

Revisiting radiative transitions of charmonium states in a covariant confined quark model

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We calculate the amplitudes and branching fractions of $\chi_{cJ}(1P)$, $h_c(1P)$, $J/\psi(1S)$, and $\psi(2S)$ radiative decays in the framework of the covariant confined quark model. First, we explicitly show that all calculated amplitudes are gauge invariant. Next, we use experimental data for three decay channels $J/\psi \rightarrow \eta_c \gamma$, $\chi_{c0} \rightarrow J/\psi \gamma$, and $\psi(2S) \rightarrow \chi_{c0} \gamma$ to fit the model parameters including the charm quark mass and the meson size parameter. Finally, we use the obtained parameters to predict the branching fractions of the decays $h_c \rightarrow \eta_c \gamma$, $\chi_{c1(c2)} \rightarrow J/\psi \gamma$, and $\psi(2S) \rightarrow \chi_{c1(c2)} \gamma$. Our predictions agree well with experimental data.

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I. INTRODUCTION

Charmonium states with masses below the $D\bar{D}$ threshold attract much attention from both theorists and experimentalists. The main reason is that these states have dominant radiative decays with narrow decay widths. To date, radiative decays of the $\eta_c(1S)$, $J/\psi(1S)$, $\chi_{c0,c1,c2}(1P)$, $h_c(1P)$, $\eta_c(2S)$, and $\psi(2S)$ charmonium states have been observed and studied experimentally. The most comprehensive overview of the field up to 2011 was presented in Ref. [1].

Branching fractions for radiative transitions from the $\psi(2S)$ to the $\chi_{c0,c1,c2}$ states and also to the η_c were presented in Ref. [2]. These results were obtained from a detailed study using the Crystal Ball detector at SPEAR (SLAC). Additionally, the resonance parameters of the χ_{c0} were measured at the Fermilab Antiproton Accumulator [3]. The CLEO Collaboration, using the CLEO III detector at the Cornell Electron Storage Ring, has performed several key studies on charmonium states. In particular, they investigated photon transitions in $\psi(2S)$ decays to $\chi_{cJ}(1P)$ and $\eta_c(1S)$ states [4]. Their work also includes the measurement of the branching fractions for the $\chi_{cJ} \rightarrow \gamma J/\psi$ and $\psi(2S) \rightarrow \text{light hadrons}$ decays [5], as well as for $\psi(2S) \rightarrow J/\psi$

transitions with greater precision than previously achieved [6]. Furthermore, the CLEO Collaboration observed the $h_c(1P_1)$ state of charmonium in the reaction $\psi(2S) \rightarrow \pi^0 h_c \rightarrow (\gamma\gamma)(\gamma\eta_c)$ [7] and later reported a precise measurement of its mass [8]. In a separate study, they presented the most precise measurements of the radiative decays $\psi(2S) \rightarrow \eta_c \gamma$ and $J/\psi \rightarrow \eta_c \gamma$ [9].

Using photon conversions to e^+e^- pairs, the energy spectrum of inclusive photons from $\psi(2S)$ radiative decays was measured by BESII at the Beijing Electron-Positron Collider [10]. The $\chi_{cJ}(1P)$ states ($J = 0, 1, 2$) were clearly observed and their masses were determined. The process $\psi(3686) \rightarrow \pi^0 h_c, h_c \rightarrow \gamma\eta_c$ was studied in Ref. [11]. The mass and decay width of the $h_c(1P_1)$ state were determined by simultaneously fitting distributions of the π^0 recoil mass for 16 exclusive η_c decay modes. Using a sample of 106 million $\psi(3686)$ decays, the branching fractions $\psi(3686) \rightarrow \gamma\chi_{c0}$, $\psi(3686) \rightarrow \gamma\chi_{c1}$, and $\psi(3686) \rightarrow \gamma\chi_{c2}$ were measured with improved precision [12]. First measurements of the absolute branching fractions $\mathcal{B}(\psi' \rightarrow \pi^0 h_c) = (8.4 \pm 1.3 \pm 1.0) \times 10^{-4}$ and $\mathcal{B}(h_c \rightarrow \gamma\eta_c) = (54.3 \pm 6.7 \pm 5.2)\%$ were also reported by the BESIII Collaboration [13]. Using 448 million $\psi(2S)$ events, the spin-singlet P -wave charmonium state $h_c(1^1P_1)$ was studied via the $\psi(2S) \rightarrow \pi^0 h_c$ decay followed by the $h_c \rightarrow \gamma\eta_c$ transition [14]. The branching fractions were measured to be $\mathcal{B}_{\text{Inc}}(\psi(2S) \rightarrow \pi^0 h_c) \times \mathcal{B}_{\text{Tag}}(h_c \rightarrow \gamma\eta_c) = (4.22^{+0.27}_{-0.26} \pm 0.19) \times 10^{-4}$, $\mathcal{B}_{\text{Inc}}(\psi(2S) \rightarrow \pi^0 h_c) = (7.32 \pm 0.34 \pm 0.41) \times 10^{-4}$, and $\mathcal{B}_{\text{Tag}}(h_c \rightarrow \gamma\eta_c) = (57.66^{+3.62}_{-3.50} \pm 0.58)\%$, where the uncertainties are statistical and systematic,

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respectively. The $h_c(1^1P_1)$ mass and width were determined to be $M = (3525.32 \pm 0.06 \pm 0.15)$ MeV and $\Gamma = (0.78_{-0.24}^{+0.27} \pm 0.12)$ MeV. The Particle Data Group (PDG) combined their best value of $\mathcal{B}(\psi(2S) \rightarrow h_c(1P)\pi^0) = (7.4 \pm 0.5) \times 10^{-4}$ [15] with the result

$$\begin{aligned} \mathcal{B}(h_c \rightarrow \gamma\eta_c) \times \mathcal{B}(\psi(2S) \rightarrow h_c(1P)\pi^0) \\ = (4.22_{-0.26}^{+0.27} \pm 0.19) \times 10^{-4} \end{aligned}$$

reported by BESIII [14] and found

$$\mathcal{B}(h_c \rightarrow \gamma\eta_c) = \begin{cases} (60 \pm 4)\% & \text{FIT} \\ (57 \pm 5)\% & \text{AVERAGE} \end{cases}.$$

Radiative decays in charmonium play an important role in the understanding of its structure and can serve a testing ground for a number of theories and models. In Ref. [16], results for the spectrum and radiative partial widths were presented. They have been evaluated using two models, the relativized Godfrey-Isgur model and a nonrelativistic potential model. The electromagnetic transitions were evaluated using Coulomb plus linear plus smeared hyperfine wave functions, both in the nonrelativistic potential model and in the Godfrey-Isgur model. The available information on quarkonia and their transitions has been reviewed in Ref. [17] and theoretical implications have been discussed. Topics in the description of the properties of charmonium states were reviewed in Ref. [18] with an emphasis on specific theoretical ideas and methods of relating those properties to the underlying theory of quantum chromodynamics. The masses, electromagnetic decays, and E1 transitions of charmonium states were calculated in the screened potential model [19]. Study of the mass spectrum and electromagnetic processes of the charmonium system was carried out in Ref. [20] with the spin-dependent potentials fully taken into account in the solution of the Schrödinger equation. The charmonium spectrum was calculated with two nonrelativistic quark models, the linear potential model and the screened potential model in Ref. [21]. Using the obtained wave functions, the electromagnetic transitions of charmonium states were evaluated.

The transition form factors of various multipolarities between the lightest few charmonium states were computed for the first time within lattice QCD (LQCD) in Ref. [22]. In a subsequent publication [23], this method was applied to compute radiative transition rates involving excited charmonium states, states of high spin, and exotics. Charmonium radiative transitions $J/\psi \rightarrow \eta_c\gamma$, $\chi_{c0} \rightarrow J/\psi\gamma$, and $h_c \rightarrow \eta_c\gamma$ were calculated using $N_f = 2$ twisted mass lattice QCD gauge configurations [24]. Hadronic matrix elements relevant to the $h_c \rightarrow \eta_c\gamma$, $h_b \rightarrow \eta_b\gamma$, and $\chi_{c1} \rightarrow J/\psi\gamma$ decays were computed within LQCD by using the gauge configurations produced by the Extended Twisted Mass

Collaboration with $N_f = 2 + 1 + 1$ dynamical Wilson-Clover twisted mass fermions in Refs. [25,26].

Dispersion sum rules have also been applied to the calculation of radiative transition amplitudes in quarkonium, see, e.g., Ref. [27]. Within the QCD sum rule approach, the radiative decay $J/\psi \rightarrow \eta_c\gamma$ was analyzed [28] taking into account both nonperturbative and perturbative corrections. The radiative decays of heavy $\bar{Q}Q$ state were studied in Ref. [29] using an effective Lagrangian approach, which exploits spin symmetry for such states. The radiative transitions among the vector and scalar heavy quarkonium states were studied in Ref. [30] within the covariant light-front quark model. It was observed that the radiative decay widths are sensitive to the constituent quark masses and the shape parameters of the wave functions.

In the paper [31] magnetic dipole (M1) transitions between two quarkonia were studied in the framework of nonrelativistic effective field theories of QCD. This study was extended in Ref. [32] to explore electric dipole (E1) transitions in heavy quarkonium. The determination of heavy quarkonium magnetic dipole transitions were updated and improved in Ref. [33] using the potential nonrelativistic QCD.

The mass spectra and electromagnetic decay rates of charmonium, bottomonium, and B_c mesons were comprehensively investigated in the relativistic quark model [34]. A theoretical description of charmonium radiative decays was provided in Ref. [35] in the framework of a simple quark potential model beyond the so-called standard p/m approximation. Radiative decays of 0^{++} and 1^{+-} heavy mesons were studied in Ref. [36] within the light front quark model. In Ref. [37], radiative transitions between conventional charmonium states and from the lowest multiplet of $c\bar{c}$ hybrids to charmonium mesons were studied by considering the nonrelativistic limit of the QCD Hamiltonian in the Coulomb gauge.

In this study, we aim to evaluate the amplitudes and branching fractions of radiative decays of charmonium states $\chi_{c0,c1,c2} \rightarrow J/\psi\gamma$, $\psi(2S) \rightarrow \chi_{c0,c1,c2}\gamma$, $h_c \rightarrow \eta_c\gamma$, and $J/\psi \rightarrow \eta_c\gamma$ in the framework of the covariant confined quark model (CCQM) previously developed by us. Keeping in mind that some of the radiative decays mentioned above have been evaluated in the framework of the CCQM before [38], one has to emphasize how this work differs from the previous one. First, we accept here the conventional method of electromagnetic gauging of the nonlocal Lagrangian that was suggested in Refs. [39,40] and intensively used in our previous papers [41–48]. Second, we add three more modes $\psi(2S) \rightarrow \chi_{c0,1,2}\gamma$ in the present study. Finally, we provide a new fit of the charm quark mass and meson size parameters, and calculate the propagation of errors from the fit to the final theoretical predictions.

The paper is organized as follows. In Sec. II we present the theoretical framework of the study. Here we briefly discuss how charmonium states are described within the

CCQM and how the gauging process of the nonlocal quark current proceeds. We show that all calculated amplitudes are gauge invariant. In Sec. III we obtain all expressions for the decay width of radiative charmonium decays in terms of helicity amplitudes. The fitting process and numerical results are presented in Sec. IV. A brief conclusion is given in Sec. V.

II. THEORETICAL FRAMEWORK

A. Charmonium in covariant confined quark model

In this study, we consider a large set of charmonium states of different quantum numbers and titles. Therefore, for the convenience of the reader, we list in Table I a brief summary for the charmonium states treated in this paper. Their masses and widths are taken from the PDG [15]. The CCQM is based on a relativistic Lagrangian describing the interaction of a hadron with its constituent quarks. The charmonium is described by a field $\phi_{cc}(x)$, which couples to an interpolating quark current $J_{cc}(x)$. The interaction Lagrangian reads

$$\mathcal{L}_{\text{int}}(x) = g_{cc}\phi_{cc}(x) \cdot J_{cc}(x). \quad (1)$$

The quark current $J_{cc}(x)$ is a nonlocal generalization of the quark currents shown in Table I,

$$J_{cc}(x) = \int \int dx_1 dx_2 F_{cc}(x, x_1, x_2) \cdot \bar{c}(x_1) \Gamma_{cc} c(x_2), \quad (2)$$

$$F_{cc}(x, x_1, x_2) = \delta(x - wx_1 - wx_2) \Phi_{cc}((x_1 - x_2)^2), \quad (3)$$

($w = 1/2$).

The Fourier transform of the translationally invariant vertex function $\Phi_{cc}((x_1 - x_2)^2)$ in momentum space is required to fall off in the Euclidean region in order to ensure the ultraviolet convergence of the loop integrals. We use a simple Gaussian form written as

$$\tilde{\Phi}_{cc}(-p^2) = \exp(s_{cc} \cdot p^2), \quad s_{cc} \equiv 1/\Lambda_{cc}^2, \quad (4)$$

where Λ_{cc} is an adjustable charmonium size-related parameter of the CCQM. The choice of this parameter will be discussed later on. One has to note that in the case of the radial excitation $\psi(2S)$, we use an alternative form for the vertex function [48]

$$\tilde{\Phi}_{2S}(-p^2) = (1 + c_1 s_{2S} p^2) \exp(s_{2S} \cdot p^2), \quad s_{2S} \equiv 1/\Lambda_{2S}^2. \quad (5)$$

The coefficient c_1 is determined from the orthogonality condition, which was suggested and discussed in our paper [48].

The coupling constant g_{cc} in Eq. (1) is calculated from the so-called compositeness condition, which is expressed in terms of the derivative of the scalar part of the charmonium mass operator,

$$Z_{cc} = 1 - g_{cc}^2 \tilde{\Pi}'_{cc}(m_{cc}^2) = 0. \quad (6)$$

The calculation details can be found in our previous papers [38,49,50].

B. Gauging of nonlocal quark current

The gauge invariant interaction of a bound quark state with the electromagnetic field has been described in some detail in Ref. [41]. For comprehensive purposes we recall some of the key points of the gauging process.

In order to guarantee gauge invariance of the nonlocal strong interaction Lagrangian, one multiplies each quark field $q(x_i)$ by a gauge field exponential according to

$$\begin{aligned} q(x_i) &\rightarrow Q(x_i) = e^{-ie_q I(x_i, x, P)} q(x_i), \\ I(x_i, x, P) &= \int_x^{x_i} dz_\mu A^\mu(z), \\ \bar{q}(x_i) &\rightarrow \bar{Q}(x_i) = e^{ie_q I(x_i, x, P)} \bar{q}(x_i), \end{aligned} \quad (7)$$

where P is the path taken from x to x_i . It is readily seen that the neutral nonlocal quark current defined by

TABLE I. Charmonium states $^{2S+1}L_J$. We use the notation $\vec{\partial} = \vec{\partial} - \overleftarrow{\partial}$. The numerical values of masses and decay widths are taken from the PDG [15].

Quantum number	Title	Quark current	Mass (MeV)	Width (MeV)
$J^{PC} = 0^{-+}$	$^1S_0 = \eta_c(1S)$	$\bar{q}i\gamma^5 q$	2984.1(4)	30.5(5)
$J^{PC} = 1^{--}$	$^3S_1 = J/\psi(1S)$	$\bar{q}\gamma^\mu q$	3096.900(6)	0.0926(17)
$J^{PC} = 0^{++}$	$^3P_0 = \chi_{c0}(1P)$	$\bar{q}q$	3414.71(30)	10.5(8)
$J^{PC} = 1^{++}$	$^3P_1 = \chi_{c1}(1P)$	$\bar{q}\gamma^\mu \gamma^5 q$	3510.67(5)	0.84(4)
$J^{PC} = 1^{+-}$	$^1P_1 = h_c(1P)$	$\bar{q}\vec{\partial}^\mu \gamma^5 q$	3525.37(14)	0.78(28)
$J^{PC} = 2^{++}$	$^3P_2 = \chi_{c2}(1P)$	$(i/2)\bar{q}(\gamma^\mu \vec{\partial}^\nu + \gamma^\nu \vec{\partial}^\mu)q$	3556.17(7)	1.98(9)
$J^{PC} = 1^{--}$	$^3S_1 = \psi(2S)$	$\bar{q}\gamma^\mu q$	3686.097(11)	0.293(9)

$$J^{\text{em}}(x) = \int \int dx_1 dx_2 \delta(x - wx_1 - wx_2) \Phi((x_1 - x_2)^2) \times \bar{Q}(x_1) \Gamma Q(x_2), \quad (w = 1/2), \quad (8)$$

is invariant under the local gauge transformations

$$\begin{aligned} q(x_i) &\rightarrow e^{ie_q f(x_i)} q(x_i), \\ \bar{q}(x_i) &\rightarrow e^{-ie_q f(x_i)} \bar{q}(x_i), \\ A^\mu(z) &\rightarrow A^\mu(z) + \partial^\mu f(z), \end{aligned}$$

$$\text{so that } I(x_i, x, P) \rightarrow I(x_i, x, P) + f(x_i) - f(x), \quad (9)$$

if the matrix Γ has no derivative.

The first term of the electromagnetic interaction arises when one expands the gauge exponential in powers of $I(x_i, x, P)$. Superficially, the results appear to depend on the path P . However, one needs to know only derivatives of the path integrals when doing the perturbative expansion. One can make use of the formalism developed in [39,40] which is based on the path-independent definition of the derivative of $I(x, y, P)$,

$$\frac{\partial}{\partial x^\mu} I(x, y, P) = A_\mu(x), \quad (10)$$

which states that the derivative of the path integral $I(x, y, P)$ does not depend on the path P originally used in the definition. It is easy to check that such procedure of gauging the free quark Lagrangian leads to the standard form of $e_q \bar{q}(x) \hat{A}(x) q(x)$.

Expanding the gauged quark current given by Eq. (8) up to the first order in A^μ one obtains

$$\begin{aligned} J^{\text{em}}(x) &= \int \int dx_1 dx_2 \delta(x - wx_1 - wx_2) \\ &\times \Phi((x_1 - x_2)^2) \bar{q}(x_1) \Gamma q(x_2) \\ &\times (1 + ie_q I(x_1, x_2) + \mathcal{O}(e_q^2)). \end{aligned} \quad (11)$$

If a quark current contains derivatives, e.g., in the case of the h_c and χ_{c2} charmonium states (see Table I), extra terms arise in Eq. (11). For instance, for h_c one has

$$\begin{aligned} J_{h_c}^{\text{em}}(x) &= \int \int dx_1 dx_2 \delta(x - wx_1 - wx_2) \Phi_{h_c}((x_1 - x_2)^2) \\ &\times \{ [\bar{q}(x_1) \gamma_5 \partial_{x_2}^\mu q(x_2) - \partial_{x_1}^\mu \bar{q}(x_1) \gamma_5 q(x_2)] \\ &\times (1 + ie_q I(x_1, x_2)) - ie_q [A^\mu(x_1) + A^\mu(x_2)] \\ &\times \bar{q}(x_1) \gamma_5 q(x_2) \} + \mathcal{O}(e_q^2). \end{aligned} \quad (12)$$

The evaluation of Feynman diagrams involving the strong quark vertex with emitting photon leads to a typical integral

$$R(x; k_1, k_2) = \int \int dx_1 dx_2 \delta(x - wx_1 - wx_2) \times \Phi((x_1 - x_2)^2) I(x_1, x_2) e^{ik_1 x_1 - ik_2 x_2}, \quad (13)$$

which is calculated by replacing the integration variables $x_1 = z - w\rho$ and $x_2 = z + w\rho$ where $w = 1/2$. One has

$$R(x; k_1, k_2) = e^{i(k_1 - k_2)x} \int d\rho \Phi(\rho^2) I(x - w\rho, x + w\rho) e^{-ik\rho},$$

where $k = \frac{1}{2}(k_1 + k_2)$. (14)

In the next step, one can use the transformation of the vertex function given by

$$\Phi(\rho^2) = \int \frac{d^4 \ell}{(2\pi)^4} e^{-i\ell \rho} \tilde{\Phi}(-\ell^2) = \tilde{\Phi}(\partial_\rho^2) \delta^{(4)}(\rho). \quad (15)$$

Finally, we are coming to the expression that has been calculated in Refs. [41,44,45,48]. Assuming that $A_\alpha(x) = \epsilon_\alpha^* e^{iqx}$, one has

$$\begin{aligned} R(x, k_1, k_2) &= e^{i(k_1 - k_2)x} \int d^4 \rho \delta^{(4)}(\rho) \tilde{\Phi}(\partial_\rho^2) \\ &\times I(x - w\rho, x + w\rho) e^{-ik\rho} \\ &= i\epsilon_\alpha^* e^{i(k_1 - k_2 + q)x} \int_0^1 d\tau \{ \tilde{\Phi}'(-z_\tau^+) (k + w^2 q)^\alpha \\ &\quad + \tilde{\Phi}'(-z_\tau^-) (k - w^2 q)^\alpha \}, \\ z_\tau^\pm &= (k \pm wq)^2 \tau + k^2 (1 - \tau). \end{aligned} \quad (16)$$

C. Radiative decays $(\bar{c}c)_1 \rightarrow (\bar{c}c)_2 + \gamma$: Feynman diagrams and loop integrals

In this paper we will consider eight modes of charmonium radiative decays,

- (a) $\chi_{cJ} \rightarrow J/\psi + \gamma$,
- (b) $\psi(2S) \rightarrow \chi_{cJ} + \gamma$, where $J = 0, 1, 2$,
- (c) $h_c \rightarrow \eta_c + \gamma$ and $J/\psi \rightarrow \eta_c + \gamma$.

The relevant Feynman diagrams are shown in Fig. 1. Due to the gauging technique described above, there are three topologies, first, the diagram with photon emitted from quark line, and second, two diagrams with photons emitted from the nonlocal vertices.

We start with the group of the $\chi_{cJ}(p_1) \rightarrow J/\psi(p_2) + \gamma(q)$ ($J = 0, 1, 2$) decays. The invariant matrix elements describing these decays are written as

$$M_{\chi_{cJ} \rightarrow J/\psi + \gamma} = e \epsilon_{2\beta}^*(p_2) \epsilon_{\gamma\alpha}^*(q) M^{\beta\alpha}, \quad (17)$$

where the amplitude $M^{\beta\alpha}$ is written via loop integrals corresponding to the Feynman diagrams. One has

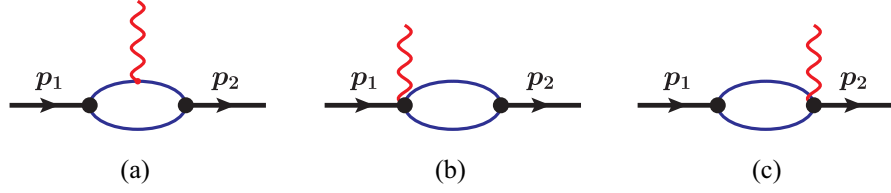


FIG. 1. Diagrams describing the $(\bar{c}c_1) \rightarrow (\bar{c}c)_2 + \gamma$ transition. (a) Triangle diagram where the photon is emitted from the quark line. (b) and (c) Bubble diagrams where the photon is emitted from the nonlocal vertex.

$$M^{\beta\alpha} = M_{\Delta a}^{\beta\alpha} + M_{\bigcirc b}^{\beta\alpha} + M_{\bigcirc c}^{\beta\alpha}, \quad (18)$$

$$M_{\Delta a}^{\beta\alpha} = -N \int \frac{d^4\ell}{(2\pi)^4 i} \tilde{\Phi}_{cJ}(-(\ell + wp_1)^2) \tilde{\Phi}_{J/\psi}(-(\ell + wp_2)^2) \times \text{tr}[\gamma^\beta S(\ell + p_2) \gamma^\alpha S(\ell + p_1) \tilde{\Gamma}_{cJ} S(\ell)], \quad (19)$$

$$M_{\bigcirc b}^{\beta\alpha} = +N \int \frac{d^4\ell}{(2\pi)^4 i} \int_0^1 d\tau \tilde{\Phi}'_{cJ}(-z_\tau^+) \tilde{\Phi}_{J/\psi}(-\ell^2) \ell^\alpha \times \text{tr}[\gamma^\beta S(\ell + wp_2) \tilde{\Gamma}_{cJ} S(\ell - wp_2)], \quad (20)$$

$$M_{\bigcirc c}^{\beta\alpha} = +N \int \frac{d^4\ell}{(2\pi)^4 i} \tilde{\Phi}_{cJ}(-\ell^2) \int_0^1 d\tau \tilde{\Phi}'_{J/\psi}(-z_\tau^+) \ell^\alpha \times \text{tr}[\gamma^\beta S(\ell + wp_1) \tilde{\Gamma}_{cJ} S(\ell - wp_1)], \quad (21)$$

where $z_\tau^+ = (\ell + wq)^2 \tau + \ell^2(1 - \tau)$, $\tilde{\Gamma}_{c0} = I$, $\tilde{\Gamma}_{c1} = \epsilon_\mu(p_1) \gamma^\mu \gamma_5$, and $\tilde{\Gamma}_{c2} = 2\epsilon_{\mu\nu}(p_1) \ell^\mu \gamma^\nu$. The overall factor N in these equations reads $N = 6g_{\chi_{cJ}} g_{J/\psi} e_c$, where $e_c = 2/3$ is the charge of charm quark in units of proton charge. We have used the properties of transversality and symmetry of the polarization vectors: $\epsilon_\alpha^* q^\alpha = 0$, $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$, and $\epsilon_{\mu\nu} p_1^\mu = 0$. Note that there is an additional diagram in the spin-2 case due to the derivative in the Lagrangian. One has

$$M_{\bigcirc d}^{\beta\alpha} = -N \int \frac{d^4\ell}{(2\pi)^4 i} \tilde{\Phi}_{cJ}(-(\ell + wq)^2) \times \tilde{\Phi}_{J/\psi}(-\ell^2) \epsilon_{\mu\nu}(p_1) g^{\nu\alpha} \times \text{Tr}[\gamma^\beta S(\ell + wp_2) \gamma^\mu S(\ell - wp_2)]. \quad (22)$$

The invariant matrix element describing the decay $h_c \rightarrow \eta_c + \gamma$ is written as

$$M_{h_c \rightarrow \eta_c + \gamma} = e \epsilon_\mu(p_1) \epsilon_{\gamma\alpha}^*(q) M^{\mu\alpha}, \quad (23)$$

where the amplitude $M^{\mu\alpha}$ is given by

$$M^{\mu\alpha} = M_{\Delta a}^{\mu\alpha} + M_{\bigcirc b}^{\mu\alpha} + M_{\bigcirc c}^{\mu\alpha} + M_{\bigcirc d}^{\mu\alpha}, \quad (24)$$

$$M_{\Delta a}^{\mu\alpha} = -N_{h_c} \int \frac{d^4\ell}{(2\pi)^4 i} \tilde{\Phi}_{h_c}(-(\ell + wp_1)^2) \tilde{\Phi}_{\eta_c}(-(\ell + wp_2)^2) \times \text{tr}[\gamma_5 S(\ell + p_2) \gamma^\alpha S(\ell + p_1) (2\ell^\mu) \gamma_5 S(\ell)], \quad (25)$$

$$M_{\bigcirc b}^{\mu\alpha} = +N_{h_c} \int \frac{d^4\ell}{(2\pi)^4 i} \int_0^1 d\tau \tilde{\Phi}'_{h_c}(-z_\tau^+) \ell^\alpha \tilde{\Phi}_{\eta_c}(-\ell^2) \times \text{tr}[\gamma_5 S(\ell + wp_2) \gamma_5 (2\ell^\mu) S(\ell - wp_2)], \quad (26)$$

$$M_{\bigcirc c}^{\mu\alpha} = +N_{h_c} \int \frac{d^4\ell}{(2\pi)^4 i} \tilde{\Phi}_{h_c}(-\ell^2) \int_0^1 d\tau \tilde{\Phi}'_{\eta_c}(-z_\tau^+) \ell^\alpha \times \text{tr}[\gamma_5 S(\ell + wp_1) \gamma_5 (2\ell^\mu) S(\ell - wp_1)], \quad (27)$$

$$M_{\bigcirc d}^{\mu\alpha} = -N_{h_c} \int \frac{d^4\ell}{(2\pi)^4 i} \tilde{\Phi}_{h_c}(-(\ell + wq)^2) \tilde{\Phi}_{J/\psi}(-\ell^2) \times \text{tr}[\gamma_5 S(\ell + wp_2) \gamma_5 g^{\mu\alpha} S(\ell - wp_2)], \quad (28)$$

where $z_\tau^+ = (\ell + wq)^2 \tau + \ell^2(1 - \tau)$, and $N_{h_c} = 6g_{h_c} g_{\eta_c} e_c$.

The first step is to check the gauge invariance before calculating the loop integration, i.e., to check

$$M^{\beta\alpha} q_\alpha = 0. \quad (29)$$

It is done by using two identities,

$$S(\ell + p_2) \not{q} S(\ell + p_1) = S(\ell + p_1) - S(\ell + p_2), \quad (30)$$

$$\int_0^1 d\tau \tilde{\Phi}'(-z_\tau) (\ell + w^2 q)^\alpha q_\alpha = \tilde{\Phi}(-\ell^2) - \tilde{\Phi}(-(\ell + wq)^2). \quad (31)$$

The second step is to reduce the loop integrals to threefold integrals that are evaluated numerically. The calculation of the matrix elements of the decays $\psi(2S) \rightarrow \chi_{cJ} + \gamma$ and $J/\psi \rightarrow \eta_c + \gamma$ is performed in a similar manner.

III. DECAY WIDTHS

We assume that the charmonium and photon are on their mass shells. The spin-1 polarization vectors $\epsilon_\beta^{(\lambda)}(p)$ and $\epsilon_\alpha^{(\lambda)}(q)$ satisfy the conditions,

$$\begin{aligned}
\epsilon_{\beta}^{(\lambda)}(p)p^{\beta} &= 0, \\
\epsilon_{\alpha}^{(\lambda)}(q)q^{\alpha} &= 0 \quad \text{transversality,} \\
\sum_{\lambda=0,\pm} \epsilon_{\beta}^{(\lambda)}(p)\epsilon_{\beta'}^{\dagger(\lambda)}(p) &= -g_{\beta\beta'} + \frac{p_{\beta}p_{\beta'}}{m^2}, \\
\sum_{\lambda=\pm} \epsilon_{\alpha}^{(\lambda)}(q)\epsilon_{\alpha'}^{\dagger(\lambda)}(q) &= -g_{\alpha\alpha'} \quad \text{completeness,} \\
\epsilon_{\mu}^{\dagger(\lambda)}\epsilon^{(\lambda')\mu} &= -\delta_{\lambda\lambda'} \quad \text{orthonormality.}
\end{aligned}$$

And the spin-2 polarization vector $\epsilon_{\mu\nu}^{(\lambda)}(p)$ satisfies the conditions,

$$\begin{aligned}
\epsilon_{\mu\nu}^{(\lambda)}(p) &= \epsilon_{\nu\mu}^{(\lambda)}(p) \quad \text{symmetry,} \\
\epsilon_{\mu\nu}^{(\lambda)}(p)p^{\mu} &= 0 \quad \text{transversality,} \\
\epsilon_{\mu\mu}^{(\lambda)}(p) &= 0 \quad \text{tracelessness,} \\
\sum_{\lambda=0,\pm 1,\pm 2} \epsilon_{\mu\nu}^{(\lambda)}\epsilon_{\alpha\beta}^{\dagger(\lambda)} &= \frac{1}{2}(S_{\mu\alpha}S_{\nu\beta} + S_{\mu\beta}S_{\nu\alpha}) \\
&\quad - \frac{1}{3}S_{\mu\nu}S_{\alpha\beta} \quad \text{completeness,} \\
\epsilon_{\mu\nu}^{\dagger(\lambda)}\epsilon^{(\lambda')\mu\nu} &= \delta_{\lambda\lambda'} \quad \text{orthonormality,}
\end{aligned}$$

where $S_{\mu\nu} = -g_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{m^2}$.

A. Transition $\chi_{c0} \rightarrow J/\psi + \gamma$

To calculate the decay width of the transition $\chi_{c0} \rightarrow J/\psi + \gamma$, one starts with the invariant matrix element

$$M_{\chi_{c0} \rightarrow J/\psi + \gamma} = e\epsilon_{2\beta}^*(p_2)\epsilon_{\gamma\alpha}^*(q)M_{\chi_{c0}}^{\beta\alpha}. \quad (32)$$

The amplitude $M_{\chi_{c0}}^{\beta\alpha}$, on one hand, is calculated based on the Feynman diagrams in Fig. 1. On the other hand, one can parametrize it in terms of one coefficient [38], namely A_1 , as follows:

$$M_{\chi_{c0}}^{\beta\alpha} = M_{\Delta a}^{\beta\alpha} + M_{\bigcirc b}^{\beta\alpha} + M_{\bigcirc c}^{\beta\alpha} \equiv (m_1 A_1) \left(g^{\beta\alpha} - \frac{q^{\beta} p_2^{\alpha}}{p_2 q} \right), \quad (33)$$

where $p_2 q = (m_1^2 - m_2^2)/2$. The coefficient A_1 is calculated in the CCQM using the loop integrals given in Eqs. (18)–(21). The techniques of calculating one-loop integrals in the CCQM have been described in our previous papers (see, e.g., Refs. [51,52]). In particular, the loop calculation for charmonium states was discussed in details in our recent paper [38]. After squaring the matrix element and summing over polarizations, one obtains the decay width

$$\Gamma(\chi_{c0} \rightarrow J/\psi + \gamma) = \alpha |\mathbf{q}| A_1^2, \quad |\mathbf{q}| = \frac{m_1^2 - m_2^2}{2m_1}. \quad (34)$$

B. Transition $\chi_{c1} \rightarrow J/\psi + \gamma$

There are two independent helicity amplitudes $H_{\lambda_1; \lambda_2 \lambda}$, which we denote by $H_i (i = L, T)$ according to the helicity of the final meson state J/ψ , where $\lambda_2 = 0$ and $\lambda_2 = \pm 1$ stand for the longitudinal and transverse helicities of J/ψ . From parity one has $H_{+;0-} = -H_{-;0+} = H_L$ and $H_{0;+-} = -H_{0;-+} = H_T$.

The polarization vectors and momenta in the χ_{c1} rest frame are defined as

$$\begin{aligned}
\epsilon_{1\mu}(\pm) &= \frac{1}{\sqrt{2}}(0; \pm 1, i, 0), \quad p_{\mu} = (m_1; 0, 0, 0), \\
\epsilon_{1\mu}(0) &= (0; 0, 0, -1), \\
\epsilon_{2\beta}^{\dagger}(\pm) &= \frac{1}{\sqrt{2}}(0; \pm 1, -i, 0), \quad p_{2\beta} = (E_2; 0, 0, -|\mathbf{q}|), \\
\epsilon_{2\beta}^{\dagger}(0) &= \frac{1}{m_2}(|\mathbf{q}|; 0, 0, -E_2), \quad E_2 = \frac{m_1^2 + m_2^2}{2m_1}, \\
\bar{\epsilon}_{\alpha}^{\dagger}(\pm) &= \frac{1}{\sqrt{2}}(0; \mp 1, -i, 0), \quad q_{\alpha} = |\mathbf{q}|(1; 0, 0, 1). \quad (35)
\end{aligned}$$

The z direction is defined by the momentum of J/ψ . The bars in the polarization four-vectors $\bar{\epsilon}_{\alpha}(\lambda)$ of the photon are a reminder that the photon helicities are defined relative to the negative z direction.

The invariant matrix element is written in the form

$$M_{\chi_{c1} \rightarrow J/\psi + \gamma} = e\epsilon_{1\mu}(p_1)\epsilon_{2\beta}^*(p_2)\epsilon_{\gamma\alpha}^*(q)M_{\chi_{c1}}^{\mu\beta\alpha}. \quad (36)$$

Similar to case of $\chi_{c0} \rightarrow J/\psi + \gamma$, the amplitude $M_{\chi_{c1}}^{\mu\beta\alpha}$ is calculated based on the Feynman diagrams in Fig. 1. However, $M_{\chi_{c1}}^{\mu\beta\alpha}$ is parametrized by two coefficients, namely D_E and D_M as follows:

$$\begin{aligned}
M_{\chi_{c1}}^{\mu\beta\alpha} &= \epsilon_1^{\mu}(p_1)(M_{\Delta a}^{\beta\alpha} + M_{\bigcirc b}^{\beta\alpha} + M_{\bigcirc c}^{\beta\alpha}) \\
&\equiv p_{2\delta} q_{\nu} (\epsilon^{\delta\nu\mu\alpha} q^{\beta} D_E + \epsilon^{\delta\nu\beta\alpha} p_2^{\mu} D_M), \quad (37)
\end{aligned}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the four-dimensional antisymmetric Levi-Civita tensor.

It is convenient to present the decay width via the helicity amplitudes. One has

$$\begin{aligned}
H_L &= H_{+;0-} = -H_{-;0+} = \epsilon_{1\mu}(+) \epsilon_{2\beta}^{\dagger}(0) \bar{\epsilon}_{\alpha}^{\dagger}(-) M_{\chi_{c1}}^{\mu\beta\alpha} \\
&= i \frac{m_1^2}{m_2} |\mathbf{q}|^2 D_E, \\
H_T &= H_{0;+-} = -H_{0;-+} = \epsilon_{1\mu}(0) \epsilon_{2\beta}^{\dagger}(+) \bar{\epsilon}_{\alpha}^{\dagger}(+) M_{\chi_{c1}}^{\mu\beta\alpha} \\
&= -im_1 |\mathbf{q}|^2 D_M. \quad (38)
\end{aligned}$$

Then the decay width is written as

$$\Gamma(\chi_{c1} \rightarrow J/\psi + \gamma) = \frac{\alpha}{3} |\mathbf{q}| (|H_L|^2 + |H_T|^2). \quad (39)$$

C. Transition $\chi_{c2} \rightarrow J/\psi + \gamma$

There are three independent helicity amplitudes $H_{\lambda_1; \lambda_2 \lambda}$ characterizing the decay $\chi_{c2} \rightarrow J/\psi + \gamma$,

$$\begin{aligned} H_{+2; +1-1} &= H_{-2; -1+1} = \epsilon_{1\mu\nu} (+2) \epsilon_{2\beta}^* (+) \bar{\epsilon}_\alpha^* (-) M_{\chi_{c2}}^{\mu\nu\beta\alpha}, \\ H_{+1; 0-1} &= H_{-1; 0+1} = \epsilon_{1\mu\nu} (+1) \epsilon_{2\beta}^* (0) \bar{\epsilon}_\alpha^* (-) M_{\chi_{c2}}^{\mu\nu\beta\alpha}, \\ H_{0; +1+1} &= H_{0; -1-1} = \epsilon_{1\mu\nu} (0) \epsilon_{2\beta}^* (+) \bar{\epsilon}_\alpha^* (+) M_{\chi_{c2}}^{\mu\nu\beta\alpha}. \end{aligned} \quad (40)$$

The polarization vectors of χ_{c2} are given by

$$\begin{aligned} \epsilon_{1\mu\nu}(\pm 2) &= \epsilon_{2\mu}(\pm) \epsilon_{2\nu}(\pm), \\ \epsilon_{1\mu\nu}(\pm 1) &= \frac{1}{\sqrt{2}} (\epsilon_{2\mu}(\pm) \epsilon_{2\nu}(0) + \epsilon_{2\mu}(0) \epsilon_{2\nu}(\pm)), \\ \epsilon_{1\mu\nu}(0) &= \frac{1}{\sqrt{6}} (\epsilon_{2\mu}(+) \epsilon_{2\nu}(-) + \epsilon_{2\mu}(-) \epsilon_{2\nu}(+)) \\ &\quad + \sqrt{\frac{2}{3}} \epsilon_{2\mu}(0) \epsilon_{2\nu}(0), \end{aligned} \quad (41)$$

where $\epsilon_{2\mu}(r)$ are defined in Eq. (35).

The invariant matrix element of the transition $\chi_{c2} \rightarrow J/\psi + \gamma$ is written as

$$M_{\chi_{c2} \rightarrow J/\psi + \gamma} = e \epsilon_{1\mu\nu}(p_1) \epsilon_{2\beta}^*(p_2) \epsilon_{\gamma\alpha}^*(q) M_{\chi_{c2}}^{\mu\nu\beta\alpha}. \quad (42)$$

The amplitude $M_{\chi_{c2}}^{\mu\nu\beta\alpha}$ is calculated in the standard way and parametrized in terms of structural coefficients. By using the gauge invariance the number of the coefficients is reduced to three, namely F_i ($i = 1, 2, 3$). One has

$$\begin{aligned} M_{\chi_{c2}}^{\mu\nu\beta\alpha} &= \epsilon_1^{\mu\nu}(p_1) (M_{\Delta a}^{\beta\alpha} + M_{\bigcirc b}^{\beta\alpha} + M_{\bigcirc c}^{\beta\alpha} + M_{\bigcirc d}^{\beta\alpha}) \\ &\equiv F_1(p_2^\alpha q^\beta - p_2^\beta q^\alpha) q^\mu q^\nu + F_2(p_2^\alpha q^\nu - p_2^\nu q^\alpha) g^{\mu\beta} \\ &\quad + F_3(g^{\mu\alpha} q^\nu q^\beta - g^{\alpha\beta} q^\mu q^\nu). \end{aligned} \quad (43)$$

It is convenient to present the decay width via helicity amplitudes. They are calculated by using the definitions given in Eqs. (40)–(43). One gets

$$\begin{aligned} \Gamma(\chi_{c2} \rightarrow J/\psi + \gamma) &= \frac{\alpha |\mathbf{q}|}{5 m_1^2} (|H_{+2; +1-1}|^2 + |H_{+1; 0-1}|^2 \\ &\quad + |H_{0; +1+1}|^2), \\ H_{+2; +1-1} &= -m_1 |\mathbf{q}| F_2, \\ H_{+1; 0-1} &= -\frac{1}{\sqrt{2}} \frac{m_1}{m_2} |\mathbf{q}| (E_2 F_2 + |\mathbf{q}| F_3), \\ H_{0; +1+1} &= -\sqrt{\frac{2}{3}} m_1 |\mathbf{q}| \left(|\mathbf{q}|^2 F_1 + \frac{1}{2} F_2 + \frac{|\mathbf{q}|}{m_1} F_3 \right). \end{aligned} \quad (44)$$

D. Transition $h_c \rightarrow \eta_c + \gamma$

The invariant matrix element of the transition $h_c \rightarrow \eta_c + \gamma$ can be written in the form

$$M_{h_c \rightarrow \eta_c + \gamma} = e \epsilon_{1\mu}(p_1) \epsilon_{\gamma\alpha}^*(q) M_{h_c}^{\mu\alpha}. \quad (45)$$

The amplitude $M_{h_c}^{\mu\alpha}$ is calculated using the loop integrals given in Eqs. (25)–(28). It can be parametrized by one coefficient, namely W_1 , as follows:

$$\begin{aligned} M_{h_c}^{\mu\alpha} &= M_{\Delta a}^{\mu\alpha} + M_{\bigcirc b}^{\mu\alpha} + M_{\bigcirc c}^{\mu\alpha} + M_{\bigcirc d}^{\mu\alpha} \\ &\equiv W_1 (p_1^\alpha q^\mu - p_1^\mu q^\alpha), \end{aligned} \quad (46)$$

where $p_1 q = (m_1^2 - m_2^2)/2$. Finally, by squaring the matrix element and summing over polarizations, one obtains

$$\Gamma(h_c \rightarrow \eta_c + \gamma) = \frac{\alpha}{3} |\mathbf{q}|^3 W_1^2, \quad |\mathbf{q}| = \frac{m_1^2 - m_2^2}{2m_1}. \quad (47)$$

IV. NUMERICAL RESULTS

The CCQM includes several free parameters: the constituent quark masses m_q , the hadron size parameters Λ_H , and the universal infrared cutoff parameter λ . In this paper, we apply a new strategy for fitting the model parameters. First, we have observed before that physical observables are rather insensitive to the concrete value of the cutoff parameter λ . In particular, in a recent study [38], we checked this observation by gradually decreasing λ and proved that the results did not change for any $\lambda < 0.181$ GeV up to the deconfinement limit. Therefore, in this study, we use again the value $\lambda = 0.181$ GeV obtained in our early fit [44]. Additionally, by varying λ by $\pm 10\%$ around the central value 0.181 GeV we found that the results in this paper were not changed. We therefore ignore the error from this parameter in our analysis. The remaining free parameters include the charm quark mass m_c and the size parameters Λ_{cc} of the charmonium states. We observe that the charmonium radiative decay widths depend rather slowly on their size parameters Λ_{cc} in the interval 2–4 GeV. Conversely, the dependence on the charm quark mass m_c running in the loop is strong. Therefore, we assume that the size parameter of a charmonium state is proportional to its mass, i.e., $\Lambda_{cc} = \rho M_{cc}$. We then determine m_c and ρ by fitting the calculated branching fractions of the decays $J/\psi \rightarrow \eta_c \gamma$, $\chi_{c0} \rightarrow J/\psi \gamma$, and $\psi(2S) \rightarrow \chi_{c0} \gamma$ to experimental data.

We use Bayesian inference for parameter estimation and Monte Carlo error propagation for predictions. We validate the Bayesian results using an independent, frequentist statistical approach. For the Bayesian analysis, we build a 16-dimensional parameter space that comprises two primary parameters of interest—the charm quark mass m_c and the hadron size parameter ρ —and 14 nuisance parameters $\{\alpha_j\}$ corresponding to the experimentally measured masses and widths of the charmonium states given in Table I. The details

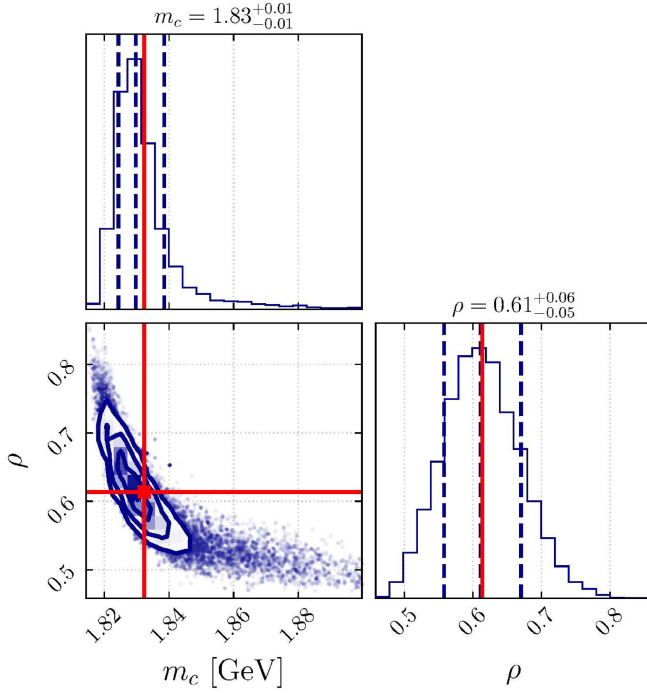


FIG. 2. Posterior probability distributions for the charm quark mass m_c and the size parameter ρ .

of the fitting procedure and error propagation calculation are given in the Appendix. Here we present only the main results.

A. Parameter constraints and model degeneracy

The Bayesian analysis framework yields robust constraints on the model's two principal parameters. The median values and 68% credible intervals, derived from the marginal posterior distributions, are determined to be,

$$\begin{aligned} m_c &= 1.83^{+0.01}_{-0.01} \text{ GeV}, \\ \rho &= 0.61^{+0.06}_{-0.05}. \end{aligned} \quad (48)$$

The full posterior probability distributions for these parameters are presented in Fig. 2. As illustrated, the diagonal panels show the one-dimensional marginalized posterior for each parameter, while the off-diagonal panel shows their two-dimensional joint probability. The contours on the joint distribution, which represent 0.5, 1.0, 1.5, and 2.0 σ credible regions, reveal a strong anti-correlation. This relationship is quantified by a Pearson coefficient of $r = -0.772$ and indicate a significant physical degeneracy within the model, where a smaller m_c value can be compensated for by a larger ρ to produce similar physical observables.

B. Goodness-of-fit and model tension

While the parameters are well constrained, the model's overall performance exhibits a quantifiable tension with the data. The global goodness of fit, evaluated at the posterior's

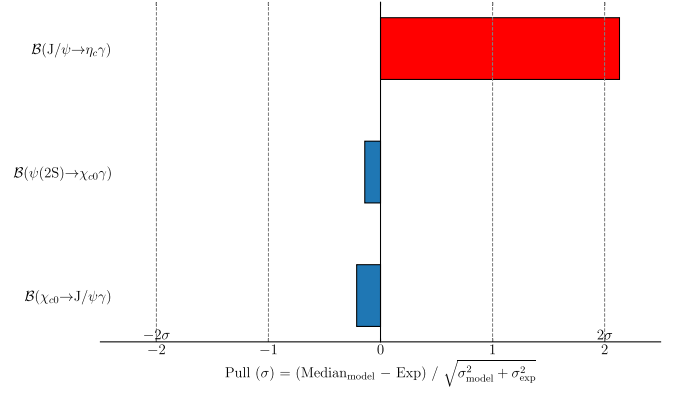


FIG. 3. Pull summary for the three fitted decay channels.

point of maximum likelihood, is characterized by,

$$\chi^2_{\min}/\text{d.o.f.} = 5.16/1. \quad (49)$$

This value indicates that the model as a whole is a poor descriptor of the data. To diagnose the source of this tension, we examine the pull of each individual channel (Fig. 3). This decomposition makes it clear that the poor global fit is not a general failure of the model, but is instead localized to a single, large discrepancy in the $J/\psi \rightarrow \eta_c \gamma$ channel. A detailed posterior predictive check for the fitted channels, shown in Fig. 4, visually confirms this finding. In the plot, the model's posterior predictive distribution (blue histogram) for $\mathcal{B}(J/\psi \rightarrow \eta_c \gamma)$ is systematically offset from the experimental data (red line and 1 σ band), resulting in a large pull of +2.13 σ . In contrast, the model demonstrates excellent agreement with experiments for the other two fitted channels. The model's results for $\psi(2S) \rightarrow \chi_{c0} \gamma$ and $\chi_{c0} \rightarrow J/\psi \gamma$ align remarkably well with the experimental data. Their pulls of -0.14σ and -0.21σ , respectively, are negligible. This finding strongly suggests that the model's deficiency is localized to the description of the $J/\psi \rightarrow \eta_c \gamma$ transition, rather than being a global failure.

C. Validation via frequentist cross-check

To ensure the robustness of our findings, we performed an independent cross-check using the frequentist profile likelihood method. The resulting profile curves are shown in Fig. 5. The 1 σ (68.3%) confidence level (CL) intervals are defined by the standard criterion $\Delta\chi^2 = \chi^2 - \chi^2_{\min} \leq 1$. In the figure, the horizontal lines mark the 1 σ , 2 σ , and 3 σ confidence levels, corresponding to $\Delta\chi^2$ values of 1, 4, and 9, respectively. This procedure yields the following frequentist estimates:

$$\begin{aligned} m_c &\in [1.82, 1.84] \text{ GeV} \quad (68.3\% \text{ CL}), \\ \rho &\in [0.56, 0.66] \quad (68.3\% \text{ CL}). \end{aligned} \quad (50)$$

These confidence intervals are in excellent agreement with the Bayesian credible intervals reported in Eq. (48).

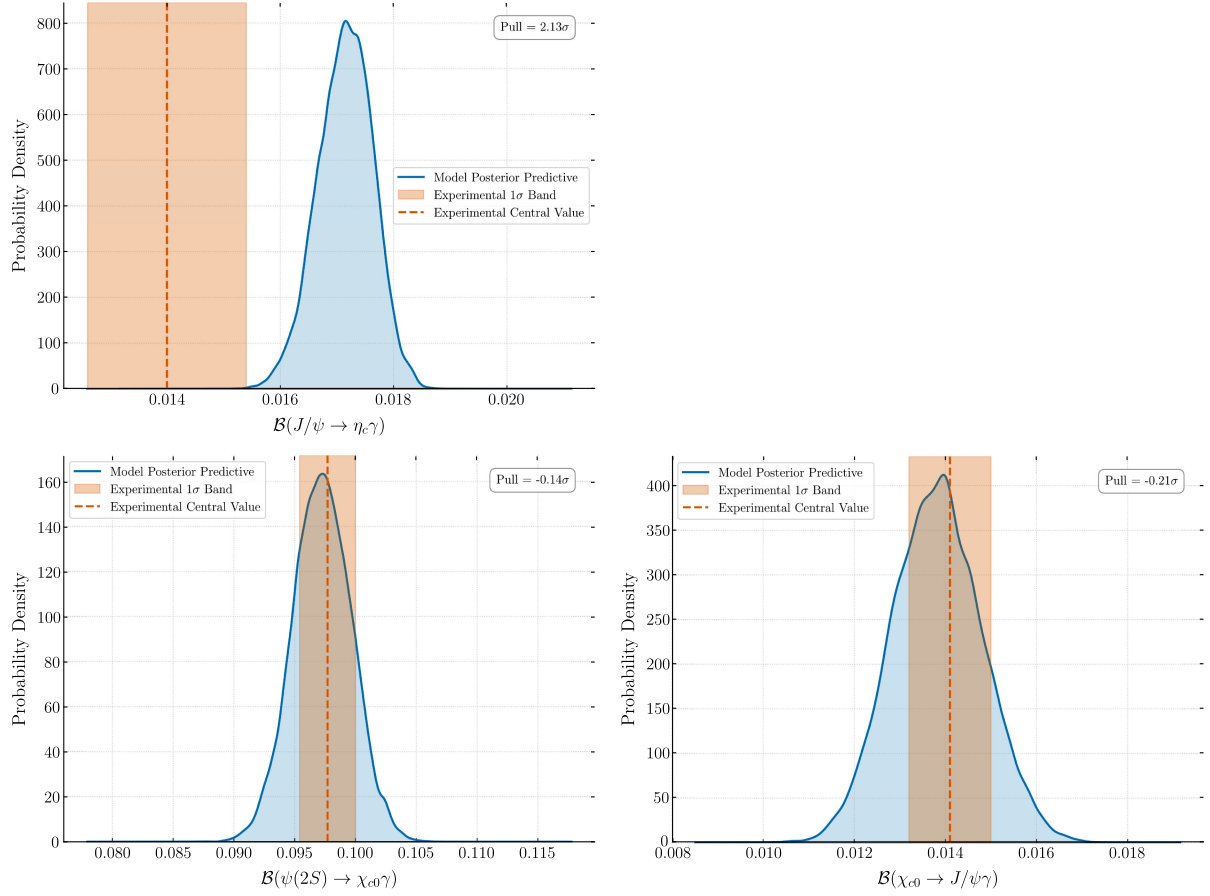
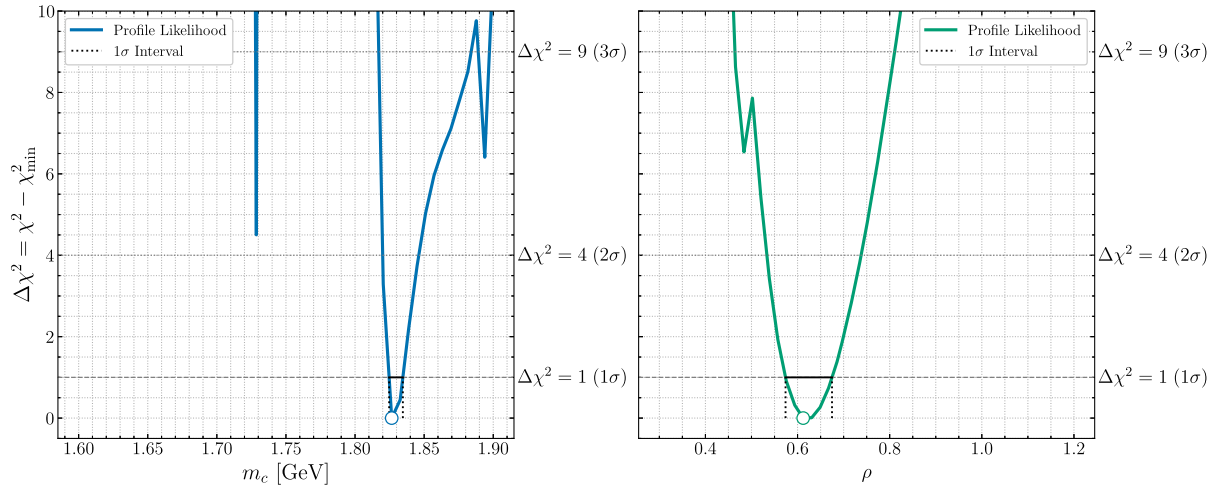


FIG. 4. Posterior predictive check for the fitted channels.

FIG. 5. Frequentist profile likelihood scans for the m_c and ρ parameters.

This strong consistency between two distinct statistical paradigms confirms that our parameter estimates are robust and not an artifact of the chosen methodology.

D. Theoretical prediction

In Fig. 6 we provide a comprehensive summary of the predicted branching fractions, visually distinguishing between

fitted (blue) and prediction-only (purple) channels. A detailed breakdown of these predictions, along with a decomposition of their theoretical uncertainty, is summarized in Table II.

The variance decomposition (see Sec. A 2 c for more details) in Table II helps disentangle uncertainty sources. The channel $h_c \rightarrow \eta_c \gamma$ is dominated by systematic uncertainty (89.1%). The class of $\psi(2S) \rightarrow \chi_{cJ} \gamma$ decays possess a

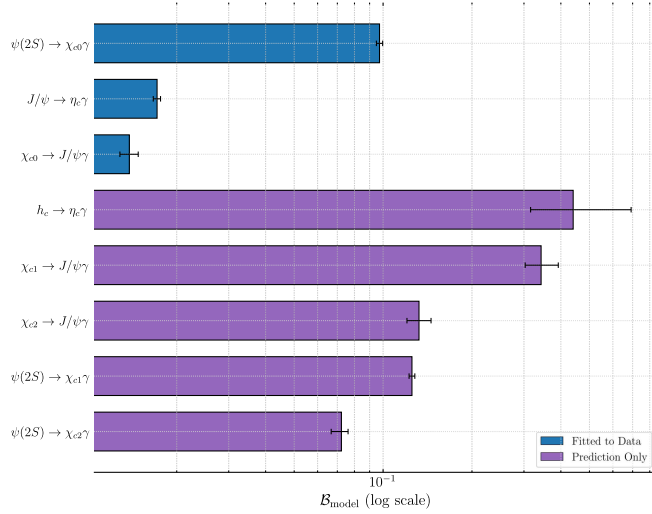


FIG. 6. Model predictions for branching fractions across all considered channels.

high sensitivity to m_c (92.8% and 62.5% for $J = 1, 2$) but a minimal dependence on ρ . In contrast, the decays $\chi_{c1,c2} \rightarrow J/\psi\gamma$ are more sensitive to ρ (79.3% and 74.5% for χ_{c1}, χ_{c2}).

Table II shows good agreement between our predictions and experimental data. To be more specific, our predictions for the branching fractions $\mathcal{B}(h_c \rightarrow \eta_c\gamma)$, $\mathcal{B}(\chi_{c1} \rightarrow J/\psi\gamma)$, and $\mathcal{B}(\psi(2S) \rightarrow \chi_{c1}\gamma)$ agree with the corresponding experimental data within 1σ . For the remaining decays $\chi_{c2} \rightarrow J/\psi\gamma$ and $\psi(2S) \rightarrow \chi_{c2}\gamma$, our results agree with data within 2σ . However, the theoretical uncertainties of our predictions in this study are large—much larger than the estimated value of $\approx 10\%$ – 20% in our previous works (see, e.g., [53,54]). This difference is the consequence of (i) the full consideration of uncertainties in the mass and width of charmonium states (as can be seen in the case of $h_c \rightarrow \eta_c\gamma$), and (ii) the distinct fitting strategy used in this study. In the traditional version of our model, each hadron H is characterized by one size parameter Λ_H , and all size parameters are assumed to be independent of each other.

This means that the traditional description would require fitting eight free parameters: the charm quark mass and seven size parameters for seven different charmonium states. In this study, we assumed that the size parameters of the charmonium states are proportional to their masses. This assumption reduces the number of free parameters from 8 to 2, which leads to improved predictive power but, as a trade-off, larger uncertainties in the theoretical predictions.

V. SUMMARY

We present an analysis of the radiative decays of the charmonium states $h_c(1P)$, $J/\psi(1S)$, $\psi(2S)$, and $\chi_{cJ}(1P)$ ($J = \{0, 1, 2\}$) within the framework of our quark model (CCQM). We proved the gauge invariance of the amplitudes and obtained expressions for the decay width in terms of the helicity amplitudes. Assuming that the size parameters of charmonium states are proportional to their mass, $\Lambda_{cc} = \rho M_{cc}$, we performed a Bayesian fit for the charm quark mass m_c and the coefficient of proportionality ρ . Three radiative channels $\psi(2S) \rightarrow \gamma\chi_{c0}$, $\chi_{c0} \rightarrow \gamma J/\psi$, and $J/\psi \rightarrow \gamma\eta_c$ were used to constrain the two parameters. The obtained values of m_c and ρ were then used to predict the branching fractions of the decays $h_c \rightarrow \gamma\eta_c$, $\chi_{c1,c2} \rightarrow \gamma J/\psi$, and $\psi(2S) \rightarrow \gamma\chi_{c1,c2}$. A Monte Carlo analysis was used to estimate the uncertainties of our theoretical predictions. Our predictions are in good agreement with available experimental data within uncertainty, which demonstrates the validity of the physical description of charmonium states in the CCQM.

ACKNOWLEDGMENTS

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TABLE II. Model predictions and variance decomposition.

Decay channel	Prediction ($\mathcal{B}(10^{-2})$)		Variance contribution (%) ^a			
	Value	σ_{tot}	m_c	ρ	Sys.	Expt. [15]
$h_c \rightarrow \eta_c\gamma$	39	32	0.1	10.8	89.1	60(4)
$\chi_{c1} \rightarrow J/\psi\gamma$	32	9	0.5	79.3	20.2	34.3(1.3)
$\chi_{c2} \rightarrow J/\psi\gamma$	13	3	1.8	74.5	23.7	19.5(8)
$\psi(2S) \rightarrow \chi_{c1}\gamma^b$	12	11	92.8	5.9	1.3	9.75(27)
$\psi(2S) \rightarrow \chi_{c2}\gamma$	7	1	62.5	33.4	4.1	9.36(23)

^aPercentage contribution to the total predictive variance, derived from the uncertainties in m_c , ρ , and all systematic (Sys.) nuisance parameters (Bold text indicates the largest contribution).

^bThis channel requires high-precision numerical integration that proved unstable in the main analysis. This result was obtained via a dedicated recalculation using a robust error-handling framework that successfully managed isolated integration failures.

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: FITTING PROCEDURE AND ERROR ESTIMATION

The whole procedure is structured as a multistage analysis pipeline, designed to systematically constrain model parameters, evaluate model performance, and ensure the robustness of our predictions. This section details the methodology of each consecutive stage.

1. Stage 1: Bayesian inference for parameter estimation

The objective of this stage is to perform a comprehensive exploration of the parameter space and determine the full joint posterior probability distribution for all model parameters. Our model operates in a 16-dimensional parameter space, denoted by the vector θ . This space comprises two primary parameters of interest—the charm quark mass m_c and hadron size parameter ρ —and 14 nuisance parameters $\{\alpha_j\}$, corresponding to the measured masses and widths of the involved particles.

To navigate this high-dimensional space, we employ the DYNesty [55] package, a powerful Python implementation of the Nested Sampling algorithm. This method is particularly well suited for this problem as it efficiently maps the posterior landscape, even in the presence of complex correlations, and simultaneously calculates the Bayesian evidence, which is crucial for model comparison. The Bayesian framework requires two fundamental components: a likelihood function and a prior distribution.

a. Likelihood function

The likelihood function $\mathcal{L}(\theta|D)$ quantifies the probability of observing the experimental data D given a specific set of model parameters θ . In statistical analyses, it is often more convenient to work with the log-likelihood, which we define in terms of a total chi-squared function χ_{total}^2 ,

$$\ln \mathcal{L}(\theta|D) = -\frac{1}{2} \chi_{\text{total}}^2(\theta) + \text{const.} \quad (\text{A1})$$

The χ_{total}^2 is constructed from two distinct contributions: a goodness-of-fit term and a penalty term for the nuisance parameters,

$$\chi_{\text{total}}^2(\theta) = \chi_{\text{data}}^2(\theta) + \chi_{\text{penalty}}^2(\theta). \quad (\text{A2})$$

The first term, χ_{data}^2 , is the conventional goodness-of-fit metric. It compares the theoretical branching fractions predicted by our model, $\mathcal{B}_{\text{th},i}(\theta)$, with the experimentally measured values, $\mathcal{B}_{\text{exp},i}$, across all N decay channels used in the fit,

TABLE III. Experimental inputs for the fitted decay channels.

Decay channel	$\mathcal{B}_{\text{exp}}$	σ_{exp}
$J/\psi \rightarrow \eta_c \gamma$	0.0140	0.0014
$\chi_{c0} \rightarrow J/\psi \gamma$	0.0141	0.0009
$\psi(2S) \rightarrow \chi_{c0} \gamma$	0.0977	0.0023

$$\chi_{\text{data}}^2(\theta) = \sum_{i=1}^N \frac{(\mathcal{B}_{\text{th},i}(\theta) - \mathcal{B}_{\text{exp},i})^2}{\sigma_{\text{exp},i}^2}, \quad (\text{A3})$$

where $\sigma_{\text{exp},i}$ is the uncertainty of the experimental data for the i -th channel. The experimental values for $\mathcal{B}_{\text{exp},i}$ and $\sigma_{\text{exp},i}$ are taken from the PDG [15] and listed in Table III.

The second term, χ_{penalty}^2 , constrains the 14 nuisance parameters $\{\alpha_j\}$ to their established values from the PDG [15]. It takes the form of a sum of squared pulls, effectively incorporating external experimental information as Gaussian constraints within the likelihood itself,

$$\chi_{\text{penalty}}^2(\theta) = \sum_{j=1}^{14} \frac{(\alpha_j - \bar{\alpha}_j)^2}{\sigma_{\alpha_j}^2}, \quad (\text{A4})$$

where $\bar{\alpha}_j$ and σ_{α_j} are the central value and uncertainty for the j th nuisance parameter, respectively (see Table I).

b. Prior distributions

A key aspect of our Bayesian approach is the choice of prior probability distributions $p(\theta)$, which represent our state of knowledge about the parameters before considering the data. For all 16 parameters in the model, we assigned uninformative, uniform (or “flat”) priors over a broad but physically plausible range $[\theta_{k,\text{min}}, \theta_{k,\text{max}}]$,

$$p(\theta_k) = \begin{cases} \frac{1}{\theta_{k,\text{max}} - \theta_{k,\text{min}}} & \text{if } \theta_k \in [\theta_{k,\text{min}}, \theta_{k,\text{max}}], \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A5})$$

The motivation for this choice is to adopt a maximally agnostic stance, allowing the data to dominate the inference process. Therefore, the final shape of the posterior distribution is dictated almost entirely by the likelihood function. By enforcing the strong constraints on the nuisance parameters via the χ_{penalty}^2 term rather than through narrow priors, we improve the sampler’s efficiency in exploring the valid regions of high likelihood.

c. Outcome

The successful execution of the Nested Sampling run yields a set of weighted samples that provides a discrete representation of the full 16-dimensional posterior probability distribution, $P(\theta|D) \propto \mathcal{L}(\theta|D)p(\theta)$. These results, which encapsulate all information about parameter correlations,

uncertainties, and best-fit values, are serialized and saved to a file (DYNesty_RESULTS.PKL). This data object serves as the direct and complete input for Stage 2 of the analysis pipeline, where we perform postprocessing, error propagation, and goodness-of-fit checks.

2. Stage 2: Postprocessing and error propagation

The output of the Nested Sampling algorithm is a discrete, weighted representation of the joint posterior probability distribution. The purpose of Stage 2 is to translate these raw statistical results into meaningful physical quantities, including central values and asymmetric uncertainties for the model parameters, and a comprehensive uncertainty budget for the model's predictions.

a. Parameter uncertainty estimation

The properties of each parameter are extracted from its one-dimensional marginal posterior distribution, $P(\theta_k|D)$, which is obtained by integrating the full joint posterior over all other parameters. This nonparametric approach correctly captures any asymmetries or non-Gaussian features present in the distributions.

For each parameter θ_k , the central value, or best fit point, is taken to be the median ($\hat{\theta}_k$) of its marginal posterior. The median, corresponding to the 50th percentile, is chosen for its robustness against skewed distributions, providing a more stable estimator of location compared to the mean. It is formally defined by the value $\hat{\theta}_k$ that satisfies

$$\int_{-\infty}^{\hat{\theta}_k} P(\theta'_k|D) d\theta'_k = 0.5. \quad (\text{A6})$$

The uncertainty is reported as the 68.3% credible interval, corresponding to a 1σ range for a Gaussian distribution. This interval is defined by the 15.87th and 84.13th percentiles of the distribution, denoted $\theta_k^{(0.1587)}$ and $\theta_k^{(0.8413)}$, respectively. The asymmetric errors are then calculated as

$$\text{Upper Error : } +\sigma_k^{\text{up}} = \theta_k^{(0.8413)} - \hat{\theta}_k. \quad (\text{A7})$$

$$\text{Lower Error : } -\sigma_k^{\text{down}} = \hat{\theta}_k - \theta_k^{(0.1587)}. \quad (\text{A8})$$

The final result for each parameter is reported as $\hat{\theta}_k^{+\sigma_k^{\text{up}}}_{-\sigma_k^{\text{down}}}$. This method provides an honest representation of the parameter's uncertainty by faithfully reflecting the true shape of its posterior distribution.

b. Monte Carlo error propagation for predictions

To determine the theoretical uncertainty of a predicted quantity y (e.g., a branching fraction not used in the fit), which is a function of the model parameters, $y = f(\theta)$, we must propagate the uncertainties from the full 16-dimensional parameter space. A Monte Carlo approach is ideally suited

for this task as it naturally accounts for all parameter correlations. The procedure is executed as a formal algorithm:

- (1) Sampling from the posterior: We draw a large number of random samples, $\{\theta^{(i)}\}_{i=1}^N$ with $N = 20000$, from the joint posterior distribution $P(\theta|D)$ stored in DYNesty_RESULTS.PKL. Each sample $\theta^{(i)}$ is a 16-dimensional vector representing a complete, physically plausible parameter set.
- (2) Iterative calculation: For each sample vector $\theta^{(i)}$, we execute the physics model f to calculate the corresponding predicted value: $y^{(i)} = f(\theta^{(i)})$.
- (3) Generating the prediction distribution: This process yields a set of N predicted values, $\{y^{(i)}\}_{i=1}^N$. This set forms an empirical representation of the posterior predictive distribution for the quantity y .
- (4) Extracting the final result: The central value of the prediction $\hat{y}$ is calculated using the best fit (median) parameter vector $\hat{\theta}$,

$$\hat{y} = f(\hat{\theta}). \quad (\text{A9})$$

The total model uncertainty σ_y for this prediction is taken as the standard deviation of the distribution of predicted values,

$$\sigma_y = \text{std}(\{y^{(i)}\}_{i=1}^N). \quad (\text{A10})$$

This propagated error comprehensively includes all sources of uncertainty and the complex, nonlinear correlations between them.

c. Decomposition of the uncertainty budget

To understand the relative impact of different sources of uncertainty of our predictions, we decompose the total uncertainty into its primary constituents using a parameter-freezing technique. To isolate the uncertainty contribution from a single parameter, e.g., m_c , we modify the Monte Carlo procedure as follows. In each of the N sampled vectors $\theta^{(i)} = (m_c^{(i)}, \rho^{(i)}, \{\alpha_j^{(i)}\})$, only the m_c component is allowed to vary according to its posterior draws, while all other 15 parameters are held fixed at their best-fit (median) values ($\hat{\rho}, \{\hat{\alpha}_j\}$). The uncertainty contribution from m_c alone, denoted σ_{m_c} , is the standard deviation of the resulting distribution of predictions,

$$\sigma_{m_c} = \text{std}(\{f(m_c^{(i)}, \hat{\rho}, \{\hat{\alpha}_j\})\}_{i=1}^N). \quad (\text{A11})$$

This process is repeated for the parameter ρ (yielding σ_ρ) and for the block of 14 experimental nuisance parameters (yielding the systematic uncertainty σ_{sys}). Because the posterior distributions are not perfectly Gaussian and the model may be nonlinear, the variances do not sum exactly. However, they provide an excellent approximation of the total uncertainty budget, which can be checked for closure,

$$\sigma_y^2 \approx \sigma_{m_c}^2 + \sigma_\rho^2 + \sigma_{\text{syst}}^2. \quad (\text{A12})$$

This decomposition allows us to quantify the percentage contribution of each source, thereby identifying which parameters are the dominant drivers of uncertainty in our theoretical predictions.

3. Stage 3: Frequentist cross-check with profile likelihood

While the Bayesian analysis in Stage 1 was used for fitting the model parameters, we perform a validation using an independent, frequentist statistical approach. The objective is to corroborate the Bayesian credible intervals with frequentist confidence intervals calculated via the profile likelihood method. A strong agreement between the results from these two distinct paradigms significantly enhances the reliability of our conclusions. This stage relies on numerical optimization routines, for which we employ the SCIPY.OPTIMIZE library [56].

a. The profile likelihood method

The profile likelihood method provides a rigorous framework for constructing confidence intervals for a single parameter of interest (POI) while systematically accounting for the uncertainties associated with all other nuisance parameters. Let the full 16-dimensional parameter vector be denoted θ . To construct an interval for a POI, e.g., $\Psi = m_c$, we partition the vector as $\theta = (\Psi, \lambda)$, where λ represents the remaining 15 nuisance parameters (including ρ and the 14 experimental nuisances).

First, the global minimum of the χ_{total}^2 function is found by minimizing with respect to all parameters simultaneously. The best fit parameter vector $\hat{\theta}$ and the minimum χ^2 value are given by,

$$\hat{\theta} = \arg \min_{\theta} \chi_{\text{total}}^2(\theta), \quad (\text{A13})$$

$$\chi_{\text{global,min}}^2 = \chi_{\text{total}}^2(\hat{\theta}). \quad (\text{A14})$$

Next, the profile χ^2 function, $\chi_{\text{prof}}^2(\Psi)$, is constructed. For each fixed value of the POI Ψ on a predefined grid, the function $\chi_{\text{total}}^2(\Psi, \lambda)$ is minimized with respect to all nuisance parameters λ . This procedure, known as profiling, defines the conditional minimum,

$$\chi_{\text{prof}}^2(\Psi) = \min_{\lambda} \chi_{\text{total}}^2(\Psi, \lambda). \quad (\text{A15})$$

By tracing these conditional minima across the grid of Ψ values, we obtain the profile χ^2 curve. The quantity of interest is the delta chi squared, $\Delta\chi^2(\Psi)$, which measures the increase in χ^2 from its global minimum as the POI is varied,

$$\Delta\chi^2(\Psi) = \chi_{\text{prof}}^2(\Psi) - \chi_{\text{global,min}}^2. \quad (\text{A16})$$

This entire process is performed independently for each POI, such as m_c and ρ .

b. Interval estimation and validation

According to Wilks's theorem [57], for a single parameter of interest, the $\Delta\chi^2$ statistic asymptotically follows a χ^2 distribution with one degree of freedom ($k = 1$). This provides a direct mapping between values of $\Delta\chi^2$ and standard confidence levels (CL) in the frequentist framework. The $N\sigma$ confidence interval is defined by the range of POI values for which the $\Delta\chi^2$ curve remains below N^2 . Specifically:

- (i) The 1σ (68.3% CL) interval is found by solving $\Delta\chi^2(\Psi) = 1.0$.
- (ii) The 2σ (95.4% CL) interval is found by solving $\Delta\chi^2(\Psi) = 4.0$.

The final validation step consists of a direct comparison between these frequentist confidence intervals and the Bayesian 68.3% credible intervals derived in Stage 2. A close agreement between these independently derived uncertainty bounds confirms that our results are not an artifact of the chosen statistical methodology and that our uncertainty quantification is robust.

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