below.

The boundary condition for the $1+1$ RWZ evolution is prescribed through the characteristic flowing into the computational domain,

$$\Psi_t - \Psi_{r_*} = g(t) \quad \text{at } \bar{r}_* = r_*(\bar{r}), \quad (3)$$

where $g(t)$ is the given data at the extraction radius $\bar{r}$. The RWZ equation (2) is then solved on a “wave-zone” domain with $r_* \geq \bar{r}_* = r_*(\bar{r})$.

The time-domain solution at $\mathcal{I}^+$ is obtained using the hyperboloidal layer technique introduced by Zenginoğlu [39,54]. We use the setup developed in [40], to which we refer for all the details. Hyperboloidal coordinates (τ, ρ) are obtained from (t, r_*) by the transformation

$$\tau = t - h(r_*), \quad r_* = \frac{\rho}{\Omega(\rho)}, \quad (4)$$

where $h(r_*)$ is the height function and the function $\Omega(\rho)$ implements a compactification of the unbounded domain $r_* \in [r_*^L, +\infty)$ to $\rho \in [\rho_*, \rho_S]$. The coordinate $\rho_* = r_*^L > \bar{r}_*$ locates the position of the layer, while ρ_S is the coordinate location of $\mathcal{I}^+$. The time coordinate τ is fixed by demanding that outgoing null rays are invariant in the layer, i.e.,

$$u = t - r_* = \tau - \rho, \quad (5)$$

which implies $d\rho/dr_* = 1 - h'(r_*) =: 1 - H(\rho)$, where $H(\rho)$ is called the boost function. The transformation is completed with a choice of $\Omega(\rho)$ such that $\Omega = 1$ for all $\rho < \rho_*$ and some differentiability across the layer is guaranteed [40].

The RWZ equation on the hyperboloidal slices is

$$\begin{aligned} & - (1 + H)\Psi_{\tau\tau} - 2H\Psi_{\tau\rho} - H_\rho(\Psi_\tau + \Psi_\rho) \\ & + (1 - H)\Psi_{\rho\rho} + \frac{V}{1 - H}\Psi = 0, \end{aligned} \quad (6)$$

and it reduces to Eq. (2) for $\rho \leq \rho_*$. The equation requires no boundary condition at $\mathcal{I}^+$. Equation (3) is imposed at the timelike inner boundary located at the same Schwarzschild radius of the $3 + 1$ extraction sphere, $\bar{r}_* < r_*^L$.

The numerical implementation follows closely the RWZHyp code [40]. The RWZ equation for each mode is solved in first order in time and second order in space form adopting the method of lines and the Runge-Kutta fourth-order scheme. The right-hand side is discretized with an evenly space grid of N nodes in $\rho \in X = [\rho_{\min}, \rho_S]_{\rho_*}$, where $\rho_{\min} = \bar{r}_*$ and $\rho_* = r_*^L > \bar{r}_*$ denotes the interface to the hyperboloidal layer. Fourth-order finite differences are employed for the derivatives: centered stencils are used in the bulk of the domain, lopsided or sided stencils for the outermost points. No boundary data nor ghost points are used around $\mathcal{I}^+$. The inner boundary data with Eq. (3) are implemented following the fourth-order prescription of Calabrese and Gundlach [55]; the discrete initial-boundary value problem is well posed in standard energy norms. Initial data for Ψ and Ψ_t are zero.

The $3 + 1$ NR data employed here were presented in recent work by some of us [7]. The simulations are performed with GR-Athena++ [56,57] using the Z4c free-evolution scheme

and the puncture gauge [58,59] and general relativistic hydrodynamics. We refer the reader to the above papers for all the details. Gravitational waves are extracted at spheres of coordinate (isotropic) radii in the range $R \sim [100, 800]$ using a covariant and gauge-invariant RWZ metric extraction algorithm that provides Ψ^{NR} at the extraction sphere [7]. This choice of world tubes is typical for astrophysical simulations; it strikes a balance between large extraction radii and the resolution of the (derefined) 3D mesh in the wave zone.

The PHE of these NR data is computed with a grid $X = [\rho_{\min}, \rho_S]_{\rho_*} = [\bar{r}_*, 2\bar{r}_*]_{3\bar{r}_*/2}$ where the innermost point is located at the extraction sphere $\rho_{\min} = \bar{r}_* = r_*(\bar{r}(R))$, the layer is located at $\rho_* = 3\bar{r}_*/2$, and the outermost point ($\mathcal{I}^+$) is located at $\rho_S = 2\bar{r}_*$. The resolutions employed in this work are $N = 101, 201, 401, 801, 1601, 3200$. Simulations with $N \gtrsim 1601$ are well converged and truncation errors are practically negligible in the comparison with other extraction methods. The RWZ waveforms are also extrapolated in $1/R$; we refer to these waveforms to as “ R extrapolated.” Fontbuté *et al.* [7] experimented with different extrapolation formulas and comparison with CCE data. Best results are obtained using a variant of Eq. (4) in Nakano *et al.* [12],

$$\Psi(u) = \left(1 - \frac{2M}{\bar{r}}\right) \left(\Psi(u, \bar{r}) - \frac{\ell(\ell+1)}{2\bar{r}} \int_0^u dt \Psi(t, \bar{r}) \right), \quad (7)$$

which analytically accounts for $\mathcal{O}(1/r^2)$ terms. CCE waveforms are computed from the same simulations using the PittNull code [22,60]. Note that these CCE data are in terms of Weyl’s scalar or News function and thus require integration to compute the RWZ multipoles (or, equivalently, the strain). Therefore, CCE waveforms have been further crossed-checked with the spECTRE CCE implementation [19,24], which additionally provides the strain. We find good agreement between the two outputs and show the former data for most of the cases.

III. RESULTS

A. Toy boundary data

As a first toy problem, we consider a Schwarzschild spacetime with mass $M = 1$ and the boundary function

$$g(t) = B_{ab}(t) \sin(t/8), \quad (8)$$

where $B_{ab}(t)$ is a smooth bump (or activation) function between $t = a$ and $t = b$ such that $B_{ab}(t \leq a) = 0$ and $B_{ab}(t \geq b) = 1$. The grid is $X = [100, 200]_{150}$ in units of M and the resolutions employed are $N = 101, 201, 401$.

Figure 2 shows the convergence of the RWZ waveform at $\mathcal{I}^+$. The differences between waveforms computed at different resolutions overlap once rescaled by the appropriate convergence factor for fourth-order accuracy. Some noise is introduced in the convergence plot if the activation

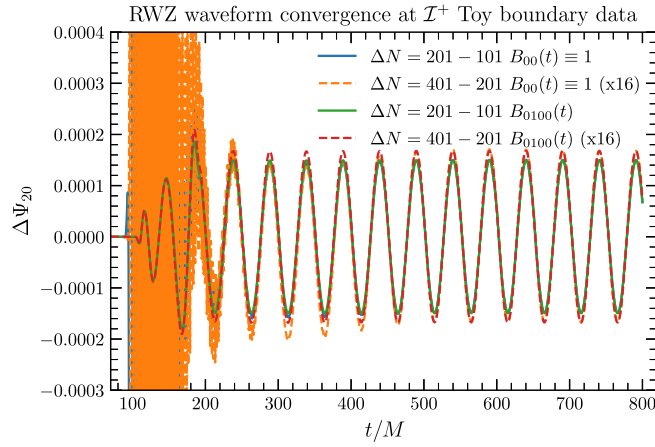


FIG. 2. Convergence of RWZ waveforms at $\mathcal{I}^+$ with the toy boundary data. Differences between a pair of waveforms at resolutions $N = 101, 201, 401$ with and without using the smooth bump function in the boundary data. The difference between data at the two highest resolutions is rescaled by a factor 16, corresponding to fourth-order convergence (dashed lines).

function is not used [$B_{00}(t) \equiv 1$]. This is due to the nonsmooth injection of boundary data. The issue can be cured by using the activation function, for example, with parameters $(a, b) = (0, 100)$. In the next applications, the bump function is not used since the inner boundary data are sufficiently smooth and do not introduce noise.

B. Test-mass effective-one-body insplunge

The next test performed to validate the PHE method employs the RWZHyp code to generate a waveform from the inspiral and plunge of a test-mass source subject to an effective-one-body radiation reaction, e.g., [40,61,62]. The radiation force is triggered by setting a symmetric mass ratio of $\nu = 10^{-2}$; the background spacetime mass is $M = 1$. RWZ waveforms are extracted both in the Cauchy region of the computational domain, specifically at $\bar{r}_* = 100$, and at $\mathcal{I}^+$. The waveform extracted at $\bar{r}_* = 100$ is injected in a new set of PHE simulations¹ as boundary data,

$$g(t) = \Psi_t^{\text{RWZ}} - \Psi_{r_*}^{\text{RWZ}}. \quad (9)$$

The waveform propagates (again) to $\mathcal{I}^+$ and is compared to the result of the full RWZHyp simulation (reference data). This is a self-consistency check of the method. The PHE grid is resolved with $N = 101, 201, 401, 801$ points, which are at least a factor 10 lower than the reference RWZHyp data (the latter uses $X = [-50, 200]_{150}$ and $N = 12501$).

Figure 3 shows the $\ell m = 22$ PHE waveform at $\mathcal{I}^+$ from various resolutions compared to the reference data. The waveforms are correctly converging to the reference data

¹The PHE simulations are performed with an independent code, implementing the same methods.

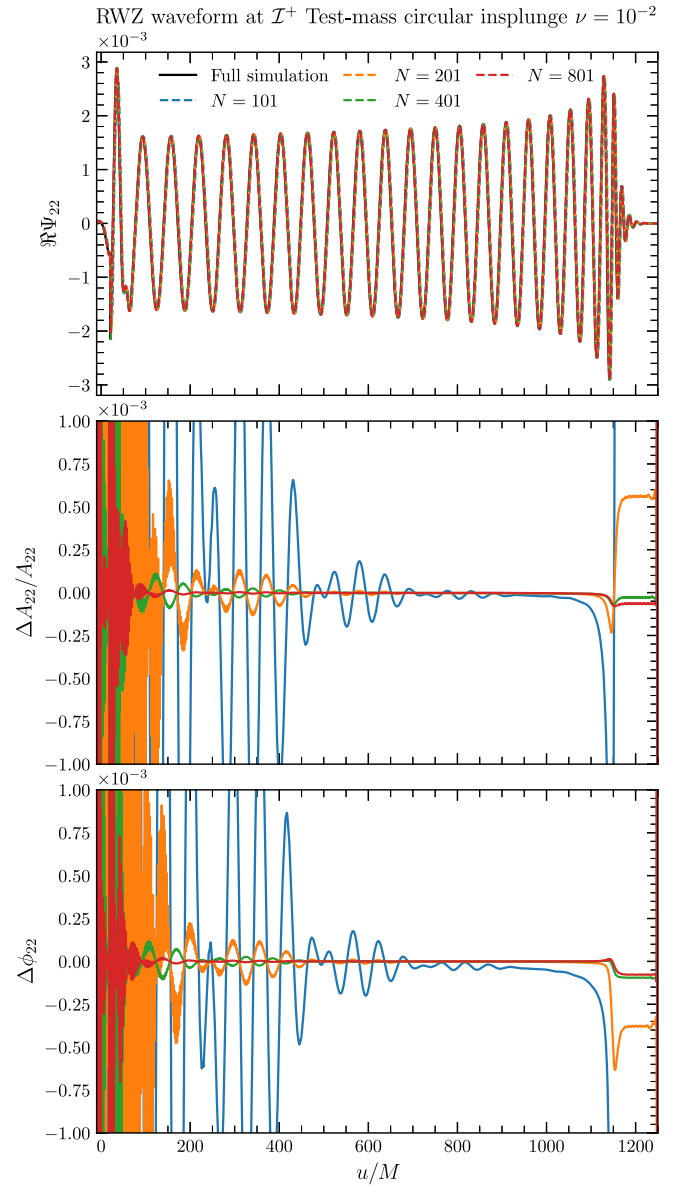


FIG. 3. RWZ waveform at $\mathcal{I}^+$ from a test-mass insplunge simulation. Top: real part of Ψ_{22} from the original simulation (black solid line) and from PHE using data $r_* = 100$ and increasing resolutions $N = 101, 201, 401, 801$. Middle: relative amplitude differences to the full simulation for the different resolutions. Bottom: phase differences to the full simulation for the different resolutions.

with phase differences of $\Delta\phi_{22} \lesssim 10^{-4}$ and relative amplitude differences $\Delta A_{22}/A_{22} \sim 2 \times 10^{-4}$. Note that the phase differences between the (2,2) waveforms extracted at $r_* = 100$ and $\mathcal{I}^+$ are larger than $\Delta\phi_{22} \gtrsim 0.1$ rad and out of the plot scale (see, e.g., Fig. 9 of [40]). Similarly, typical amplitude errors due to finite extraction are $\Delta A_{22}/A_{22} \gtrsim 2 \times 10^{-3}$.

C. Gravitational collapse

Moving to applications of PHE to (3 + 1)D NR data, we consider waveforms from the gravitational collapse of an

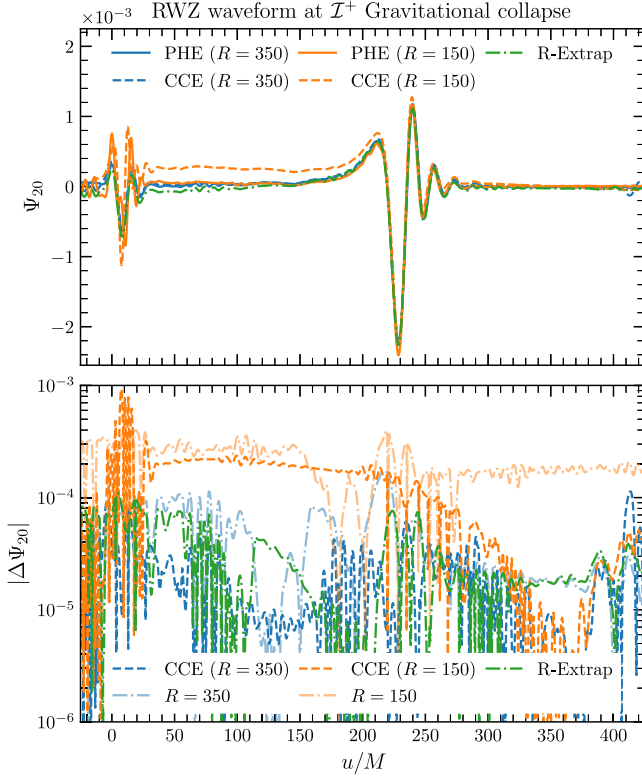


FIG. 4. RWZ waveform at $\mathcal{I}^+$ from the gravitational collapse of a rotating neutron star simulation. Top: Ψ_{20} extrapolated with PHE (solid) and CCE (dashed) from world tube data at $R = 350M_\odot$ (blue) and $150M_\odot$ (orange) and best finite radius extrapolation (dash-dotted green, at $R = 150M_\odot$). Bottom: absolute differences between the PHE waveform and others: $|\Delta\Psi_{20}| := |\Psi_{20}^{\text{PHE}} - \Psi_{20}^X|$, where the subtrahend X waveforms are indicated in the legend. Light dash-dotted lines refer to differences to RWZ waveforms extracted at $R = 350M_\odot$ (blue) and $150M_\odot$ (orange). These RWZ waveforms are not shown in the top panel.

unstable rotating neutron star with mass $M \simeq 1.86$ and uniform rotation close to the mass-shedding (Kepler) limit [7], cf. [57,63,64] for similar simulations. The waveforms $\Psi^{\text{NR}}(t)$ extracted at finite radius are injected in a set of PHE simulations using

$$g(t) = \Psi_t^{\text{NR}} - \Psi_{r_*}^{\text{NR}} \quad (10)$$

and propagated to $\mathcal{I}^+$.

Figure 4 compares the dominant RWZ $\ell m = 20$ mode at $\mathcal{I}^+$ from different propagation (extrapolation) methods. The top panel shows the RWZ waveforms: PHE from world tubes at $R = 350, 150$ (blue and orange solid lines), CCE from world tubes at $R = 350, 150$ (blue and orange dashed lines), and a “best” R extrapolated using $R = 150$ (green dash-dotted line). All waveforms have the characteristic “precursor-burst-ringdown” morphology [65] and are broadly compatible with each other. The bottom

panel shows absolute differences $|\Delta\Psi_{20}| := |\Psi_{20}^{\text{PHE}} - \Psi_{20}^X|$ between the PHE waveform and the others relative to the same world tube; the legend indicates the subtrahend X waveform. The bottom panel also shows in lighter colors the differences with respect the two waveforms extracted at $R = 350$ (blue dash-dotted) and $R = 150$ (orange dash-dotted).

The differences between PHE and CCE data converge to zero, indicating that the waveforms computed with the two methods converge as the world tubes are placed at larger R . PHE-CCE differences in the precursor for the choice $R = 150$ are of the same order as the differences between PHE and RWZ at $R = 150$. The CCE extraction at $R = 150$ is therefore not sufficiently accurate due to the choice of the world tube. The differences PHE-CCE reduce about an order of magnitude for the choice $R = 350$. Some unphysical (and small) oscillatory features in the precursor of the RWZ waveforms at finite radius are propagated to $\mathcal{I}^+$ by the PHE. These features are reduced for world tubes with larger radii and are not present in the CCE waveforms. The R -extrapolated waveform agrees well with both PHE and CCE.

D. Binary black hole mergers

Next we consider three waveforms from nonspinning equal-mass binary black hole simulations with very different morphology and generated in a circular two-to-three orbit merger, a dynamical capture, and a scattering simulation.

1. Circular merger

The binary mass is $M \simeq 1$ and the initial separation is such that the binary undergoes about two orbits to merger, see, e.g., [56,66] for similar simulations. Gravitational waves are extracted at coordinate (isotropic) radii $R = 400, 220, 100, 140$. Waveforms from the largest extraction radii are significantly affected by the resolution of the Cartesian grid and we thus focus on the smallest two radii. The waveforms $\Psi^{\text{NR}}(t)$ extracted at finite radius are injected in a set of PHE simulations using Eq. (10) and propagated to $\mathcal{I}^+$. The PHE grid is resolved with $N = 201, 401, 801, 1601, 3201$ points. PHE waveforms show convergence similar to that discussed above; only the highest resolution is discussed below.

Figure 5 compares the dominant RWZ $\ell m = 22$ mode at $\mathcal{I}^+$ from different propagation (extrapolation) methods. The top panel shows the RWZ waveforms: PHE from world tubes at $R = 140, 100$ (blue and orange solid lines), CCE from world tubes at $R = 140, 100$ (blue and orange dashed lines), and a best R extrapolated using $R = 140$ (green dash-dotted line). All waveforms have the characteristic “inspiral-merger-ringdown” morphology. The bottom panel shows the relative amplitude differences $\Delta A_{22}/A_{22} := (A_{22}^{\text{PHE}} - A_{22}^X)/A_{22}^{\text{PHE}}$ and the phase differences $\Delta\phi_{22} := \phi_{22}^{\text{PHE}} - \phi_{22}^X$; the legend indicates the subtrahend X waveform. The bottom panel also shows in lighter colors the

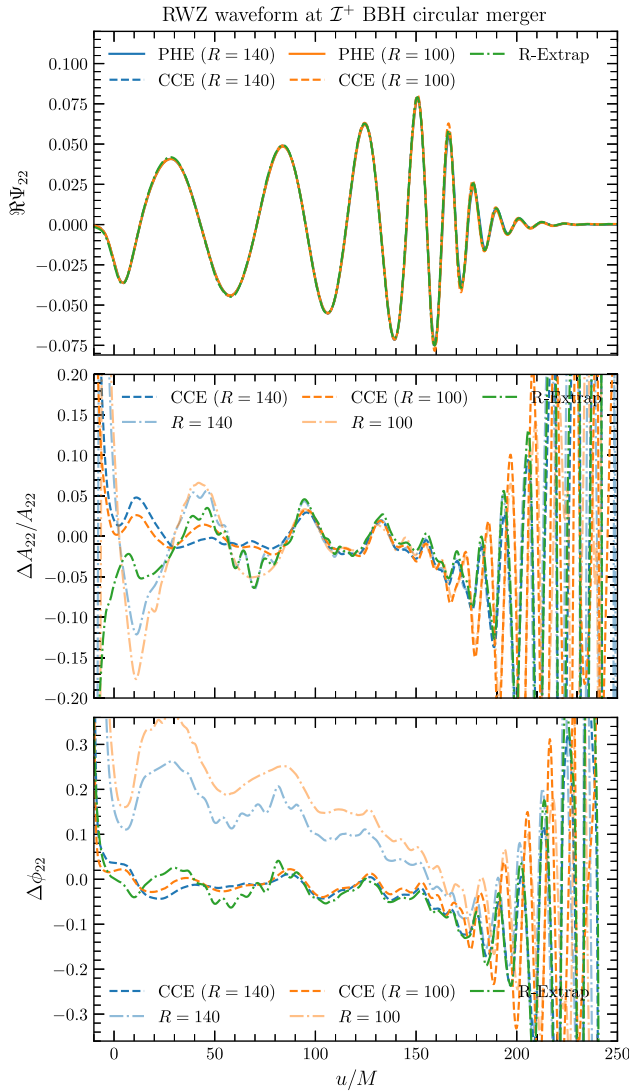


FIG. 5. RWZ waveform at $\mathcal{I}^+$ from a nonspinning equal-mass binary black hole circular merger. Top: real part of Ψ_{22} extrapolated with PHE (solid) and CCE (dashed) from world tube data at $R = 140M_\odot$ (blue) and $100M_\odot$ (orange) and best finite radius extrapolation (dash-dotted green, $R = 140M_\odot$). Middle: relative amplitude differences between the PHE waveform and others: $\Delta A_{22}/A_{22} := (A_{22}^{\text{PHE}} - A_{22}^X)/A_{22}^{\text{PHE}}$, where the subtrahend X waveforms are indicated in the legend. Bottom: phase differences between the PHE waveform and others: $\Delta\phi_{22} := \phi_{22}^{\text{PHE}} - \phi_{22}^X$, where the subtrahend X waveforms are indicated in the legend. Light dash-dotted lines refer to differences to RWZ waveforms extracted at $R = 140M_\odot$ (blue) and $100M_\odot$ (orange). These RWZ waveforms are not shown in the top panel.

differences with respect to the two waveforms extracted at $R = 140$ (blue dash-dotted lines) and $R = 100$ (orange dash-dotted lines).

Amplitude relative differences PHE-CCE up to merger are below $\lesssim 0.5$ and are compatible with the truncation error of NR data at the considered resolution. In the late ringdown, the differences increase significantly but this

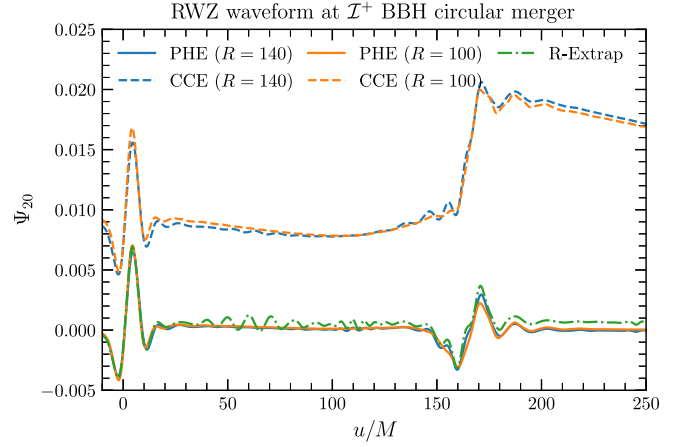


FIG. 6. RWZ waveform at $\mathcal{I}^+$ from a nonspinning equal-mass binary black hole circular merger. Real part of Ψ_{20} extrapolated with PHE (solid) and CCE (dashed) from radii $R = 140, 100M_\odot$ and best finite radius extrapolation (dash-dotted green, $R = 140M_\odot$). Nonlinear memory is captured only by the PHE linear propagation to $\mathcal{I}^+$. The unphysical drift in the late ringdown of CCE is an artifact of the short-duration simulation.

is mostly due to the amplitude approaching zero and small relative dephasing of waveforms around their zeros. The R -extrapolated waveforms well agree with the $\mathcal{I}^+$ extrapolated waveforms. Finite radii extracted RWZ multipoles show differences of $\lesssim 20\%$ in the early cycles but better agree at later times to merger. Phase differences PHE-CCE and PHE- R extrapolated are $\lesssim \pm 0.05$ rad, indicating a good agreement up to merger and early ringdown. Finite radius RWZ waveforms have instead differences as large as ~ 0.2 – 0.3 rad at early times over the two-orbit evolution. This is in agreement with previous work that indicated that the finite extraction effect is more significant at early simulation times, see also below.

Figure 6 compares the RWZ $\ell m = 20$ mode at $\mathcal{I}^+$ from different propagation (extrapolation) methods. This mode has nonlinear memory [67,68] that can be captured only by means of a nonlinear propagation to $\mathcal{I}^+$. Consequently, neither PHE nor extrapolation can capture the correct morphology of the signal. We stress that the CCE signals shown in the figure are strongly affected by short duration of the considered simulation. The CCE waveforms in the late ringdown are unphysically drifting to lower value instead of being constant. This is not a drawback of CCE, but simply a consequence that the (integral) flux of gravitational waves is not entirely captured by the simulation considered here, cf. [69].

2. Dynamical capture

A different waveform morphology is given by the dynamical capture of two black holes with total mass $M = 1$ [7]. The punctures evolve in a close encounter; they separate and then come together again to merge. Correspondingly, the waveform's amplitude is characterized

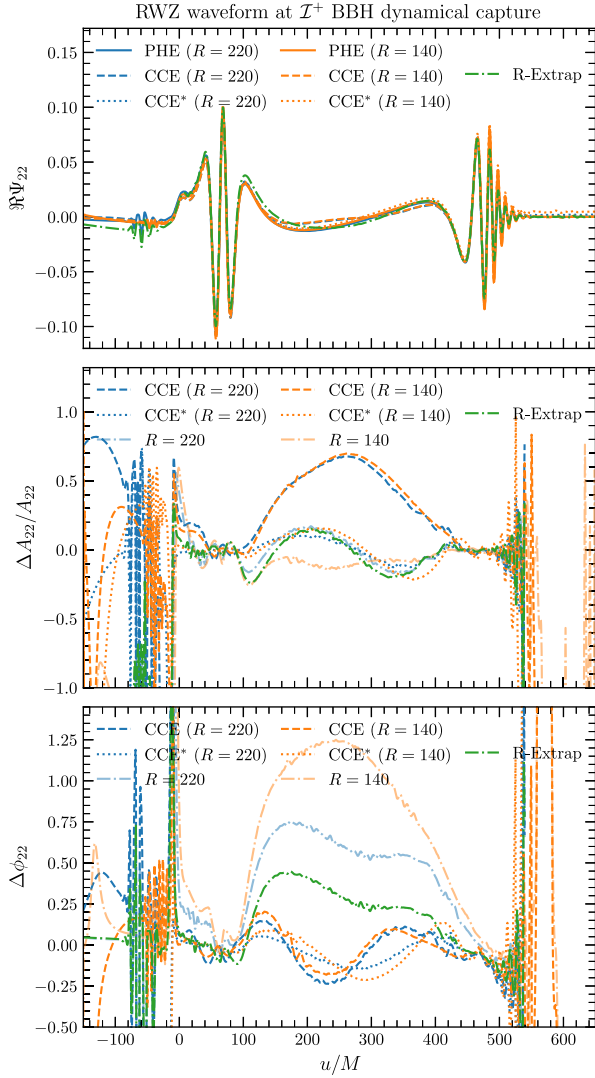


FIG. 7. RWZ waveform at $\mathcal{I}^+$ from a nonspinning equal-mass black hole dynamical encounter. Top: real part of Ψ_{22} extrapolated with PHE (solid), PittNull CCE (dashed), and SpECTRE CCE (CCE*, dotted) from world tube data at $R = 140M_\odot$ (blue) and $100M_\odot$ (orange) and best finite radius extrapolation (dash-dotted green, $R = 220M_\odot$). Middle: relative amplitude differences between the PHE waveform and others: $\Delta A_{22}/A_{22} := (A_{22}^{\text{PHE}} - A_{22}^X)/A_{22}^{\text{PHE}}$, where the subtrahend X waveforms are indicated in the legend. Bottom: phase differences between the PHE waveform and others: $\Delta \phi_{22} := \phi_{22}^{\text{PHE}} - \phi_{22}^X$, where the subtrahend X waveforms are indicated in the legend. Light dash-dotted lines refer to differences to RWZ waveforms extracted at $R = 220M_\odot$ (blue) and $140M_\odot$ (orange). These RWZ waveforms are not shown in the top panel.

by two peaks connected by a lower frequency signal and the final ringdown. PHE data are computed at the same resolutions as for the circular merger and we focus on the highest resolution $N = 3201$.

Figure 7 compares the dominant RWZ $\ell m = 22$ mode at $\mathcal{I}^+$ from different propagation (extrapolation) methods. PHE (from world tubes at $R = 220, 140$, blue and orange

solid lines), CCE (from world tubes at $R = 200, 140$, blue and orange dashed lines), CCE obtained with SpECTRE (from world tubes at $R = 220, 140$, blue and orange dotted lines), and the R extrapolated (from $R = 220$, green dash-dotted line). After computing CCE SpECTRE data, we apply a Bondi–Van der Burg–Metzner–Sachs (BMS) transformation and map the strain to the superrest frame [70].

The figure illustrates that all waveforms have qualitatively similar morphology with the double-peaked amplitude, the low-frequency signal, and the ringdown. The largest differences are in the low-frequency signal between the peaks and in the late ringdown.

Relative amplitude differences PHE-CCE show a good agreement during the encounter and merger, while the transient before merger is not well captured by the PittNull CCE extraction. The maximum difference reaches up to $\Delta A_{22}/A_{22} \sim 0.7$. This feature is due to the integration of the News function to the strain (or RWZ). Indeed, the SpECTRE CCE waveform shows significantly better performances here and agrees with the PHE with maximal amplitude differences of $\Delta A_{22}/A_{22} \sim 0.1$.

Phase differences PHE-CCE remain relatively flat with deviations of order $\lesssim 0.25$ rad, which is comparable to truncation errors of the NR simulation. Finite extraction and R -extrapolated RWZ waveform suffer from larger phase uncertainties up to ~ 1.25 and ~ 0.75 rad, respectively, accumulated in between the two amplitude peaks. This highlights the need for computing the waveform at $\mathcal{I}^+$ in order to accurately capture the entire signal for waveform modeling purposes [71].

3. Scattering

We further consider a scattering problem with two nonspinning black holes of total mass $M = 1$ and initial energy (angular momentum) of $E_{\text{in}}/M \simeq 1.023$ ($J_{\text{in}}/M^2 \simeq 1.26$). The initial data setup follows closely [72] with an initial separation $d/M = 100$. Wave extraction in these types of simulations is challenging because the punctures start at large separation and return quickly to the wave zone of the computational domain. We probe this scenario with a simulation using the same grid setup as the circular and dynamical capture [7]. PHE data are computed at the same resolution as for the circular merger and we focus on the highest resolution $N = 3201$.

Figure 8 shows the real part of the $\ell m = 22$ mode at $\mathcal{I}^+$ from different propagation (extrapolation) methods. The R -extrapolated waveform (dash-dotted green) carries the typical systematics of finite radius extraction: an unphysical drift is visible in both the precursor and the late tail and it is due to the close extraction radii. These features are significantly amplified in waveforms obtained at finite extraction radii (not shown). They cannot be entirely removed by considering larger extraction radii since those NR waveform are additionally affected by the progressively lower resolution of the Cartesian boxes. Interestingly, the

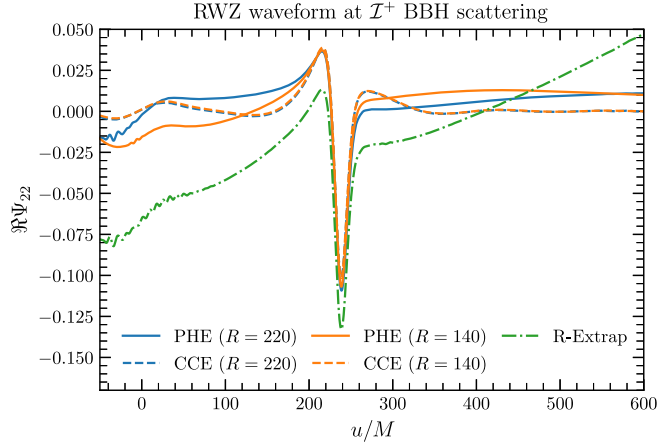


FIG. 8. RWZ waveform at $\mathcal{I}^+$ from a nonspinning equal-mass black hole scattering process. Real part of Ψ_{22} extrapolated with PHE (solid) and CCE (dashed) from world tube data at $R = 200M_\odot$ (blue) and $140M_\odot$ (orange) and best finite radius extrapolation (dash-dotted green, $R = 220M_\odot$).

PHE removes most of the drift and delivers waveforms in agreement with CCE around the amplitude's peak, i.e., the portion of waveform contributing to the largest luminosity.

The PittNull CCE waveform is also affected by systematics due to the reconstruction of the strain from the News function. Here the integration is performed with the fixed-frequency integration algorithm [73] with a cutting frequency of $f_0 = 0.007$, which appears close to optimal after manual experiments. As discussed in detail in the Appendix of Albanesi *et al.* [71], the reconstruction is ambiguous due to the difficulty of identifying a cutting frequency for the fixed-frequency integration algorithm (or a suitable interval for the polynomial drift in the time-domain integral). As a consequence, the reconstruction procedure introduces systematics that are more pronounced in the amplitude of the precursor and of the tail. We have also checked spECTRE CCE data (not shown) and found a large unphysical overall drift that cannot be straightforwardly cured by a BMS frame rotation. In summary, an improved NR setup with a more extended wave zone and higher resolutions than those considered here are required to mitigate waveform extraction effects in scattering simulations.

E. Binary neutron star merger

As a final application, we consider a circular equal-mass binary neutron star merger. The binary mass is $M \simeq 2.7$ and the initial separation is such that the binary undergoes about ten orbits to merger [7,74,75]. Gravitational waves are extracted at coordinate (isotropic) radii $R = 800, 600, 400, 200$. PHE data are injected using Eq. (10) and we focus on the highest resolution $N = 3201$.

Figure 9 shows the real part of the $\ell m = 22$ mode at $\mathcal{I}^+$ from different propagation (extrapolation) methods. Relative amplitude differences PHE-CCE are essentially

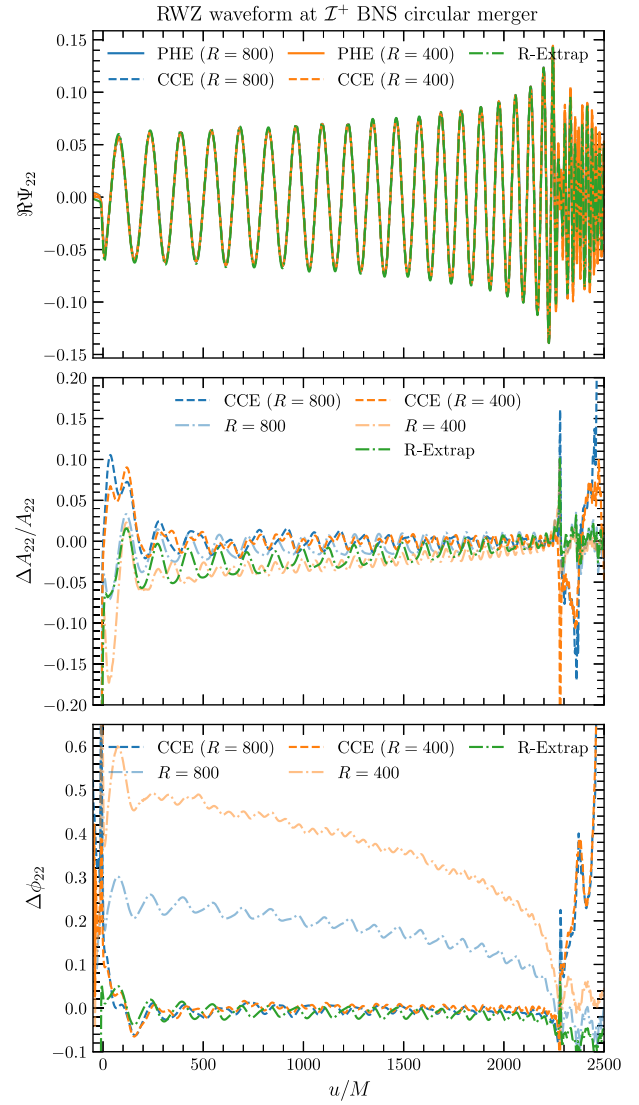


FIG. 9. RWZ waveform at $\mathcal{I}^+$ from a nonspinning equal-mass binary neutron star (BNS) circular merger. Top: real part of Ψ_{22} extrapolated with PHE (solid) and CCE (dashed) from world tube data at $R = 800M_\odot$ (blue) and $400M_\odot$ (orange) and best finite radius extrapolation (dash-dotted green, $R = 800M_\odot$). Middle: relative amplitude differences between the PHE waveform and others: $\Delta A_{22}/A_{22} := (A_{22}^{\text{PHE}} - A_{22}^X)/A_{22}^{\text{PHE}}$, where the subtrahend X waveforms are indicated in the legend. Bottom: phase differences between the PHE waveform and others: $\Delta \phi_{22} := \phi_{22}^{\text{PHE}} - \phi_{22}^X$, where the subtrahend X waveforms are indicated in the legend. Light dash-dotted lines refer to differences to RWZ waveforms extracted at $R = 800M_\odot$ (blue) and $400M_\odot$ (orange). These RWZ waveforms are not shown in the top panel.

flat with oscillations around $\lesssim 5\%$ that reduce to $\lesssim 1\%$ to merger. The finite extraction RWZ and the R -extrapolated waveforms are also consistent within these levels, thus providing a very good representation of the waveform at $\mathcal{I}^+$. Accumulated phase differences PHE-CCE (and PHE- R extrapolated) to merger are also flat and below $\lesssim 0.05$ rad. These differences are about an order magnitude smaller

than those due to truncation errors of the NR hydrodynamics. While the latter typically increase toward merger, finite extraction phase errors are of the same order ~ 0.5 rad at early times and reduce toward merger [11]. This highlights the need for $\mathcal{I}^+$ propagation or extrapolation in high precision binary neutron star simulations [76].

Figure 9 also highlights significant differences in both amplitude and phase in the postmerger waveform. This is mainly due to the reconstruction of the strain from the News function in CCE data. We stress that, in general, the postmerger signal is less accurate than the inspiral-merger signal and typically shows slower convergence in the grid resolution due to the development of shocks in the remnant. Therefore, these errors are also amplified by the strain reconstruction procedure.

IV. CONCLUSION

PHE is a framework based on perturbative hyperboloidal evolutions for the propagation to $\mathcal{I}^+$ of gravitational waveforms extracted at finite radius in $3+1$ numerical relativity. The benchmark problems considered in this work include $(3+1)$ D simulations of gravitational collapse of rotating neutron stars, binary black holes, and binary neutron star mergers. They demonstrate that PHE is a simpler and yet robust alternative approach to Cauchy-characteristic extraction for astrophysical simulations in strong gravity.

Nonlinear effects in the waveform propagation are not captured by PHE. Thus, PHE cannot handle, for example, nonlinear tails and memory effects, e.g., [67,68,77,78]. We plan to extend the linear PHE to second order to capture nonlinear waveform interactions, such as mode coupling and memory. At second order, the master equations acquire quadratic source terms built from products of first-order RWZ modes, which drive the generation of new multipoles and nonoscillatory offsets in the strain. The general covariant and gauge-invariant formalism with explicit source constructions for Schwarzschild perturbations is already developed [79–81], see, e.g., [82,83] for recent applications. Implementing these techniques in a $(1+1)$ D hyperboloidal code would allow us to propagate second-order corrections to $\mathcal{I}^+$, delivering perturbative predictions of nonlinear memory, tail, and mode coupling in astrophysically relevant gravitational-wave signals.

Similar to CCE, PHE waveforms depend on the identification of an optimal radius for the world tube. When Cartesian coordinates are employed in the $3+1$ simulation, the computational mesh is typically a hierarchy of Cartesian boxes which progressively derefine the wave zone. Therefore, the radius of the extraction world tube cannot be chosen arbitrarily large: resolution effects can severely affect the waveform quality. Our work has shown that PHE waveforms robustly match CCE for “optimal” choices of the world tube. PHE can sometimes provide even more robust prediction than CCE (see, e.g., the

rotational collapse simulations) and reduce some of the unphysical features that are present in waveforms extracted at finite radii.

RWZ waveforms extracted at finite radii are more prone to gauge effects and numerical noise than Weyl ψ_4 waveforms [7]. Improved PHE waveforms at $\mathcal{I}^+$ may be obtained by propagating ψ_4 using the Teukolsky equation. Such variant of PHE can be straightforwardly implemented using the time-domain hyperboloidal approach of Harms *et al.* [84] in either $(1+1)$ D or $(2+1)$ D. These ψ_4 PHE waveforms would still require double integration to the strain and would thus carry some of the systematics discussed in this work, see also [7]. Moreover, a quantitative investigation of the impact of the (choice of the) background spin is required.

The PHE proposed here is another step toward the use of hyperboloidal methods for the computation of gravitational radiation at null infinity. Future work may also consider a Cauchy hyperboloidal extraction that incorporates nonlinearities in the waveform extraction by leveraging recent advances in numerical relativity with hyperboloidal foliations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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