Notes on Fourier transform and its application to three-point momentum-space integrals

Xuhang Jiang*

Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

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The Fourier transform of two-point momentum-space Feynman integrals with massless propagators and two off-shell legs can be used to prove identities between their periods, exemplified by the glue-and-cut identity. We generalize this framework to massless momentum-space Feynman integrals with three off-shell legs and obtain a similar family of identities that can be used to calculate these integrals, especially for a nonplanar subset of them, which naturally arise in the off-shell Sudakov form factors.

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I. INTRODUCTION

Feynman integrals (FIs), which form the foundation of perturbative calculations in quantum field theory, satisfy a wide variety of internal identities among themselves. Such identities play a crucial rule in the reduction and evaluation of FIs. The most common and universal identities among general FIs are the integration-by-part (IBP) relations [1,2], which are linear relations among FIs and have been explored in great depth by many different approaches in the last few decades [3–11]. Besides the IBP relations, additional nontrivial identities exist, especially for integrals with additional symmetries. One such example is the (generalized) magic identity for conformal integrals [12–14], which relate different four-point conformal integrals in momentum space with each other. Many identities between four-point conformal integrals, e.g., twist identities [15] and planar duality, have been investigated systematically in the framework of graphical function [16–20]. They are mostly used to study the periods of Feynman integrals, which are interesting both physically and mathematically [21–26]. Another example of amazing identities among conformal integrals is the Basso-Dixon formula for fishnet family, which evaluates to the determinant of ladders [27–29]. These identities are not linear and thus highly nontrivial. The third example is the glue-and-cut identity [30] among p-integrals (massless Feynman integrals with two off-shell legs), which relates different p-integrals through graph operations as the name indicates. By exploiting all these identities, one can not only reduce the integrals to a known basis, but also use them as constraints to obtain analytic results [14,30]. For example, the glue-and-cut identity [30] has been used to calculate p-integrals up to 5 loops [31].

Interestingly, by Fourier transform, the glue-and-cut identity in momentum space can be translated into the periods identity of graphical functions in position space. This can be proved by first Fourier transforming the momentum-space integral with two off-shell legs to position space, completing it to conformal integrals and then using the conformal property to fix any two adjacent vertices to 0 and 1. We give an example of this correspondence in Table I, while referring the proof by Fourier transform to the appendix of [32].

In this work, we extend this correspondence to integrals with three off-shell legs and refer to it as the generalized Fourier identity, which is distinguished from the Fourier identity of periods introduced in [16,21,33]. The generalized Fourier identities are also formulated by a set of graph rules. They relate Feynman integrals with massless propagators and three off-shell legs in momentum space to their counterparts in position space, which are conformal integrals with four external vertices fixed. The graphs in momentum space and position space follow different Feynman rules as listed in Table II. The normalization factor $1/\pi^2$ in the integration measure is for the convenience of Fourier transform, which is also a common convention in the literature. The propagators in these two spaces are related by the general Fourier transform in $d = 4 - 2\epsilon$ dimension

$$\frac{1}{(k^2)^\alpha} = \frac{\Gamma(2-\alpha-\epsilon)}{\Gamma(\alpha)} \int \frac{\mathrm{d}^d x}{\pi^{d/2}} \frac{e^{2ik\cdot x}}{(x^2)^{2-\alpha-\epsilon}}, \tag{1}$$

which simplifies to

$$\frac{1}{k^2} = \int \frac{\mathrm{d}^4 x}{\pi^2} \frac{e^{2ik\cdot x}}{x^2} \tag{2}$$

*Contact author: xhjiang@itp.ac.cn

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TABLE I. The same diagram has different meanings in momentum space and position space. They are related by Fourier transform. Note that we can set any two vertices of an edge to 0 and 1 in position space, which corresponds to cutting the edge and assigning an off-shell momentum to it in momentum space. Note that in Euclidean space $q^2 = 1$, while in Minkowski space, $q^2 = -1$.

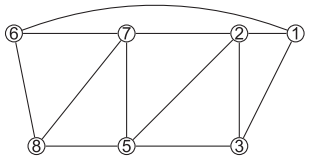
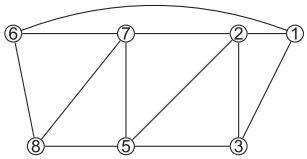
Position space	Loop momentum space
	
Fix $x_6 = 0$ and $x_1 = 1$	Cut line 61 and assign it an off-shell momentum q^2
x_4 has been fixed to infinity and the edges joining degree-3 vertices and x_4 have been removed	The external momenta attached to degree-3 vertices have been set to 0
Result: $168\zeta_9$	Result: $168\zeta_9$

TABLE II. The Feynman rules and integration measures in position space and momentum space.

	Feynman rule	Integration measure
Position space	x_i — x_j $\quad \frac{1}{x_{ij}^2}$	$\int \frac{\mathrm{d}^4 x}{\pi^2}$
Momentum space	$\xrightarrow{\quad l \quad}$ $\quad \frac{1}{l^2}$	$\int \frac{\mathrm{d}^4 l}{\pi^2}$

in four dimension for $\alpha = 1$. The generalized Fourier identity is closely related to the planar duality [17,18,34], which relates a graphical function to the one associated with its dual graph. These two concepts coincide in many cases but different in general. While the main result of this paper might be already known to some experts, we are not aware of any previous publication where it has been explicitly formulated.

This paper is organized as follows. In Sec. II, we introduce the generalized Fourier identities, which relate momentum-space integrals to their position-space counterparts. They can be systematically generated by graph rules. We give a proof of these identities by Fourier transform, Mellin transform and star-triangle relations [35–38]. We will mainly apply it to a particular family of nonplanar momentum-space integrals, which naturally arise in the off-shell Sudakov form factors [39,40]. In Sec. III, we discuss the relations between these identities and the planar duality of graphical functions. In Sec. IV, we summarize and conclude.

II. GENERALIZED FOURIER IDENTITY AND ITS APPLICATION

In this section, we discuss the main result of this article. It states a kind of identities between momentum-space integrals with massless propagators and three off-shell legs with graphical functions [17,41,42], which also arise as four-point conformal integrals. We call such identities the generalized Fourier identities to distinguish it from the Fourier identities for periods of Feynman integrals [16]. Then we apply them to a set of nonplanar integrals appearing in the off-shell Sudakov form factors.

A. Generalized Fourier identity

We begin by setting up our notations here. The propagators of all the integrals considered below are massless and depicted as thin lines. To distinguish momentum-space integrals from position-space integrals, the former will be dressed up with thick lines indicating off-shell external momenta that always flow outward. Thus, a three-point momentum-space integral will carry three thick lines attached to distinct vertices. For position space integrals, the external points (positions that are

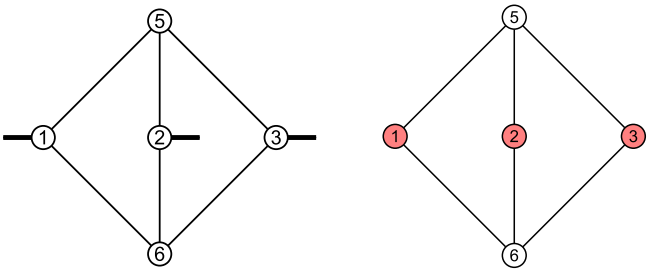


FIG. 1. Two examples of integrals in different representations. The left diagram is in momentum-space with three external legs, while the right diagram is in position space with three external vertices colored pink.

fixed) are drawn as pink vertices. All vertices in both the momentum- and position-space integrals are labeled by numbers. Momentum-space integrals are integrated over independent loop momenta flowing along *edges*, while position-space integrals are integrated over all internal *vertices* (which are not colored). An illustrative example of the notation is provided in Fig. 1. Although we will show later that the two integrals in Fig. 1 equal to each other after identification of kinematics, they are actually very different according to the defined Feynman rules.

The identities we will discuss are most easily formulated by graph rule. The graphs we exploit satisfy the following properties:

(1) The edges can be solid or dashed. An edge must be attached to two distinct vertices (no tadpole). Two solid edges cannot be attached to the same two vertices (no bubble for solid lines).
(2) The degree of a vertex is the number of solid lines minus the number of dashed lines attached to the vertex. All vertices in the graph have degree 4.
(3) The dashed lines of the graph (if there are any dashed lines in it) can all be removed by removing a single vertex of the graph.

We denote the set of such graphs as $\mathcal{G}$. These graphs are inspired and closely related to a set of general f-integrands or f-graphs used to bootstrap the four-point correlators in $\mathcal{N}=4$ super-Yang-Mills theory [43–49]. However, our discussion is independent of the knowledge of f-graphs. Graphs satisfying the first two conditions naturally encode the integrands of conformal integrals in position space where one integrates over all vertices. Solid lines denote propagator denominators, while dashed lines represent the numerators. The degree-4 vertex condition ensures the conformal invariance. The integral is well-defined after fixing the conformal invariance by sending three vertices to 0,1 and ∞. The vertex mapped to ∞ and all edges attached it can be removed. Then by the third condition above, one can always choose some vertex to be placed at ∞ and the remaining graph is free of dashed lines. This is a necessary condition for the identities to hold. Then we are ready to present the graph rule as follows

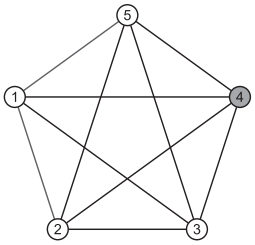
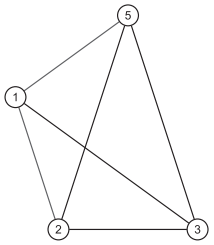
Definition 1. Considering a graph $G\in\mathcal{G}$, choose a vertex in G and send it to ∞ so that after removing this vertex and edges attached to it, the reduced graph G' is free of dashed lines.

Definition 2. Cut two consecutive edges $[v_1,v_2]$ and $[v_2,v_3]$ of G' and give the edge $[v_1,v_2]$ an external momentum flow p_1, the edge $[v_2,v_3]$ another momentum flow p_2. Take this cut graph as *momentum-space* integral and denote it as $G'_{\rm ms}(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2})$ with $q=-p_1-p_2$. In the same time, consider original G', fix the coordinate of above three vertex [50] $x_1=1$, $x_2=z$, $x_3=0$ and remove edges $[v_1,v_2]$, $[v_2,v_3]$. Take this reduced graph as *position-space* integral and denote it as $G'_{\rm ps}(z,\bar{z})$.

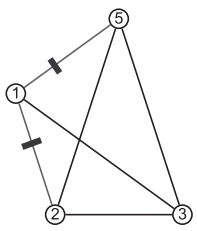
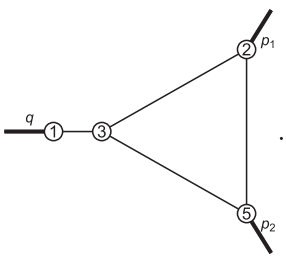
Proposition 1. If $G'_{\rm ms}(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2})$ is finite for generic kinematics $p_1^2,p_2^2,q^2\neq 0$ (no subdivergence), then $G'_{\rm ms}(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2})=\frac{1}{(q^2)^2}G'_{\rm ps}(z,\bar{z})$ where $\frac{p_1^2}{q^2}=z\bar{z},\frac{p_2^2}{q^2}=(1-z)(1-\bar{z})$.

In position space, we denote the distance between two vertices v_i and v_j as $x_{ij}^2=(x_i-x_j)^2$. Fixing $x_1=1,x_2=z$ and $x_3=0$ represents $x_{12}^2=(1-z)(1-\bar{z})$ and $x_{23}^2=z\bar{z}$. Note that $p_1^2/p_2^2=x_{23}^2/x_{12}^2$, which makes explicit the duality between momentum space and position space. It also indicates a proof by Fourier transform. Furthermore, this result can be generalized to d dimension while keeping the conformal property of the integral by deforming the edge weights [51]. In this case, an additional constraint must be imposed on the edge weights of the two edges being cut, as we will discuss later. Before proving this proposition at Sec. II B, we present some explicit examples.

Example 1. Let us start from the five-point complete graph G. The reduced graph of G is G' by sending v_4 dark-colored vertex to ∞,

$G=$  , $G'=$  . (3)

Then $G'_{\rm ms}(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2})$ is

 $\Rightarrow G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right)=$  . (4)

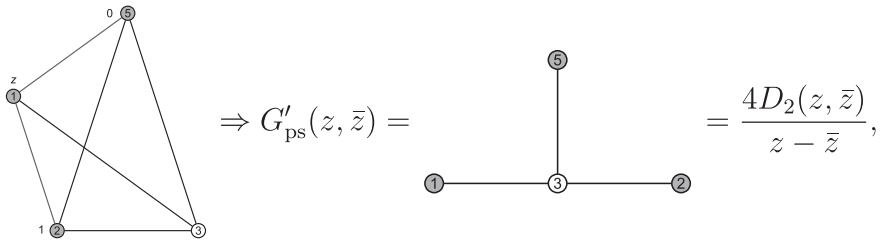
Its result is well-known [52,53] to be

$$G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right)=\frac{1}{(q^2)^2}\frac{4D_2(z,\bar{z})}{z-\bar{z}}, \tag{5}$$

where $D_2(z,\bar{z})$ is the famous Bloch-Wigner function whose expression is

$$D_2(z,\bar{z})=\log|z|\,\mathrm{Im}\;\log(1-z)+\mathrm{Im}\,\mathrm{Li}_2(z). \tag{6}$$

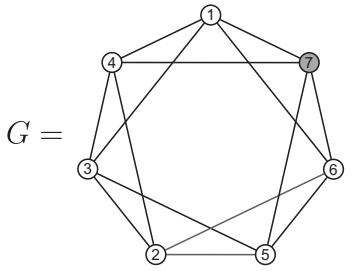
Now we consider the position space



$$\Rightarrow G'_{\rm ps}(z,\bar{z})= \qquad =\frac{4D_2(z,\bar{z})}{z-\bar{z}}, \tag{7}$$

where the edge $[v_2,v_5]$ is also removed since the distance $x_{25}^2=1$. It is the same with one-loop triangle momentum-space integral and can be computed as graphical function [41]. It is then obvious that $G'_{\rm ms}(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2})=\frac{1}{(q^2)^2}G'_{\rm ps}(z,\bar{z})$ in the example above. Due to the symmetry of G', taking any two consecutive edges will arrive at the same result.

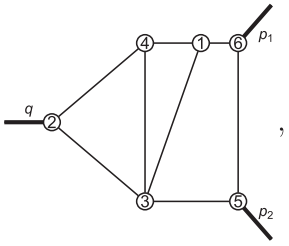
Example 2. We give another example to show that the generalized Fourier identity is related to the planar duality when the planar dual of the momentum-space integral exists. Consider the following graph G,

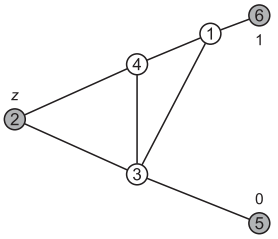


$$G= \qquad . \tag{8}$$

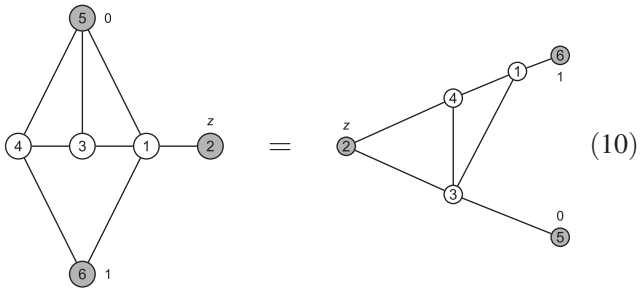
We will send x_7 dark-colored vertex to ∞ and cut edges $[v_5,v_2]$ and $[v_2,v_6]$. The resulting integrals in momentum space and position space are

$$G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right)= \qquad , \quad G'_{\rm ps}(z,\bar{z})= \tag{9}$$





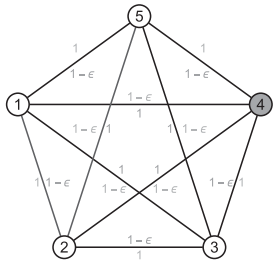
where in the right graph we have omitted the edge between v_5 (0) and v_6 (1). $G'_{\rm ms}(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2})$ can be evaluated by the package HyperInt [54] and $G'_{\rm ps}(z,\bar{z})$ can be evaluated by another package HyperlogProcedures [55]. The two results agree with each other and satisfy the proposition. Note that $G'_{\rm ps}$ is *not* the dual graph of $G'_{\rm ms}$. Since $G'_{\rm ms}$ can also be represented by its dual graph, setting $q^2=-1$, we derive the following identity between two integrals in position space,



(10)

where the left diagram is the dual representation of $G'_{\rm ms}(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2})$. This is exactly the planar duality relation for graphical functions [17,18,34,56]. We will discuss more in Sec. III about the relations between the generalized Fourier identity and planar duality which is proved in Feynman parametrization [34].

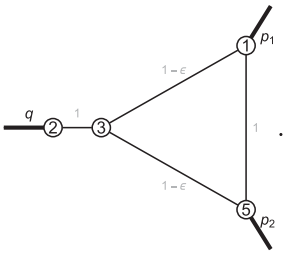
Example 3. Let us still take the diagram in Eq. (3) as an example, but with the edge weights being deformed from 1. That is, the Feynman rule for propagators will be $1/(l^2)^\nu$ with ν being general complex parameter instead of 1. The edge weights are different in momentum space and in position space as the general Fourier transform Eq. (1) indicates. They have been marked at different locations in the following diagram (on the outer side of each edge for momentum space, and on the interior side of each edge for position space).

$$G = $$  (11)

We denote the weights of the two edges being cut $[[v_1,v_2],[v_2,v_5]$ in Eq. (11)] in momentum space as ν'_1 and ν'_2. $\nu'_1+\nu'_2=d/2=2-\epsilon$ is a necessary condition for the generalized identity to hold. The corresponding edge weights in position space are $\nu_i=d/2-\nu'_i$. So it follows that this condition holds in position space as well. This is why we cut $[v_1,v_2],[v_2,v_5]$ instead of $[v_2,v_1],[v_1,v_5]$. It can also be seen that, the total edge weight of the edges attached to each vertex equals to $d=4-2\epsilon$, which is another necessary condition which guarantees the conformal property of the integral. Now this identity can be stated as

$$G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right)=\frac{1}{(q^2)^{d/2}}G'_{\rm ps}(z,\bar z) \tag{12}$$

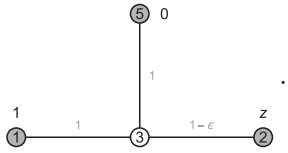
Let us check it by explicit evaluation in momentum space and position space. First, in momentum space,

$$G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right)=$$  (13)

It can be evaluated in Feynman parametrization as

$$\begin{aligned}&G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right)\\&=\frac{1}{(q^2)^{2-\epsilon}}\frac{1}{\Gamma(1-\epsilon)}\int[{\rm d}\alpha_1{\rm d}\alpha_2{\rm d}\alpha_3]\\&\quad\times\frac{\alpha_2^{-\epsilon}\alpha_3^{-\epsilon}}{(\alpha_1+\alpha_2+\alpha_3)(\alpha_2\alpha_3+\frac{p_1^2}{q^2}\alpha_1\alpha_3+\frac{p_2^2}{q^2}\alpha_1\alpha_2)^{1-\epsilon}}\end{aligned} \tag{14}$$

with $z\bar z\equiv p_1^2/q^2$, $(1-z)(1-\bar z)\equiv p_2^2/q^2$. $[{\rm d}\alpha_1{\rm d}\alpha_2{\rm d}\alpha_3]$ indicates the integration with a delta function to fix the scaling. Now we turn to the position space,

$$G'_{\rm ps}(z,\bar z)=$$  (15)

It can also be evaluated in Feynman parametrization as

$$\begin{aligned}&G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right)\\&=\frac{1}{\Gamma(1-\epsilon)}\int[{\rm d}\alpha_1{\rm d}\alpha_2{\rm d}\alpha_3]\\&\quad\times\frac{\alpha_1^{-\epsilon}}{(\alpha_1+\alpha_2+\alpha_3)^{1-\epsilon}(\alpha_2\alpha_3+\frac{p_1^2}{q^2}\alpha_1\alpha_3+\frac{p_2^2}{q^2}\alpha_1\alpha_2)}\end{aligned} \tag{16}$$

Performing a Cremona transformation $\alpha_i\to 1/\alpha_i$ to Eq. (16) and then rescaling $\alpha_2\to\alpha_2 q^2/p_1^2$, $\alpha_3\to\alpha_3 q^2/p_2^2$, it can be shown straightforwardly that

$$G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right)=\frac{1}{(q^2)^{2-\epsilon}}G'_{\rm ms}\left(\frac{p_1^2}{q^2},\frac{p_2^2}{q^2}\right). \tag{17}$$

we will prove the general case in Sec. II B.

B. Proof of the generalized Fourier identities

Proof. Now we prove Proposition 1 by Fourier transform, Mellin transform and star-triangle relation (or called the method of uniqueness) [35–38]. Readers mainly interested in the applications can skip this section.

We first keep ourselves in four dimensions and the Feynman rules are given in Table II. Starting from $G'_{\rm ms}$ with edge $[v_1, v_2]$ and $[v_2, v_3]$ cut, we perform the Fourier transform to every propagator of the momentum-space integral by Eq. (2). Integrating out all the loop momenta will result in a set of delta functions for coordinates by

$$\int \mathrm{d}^4 k e^{2ik\cdot x} = \pi^4 \delta^{(4)}(x). \tag{18}$$

These delta functions will be satisfied if we identify the dual coordinates of loop momentum l flowing along edge $[v_i, v_j]$ with $x_i - x_j$. Thus the resulting position-space integral share the same graph with original momentum-space integral, except that the integrand related to the three vertices v_1, v_2, v_3 where the three external legs are attached is modified. Due to the translation invariance of integral in position space, we fix x_3 to 0. The resulting integral is then

$$G'_{\rm ms}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \int \frac{\mathrm{d}^4 x_1}{\pi^2} \int \frac{\mathrm{d}^4 x_2}{\pi^2} e^{2ip_1\cdot x_1} e^{2iq\cdot x_2} \times \frac{1}{(x_1^2)^2} I_{G'_{\rm ps}}\left(\frac{x_{12}^2}{x_1^2}, \frac{x_2^2}{x_1^2}\right). \tag{19}$$

$I_{G'_{\rm ps}}$ is the integral of $G'_{\rm ps}$ but with three "external" vertices x_1, x_2, x_3 where x_3 has been fixed to 0. The transformation between momentum space and position space can be derived straightforwardly. We refer to [57] for some pedagogical examples. Note that x_1 and x_2 need to be further integrated out, so they are not the genuine external points. $I_{G'_{\rm ps}}$ depends only on two cross ratios $\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = \frac{\frac{x_{12}^2}{x_1^2}}{\frac{x_{13}^2}{x_1^2}\frac{x_{24}^2}{x_1^2}}, \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = \frac{\frac{x_2^2}{x_1^2}}{\frac{x_{13}^2}{x_1^2}\frac{x_{24}^2}{x_1^2}}$ after we extract a normalization factor $1/(x_1^2)^2$ out [58], because the diagram has conformal property and we can always complete this graph to conformal integrals by adding the $x_4 = \infty$ back. Now we rewrite $I_{G'_{\rm ps}}$ by its Mellin transform

$$I_{G'_{\rm ps}}\left(\frac{x_{12}^2}{x_1^2}, \frac{x_2^2}{x_1^2}\right) = \int \frac{\mathrm{d}s}{2\pi i} \int \frac{\mathrm{d}t}{2\pi i} \left(\frac{x_{12}^2}{x_1^2}\right)^{-s} \left(\frac{x_2^2}{x_1^2}\right)^{-t} \tilde{I}_{G'_{\rm ps}}(s,t), \tag{20}$$

where $\tilde{I}_{G'_{\rm ps}}(s,t)$ is the Mellin transform of $I_{G'_{\rm ps}}(\frac{x_{12}^2}{x_1^2}, \frac{x_2^2}{x_1^2})$. In Eqs. (19) and (20), we assume that $G'_{\rm ms}(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2})$ [and thus $I_{G'_{\rm ps}}(\frac{x_{12}^2}{x_1^2}, \frac{x_2^2}{x_1^2})$ which is the Fourier transform of $G'_{\rm ms}(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2})$] is finite for general kinematics so that its Mellin transform $\tilde{I}_{G'_{\rm ps}}(s,t)$ and the multiple integration is well-defined. Then we can exchange the integration order,

$$G'_{\rm ms}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \int \frac{\mathrm{d}s}{2\pi i} \int \frac{\mathrm{d}t}{2\pi i} \tilde{I}_{G'_{\rm ps}}(s,t) \int \frac{\mathrm{d}^4 x_1}{\pi^2} \int \frac{\mathrm{d}^4 x_2}{\pi^2} e^{2ip_1\cdot x_1} e^{2iq\cdot x_2} \frac{1}{(x_1^2)^2} \left(\frac{x_{12}^2}{x_1^2}\right)^{-s} \left(\frac{x_2^2}{x_1^2}\right)^{-t}. \tag{21}$$

Now we transform the power of x_1^2, x_2^2, x_{12}^2 back to momentum space,

$$\frac{1}{(x_1^2)^{2-s-t}} = \frac{\Gamma(s+t)}{\Gamma(2-s-t)} \int \frac{\mathrm{d}^4 l_1}{\pi^2} \frac{e^{2il_1\cdot x_1}}{(l_1^2)^{s+t}}, \qquad \frac{1}{(x_2^2)^t} = \frac{\Gamma(2-t)}{\Gamma(t)} \int \frac{\mathrm{d}^4 l_2}{\pi^2} \frac{e^{2il_2\cdot x_2}}{(l_2^2)^{2-t}}$$
$$\frac{1}{(x_{12}^2)^s} = \frac{\Gamma(2-s)}{\Gamma(s)} \int \frac{\mathrm{d}^4 l_3}{\pi^2} \frac{e^{2il_3\cdot x_{12}}}{(l_3^2)^{2-s}}. \tag{22}$$

Note that s, t are complex variables and the Fourier transforms above are defined by analytically continuing in α while keeping $d = 4$ fixed in Eq. (1). After integrating x_1, x_2 out, l_1 is localized to $-l_3 - p_1$ and l_2 is localized to $l_3 - q$ by delta functions. Thus,

$$G'_{\rm ms}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \int \frac{\mathrm{d}s}{2\pi i} \int \frac{\mathrm{d}t}{2\pi i} \tilde{I}_{G'_{\rm ps}}(s,t) c_\Gamma \int \frac{\mathrm{d}^4 l_3}{\pi^2} \frac{1}{[(l_3+p_1)^2]^{s+t}[(l_3-q)^2]^{2-t}[l_3^2]^{2-s}} \tag{23}$$

where

$$c_\Gamma = \frac{\Gamma(s+t)\Gamma(2-t)\Gamma(2-s)}{\Gamma(s)\Gamma(t)\Gamma(2-s-t)}. \tag{24}$$

Recall the star-triangle relations for general conformal scalar integrals [53],

$$\int \frac{\mathrm{d}^D x_0}{\pi^{D/2}} \frac{1}{(x_{01}^2)^a (x_{02}^2)^b (x_{03}^2)^c} = \frac{\Gamma(a')\Gamma(b')\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c)} \frac{1}{(x_{12}^2)^{c'} (x_{23}^2)^{a'} (x_{13}^2)^{b'}}, \quad \text{when } a+b+c = D \tag{25}$$

where $a' = D/2 - a, b' = D/2 - b, c' = D/2 - c$. Therefore,

$$\int \frac{\mathrm{d}^4 l_3}{\pi^2} \frac{1}{[(l_3+p_1)^2]^{s+t}[(l_3-q)^2]^{2-t}[l_3^2]^{2-s}} = \frac{1}{c_\Gamma} \frac{1}{(q^2)^{2-s-t}(p_1^2)^t (p_2^2)^s} \tag{26}$$

We finally arrive at

$$G'_{\mathrm{ms}}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \frac{1}{(q^2)^2} \int \frac{\mathrm{d}s}{2\pi i} \int \frac{\mathrm{d}t}{2\pi i} \tilde{I}_{G'_{\mathrm{ps}}}(s,t) \left(\frac{p_2^2}{q^2}\right)^{-s} \left(\frac{p_1^2}{q^2}\right)^{-t} \tag{27}$$

Compared with Eq. (20), it can be recognized that the right-hand side is $1/(q^2)^2 I_{G'}(z,\bar{z})$ with $x_1 = 1, x_2 = z, x_3 = 0$, given $p_1^2/q^2 = z\bar{z}$ and $p_2^2/q^2 = (1-z)(1-\bar{z})$. Then we arrive at our proposition

$$G'_{\mathrm{ms}}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \frac{1}{(q^2)^2} G'_{\mathrm{ps}}(z,\bar{z}). \tag{28}$$

Next we can generalize above result to d dimension while keeping the conformal property of the integrals by deforming the edge weights of propagators. The author thanks Oliver Schnetz for pointing the existence of this generalization out. Supposing the edge weight for each edge e_i is ν_i in *position space*, and the set of edges attached to vertex v_i is defined as E_i, we require

$$\sum_{e_i \in E_j} \nu_i = d = 4 - 2\epsilon, \quad \forall\ j. \tag{29}$$

for the conformal property to hold. After performing the general Fourier transform Eq. (1) to original momentum-space integral, one can generalize Eq. (19) to

$$G'_{\mathrm{ms}}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \int \frac{\mathrm{d}^d x_1}{\pi^{d/2}} \int \frac{\mathrm{d}^d x_2}{\pi^{d/2}} e^{2ip_1\cdot x_1} e^{2iq\cdot x_2} \times \frac{1}{(x_1^2)^{d-\nu_1-\nu_2}} I_{G'_{\mathrm{ps}}}\left(\frac{x_{12}^2}{x_1^2}, \frac{x_2^2}{x_1^2}\right). \tag{30}$$

where ν_1 and ν_2 are the weights of the two cut edges. Then Eq. (23) becomes

$$G'_{\mathrm{ms}}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \int \frac{\mathrm{d}s}{2\pi i} \int \frac{\mathrm{d}t}{2\pi i} \tilde{I}_{G'_{\mathrm{ps}}}(s,t) c'_\Gamma \times \int \frac{\mathrm{d}^d l_3}{\pi^{d/2}} \frac{1}{[(l_3+p_1)^2]^{s+t+\nu_1+\nu_2-2+\epsilon}[(l_3-q)^2]^{2-t-\epsilon}[l_3^2]^{2-s-\epsilon}} \tag{31}$$

where

$$c'_\Gamma = \frac{\Gamma(s+t+\nu_1+\nu_2-2+\epsilon)\Gamma(2-t-\epsilon)\Gamma(2-s-\epsilon)}{\Gamma(s)\Gamma(t)\Gamma(d-\nu_1-\nu_2-s-t)}. \tag{32}$$

Subtleties arise when dealing with the integration of l_3 with star-triangle relation, which requires $2 - \epsilon + \nu_1 + \nu_2 = d$, that is, $\nu_1 + \nu_2 = d/2$. In four dimension, $\nu_i = 1$ and $d = 4$, this requirement is naturally satisfied. We assume $\nu_1 + \nu_2 = d/2$ for general case, then l_3 can still be directly integrated out as before. We arrive at the generalization of Eq. (27)

$$G'_{\mathrm{ms}}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \frac{1}{(q^2)^{d-\nu_1-\nu_2}} \int \frac{\mathrm{d}s}{2\pi i} \int \frac{\mathrm{d}t}{2\pi i} \times \tilde{I}_{G'_{\mathrm{ps}}}(s,t) \left(\frac{p_2^2}{q^2}\right)^{-s} \left(\frac{p_1^2}{q^2}\right)^{-t} \tag{33}$$

As a result,

$$G'_{\mathrm{ms}}\left(\frac{p_1^2}{q^2}, \frac{p_2^2}{q^2}\right) = \frac{1}{(q^2)^{d/2}} G'_{\mathrm{ps}}(z,\bar{z}). \tag{34}$$

We finish the proof of proposition 1 and its generalization. ∎

A lot of identities can be systematically generated from proposition above, as long as the graph set $\mathcal{G}$ is given. However, in the following we will focus on its application to a particular class of nonplanar momentum-space integrals, which has not been discussed in general in the literature before.

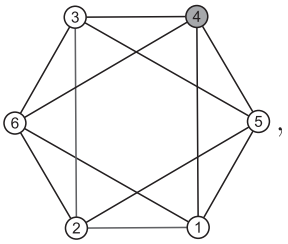
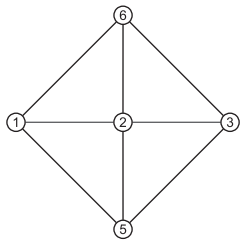
C. The application to the off-shell form factor

In this section, we consider a particular family of nonplanar integrals relevant to the off-shell form factor [39,40]. Their evaluation in Feynman parametrization is usually time-consuming or, in some cases, infeasible. We note here that the calculations in Feynman parametrization are done in HyperInt [54]. Recently, a new package HyperForm [59] has been presented. It is more efficient in nontrivial examples. Nevertheless, for the integration in Feynman parametrization, another difficulty lies in identifying an efficient linear reduction order, particularly in

high-loop cases. By contrast, their position-space counterparts, obtained via proposition 1, are readily computable or already known. Consequently, proposition 1 serves as an effective tool to handle this class of integrals.

This nonplanar family starts from two loops. The first example has been presented in Fig. 1. Starting from the following G and G',

$$G = \quad , \qquad G' = \qquad (35)$$

taking $x_4 = \infty$ and cutting two consecutive edges $[v_1, v_2], [v_2, v_3]$, it follows that the integral in momentum space (left in Fig. 1), denoted as $N_1^{(2)}$, equals to the integral in position space (right in Fig. 1) up to a factor $1/(q^2)^2$, where $x_1 = 1, x_2 = z, x_3 = 0$ and $z, \bar{z}$ are defined by $p_1^2 = q^2 z\bar{z}$ flowing out of v_1 and $p_2^2 = q^2(1-z)(1-\bar{z})$ flowing out of v_3. Furthermore, the integral in position space factorizes into the product of two one-loop conformal box [60]. As a result, $N_1^{(2)}$ equals to the square of conformal box, $F_1^2(z, \bar{z})$, where F_L denotes the L-loop conformal ladder integral [61–63],

$$F_L(z,\bar{z}) = \frac{1}{(z-\bar{z})L!}\sum_{j=L}^{2L}\frac{j!}{(j-L)!(2L-j)!} \times (-\log z\bar{z})^{2L-j}(\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar{z})). \qquad (36)$$

This is also pointed out in [40].

At three loops, there is also one such integral [40], denoted as $N_1^{(3)}$. We depict this integral and its position-space counterpart in Fig. 2. The kinematic setting is the same as two-loop case. It factorizes into the product of a two-loop conformal ladder and a one-loop conformal box, F_1F_2. The graph G which generates these two integrals is depicted in Eq. (37)

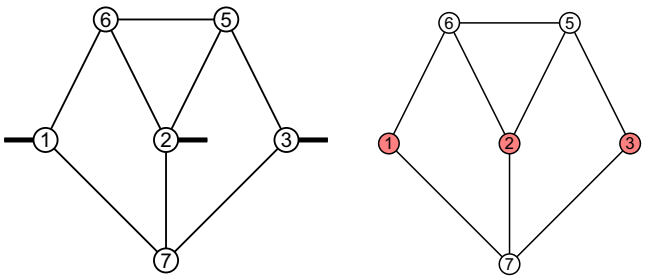
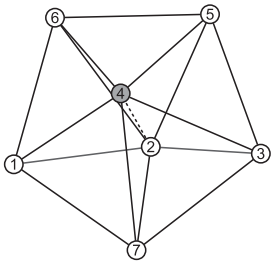


FIG. 2. The three-loop nonplanar integral which will contribute to the off-shell Sudakov form factor (left) and its counterpart in position space (right).

$$G = \qquad (37)$$

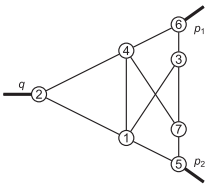
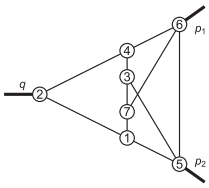
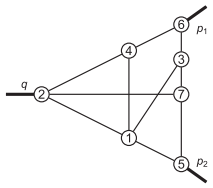
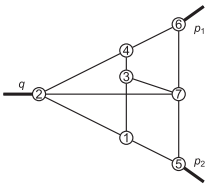
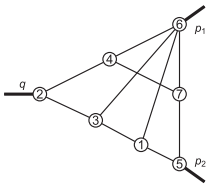


where the convention is also the same. Since the conformal ladder family has been calculated to arbitrary loops, we can easily extend above identities between such a nonplanar family and product of ladders by generalizing the pentagon in Eq. (37) to n-gon and cut any two spokes. It can be even further generalized to cases where other finite integrals instead of ladders involve in the factorization, in which case the identities follow in the same way.

At four loops, there are five such integrals in total [64] once the massless external legs are taken off-shell, in direct analog with the procedure employed at three loops [40]. We use the same number to denote the off-shell version of the five integrals in [64]

$$N_{19}^{(4)} \sim I_{19}^{(22)}, \quad N_{20}^{(4)} \sim I_{20}^{(22)}, \quad N_{21}^{(4)} \sim I_{21}^{(24)}, \quad N_{22}^{(4)} \sim I_{22}^{(24)}, \quad N_{23}^{(4)} \sim I_{23}^{(28)}, \qquad (38)$$

which are depicted in Eq. (39),

$$N_{19}^{(4)} = \text{}, \quad N_{20}^{(4)} = \text{}, \quad N_{21}^{(4)} = \text{}$$
$$N_{22}^{(4)} = \text{}, \quad N_{23}^{(4)} = \text{} \tag{39}$$

$N_{23}^{(4)}$ belongs to the special nonplanar family and we can easily derive that it equals to the square of two-loop ladders, F_2^2. This is checked by integrating $N_{23}^{(4)}$ in momentum space by HyperInt and its position-space counterpart by HyperlogProcedures, respectively. If we further consider the small mass limit of this integral with $q^2 = -1$, that is, taking the off-shell legs back to on-shell,

$$z = -r, \qquad \bar{z} = 1 + r, \quad \text{with} \quad r \to 0, \tag{40}$$

the asymptotic behavior of $N_{23}^{(4)}(z, \bar{z})$ is then

$$N_{23}^{(4)}(z,\bar{z}) = \frac{\pi^4}{144}\log^4 r - \frac{\pi^2}{2}\zeta_3 \log^3 r + \left(\frac{\pi^6}{90} + 9\zeta_3^2\right)\log^2 r - \frac{2\pi^4}{5}\zeta_3 \log r + \frac{\pi^8}{225} + \mathcal{O}(r). \tag{41}$$

The coefficient of leading order divergence, $\frac{\pi^4}{144} = \frac{5\zeta_4}{8}$, agrees with the result calculated from dimensional regularization [64], which is expected [65].

$N_{22}^{(4)}$ can be calculated from its position-space counterpart by HyperlogProcedures. It equals to 2×2 fishnet, which is the determinant of ladders [27–29], $F_1F_3 - 1/3F_2^2$. We note that the arguments of F_L here is $z/(z-1)$ and $\bar{z}/(\bar{z}-1)$ instead of $z, \bar{z}$, that is, a Möbius transformation is needed. It shows a more complex structure than $N_{22}^{(4)}$ and somehow explains the missing of function basis F_1F_3 in the results. However, for the direct integration of $N_{22}^{(4)}$ in momentum space, HyperInt fails to find a linear reduction order. Nevertheless, we have checked the analytic expression with numerical results evaluated by FEYNTROP [66] in momentum space. They agree well with each other. The leading order coefficient under the limit Eq. (40) also agrees with the result in dimensional regularization.

The calculation of $N_{20}^{(4)}$ is similar to $N_{22}^{(4)}$. Its position-space counterpart can be calculated by HyperlogProcedures. However, it now involves more complicated multiple polylogarithm with letter $z - \bar{z}$, like $G_{1,\bar{z},\bar{z},\bar{z},0,0,1,1}(z)$ where G is multiple polylogarithm function [67,68] defined recursively as

$$G_{a_1,\ldots,a_n}(z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G_{a_2,\ldots,a_n}(t). \tag{42}$$

Again, the analytic expression [69] agrees well with numerical results evaluated by FEYNTROP in momentum space.

Finally, $N_{19}^{(4)}$ and $N_{21}^{(4)}$ are the two most complicated integrals. Neither of them nor their position space counterparts can be calculated analytically by the current tools. Furthermore, a naive leading singularity analysis of the position-space counterpart of $N_{19}^{(4)}$ shows that $N_{19}^{(4)}$ may involve elliptic integrals. The position-space counterpart of $N_{19}^{(4)}$ is

$$N_{19}^{(4)} = \int \frac{\mathrm{d}^4x_1 \mathrm{d}^4x_3 \mathrm{d}^4x_4 \mathrm{d}^4x_7}{(\pi^2)^4} \times \frac{x_{28}^4 x_{56}^4}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{24}^2 x_{36}^2 x_{37}^2 x_{46}^2 x_{47}^2 x_{57}^2 x_{26}^2 x_{25}^2 x_{58}^2 x_{68}^2 x_{38}^2 x_{78}^2}, \tag{43}$$

where the integrand is completed to a conformal integral. x_2, x_5, x_6, x_8 are four external points and x_8 is at ∞ [70]. We first cut all propagators for x_1 and x_7, which results in two leading singularities λ_{2345} and λ_{3458} where $\lambda_{ijkl} = \sqrt{\det(x_{mn}^2)_{m,n=i,j,k,l}}$. They are both the leading singularity of conformal boxes [71], which arise from the Jacobian transformation. The remaining cut integral is

$$\tilde{N}_{19} = \int \frac{\mathrm{d}^4x_3 \mathrm{d}^4x_4}{(\pi^2)^2} \frac{x_{28}^4 x_{56}^4}{x_{24}^2 x_{36}^2 x_{46}^2 \lambda_{2345} \lambda_{3458} x_{26}^2 x_{25}^2 x_{58}^2 x_{68}^2 x_{38}^2}. \tag{44}$$

Then we consider the cut of x_3 and x_4. Under the following cut of propagators

$$\text{cut}:\ x_{36}^2 = x_{38}^2 = x_{24}^2 = x_{46}^2 = 0, \tag{45}$$

λ_{2345} and λ_{3458} will degenerate to

$$\lambda_{2345}|_{\text{cut}} = x_{25}^2 x_{34}^2 - x_{23}^2 x_{45}^2, \quad \lambda_{3458}|_{\text{cut}} = x_{35}^2 x_{48}^2 - x_{34}^2 x_{58}^2. \tag{46}$$

Then the cut of $x_{36}^2, x_{38}^2, \lambda_{2345}|_{\text{cut}}, \lambda_{3458}|_{\text{cut}}$ for x_3^μ results in another Jacobian factor which is,

$$\lambda_{68\{245\}\{458\}}|_{\text{cut}} = \sqrt{x_{48}^2(x_{28}^2x_{45}^2 - x_{25}^2x_{48}^2)(x_{28}^2x_{56}^4x_{45}^2x_{48}^2 - x_{25}^2x_{56}^4x_{48}^4 - 4x_{26}^2x_{58}^2x_{68}^2x_{45}^4)}. \tag{47}$$

The corresponding maximal cut integral is

$$\tilde{N}_{19}^{\text{max cut}} \propto \int \frac{\mathrm{d}x_{45}^2 \mathrm{d}x_{48}^2}{\lambda_{2568}\sqrt{x_{48}^2(x_{28}^2x_{45}^2 - x_{25}^2x_{48}^2)(x_{28}^2x_{56}^4x_{45}^2x_{48}^2 - x_{25}^2x_{56}^4x_{48}^4 - 4x_{26}^2x_{58}^2x_{68}^2x_{45}^4)}}. \tag{48}$$

The polynomial under square root is homogeneous in x_{45}^2 and x_{48}^2. We can perform a variable transformation to $r \equiv x_{45}^2/x_{48}^2$ and x_{48}^2, then cut x_{48}^2. The polynomial becomes cubic in the remaining integration variable r. This signals a elliptic cut of the integral. Performing a similar analysis for $N_{21}^{(4)}$, no elliptic cut has been found. Instead, we find the following leading singularity under the maximal cut of all variables:

$$\frac{1}{(z-\bar{z})\sqrt{1+z^2+\bar{z}^2-2z-2\bar{z}-2z\bar{z}}}, \qquad \frac{x_{26}^2x_{58}^2}{x_{28}^2x_{56}^2} \equiv \frac{z\bar{z}}{(1-z)(1-\bar{z})}, \qquad \frac{x_{25}^2x_{68}^2}{x_{28}^2x_{56}^2} \equiv \frac{1}{(1-z)(1-\bar{z})}. \tag{49}$$

This leading singularity involves a square root of polynomial in z and $\bar{z}$, which shows the complexity of this integral. Nevertheless, we expect this integral to be multiple polylogarithm function after the following variable transformation [72,73]:

$$z \to -\bar{y}(1+y), \qquad \bar{z} \to -y(1+\bar{y}). \tag{50}$$

The leading singularity in Eq. (49) simplifies to

$$\frac{1}{y-\bar{y}} \tag{51}$$

after considering the Jacobian factor from this variable transformation.

III. RELATION TO PLANAR DUALITY BY FEYNMAN PARAMETRIZATION

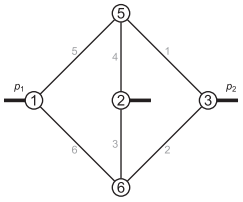
In Example 2, we show the close connection between the generalized Fourier identity and planar duality in the literature [17,18,34]. In this section, we discuss their relations in greater depth.

Generally speaking, the generalized Fourier identity developed in this work relates momentum-space integrals to their position-space counterparts (graphical functions) regardless of the possible underlying topology. It agrees with planar duality only when the momentum-space integrals under consideration admit planar duals. $N_{20}^{(4)}$ in Sec. II C is an example which does not admit planar duals, in which case these two concepts are different. In what follows, we restrict ourselves to cases when the planar duals *can* exist.

When the momentum-space integrals are planar graphs, these two are equivalent by definition. Since if a graphical function (which is finite in four dimensions) admits a planar dual, the we can draw its planar dual, which represents a dual integral in momentum space. Then we can use proposition 1 to identify this dual integral with its position-space counterpart which shares the same graph. Then the original graphical function is related to the new one which is dual to it in the graph level. The reverse argument from planar duality to generalized Fourier identity is straightforward in the same way.

The more nontrivial case is when the planar dual is not straightforward like the example in Figs. 1, 2 and $N_{23}^{(4)}$ in Eq. (39). It is not obvious what the planar dual of these integrals would be and how it is related to its planar dual, since the integrals in the momentum space are nonplanar, due to the external legs. Now we show how such cases can also be discussed using language of planar duality for momentum-space integrals, taking integrals in Fig. 1 as an example. The key idea is to join the external legs and study the resulting graph. For this example, there *exists* such a way that after joining two external legs with the last one, the resulting graph G admits a planar embedding.

We first write down the Feynman parametrization for the momentum-space integral in Fig. 1,



$$= N_1^{(2)} = \int [\mathrm{d}\alpha] \frac{1}{Q^2}. \tag{52}$$

where $[\mathrm{d}\alpha]$ is a shorthand for the integration measure for projective space of α_i and

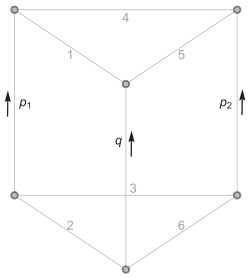
$$\begin{aligned} Q = {} & p_1^2(\alpha_5\alpha_6(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_1\alpha_4\alpha_6+\alpha_2\alpha_3\alpha_5) + p_2^2(\alpha_1\alpha_2(\alpha_3+\alpha_4+\alpha_5+\alpha_6)+\alpha_1\alpha_3\alpha_6+\alpha_2\alpha_4\alpha_5) \\ & + q^2(\alpha_3\alpha_4(\alpha_1+\alpha_2+\alpha_5+\alpha_6)+\alpha_2\alpha_4\alpha_6+\alpha_1\alpha_3\alpha_5). \end{aligned} \tag{53}$$

Performing Cremona transformation $\alpha \to 1/\alpha$, we will arrive at a new representation for the same integral

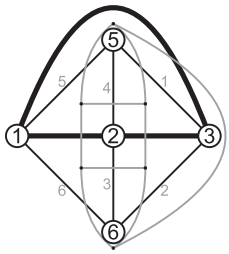
$$\begin{aligned} N_1^{(2)} = {} & \int [\mathrm{d}\alpha] \frac{1}{\tilde{Q}^2}. \\ \tilde{Q} = {} & p_1^2(\alpha_1\alpha_4(\alpha_2+\alpha_3+\alpha_6)+\alpha_2\alpha_3(\alpha_1+\alpha_4+\alpha_5)) + p_2^2(\alpha_3\alpha_6(\alpha_1+\alpha_4+\alpha_5)+\alpha_4\alpha_5(\alpha_2+\alpha_3+\alpha_6)) \\ & + q^2(\alpha_2\alpha_6(\alpha_1+\alpha_4+\alpha_5)+\alpha_1\alpha_5(\alpha_2+\alpha_3+\alpha_6)). \end{aligned} \tag{54}$$

It corresponds to the following integrals in momentum space

$$\int [\mathrm{d}\alpha] \frac{1}{\tilde{Q}^2} = \tag{55}$$



This is actually the dual graph of $N_1^{(2)}$ by first joining three external legs and drawing its dual as follows



(56)

This is the planar duality directly for the integrals in momentum space. It serves as another way to understand the factorization of nonplanar integrals. We do not aim to explain all such identities by using planar duality, but only exploit this example to show that such identities may also be understood by planar duality for the momentum-space integrals which can be proved in Feynman parametric space.

IV. CONCLUSION AND OUTLOOK

In this work, we have proved generalized Fourier identities between massless momentum-space integrals with three off-shell legs and graphical functions (conformal integrals) in position space. Such identities implies the planar duality when the momentum-space integrals admit planar duals and can be generated systematically by graph rules. Especially, We obtained the factorization of a particular family of nonplanar integrals, which contributes to the off-shell Sudakov form factors.

This work is part of the broader effort to uncover the hidden relations among conformal integrals, thereby facilitating their calculation. It is interesting to further explore the implications of such identities. For example, nonplanar integral like $N_{23}^{(4)}$ encodes the product of two ladder integrals. Then a natural question is, can the triple products or higher power of ladders be encoded in one single momentum space integral? Since we know the $m \times n$ $(m \leq n)$ fishnet in momentum space can generate determinant of ladders with m powers. Another related question is how to understand the fact that $N_{22}^{(4)}$ equals to the 2×2 fishnet. Such an equivalence between nonplanar integrals and planar integrals is interesting. Is this a coincidence, or is this a consequence of some deep connections between their graph structures which can be exploited to construct similar identities systematically?

Another possible direction is to extend the identities to more general cases. For example, one may ask how to extend the identities to integrals with deformed edge weights when the condition $\nu_1 + \nu_2 = d/2$ is no long imposed or whether analogous identities exist for higher-point conformal integrals. Finally, in this work, we have

focused on conformal integrals, it would also be interesting to investigate whether these identities can be extended to more general integrals without this symmetry. It would be of great interest if more relations are discovered and understood.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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