

Geometrical reconstruction of Dirac solutions: Gravitational interactions

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We develop a technique to construct solutions of the Dirac equation employing the underlying group symmetries and the classical-quantum relationship within the Clifford algebra formalism. In particular, we construct a novel solution of the Dirac equation in the form of an electron vortex beam in a gravitational plane wave and show its relationship to the solution of the classical analog of the Dirac equation—rotor equation—for a particle in a gravitational wave. Our approach put forward a new paradigm in treating dynamical problems based on the underlying group symmetries.

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I. INTRODUCTION

The dynamics of physical systems is governed by equations of motion, typically formulated as differential equations whose solutions are often difficult to obtain. There are two main approaches to solving those equations of motion. The direct and inverse methods. In the former a force acting on a particle is given, and the task is to find the particle's motion caused by the force field, while the latter consists in finding the force acting on a particle given its motion (i.e., its trajectories). Direct problems are solved by integrating the differential equations of motion. However, even after identifying the symmetries of a given problem, considerable freedom remains in the explicit form of the solutions. On the other hand, the inverse approach consists in differentiating the trajectory of the particle to find the force.

Recently, the relativistic dynamical inversion (RDI) method was introduced [1] to solve the inverse problem for the Dirac equation. It relies on the fact that relativistic interactions and the corresponding equations of motion are inextricably intertwined with geometry; more specifically group theory and the Poincaré group. This in turn fixes the form the spinor can take, thus imposing strict constraints and facilitating the solution finding. The RDI method is heavily based on the spinor representation via Clifford algebra [2,3], which employs the matrix spinor objects to describe not only the quantum dynamics but also the classical ones, and provides a transparent relationship between the two descriptions. In RDI, the spinor can be

parametrized as a combination of the Lorentz boost and rotation transformations which are applied to the electron rest frame state, transforming it to the laboratory frame at each coordinate and time. For a given dynamical problem the specific boost and rotation transformations can be inferred from the consideration of the classical problem. In fact, the classical trajectory points out the rest frame properties and in the Clifford algebra representation provides the matrix spinor which is identical to that of the quantum case.

In this work we put forward a new paradigm in treating dynamical problems: rather than solving the Dirac equation, we construct its solution by employing the Lorentz boost and rotation operators derived from the corresponding classical dynamics to the initial matrix spinor. This procedure of the relativistic dynamical construction (RDC), which capitalizes on the knowledge gained from the RDI parametrization, then consists in first finding the trajectories of the electron for the correspondingly classical problem. For this it is sufficient to solve the so-called classical rotor equation [2,3] for the classical matrix spinor. The next step depends on whether the desired solution is quasiclassical or fully quantum. In the former case, a WKB wavefunction built from the classical spinor and with the additional phase via the classical action is enough to fully construct the solution to the Dirac equation. However, in the latter case extra ingredients are necessary for the classical to quantum transition which incorporate the remaining degrees of freedom of the Dirac spinor via two functions, the scalar (density) and the pseudoscalar (spin degree of freedom), constrained to obey two continuity equations. We employ the developed RDC technique to find the exact quantum solutions for a relativistic electron vortex beam (EVB) interacting with a gravitational plane wave. Finally, we anticipate the extension of the RDC method beyond gravitational and electromagnetic interactions by bridging the internal symmetries of the standard model with the

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Lorentz symmetry of the Dirac equation. For instance, such a connection has already been demonstrated for the electro-weak interaction in [4]. Thus one may be able to apply the RDC technique to the weak interaction in the language of matrix spinors. As a final note, for the RDC technique, the Clifford algebra formalism is crucial since it deals with Lorentz transformations, requiring the use of matrices obeying the Pauli algebra which is an essential feature of Clifford's framework.

II. RDC TECHNIQUE

The RDC technique is based on the Dirac equation reformulation as the Hestenes-Dirac equation [5]

$$\hbar\gamma^\mu\partial_\mu\Psi\gamma_2\gamma_1 - e\mathcal{A}\Psi - mc\Psi\gamma_0 = 0, \quad (1)$$

where the Dirac 4-component vector spinor ψ is replaced by the 4×4 matrix spinor

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}. \quad (2)$$

Here the Greek indices are used for the spacetime manifold and Latin indices for the tangent space. The vector spinor can be easily recovered from the matrix spinor,

$$\psi = \Psi \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3)$$

The matrix spinor Ψ can be parametrized in the so-called polar form [6], which enables a geometrical interpretation of the spinor and which is the cornerstone of RDC,

$$\Psi = \sqrt{\rho} \exp(i\beta/2)\mathcal{R}. \quad (4)$$

In this parametrization, ρ is a non-negative scalar function that represents the probability density, β is a pseudoscalar, the so-called Yvon-Takabayashi angle [7,8], describing the spin degree of freedom, and $\mathcal{R}$ is a matrix that is an element of the Lorentz group. The Lorentz transformation $\mathcal{R}$ consisting of boosts and rotations connects the rest frame of the electron with the laboratory frame. The identification of the matrix spinor with local, spacetime dependent Lorentz transformations given by Eq. (4) is the key element of the RDC technique and its construction relies on knowing the trajectory of the electron. One builds $\mathcal{R}$ from its classical counterpart of the Lorentz transformation R . The latter is derived from the classical Rotor equation [2]

$$\frac{dR}{d\tau} = \frac{1}{2} \left[\frac{e}{m} F - \omega(v) \right] R, \quad (5)$$

where F is the Lorentz force, and $\omega(v) = v^\mu \omega_\mu = 2v^\mu \Omega_\mu$ where v is the electron's proper velocity. Using the covariant derivative $D_\mu = \partial_\mu + \Omega_\mu$, with Ω_μ being the spinorial connection, instead of only the partial derivative ∂_μ , we incorporate gravity into the Dirac as well as the Hestenes-Dirac equation, if we additionally use the tetrads e_a^μ to transform the gamma matrices according to $\gamma^\mu = \gamma^a e_a^\mu$ [9]. For the transformed gamma matrices $\gamma_\mu, \gamma_\nu = 2g_{\mu\nu}\mathbf{1}$ applies.

III. RDC GENERAL RECIPE

Here we provide the general recipe, with specific application to the gravitational wave case given in the Appendix E. This recipe applies equally when $F_{ab} \neq 0$ and/or $\Omega_{AB} \neq 0$. It serves as a guide in the construction of an ansatz for the matrix spinor.

IV. STEP-BY-STEP PROCEDURE

- (1) Step 1: Choose a tetrad. Pick $e_a^\mu(x)$ satisfying $g_{\mu\nu} = e_a^\mu e_b^\nu \eta_{ab}$ and set $\gamma^\mu(x) = e_a^\mu \gamma^a$.
- (2) Step 2: Find Killing fields. Solve

$$\nabla_{(\mu} K_{\nu)} = 0$$

to list all residual isometries $\{K_A\}$. Next separate the trivial ones (i.e., which follow trivially from the metric) from the others.

- (3) Step 3: Impose spinorial Lie-derivative constraints. Use the nontrivial Killing vector fields K_A to determine the structure of the rotor R by applying the finite transformation of the combined Lie derivatives $\mathcal{L}_{K_A}\psi = K_A^\mu(\nabla_\mu)\psi - \frac{1}{8}(\nabla_\mu K_{A,\nu} - \nabla_\nu K_{A,\mu})\gamma^\mu\gamma^\nu\psi$ to a constant reference spinor ψ_0

$$\begin{aligned} \psi_0 &\mapsto \exp\left(\sum_A \alpha_A p^A \mathcal{L}_{K_A}\right)\psi_0 \\ &= R\psi_0. \end{aligned}$$

The α_A are fixed using the Rotor equation. Impose the constraints $K_A^\mu \partial_\mu S = 0$ to fix the eikonal phase S of the solution.

- (4) Step 4: Write the classical rotor equation. This step should be considered as complimentary to the Lie derivative. The spin connection is $\Omega_\mu = \frac{1}{4}\Omega_{\mu ab}\gamma^a\gamma^b$. The classical rotor $R(\tau)$ obeys

$$\begin{aligned} \frac{dR}{d\tau} &= \frac{1}{2} \left[\frac{e}{m} F_{ab}(x) \gamma^a \gamma^b - \omega(v) \right] R, \\ \omega(v) &= 2v^\mu \Omega_\mu(x). \end{aligned}$$

- (5) Step 5: Change variable to a symmetry parameter ϕ . (E.g., for plane-wave backgrounds set $\phi = k \cdot x$), giving

$$\frac{dR}{d\phi} = -\frac{1}{2} \left[\frac{e}{m} F_{ab}(\phi) \gamma^a \gamma^b - \Omega_{ab}(\phi) \gamma^a \gamma^b \right] R,$$

where Ω_{ab} are the components of the bivector $\omega(v)$.

- (6) Step 6: Check that the bivector commutes at different parameter values. Define the total generator

$$B(\phi) = \frac{e}{m} F_{ab}(\phi) \gamma^a \gamma^b - \Omega_{ab}(\phi) \gamma^a \gamma^b.$$

Why change from τ to ϕ ?

- (a) If $[B(\phi), B(\phi')] = 0$, then

$$R(\phi) = \exp \left[-\frac{1}{2} \int_0^\phi B(\phi') d\phi' \right].$$

- (b) If no such parameter exists, one must use a path-ordered or Magnus expansion, which generally has no simple analytic form.

- (c) Most high-symmetry backgrounds (Coulomb, electromagnetic or gravitational plane waves, Kerr-Schild) do have such parameter.

- (7) Step 7: Construct the polar spinor.

$$\Psi(x) = \sqrt{\rho(x)} e^{\frac{i}{2}\beta(x)} \mathcal{R}(x) e^{-\gamma^2 \gamma^1 S(x)/\hbar}$$

- (8) Step 8: (Optional) Factor into physical boosts and rotations.

$$\mathcal{R} = \exp \left[-\frac{1}{2} \theta \hat{n} \cdot \gamma \gamma^3 \right] \times \exp \left[-\frac{1}{2} \omega \hat{v} \cdot \gamma^0 \gamma \right],$$

isolating spin-precession $[SO(3)]$ from kinematic boost $(\mathbb{R}^3)$.

Examples: Hydrogen ground state ($SO(3) \times U(1)$), Volkov EM solution (plane wave), gravitational plane wave (null Heisenberg algebra).

V. ILLUSTRATION FOR THE GROUND STATE OF A HYDROGENLIKE ION

We illustrate the RDC method by constructing the exact solution of the Dirac equation for the ground state of a hydrogenlike ion, a detailed explanation of this can be found in Appendix D. We begin with the classical rotor equation for a charged particle in the Coulomb field $F = Ze^2/r^2$ and $\omega = 0$, with the nuclear charge Z , and the electron charge e . The solution to Rotor equation (5) in this case reads [10,11]

$$R = e^{\frac{1}{2}w\hat{\phi}} e^{\frac{i\phi}{2\gamma^1\gamma^2}} e^{-\frac{1}{2}w\hat{\phi}} e^{\gamma^2\gamma^1\phi/2}, \quad (6)$$

where $w = \text{arctanh}(\frac{Z\alpha}{l})$ is the rapidity, $\hat{\phi} = e^{-\gamma^1\gamma^2\phi/2} \gamma_2 e^{\gamma^1\gamma^2\phi/2}$, $\gamma^{-1} = \sqrt{l^2 - Z^2\alpha^2}/l$, l is the angular momentum, and α is the fine structure constant. Note the

appearance of a boost with rapidity w and its inverse with a rotation around the z direction by the irrational angle ϕ/γ in between. This comes from the Thomas precession. In the quantum case, the Thomas precession leads to spin-orbit coupling which is then described by the Takabayashi angle β .

In the quantum case one associates l with the total quantum angular momentum and takes it to be $l = 1$. Using the classical information for R via Eq. (6), we retrieve the quantum spinor in the following form:

$$\Psi = \sqrt{\rho} e^{\frac{i}{2}\beta} \mathcal{R} e^{-\gamma^2 \gamma^1 \epsilon t / \hbar}, \quad (7)$$

$$\mathcal{R} = e^{\hat{\phi} \frac{\phi}{2}} U, \quad (8)$$

$$U = e^{-\gamma^2 \gamma^1 \phi/2} e^{-\gamma^3 \gamma^1 \Theta} e^{\gamma^2 \gamma^1 \phi/2}, \quad (9)$$

$$\Theta = \cos^{-1} \left(\frac{\tan \beta \sin \beta}{1 + \gamma} + \cos \beta \right) \quad (10)$$

where $\hat{\phi} = e^{-\gamma^2 \gamma^1 \phi/2} \gamma_2 e^{\gamma^2 \gamma^1 \phi/2}$, $\beta = \tan^{-1}(\gamma Z \alpha \cos \theta)$, $w = \tanh^{-1}(Z \alpha \sin \theta)$ and $\gamma^{-1} = \sqrt{1 - Z^2 \alpha^2}$.

The origin of this expression can be understood from Fig. 1. It is known that the effect of the pseudoscalar is to perform a spacetime reflection (duality rotation). Hence,

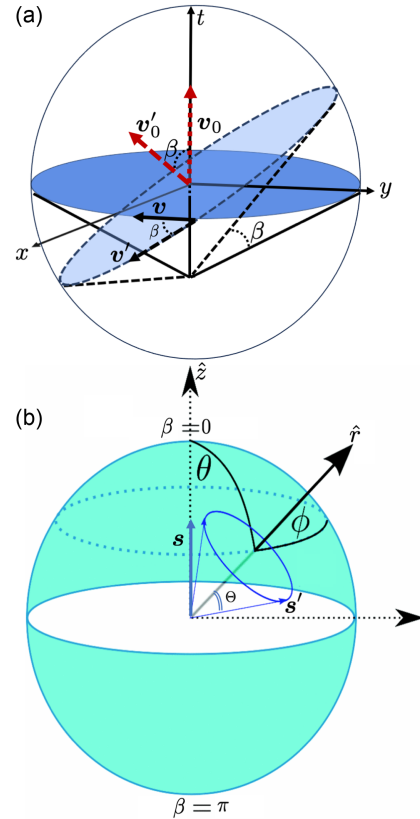


FIG. 1. Illustration of the effect of β for the electron's (a) light cone, and (b) spin vector.

the velocity and spin of the electron should change sign. Figure 1(a) is a lower dimension representation (we omit the z axis) of the electron's light cone. The dark blue disc (which shall be understood as a sphere) is the intersection of the hyperplane $t = 0$ with the electron's light cone for $\beta = 0$. That is, it represents the classical velocity space. The effect of the pseudoscalar is to tilt the electron's light cone by the angle β (light-blue disc). The dashed circle indicates that the electron's velocity in $3D$ space no longer lies on the equator of the sphere (i.e., $z = 0$) but rather along the ϕ axis on the upper hemisphere. This tells us how to get the expression for the quantum velocity vector, as discussed in the Appendix D.

The spin induced rotation is illustrated in Fig. 1(b). As previously mentioned, a spacetime reflection changes the direction of the spin. For $\beta = 0$ we have the spin pointing up (north pole) and completely flipping and pointing down (south-pole) for $\beta = \pi$. For the case of the hydrogen atom, the spin vector precesses on the upper hemisphere since the velocity is also moving along the lines of latitude above the equator. Since originally (i.e., for $\beta = 0$) the spatial components of the velocity and spin vectors are perpendicular, they must remain so after the duality rotation. Hence the spin vector gets an extra component along the r axis, with a nonzero projection on the z axis which should depend on $\cos \beta$. From such considerations, the spin vector can also be fully derived from our geometrical considerations, as shown in the Appendix D. The missing component is ρ , which comes from the continuity equation for the current density.

VI. ELECTRON IN A GRAVITATIONAL PLANE WAVE: CLASSICAL TREATMENT

The gravitational plane wave [12] is described by the exact solution of Einstein's field equations with the following metric [13]:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -f^2(\phi) & 0 & 0 \\ 0 & 0 & -g^2(\phi) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (11)$$

where $k^\mu = (k, 0, 0, k)^T$, $\phi = k \cdot x = k(ct - z)$ and $f''(\phi)/f(\phi) + g''(\phi)/g(\phi) = 0$. This results in the tetrads

$$e_a^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{f(\phi)} & 0 & 0 \\ 0 & 0 & \frac{1}{g(\phi)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (12)$$

and the spinorial connections $\Omega_t = 0, \Omega_x = kf'(\phi)(\sigma^{01} + \sigma^{13}), \Omega_y = kg'(\phi)(\sigma^{02} + \sigma^{23}), \Omega_z = 0$, where $\sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$ and $\alpha^k = \gamma^k \gamma^0$.

We will now solve the classical Rotor equation for an electron interacting with a gravitational plane wave, when $F = 0$ in Eq. (5) and

$$\omega(v) = k \left(v_0^x \frac{f'(\phi)}{f^2(\phi)} (\gamma^0 - \gamma^3) \gamma^1 + v_0^y \frac{g'(\phi)}{g^2(\phi)} (\gamma^0 - \gamma^3) \gamma^2 \right). \quad (13)$$

We have $v_0^x = v^x(\phi_0)f^2(\phi_0)$ and $v_0^y = v^y(\phi_0)g^2(\phi_0)$, with ϕ_0 being the initial value of the angle ϕ . $\omega(v)$ can be derived by solving the geodesic equations in the background of the metric (11) and substituting the resulting proper velocity v^μ into the definition $\omega(v) = 2v^\mu \Omega_\mu$. For a detailed derivation, see Ref. [14]. In order to solve the Rotor equation using Eq. (13), it remains to establish a relation between ϕ and τ . We first note that $\omega(v)$ is orthogonal to $\not{k}$, i.e., $\not{k}\omega(v) = 0 = \omega(v)\not{k}$. Using the rotor equation, where dotted quantities denote derivatives with respect to the proper time τ , this yields $\not{k}\dot{R} = 0 = \dot{R}\not{k}$, where $\tilde{R} = \gamma^0 R^\dagger \gamma^0$ is the reverse of R . This implies that the wave vector of the gravitational wave in the rest frame of the electron, $\not{k}_{\text{rest}} = R\not{k}\tilde{R}$, is constant, $\frac{d\not{k}_{\text{rest}}}{d\tau} = \dot{R}\not{k}\tilde{R} + R\not{k}\dot{\tilde{R}} = 0 \Rightarrow \frac{d\not{k}_{\text{rest}}^\mu}{d\tau} = 0$. Therefore, the rest frame frequency $\omega_{\text{rest}} = c k_{\text{rest}}^0$, which is the frequency of the gravitational wave in the rest frame of the electron, is also constant. Since the inner product is invariant under Lorentz transformations, here mediated by the rotor R , one obtains the relation $\frac{d\phi}{d\tau} = \frac{d(k \cdot x)}{d\tau} = \frac{d(k_{\text{rest}} \cdot x_{\text{rest}})}{d\tau} = \frac{d(k_{\text{rest}}^0 c \tau)}{d\tau} = \omega_{\text{rest}}$. This establishes not only a relation between ϕ and τ , but also the additional relation $\frac{v^\mu(\phi_0)}{\omega_{\text{rest}}} = \frac{m v^\mu(\phi_0)}{m \omega_{\text{rest}}} = \frac{p^\mu}{k \cdot p}$. For a detailed derivation, we refer again to Ref. [14]. We thus obtain the following explicit form of the Rotor equation (5):

$$\frac{dR}{d\phi} = -\frac{\not{k}}{2k \cdot p} \left(p^x \frac{f^2(\phi_0)f'(\phi)}{f^2(\phi)} \gamma^1 + p^y \frac{g^2(\phi_0)g'(\phi)}{g^2(\phi)} \gamma^2 \right) R. \quad (14)$$

This can be solved directly using an exponential ansatz, which only the first two terms of the exponential series remain since $\not{k}^2 = k^2 = 0$:

$$R = \left\{ 1 + \frac{\not{k}}{2k \cdot p} \left(p^x \left[\frac{f^2(\phi_0)}{f(\phi)} - f(\phi_0) \right] \gamma^1 + p^y \left[\frac{g^2(\phi_0)}{g(\phi)} - g(\phi_0) \right] \gamma^2 \right) \right\} R(\phi_0), \quad (15)$$

where $R(\phi_0)$ is a constant Rotor corresponding to the free solution to the Rotor equation. By choosing the initial conditions $f(\phi_0) = g(\phi_0) = 1$ and $R(\phi_0) = 1$, which means that there is no plane gravitational wave and the

electron is free, the Rotor takes the simple form

$$R = \left(1 - \frac{H_i^\mu k \gamma^i p_\mu}{2k \cdot p}\right). \quad (16)$$

Let us have a closer look at R . By using the definition of $H_i^\mu = e_i^\mu - \mathbf{1}$, we can split R into a rotation U_v and a boost $\mathcal{B}_v$,

$$\begin{aligned} R &= U_v \mathcal{B}_v, \\ U_v &= \exp\left(-\frac{\theta}{2}(\cos(\vartheta)\gamma^1\gamma^3 + \sin(\vartheta)\gamma^2\gamma^3)\right), \\ \mathcal{B}_v &= \exp\left(\frac{\omega}{2}(V_1\alpha^1 + V_2\alpha^2 + V_3\alpha^3)\right), \end{aligned} \quad (17)$$

where

$$\begin{aligned} \frac{\theta}{2} &= \arctan\left(k \frac{1}{2k_j p^j} \sqrt{(A^1)^2 + (A^2)^2}\right), \\ \cos(\vartheta) &= \frac{A^1}{\sqrt{(A^1)^2 + (A^2)^2}}, \quad \sin(\vartheta) = \frac{A^2}{\sqrt{(A^1)^2 + (A^2)^2}}, \\ \frac{V_1}{2} &= \cos\left(\frac{\theta}{2}\right) \cos(\vartheta), \quad \frac{V_2}{2} = \cos\left(\frac{\theta}{2}\right) \sin(\vartheta), \\ \frac{V_3}{2} &= \sin\left(\frac{\theta}{2}\right), \quad \omega = \operatorname{arctanh}\left(\frac{V_3}{2}\right), \\ A^1 &= -\left(\frac{1}{f(\phi)} - 1\right)p_x, \quad A^2 = -\left(\frac{1}{g(\phi)} - 1\right)p_y. \end{aligned} \quad (18)$$

The above factorization of the matrix spinor encodes full information about the electron's trajectory. A detailed explanation on how to arrive at the appropriate matrix spinor ansatz is given in Appendix E.

VII. ELECTRON VORTEX BEAM IN GRAVITATIONAL PLANE WAVE

The Dirac equation for an electron interacting with a gravitational plane wave is,

$$\hbar(\gamma^\mu \partial_\mu + \chi k) \Psi \gamma_2 \gamma_1 - mc \Psi \gamma_0 = 0, \quad (19)$$

with $\chi = \frac{1}{2}(\frac{f'(\phi)}{f(\phi)} + \frac{g'(\phi)}{g(\phi)})$. The gamma matrices of the spacetime manifold are $\gamma^t = \gamma^0, \gamma^x = \frac{1}{f(\phi)}\gamma^1, \gamma^y = \frac{1}{g(\phi)}\gamma^2, \gamma^z = \gamma^3$.

We can now construct the EVB for interaction with a gravitational plane wave. To do this we make use of the Lorentz rotor (16). A plane wave solution to Eq. (19) was first found in Ref. [15] and can be written as [16]

$$\Psi_p = \frac{\left(1 - \frac{H_i^\mu k \gamma^i p_\mu}{2k \cdot p}\right)}{\sqrt{f(\phi)g(\phi)}} \mathcal{B}_g \exp\left(-\gamma^2 \gamma^1 \frac{x^m p_m - S}{\hbar}\right), \quad (20)$$

with the classical action

$$\begin{aligned} S &= \frac{1}{2p \cdot k} \int_{\phi_0}^{\phi} d\phi' (2H^{i\mu} p_i p_\mu + H_l^\mu H^{l\nu} p_\mu p_\nu) \\ &= \frac{1}{2p \cdot k} \int_{\phi_0}^{\phi} d\phi' [(1 - f^{-2}(\phi')) p_1^2 + (1 - g^{-2}(\phi')) p_2^2] \end{aligned}$$

and the additional tensor

$$H_a^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{f(\phi)} - 1 & 0 & 0 \\ 0 & 0 & \frac{1}{g(\phi)} - 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (21)$$

was introduced, while

$$\mathcal{B}_g = \sqrt{\frac{E + mc^2}{2mc^2}} \begin{pmatrix} 1 & 0 & \frac{p^z c}{E + mc^2} & \frac{p^- c}{E + mc^2} \\ 0 & 1 & \frac{p^+ c}{E + mc^2} & \frac{-p^z c}{E + mc^2} \\ \frac{p^z c}{E + mc^2} & \frac{p^- c}{E + mc^2} & 1 & 0 \\ \frac{p^+ c}{E + mc^2} & \frac{-p^z c}{E + mc^2} & 0 & 1 \end{pmatrix}, \quad (22)$$

with $p^\pm = p^x \pm ip^y$. Now that we have recognized $\mathcal{R}_v = (1 - \frac{H_i^\mu k \gamma^i p_\mu}{2k \cdot p})$ as a Lorentz transformation, we can make the following RDC ansatz for the EVB solution,

$$\Psi_l = \frac{\mathcal{R}_v}{\sqrt{f(\phi)g(\phi)}} U_l \exp\left(-\gamma^2 \gamma^1 \frac{Et}{\hbar}\right). \quad (23)$$

Here we replace the momentum p_μ in $\mathcal{R}_v$ and S with the momentum operator $\hat{p}_\mu = i\hbar\partial_\mu$ in order to have an ansatz that is as general as possible. In addition U_l , which now replaces $\mathcal{B}_g$ in the matrix spinor (20), is a matrix spinor yet to be determined, but following RDC's considerations, it should be a slightly modified version of the free EVB matrix spinor given in Ref. [6] as shown in Appendix B. If we then use cylindrical coordinates with the transverse momentum $p_\perp$ which satisfies $E^2 = p_\perp^2 c^2 + m^2 c^4$ and being constant while assuming $p_z = 0$ for simplicity one can prove the following for the projection onto the first column of Eq. (23) as defined by Eq. (3)

$$\psi_l = \sqrt{2} \frac{e^{-i(\frac{Et}{\hbar} - \frac{a+b}{2})}}{\sqrt{f(\phi)g(\phi)}} e^{-i\frac{(a-b)\hbar^2}{2p_\perp^2}(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})} \mathcal{R}_v u_l \quad (24)$$

where

$$u_l = \begin{pmatrix} e^{il\varphi} J_l(\lambda) \\ 0 \\ 0 \\ \frac{ip_{\perp}c}{E+mc^2} e^{i(1+l)\varphi} J_{1+l}(\lambda) \end{pmatrix}. \quad (25)$$

Here l is the orbital angular momentum (OAM) quantum number and

$$a = \frac{p_{\perp}^2 c}{\hbar} \int \frac{d\phi}{2kE} \left(1 - \frac{1}{f^2(\phi)} \right), \quad (26)$$

$$b = \frac{p_{\perp}^2 c}{\hbar} \int \frac{d\phi}{2kE} \left(1 - \frac{1}{g^2(\phi)} \right), \quad (27)$$

with $r = \sqrt{x^2 + y^2}$, $\varphi = \arctan(\frac{y}{x})$, $\alpha = \frac{r}{2\hbar c}$, $\lambda = \frac{p_{\perp}}{\hbar} r$. In comparison with Eq. (23), we hence find that $\exp(-i\frac{(a-b)\hbar^2}{2p_{\perp}^2}(\frac{\partial^2}{\partial x^2}))u_l$ is the projection as defined by Eq. (3) onto the first column of U_l . Thus, using the RDC method we have constructed a novel solution to the Dirac equation for an EVB interacting with a gravitational plane wave [14].

Now we can apply the geometrical interpretation of the RDC technique to the solution of Eq. (23). Doing so reveals that the EBV solution primarily consists of the Lorentz transformation $\mathcal{R}_v$, which acts on the free EVB consisting of $\sqrt{\rho}U_l \exp(-\gamma^2\gamma^1 Et/\hbar)$ along with a rotation $\exp(\gamma^2\gamma^1 S/\hbar)$ around the spin axis. Such rotation is just a gauge transformation. For a closer look on the free EVB see for example Ref. [6]. If we compare the EVB solution with the plane wave solution (20), we can see that the overall structure of the two is very similar. We can state that our results using RDC show that solutions of the Dirac equation for the interaction with a plane gravitational wave have the form,

$$\psi = \frac{U_v \mathcal{B}_v}{\sqrt{f(\phi)g(\phi)}} U \exp\left(-\gamma^2\gamma^1 \left(\frac{Et - S}{\hbar}\right)\right), \quad (28)$$

where U is a form of a (potentially slightly modified) free spinor for the given problem, i.e. here a plane wave or EVB spinor. The free EVB solution is recovered from Eq. (23), by switching off the gravitational wave $f(\phi) = g(\phi) = 1$.

Finally, let us compare the EVB solution for the interaction with a gravitational plane wave (23) with the EVB solution for the interaction with an electromagnetic plane wave [17] (see Appendix A). These two solutions are very similar in their structure. However, the EVB solutions are not identical under the identification $-eA_i \triangleq p_{\mu}H_i^{\mu}$, in contrast to the plane wave solutions, cf. Eq. (20) with the Volkov state [18] in the electromagnetic case. The reason for that is that the “generalized vector potential” $-H_i^{\mu}$ couples to the initial kinetic momentum p_{μ} and not the electric

charge e , which leads to the momentum p_{μ} entering the phase S (20) quadratically instead of linearly. As a result in the case of EVB in the gravitational wave, we no longer have the gauge rotation S leading to a simple shift of the coordinates x and y through the Fourier shift theorem. This difference in the coupling thus provides a physical explanation for the main difference between the EVB solutions in the gravitational and electromagnetic cases.

VIII. CONCLUSION

We have developed the RDC paradigm for describing the relativistic quantum dynamics via the construction of the Dirac equation solution using group symmetries of the problem. In this way we have obtained a novel solution to the Dirac equation in form of an EVB interacting with a gravitational plane wave from the novel solution of the corresponding classical problem. The power of RDC is based on the fact that it employs similarities of the matrix spinor in the Clifford algebra representation for the classical and quantum problems. This has been shown by constructing the solution to the classical Rotor equation for interaction with a gravitational plane wave. The Lorentz transformations due to the interaction with the gravitational wave are almost identical in both cases.

Additional credibility to RDC could be given by further investigating the applicability of RDI to weak and strong interactions. For example, the lepton matrix spinor during weak interaction can be constructed from the spinor in the electromagnetic field, applying rotation transformation in the abstract space which transforms the electromagnetic interaction into the weak interaction, see Appendix C.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: EVB IN ELECTROMAGNETIC PLANE WAVE

For comparison with our result of an EVB in a gravitational plane wave, we will present here the EVB solution to the Dirac equation for interaction with an electromagnetic plane wave. This solution was first found in Ref. [17] and generalized to the form given below in Ref. [6]

$$\psi_l = \left(1 + \frac{e}{2k \cdot p} \not{A}\right) \exp\left(-i\frac{Et + G}{\hbar}\right) e^{il\varphi'} u'_l, \quad (A1)$$

with

$$x' = x + \frac{\hbar F_1}{p_{\perp}}, \quad y' = y + \frac{\hbar F_2}{p_{\perp}}, \quad r' = \sqrt{x'^2 + y'^2}, \quad (A2)$$

$$\varphi' = \arctan\left(\frac{y'}{x'}\right), \quad \lambda' = \frac{p_\perp}{\hbar} r', \quad (\text{A3})$$

$$u'_l = \begin{pmatrix} J_l(\lambda') \\ 0 \\ 0 \\ i \frac{p_\perp c}{E + mc^2} e^{i\varphi'} J_{l+1}(\lambda') \end{pmatrix}, \quad (\text{A4})$$

$$F_1 = \int_0^{k \cdot x} \frac{e p_\perp A^1(\phi)}{k \cdot p} d\phi, \quad F_2 = \int_0^{k \cdot x} \frac{e p_\perp A^2(\phi)}{k \cdot p} d\phi, \quad (\text{A5})$$

$$G = \int_0^{k \cdot x} \frac{e^2 A^2}{2k \cdot p} d\phi, \quad \lambda = \frac{p_\perp}{\hbar} r. \quad (\text{A6})$$

This representation now shows that the center of the vortex beam has been shifted to the coordinates $(\frac{\hbar F_1}{p_\perp}, \frac{\hbar F_2}{p_\perp}, 0)$ in the x-y plane by the interaction with the electromagnetic plane wave.

APPENDIX B: VECTOR SPINOR FORM OF EVB INTERACTING WITH GRAVITATIONAL PLANE WAVE

The question remains if there is a closed form for our EVB solution (23) without the summation found in Eq. (B1). Looking into this problem one will quickly notice that while there is such a closed form for the electromagnetic case, which can be obtained using Graf's addition theorem (see Ref. [6]), a comparable closed form for our solution is harder to obtain, since here the free EVB part is of order $2n + l$ instead of $n + l$ as a result of the coupling of the “generalized vector potential” $-H_i^\mu$ to the momentum p_μ . Also this closed form would allow us to verify if the center of the EVB gets shifted due to the interaction with the gravitational wave, as it is the case for the EVB interacting with the electromagnetic plane wave [17].

For a better understanding of the solution (23) and comparison with the case of interaction with an electromagnetic wave, it is presented again below as a vector spinor (Dirac spinor). The vector spinor form can be derived from Eq. (23) using (3), for example. We find,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(-\frac{i}{4})^n (a-b)^n}{n!} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e^{2i\varphi(k-n)} \frac{e^{-i\frac{a-b}{2}}}{2\pi i} \int_{-\infty}^{(0+)} \frac{dt}{t^{1+l+2k-2n}} e^{\frac{i}{2}(t-\frac{1}{t})} \\ &= \sum_{n=0}^{\infty} \frac{(-\frac{i}{4})^n (a-b)^n}{n!} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} e^{2i\varphi(k-n)} e^{-i\frac{a-b}{2}} J_{2k-2n+l}(\lambda) = \sum_{n=0}^{\infty} \left(\frac{\hbar}{p_\perp}\right)^{2n} \frac{(-i)^n (a-b)^n}{n!} e^{-i\frac{a-b}{2}} \frac{\partial^{2n}}{\partial x^{2n}} (e^{il\varphi} J_l(\lambda)) \\ &= e^{-i(a-b)[\frac{1}{2} + (\frac{\hbar}{p_\perp} \frac{\partial}{\partial x})^2]} (e^{il\varphi} J_l(\lambda)). \end{aligned} \quad (\text{B5})$$

$$\psi_l = \frac{e^{-i(\frac{Et}{\hbar} - \frac{a+b}{2})}}{\sqrt{f(\phi)g(\phi)}} \sum_{n=-\infty}^{\infty} (-i)^n J_n\left(\frac{a-b}{2}\right) \mathcal{R}_v u_{n+l}, \quad (\text{B1})$$

with

$$\mathcal{R}_v = \left[1 - \frac{c H_i^\mu k \gamma^i \hat{p}_\mu}{2Ek}\right], \quad (\text{B2})$$

$$\begin{aligned} u_{n+l} &= \begin{pmatrix} e^{i(2n+l)\varphi} J_{2n+l}(\lambda) \\ 0 \\ 0 \\ i \frac{p_\perp c}{E + mc^2} e^{i(2n+1+l)\varphi} J_{2n+1+l}(\lambda) \end{pmatrix}, \\ a &= \frac{p_\perp^2 c}{\hbar} \int \frac{d\phi}{2kE} \left(1 - \frac{1}{f^2(\phi)}\right), \\ b &= \frac{p_\perp^2 c}{\hbar} \int \frac{d\phi}{2kE} \left(1 - \frac{1}{g^2(\phi)}\right), \\ r &= \sqrt{x^2 + y^2}, \quad \varphi = \arctan\left(\frac{y}{x}\right), \quad \lambda = \frac{p_\perp}{\hbar} r. \end{aligned} \quad (\text{B3})$$

First, we start with

$$J_{2n+k}(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{dt}{t^{2n+1+k}} e^{\frac{i}{2}(t-\frac{1}{t})}.$$

Putting it in the first line of the spinor above, gives

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{dt e^{\frac{i}{2}(t-\frac{1}{t})}}{t^{1+k}} \sum_{n=-\infty}^{\infty} \left(\frac{e^{2i\varphi}}{it^2}\right)^n J_n\left(\frac{a-b}{2}\right) \\ &= \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{dt}{t^{1+k}} e^{\frac{i}{2}(t-\frac{1}{t})} e^{-i\frac{a-b}{4}(\frac{e^{2i\varphi}}{t^2} + \frac{t^2}{e^{2i\varphi}})} \\ &= \frac{e^{-i\frac{a-b}{2}}}{2\pi i} \int_{-\infty}^{(0+)} \frac{dt}{t^{1+k}} e^{\frac{i}{2}(t-\frac{1}{t}) - i\frac{a-b}{4}(\frac{t}{e^{i\varphi}} - \frac{e^{i\varphi}}{t})^2}. \end{aligned} \quad (\text{B4})$$

Then note the following identity

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \frac{(-1)^k \left(-\frac{i}{4}\right)^n (a-b)^n \binom{2n}{k} e^{2i\varphi(k-n)} t^{2n-2k}}{n!} \\ &= e^{-i\frac{a-b}{4}(\frac{t}{e^{i\varphi}} - \frac{e^{i\varphi}}{t})^2}. \end{aligned}$$

Hence

Thus, we have

$$\begin{aligned}\psi_l &= \frac{e^{-i(\frac{Et}{\hbar} - \frac{a+b}{2})}}{\sqrt{f(\phi)g(\phi)}} e^{-i(a-b)[\frac{1}{2} + (\frac{\hbar}{p_\perp} \frac{\partial}{\partial y})^2]} \mathcal{R}_v u_l \\ &= \frac{e^{-i(\frac{Et}{\hbar} - \frac{a+b}{2})}}{\sqrt{f(\phi)g(\phi)}} e^{i(a-b)[\frac{1}{2} + (\frac{\hbar}{p_\perp} \frac{\partial}{\partial y})^2]} \mathcal{R}_v u_l\end{aligned}\quad (\text{B6})$$

where

$$u_l = \begin{pmatrix} e^{il\varphi} J_l(\lambda) \\ 0 \\ 0 \\ \frac{ip_\perp c}{E+mc^2} e^{i(1+l)\varphi} J_{1+l}(\lambda) \end{pmatrix}. \quad (\text{B7})$$

Let us construct a more general solution by superposing both cases shown above. Consider the sum with equal coefficients. One then has

$$\begin{aligned}& \frac{1}{\sqrt{2}} e^{-i(a-b)[\frac{1}{2} + (\frac{\hbar}{p_\perp} \frac{\partial}{\partial y})^2]} + \frac{1}{\sqrt{2}} e^{i(a-b)[\frac{1}{2} + (\frac{\hbar}{p_\perp} \frac{\partial}{\partial y})^2]} \\ &= \sqrt{2} \cos \left[\frac{a-b}{2} \left(1 + \frac{\hbar^2}{p_\perp^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right) \right] \\ &\quad \times e^{-i \frac{(a-b)\hbar^2}{2p_\perp^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)}.\end{aligned}$$

Applying it to the wavefunction gives

$$\psi_l = \sqrt{2} \frac{e^{-i(\frac{Et}{\hbar} - \frac{a+b}{2})}}{\sqrt{f(\phi)g(\phi)}} e^{-i \frac{(a-b)\hbar^2}{2p_\perp^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)} \mathcal{R}_v u_l. \quad (\text{B8})$$

APPENDIX C: ELECTROWEAK INTERACTIONS IN THE MAJORANA MODEL AND RDI

In this section, a brief overview of the results of Ref. [4] regarding the incorporation of electroweak interactions into the Hestenes-Dirac equation is given. Note that here ψ_e describes a matrix spinor, the same holds for ψ_ν , ψ_l and ψ_r . Assuming ψ_e is an electron matrix spinor solving the Hestenes-Dirac equation (1), while ψ_ν is a neutrino matrix spinor, we can combine them together into a lepton wave function or lepton matrix spinor,

$$\Psi = \Psi_e + \Psi_\nu = \psi_e(1 - \alpha_2) + \psi_\nu(1 - \alpha_2)\alpha_3, \quad (\text{C1})$$

with

$$\begin{aligned}\Psi_e &= \frac{1}{2}(\Psi - \Psi\alpha_2), \\ \Psi_\nu &= \frac{1}{2}(\Psi + \Psi\alpha_2).\end{aligned}\quad (\text{C2})$$

Here we use $\alpha_k = \gamma_k \gamma_0$. We then assume an electroweak gauge transformation of the form,

$$\Psi \longrightarrow \Psi' = \Psi U. \quad (\text{C3})$$

In order to be covariant on the right side of the Hestenes-Dirac equation U has the form

$$U = e^{\frac{i}{2}\theta} e^{\frac{i}{2}\chi}, \quad (\text{C4})$$

where $\mathbf{i} = \gamma_0 \gamma_1 \gamma_2 \gamma_3$, $\theta = \theta_1 \alpha_1 + \theta_2 \alpha_2 + \theta_3 \alpha_3$. χ and the θ_i are functions of space and time. This is the gauge group $\text{SU}(2) \otimes \text{U}(1)$ of electroweak theory. That is, a combination of a rotation and a reflection, but now acting on the abstract isospin state. This gauge symmetry can be accounted for by the following generalized covariant derivative,

$$D_\mu \Psi = \partial_\mu \Psi - \Psi i W_\mu. \quad (\text{C5})$$

Here W_μ is an additional gauge field, which is connected to the Photon (A_μ) and the massive Z_μ and $W_\mu^\pm$ bosons as follows:

$$\begin{aligned}W_\mu &= \frac{g}{\sqrt{2}} [W_\mu^+ (1 + \alpha_2) \alpha_3 + W_\mu^- (1 - \alpha_2) \alpha_3] \\ &\quad - e A_\mu \frac{1}{2} (1 - \alpha_2) \\ &\quad - \frac{g}{\cos(\theta_W)} Z_\mu (\alpha_2 \cos^2(\theta_W) + \sin^2(\theta_W)).\end{aligned}\quad (\text{C6})$$

g denotes the weak isospin, while θ_W is the weak mixing angle, the so-called Weinberg angle, satisfying

$$\cos(\theta_W) = \frac{m_W}{m_Z}. \quad (\text{C7})$$

If we incorporate the lepton masses into a mass operator

$$\underline{m} = m_e \frac{1}{2} (1 - \alpha_2) + m_\nu \frac{1}{2} (1 + \alpha_2), \quad (\text{C8})$$

the lepton spinor Ψ satisfies the Dirac equation of the Majorana model

$$\gamma^\mu D_\mu \Psi i \gamma_3 = \gamma^\mu (\partial_\mu \Psi - \Psi i W_\mu) i \gamma_3 = \Psi \underline{m}. \quad (\text{C9})$$

If we now compare this equation with the Hestenes-Dirac equation (1) and note that the lepton spinor Ψ is also a matrix spinor, the similarities between the two equations indicate that RDI should also be applicable to this Dirac equation for the electroweak interaction. In particular, it should also be possible here to invert the equation for a given spinor Ψ and solve for the gauge field W_μ .

APPENDIX D: BUILDING HYDROGEN SOLUTION

We begin with an overview of group theory and the Poincaré group. A note on notation: Bold letters such as $\mathbf{x}^0 = x_0$ and $\mathbf{x}^k = -x_k$ correspond to the components of a 4-vector. Moreover, bold letters such as $\mathbf{x} = (x^1, x^2, x^3)^T$ represent three-dimensional vectors while letters such as (x, y) represent a 4-vector in Minkowsky spacetime.

1. Lorentz transformations and the Dirac equation

An event is something that happens at a definite time at a definite place. Events in spacetime are described with respect to an inertial frame by coordinates $x = (x^0, \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)^T \in \mathbb{R}^4$, where $x^0 = ct$. The principle of relativity states that all inertial frames are equivalent for the description of nature. The coordinate transformations $I \rightarrow I'$ between all possible inertial frames are called Poincaré transformations.

In the vector space $\mathbb{R}^4$ we define the Lorentz metric

$$\langle y, x \rangle = y^0 x^0 - y^1 x^1 - y^2 x^2 - y^3 x^3.$$

The bilinear form $\langle \cdot, \cdot \rangle$ is symmetric and nondegenerate, but not positive definite. With the 4×4 matrix

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

the inner product $\langle \cdot, \cdot \rangle$ becomes

$$\langle y, x \rangle = \eta_{\mu\nu} y^\mu x^\nu.$$

A homogenous Lorentz transformation is then defined as

$$\langle \hat{\Lambda} y, \hat{\Lambda} x \rangle = \langle y, x \rangle,$$

with

$$x'^\nu = \Lambda^\nu_\mu x^\mu = (\hat{\Lambda} \mathbf{x})^\nu. \quad (\text{D1})$$

One example of a Lorentz transformation is a rotation in $\mathbb{R}^3$

$$\hat{\Lambda}(\Theta) = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathfrak{R} \end{pmatrix}, \quad (\text{D2})$$

where $\mathfrak{R} \in \text{SO}(3)$ is a 3×3 orthogonal matrix with unit determinant and $\mathbf{0}^T = (0, 0, 0)$, while Θ is the rotation vector. For instance, for a rotation by an angle $\Theta \in [0, \pi)$ around an unit vector $\mathbf{n}$, $\Theta = \Theta \mathbf{n}$, the rotation matrix $\mathfrak{R}$ becomes

$$\mathfrak{R}_{ik} = \delta_{ik} \cos(\Theta) + \mathbf{n}_i \mathbf{n}_k (1 - \cos(\Theta)) - \epsilon_{ikm} \mathbf{n}^m \sin(\Theta). \quad (\text{D3})$$

We can write a general spatial rotation in the form

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \mathfrak{R} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Since $\mathfrak{R}^T \mathfrak{R} = \mathbf{1}_3$, rotations preserve the distance from the origin $\mathbf{x}'^2 = \mathbf{x}^2$.

Another example is a boost with $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| < 1$

$$\gamma(|\mathbf{v}|) = \frac{1}{\sqrt{1 - |\mathbf{v}|^2}}, \quad \hat{\Lambda}(\mathbf{v}) = \begin{pmatrix} \gamma(|\mathbf{v}|) & \gamma(|\mathbf{v}|) \mathbf{v}^T \\ \gamma(|\mathbf{v}|) \mathbf{v} & \mathbf{1}_3 + \frac{\gamma(|\mathbf{v}|) - 1}{v^2} \mathbf{v} \mathbf{v}^T \end{pmatrix}, \quad (\text{D4})$$

with $(\mathbf{v} \mathbf{v}^T)_{ik} = v_i v_k$ and $\mathbf{1}_3$ is the 3×3 identity matrix. A general Lorentz transformation that doesn't change the direction of time can always be decomposed as

$$\hat{\Lambda} = \hat{\Lambda}(\mathbf{v}) \hat{\Lambda}(\Theta). \quad (\text{D5})$$

It forms a group under matrix multiplication called the proper Lorentz group $\mathcal{L}$. A Poincaré transformation $\Pi = (a, \hat{\Lambda})$ is a combination of a Lorentz transformation and a spacetime translation, that is

$$\Pi(x) = \hat{\Lambda} x + a, \quad a \in \mathbb{R}^4.$$

The set of all Poincaré transformations forms a group $\mathcal{P}$ with the composition law

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2).$$

Our task now is to investigate how the Poincaré transformations could be represented in the Hilbert space of a quantum mechanical system. Since the predictions of quantum theory are given as transitions probabilities, the basic requirement for the relativistic invariance of the theory is that the transition probabilities between two quantum states should be the same in every inertial frame. The first step would be to find suitable representations of the Lorentz group.

2. $SL(2, \mathbb{C})$ and the Lorentz group

The set of complex 2×2 matrices with unit determinant forms a group with respect to matrix multiplication, the $SL(2, \mathbb{C})$. It also forms a vector space under multiplication by real numbers as well as matrix addition. This vector space is 4-dimensional because the Pauli matrices $\sigma_0 = \mathbf{1}_2$,

σ_i , $i = 1, 2, 3$, having the property $\sigma_1\sigma_2\sigma_3 = i\mathbf{1}_2$ where $i = \sqrt{-1}$, forms a basis of it.

Every matrix σ in $SL(2, \mathbb{C})$ has the representation

$$\sigma = \sigma(x) = \mathbf{x}^0\sigma_0 + \mathbf{x}^1\sigma_1 + \mathbf{x}^2\sigma_2 + \mathbf{x}^3\sigma_3 = \mathbf{x}^\mu\sigma_\mu,$$

with $x \in \mathbb{R}^4$.

The association of x with $\sigma(x) \in SL(2, \mathbb{C})$ defines an isomorphism of $\mathbb{R}^4$ onto $SL(2, \mathbb{C})$. Another isomorphism is given by

$$x \rightarrow \sigma'(x) = \mathbf{x}^0\sigma_0 - \mathbf{x}^1\sigma_1 - \mathbf{x}^2\sigma_2 - \mathbf{x}^3\sigma_3 = \sum_{\mu=0}^3 \mathbf{x}_\mu\sigma_\mu.$$

It then follows that

$$\det \sigma(x) = \det \sigma'(x) = \langle x, x \rangle = \sigma(x)\sigma'(x) = \sigma'(x)\sigma(x).$$

Next, let us consider the similarity transformation

$$\sigma(y) = A\sigma(x)A^\dagger, \quad A \in SL(2, \mathbb{C}).$$

Its components in the basis of Pauli matrices are determined from

$$\mathbf{y}^\nu = \frac{1}{2} \text{Tr}(\sigma(y)\sigma_\nu). \quad (\text{D6})$$

We obtain in this way a mapping $\Lambda_A: x \rightarrow y$ of $\mathbb{R}^4$ into itself. Moreover, we have

$$\begin{aligned} \langle y, y \rangle &= \det \sigma(y) = \det A \det \sigma(x) \det A^\dagger \\ &= \det \sigma(x) = \langle x, x \rangle. \end{aligned}$$

Thus, the mapping Λ_A defines a Lorentz transformation. The group $\tilde{\mathcal{L}} = SL(2, \mathbb{C})$ is a simply connected Lie group and hence is the universal covering group of the Lorentz group $\mathcal{L}$.

a. A representation of boosts

Let $H \in SL(2, \mathbb{C})$ be a Hermitian matrix. The ansatz

$$H = \cosh(\omega/2)\sigma_0 + \sinh(\omega/2)\mathbf{n}^k\sigma_k = e^{\frac{\omega}{2}\mathbf{n}\cdot\vec{\sigma}},$$

$\omega \in \mathbb{R}, \quad |\mathbf{n}| = 1, \quad (\text{D7})$

automatically satisfies $\det H = 1$. The matrix elements of Λ_H can be read off from Eq. (D6). Comparing with the matrix (D4), we see that Eq. (D7) is a boost with velocity

$$\mathbf{v} = \tanh(\omega)\mathbf{n}.$$

b. A representation of rotations

For any unitary matrix $U \in SU(2)$ (the group of complex 2×2 unitary matrices with unit determinant) we may write

$$U = \cos(\phi/2) - i \sin(\phi/2)\mathbf{n} \cdot \vec{\sigma} = e^{-i\frac{\phi}{2}\mathbf{n}\cdot\vec{\sigma}}, \quad \phi \in [0, 4\pi). \quad (\text{D8})$$

The matrix elements of Λ_U are again determined by Eq. (D6). Comparison with (D3) shows that Λ_U is a rotation around the axis $\mathbf{n}$ through an angle ϕ . The decomposition $A = HU \in SL(2, \mathbb{C})$ corresponds to the decomposition (D5) of a proper Lorentz transformation into boosts and rotations.

The isomorphism $x \rightarrow \sigma(x)$ defines a correspondence between $SL(2, \mathbb{C})$ and Lorentz transformations. Let us consider now the isomorphism $x \rightarrow \sigma'(x)$. The relation

$$\sigma'(x) \rightarrow \sigma'(y) = B\sigma'(x)B^\dagger, \quad B \in SL(2, \mathbb{C}),$$

also defines a Lorentz transformation, the matrices A and B corresponding to the same Lorentz transformation. They are connected by $B = (-i\sigma_2)A^*(-i\sigma_2)^\dagger = (A^\dagger)^{-1}$. The map $A \rightarrow B$ is a group automorphism of $SL(2, \mathbb{C})$. It corresponds to a space reflection

$$\Lambda_B = P\Lambda_AP^{-1}, \quad P = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{1}_3 \end{pmatrix}.$$

Since there's no element corresponding to P in $SL(2, \mathbb{C})$, the representations A and B are not equivalent. Since a space reflection is also a Lorentz transformation, the automorphism $A \leftrightarrow (A^\dagger)^{-1}$ suggests to double the dimension of the representation space in order to obtain a matrix representation of P . In the linear space $\mathbb{C}^4$, the matrices A and $(A^\dagger)^{-1}$ are combined into 4×4 matrices of the form (doubling the dimension is needed to avoid the operation of complex conjugation, which is not in the covering groups)

$$L_A = \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix}, \quad A \in SL(2, \mathbb{C}). \quad (\text{D9})$$

The automorphism $A \leftrightarrow (A^\dagger)^{-1}$ can now be represented by the matrix

$$L_P = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}.$$

The matrices L_A forms a group under matrix multiplication which is isomorphic to $SL(2, \mathbb{C})$. The mapping $A \rightarrow L_A$ (D9) defines an injective representation of $SL(2, \mathbb{C})$ which is reducible. There are only two invariant subspaces which are mapped into each other by the matrix L_P . Hence, the matrix group

$$\mathcal{L} = \{L_A, L_P L_A | A \in SL(2, \mathbb{C})\}$$

acts irreducible on $\mathbb{C}^4$.

c. Gamma matrices

In order to describe the connection of L_A with Lorentz transformations more precisely, we define a suitable 4×4 matrix $\gamma(x)$ for each $x \in \mathbb{R}^4$, such that Lorentz transformations can be described as a similarity transformation of $\gamma(x)$. A particular choice is

$$\gamma(x) = \begin{pmatrix} 0 & \sigma(x) \\ \sigma'(x) & 0 \end{pmatrix}. \quad (\text{D10})$$

Thus we obtain, with $B = (A^\dagger)^{-1}$

$$\begin{aligned} L_A \gamma(x) L_A^{-1} &= \begin{pmatrix} 0 & A\sigma(x)A^\dagger \\ B\sigma'(x)B^\dagger & 0 \end{pmatrix} = \gamma(\Lambda_A x), \\ L_P \gamma(x) L_P^{-1} &= \begin{pmatrix} 0 & \sigma'(x) \\ \sigma(x) & 0 \end{pmatrix} = \gamma(Px). \end{aligned} \quad (\text{D11})$$

The bijective map $\gamma: x \leftrightarrow \gamma(x)$, $x \in \mathbb{R}^4$ is an isomorphism of the vector space $\mathbb{R}^4$ and the 4-dimensional real vector space of matrices of the form (D10). The canonical basis $\{e_0, e_1, e_2, e_3\} \in \mathbb{R}^4$ is mapped to the gamma matrices

$$\gamma_0 = \gamma(e_0) = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad \gamma_k = \gamma(e_k) = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}. \quad (\text{D12})$$

Thus, in the canonical basis (i.e., the Chiral representation) $\gamma(x) = \gamma^0 x^0 - \gamma^1 x^1 - \gamma^2 x^2 - \gamma^3 x^3$, where $\gamma_0 = \gamma^0 = (\gamma^0)^\dagger$, $\gamma_k = -\gamma^k = (\gamma^k)^\dagger$. From the properties of the Pauli matrices it is easy to show that

$$\frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu} \mathbf{1}. \quad (\text{D13})$$

By construction there is a $L \in \tilde{\mathcal{L}}$ for every $\Lambda_L \in \mathcal{L}$ such that

$$L \langle \gamma, x \rangle L^{-1} = \langle \gamma, \Lambda_L x \rangle,$$

and the map $L \rightarrow \Lambda_L$ is onto and a homomorphism. Using the γ matrices, the matrices L corresponding to boosts and rotations become

$$L_H = e^{-\frac{\omega}{2} \mathbf{n}_k \gamma^k \gamma^0}, \quad \text{boost with velocity } \mathbf{v} = \tanh(\omega) \mathbf{n}, \quad (\text{D14})$$

$$L_U = e^{\frac{i\phi}{2} \gamma^5 \mathbf{n}_k \gamma^k \gamma^0}, \quad \text{rotation through an angle } \phi \text{ around } \mathbf{n}, \quad (\text{D15})$$

$$L_P = \gamma^0, \quad \text{space reflection}, \quad (\text{D16})$$

where $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. It can be shown that $L^{-1} = \gamma_0 L^\dagger \gamma_0$ [19,20].

3. Constructing the hydrogenlike ion ground state

We begin with the classical rotor solution given in the in the main text

$$R = e^{\frac{1}{2}w\hat{\phi}} e^{\frac{i\phi}{2\gamma_1\gamma_2}} e^{-\frac{1}{2}w\hat{\phi}} e^{\gamma_2\gamma_1\phi/2}. \quad (\text{D17})$$

Using the formula for the $SO(1,3)$ representation of a Lorentz boost [see (D4)] for $e^{\frac{1}{2}w\hat{\phi}}$ we obtain

$$\hat{\Lambda}(\mathbf{v}) = \begin{pmatrix} \gamma_{|\mathbf{v}|} & \gamma_{|\mathbf{v}|} \mathbf{v}_1 & \gamma_{|\mathbf{v}|} \mathbf{v}_2 & 0 \\ \gamma_{|\mathbf{v}|} \mathbf{v}_1 & 1 + \frac{(\gamma_{|\mathbf{v}|}-1)\mathbf{v}_1^2}{|\mathbf{v}|^2} & \frac{(\gamma_{|\mathbf{v}|}-1)\mathbf{v}_1 \mathbf{v}_2}{|\mathbf{v}|^2} & 0 \\ \gamma_{|\mathbf{v}|} \mathbf{v}_2 & \frac{(\gamma_{|\mathbf{v}|}-1)\mathbf{v}_1 \mathbf{v}_2}{|\mathbf{v}|^2} & 1 + \frac{(\gamma_{|\mathbf{v}|}-1)\mathbf{v}_2^2}{|\mathbf{v}|^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{D18})$$

where

$$\begin{aligned} \gamma_{|\mathbf{v}|} &= \frac{l}{\sqrt{l^2 - (Z\alpha)^2}}, \\ v^T &= \gamma_{|\mathbf{v}|}(1, \mathbf{v}_1, \mathbf{v}_2, 0) \\ &= \frac{l}{\sqrt{l^2 - (Z\alpha)^2}} \left(1, -\frac{Z\alpha}{l} \sin(\phi), \frac{Z\alpha}{l} \cos(\phi), 0 \right). \end{aligned} \quad (\text{D19})$$

In the above representation the electron's velocity is then $v = \hat{\Lambda}(\mathbf{v})u$, $u^T = (1, 0, 0, 0)$. Now we are in the position to construct the quantum counterpart of (D18) which we shall call $\mathcal{B}(\mathbf{v}')$. As is clear from Fig. 1(a), such a task becomes straightforward by noting that the quantum velocity is simply $v' = \cos \beta v$ with $l = 1$ in v and $\hat{\phi} = \sin \theta (-\sin(\phi), \cos(\phi), 0)$. Hence, $\mathcal{B}(\mathbf{v}') = \cos \beta \hat{\Lambda}(\mathbf{v})$.

The task is now to construct the quantum spin vector s' . In the classical case, due to angular momentum conservation, the electron's spin is fixed along the $\hat{z}$ axis and is just $s = (0, 0, 0, 1)$ which is a consequence of the Thomas rotation element $e^{\frac{1}{2}w\hat{\phi}} e^{\frac{i\phi}{2\gamma_1\gamma_2}} e^{-\frac{1}{2}w\hat{\phi}}$ that leaves the z axis unchanged. In the quantum case, however, spherical symmetry dictates that the spin vector should have its tip free to move on the surface of a sphere, the Thomas rotation element should take the following general form of the rotation spinor $U = e^{-\gamma_2\gamma_1\phi/2} e^{-\gamma_3\gamma_1\Theta}$ (see, for instance, Appendix A.3 of Ref. [11]). The effect of the rotation U on the electron spin is illustrated in Fig. 1(b), where the spatial component of the quantum spin s' wobbles around the $\hat{r}$ with an angle Θ due to spin-orbit coupling. The representation of U in $SO(3)$ from (D2) and (D3) is

$$\hat{\Lambda}(\Theta) = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathfrak{R} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\Theta) + k_1^2(1 - \cos(\Theta)) & k_1 k_2(1 - \cos(\Theta)) & k_2 \sin(\Theta) \\ 0 & k_1 k_2(1 - \cos(\Theta)) & \cos(\Theta) + k_2^2(1 - \cos(\Theta)) & -k_1 \sin(\Theta) \\ 0 & -k_2 \sin(\Theta) & k_1 \sin(\Theta) & \cos(\Theta) \end{pmatrix} \quad (\text{D20})$$

with $\mathbf{n}^T = \mathbf{k}^T = (k_1, k_2, k_3) = (-\sin(\phi), \cos(\phi), 0)$. Its effect on $\hat{z}$ is then

$$\mathbf{s}' = \mathfrak{R}(0, 0, 1)^T = (k_2 \sin(\Theta), -k_1 \sin(\Theta), \cos(\Theta))^T.$$

We now need to find expression for both β and Θ . From (D19) and the identity $|\mathbf{v}'|^2 = 1$, we get

$$\cos \beta = \frac{\sqrt{1 - Z^2 \alpha^2}}{\sqrt{1 - Z^2 \alpha^2 \sin^2 \theta}}, \quad \sin \beta = \frac{Z \alpha \cos \theta}{\sqrt{1 - Z^2 \alpha^2 \sin^2 \theta}}.$$

From Fig. 1(b) one can infer that the quantum spin vector should have the general form

$$\mathbf{s}' = \chi \sin(\beta) (\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta))^T + \cos(\beta) \hat{z}, \quad (\text{D21})$$

which is clear by considering the projection of $\mathbf{s}'$ on the radial and z axes, as well as the fact that for $\beta = 0$ we must have $\mathbf{s}' = \hat{z} = \mathbf{s}$. The proportionality factor comes from the property $|\mathbf{s}'|^2 = 1$, hence

$$\chi = \frac{Z \alpha}{1 + \sqrt{1 - Z^2 \alpha^2}}.$$

Finally,

$$\begin{aligned} \cos(\Theta) &= \frac{Z \alpha \cos(\theta) \sin(\beta)}{1 + \sqrt{1 - (Z \alpha)^2}} + \cos(\beta), \\ \sin(\Theta) &= \frac{Z \alpha \sin(\beta) \sin(\theta)}{1 + \sqrt{1 - (Z \alpha)^2}}. \end{aligned} \quad (\text{D22})$$

The final expression for the quantum Lorentz transformation is then

$$R = e^{\frac{1}{2} w \hat{\phi}} e^{\frac{i \phi}{2 \gamma_1 \gamma_2}} e^{-\frac{1}{2} w \hat{\phi}} \rightarrow \mathcal{R} = e^{\frac{1}{2} w' \hat{\phi}} e^{-\gamma_2 \gamma_1 \phi / 2} e^{-\gamma_3 \gamma_1 \Theta}, \quad (\text{D23})$$

with $w' = \tanh^{-1}(Z \alpha \sin \theta)$.

The last step is to find the scalar density ρ . In order to achieve this, one return to the Hestenes-Dirac equation and investigate the constraints it imposes on the matrix Spinor Ψ . To begin, we get both the current and spin densities $\Psi \gamma_0 \tilde{\Psi} = \rho \gamma(v')$ and $\Psi \gamma_3 \tilde{\Psi} = \rho \gamma(s')$. As discussed in [9], they should obey the following equation

$$\partial_\mu (\rho \mathbf{v}'^\mu) = 0, \quad \partial_\mu (\rho \mathbf{s}'^\mu) + \frac{2mc}{\hbar} \rho \sin \beta = 0. \quad (\text{D24})$$

Since we already found the expression for β from our geometrical considerations, all we need to solve is the continuity equation for the current to find ρ . For the case at hand, it is

$$\rho(r, \theta) = e^{-\frac{2cmrZ\alpha}{\hbar}} \left(\frac{cmrZ\alpha}{\hbar} \right)^{2\sqrt{1-(Z\alpha)^2}-2} \sqrt{\frac{1-(Z\alpha)^2 \sin^2(\theta)}{1-(Z\alpha)^2}}. \quad (\text{D25})$$

APPENDIX E: SYMMETRY ANALYSIS AND DETAILED DERIVATION

Given that in the main text we built the quantum spinor from the classical rotor solution, here we show how the solution can also be derived from the Lie derivative alone, without the need to invoke the classical rotor equation. Both ways should be seen as supplementary to each. The derivation provided here has the advantage of explicitly showing how the remaining isometries manifest themselves in the Lorentz rotor.

1. Residual isometries of the plane-wave metric

For simplicity, we work in the diagonal Rosen form

$$ds^2 = 2 dU dV - f(U)^2 dX^{12} - g(U)^2 dX^{22}, \quad (\text{E1})$$

$$U = \frac{ct - z}{\sqrt{2}}, \quad V = \frac{ct + z}{\sqrt{2}}. \quad (\text{E2})$$

The vacuum Einstein equation imposes $f''/f + g''/g = 0$. This spacetime admits the following five Killing vectors:

$$\begin{aligned} K_V &= \partial_V, & K_i &= \partial_{X^i}, & N_i &= X^i \partial_V - \mathcal{H}^{ii}(U) \partial_{X^i}, \\ \mathcal{H}^{ii} &= \int^U dU' g^{ii}(U'). \end{aligned}$$

All other Poincaré generators (e.g. full spatial rotations and boosts mixing U and X^i) are broken by the U dependence of f, g . Note that the N_i correspond to parabolic Lorentz transformations, i.e., *null rotations*. This has a deep impact as discussed below.

2. Structure of rotor R and phase fixing

Write the eikonal phase $S = p_V V - p_j X^j + a(U)$. Then setting the derivative to zero gives us the trajectories

$$N^\mu \partial_\mu S = \sum_i (p_V X^i - p_i \mathcal{H}^{ii}(U)) = 0,$$

$$X^i(U) = \frac{p_i}{p_V} \mathcal{H}^{ii}(U).$$

From the on-shell condition $g^{\mu\nu} \partial_\mu S \partial_\nu S = m^2 c^2$ we arrive at

$$a(U) = \int_0^U \frac{dU'}{2p_V} \left(\frac{p_1^2}{g_{11}(U')} + \frac{p_2^2}{g_{22}(U')} + m^2 c^2 \right).$$

3. Null-rotation rotor

We can now determine the structure of the rotor by applying the null rotations N_i to any initial and constant spinor ψ_0 . The action of a parabolic Lorentz transformation N , respectively a general vector field N , on a spinor (rotor) is

$$\delta\psi = \varepsilon \mathcal{L}_N \psi,$$

where

$$\mathcal{L}_N \psi = N^\mu \nabla_\mu \psi - \frac{1}{8} (\nabla_\mu N_\nu - \nabla_\nu N_\mu) \gamma^\mu \gamma^\nu \psi$$

is the Lie derivative of a spinor with respect to N , where ∇_μ is the covariant derivative. This infinitesimal transformation then results in the finite transformation by replacing the infinitesimal parameter ε with the finite real group parameter or flow time α of the zero rotation and exponentiating the above result

$$\psi \mapsto \exp(\alpha N) \psi.$$

In the case of a zero rotation $N^2 = 0$ the exponential truncates

$$\exp(\alpha N) = 1 + \alpha N.$$

This means the finite action is

$$\psi \mapsto \psi + \alpha \mathcal{L}_N \psi. \quad (\text{E3})$$

The general null rotation

$$N(p) = p^x N_x + p^y N_y$$

is now constructed from the two Killing vectors N_i . The parameters p_x and p_y characterize shear transformations

in the transverse (x, y) plane that preserve the lightlike direction $t - z$. Specifically, p_x governs translations along the x axis that are correlated with displacements in the light-front coordinate $t + z$, thereby inducing position-dependent shifts along x as one propagates along the null direction. Similarly, p_y induces shear deformations in the y direction, corresponding to combined transverse boosts and rotations within the $(y, t - z)$ light front. Collectively, the pair (p_x, p_y) parametrizes a two-dimensional Abelian subgroup of the Lorentz little group associated with massless particles, acting parabolically in the transverse plane orthogonal to the null propagation direction. By applying the transformation (E3) to a constant spinor ψ_0 we can then determine the shape of the rotor,

$$\psi_0 \mapsto \psi_0 + \alpha \mathcal{L}_N \psi_0 = R_v(U) \psi_0,$$

with

$$R_v(U) = 1 + \sum_i \alpha_i p^i \left(N_i^\mu \Omega_\mu - \frac{1}{8} (\nabla_\mu N_{i\nu} - \nabla_\nu N_{i\mu}) \gamma^\mu \gamma^\nu \right). \quad (\text{E4})$$

The factors α_i are then determined by the rotor equation. This nilpotent rotor factorizes exactly into a rotation U_v and boost $\mathcal{B}_v$,

$$R_v = U_v \mathcal{B}_v, \quad U_v = \exp \left[-\frac{1}{2} \theta \hat{u}_i \gamma^i \gamma^3 \right],$$

$$\mathcal{B}_v = \exp \left[-\frac{1}{2} \omega \hat{v}_j \gamma^0 \gamma^j \right].$$

The two unit vectors $\hat{u} = (\hat{u}_1, \hat{u}_2, 0)$ in the transverse plane and $\hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ in three space are fixed by matching the linear (first order) expansion of the product $U_v \mathcal{B}_v$ to the explicit rotor $R_v = (1 - \frac{1}{2} \int_0^\tau d\tau' \omega(v))$. Physically, $\hat{u}$ is the axis of the infinitesimal spatial rotation in the (X^1, X^2) plane required by the wave's tidal distortion while $\hat{v}$ is the direction of the infinitesimal boost that converts the null rotation into an equivalent boost-rotation product. The transverse pieces $\hat{v}_{1,2}$ reflect the obvious tidal “shaking” of the electron in the two transverse directions while the longitudinal piece $\hat{v}_3$ appears because a pure null rotation inevitably couples transverse motion to the light-cone direction. In spacetime language the longitudinal rapidity is what keeps the electron's light cone aligned with the wave vector of the gravitational plane wave. It represents an energy-momentum kick in the $+z$ direction required by the geometry of the null background.

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Correction: Reference [16] was fixed to reflect the correct source information.