

Dipionic transitions of $Y(4500)$ to J/ψ

Shi-Dong Liu^{1,*}, Qi Wu^{2,†} and Gang Li^{1,‡}

¹*College of Physics and Engineering, Qufu Normal University, Qufu 273165, China*

²*Institute of Particle and Nuclear Physics, Henan Normal University, Xinxiang 453007, China*



(Received 24 April 2025; accepted 9 October 2025; published 28 October 2025)

We, using an effective Lagrangian approach, investigated the dipionic transition of the $Y(4500)$ newly well established in the process $e^+e^- \rightarrow K^+K^-J/\psi$. In this study, the $Y(4500)$ was considered as a mixture of the $\psi(5S)$ and $\psi(4^3D_1)$, and its dipionic decay to the J/ψ was assumed to occur via charmed meson loops including box and two kinds of triangle loops. The calculated invariant mass spectra due to the different loop mechanisms exhibit distinct differences. These spectrum patterns are found to be nearly independent of the cutoff parameter, especially for the box and $\mathcal{R}$ loops. By means of comparing our calculated results with the future experimental measurements of the $\pi^+\pi^-$ and $\pi^\pm J/\psi$ invariant mass distributions, we could judge which loop mechanism is of more importance in this dipionic transition. The interference among the different kinds of loops is also exhibited. Despite the unknown phase angles between different kinds of loop diagrams, the partial decay width for the process $Y(4500) \rightarrow \pi^+\pi^-J/\psi$ is estimated to be 250–700 keV, aligning with the estimation obtained by combining BESIII Collaboration measurements and theoretical predictions for $Y(4500) \rightarrow K^+K^-J/\psi$. We hope that the present calculations would be tested by the future BESIII or Belle II experiments.

DOI: [10.1103/fz1m-3mdg](https://doi.org/10.1103/fz1m-3mdg)

I. INTRODUCTION

With the advancement of experimental techniques, many new resonant structures, whose masses greatly exceed the threshold of open heavy-flavor meson pairs, especially in the $c\bar{c}$ sector, have been identified in recent years [1–3]. These resonances usually show different properties and behaviors compared to the conventional heavy quarkonia. For the higher states of the heavy quarkonia, they would mainly decay into the open heavy-flavor meson pairs, while the resonances that cannot fit into the quark model usually exhibit strong coupling to the hidden heavy-flavor channels [3–7]. In terms of their special features (e.g., the near-threshold effect) or unexpected decay modes, various models have been proposed to interpret their natures, including molecules, multiquarks, hybrids, cusp effects, and so on (see reviews [8–19], and references therein). These novel resonances not only broaden our perspective of the hadron spectrum in the heavy quarkonium energy region, but also provide us with additional platforms to

understand the nonperturbative properties of strong interactions.

In the $c\bar{c}$ sector, the celebrated vector charmoniumlike state with energy larger than 4.0 GeV is the $Y(4260)$, first discovered in the processes $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ by the BABAR experiment in 2005 [20] and soon confirmed in the same process by the CLEO [21] and Belle [22] Collaborations. Higher-statistics data of the $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ from the BESIII Collaboration [7] revealed a shift in the peak position to a lower mass and a reduction in the width, thereby leading to the renaming of the state $Y(4260)$ as $Y(4230)$ or $\psi(4230)$ with a mass of (4222.1 ± 2.3) MeV and a width of (49 ± 7) MeV [3]. A series of experimental measurements demonstrate that the $Y(4230)$ decays predominantly through hidden charm modes [3,23,24] but has a minimal probability to the open charmed final states (plus one pion) [3,25], despite its mass being well above the $D\bar{D}$ threshold. This indicates that the $Y(4230)$ is likely to be an exotic state, such as the hybrid charmonium [26], the tetraquark [27–29], the mesonic molecule [30–43], and baryonium [44], but not a conventional charmonium [21,26,45]. In Refs. [5,46], the authors interpreted the $Y(4230)$ as a $4S - 3D$ mixture. Moreover, it was also interpreted as a nonresonant structure [47] or cusp effect [48], namely, not a genuine resonance.

Recent measurements by the BESIII Collaboration from 2022 to 2024 reveal plentiful massive charmoniumlike states [1,23–25,49–51]. Among these, some states are consistent with previously established resonances, while

*Contact author: liusd@qfnu.edu.cn

†Contact author: wuqi@htu.edu.cn

‡Contact author: gli@qfnu.edu.cn

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

others, such as the $Y(4500)$, $Y(4710)$, and $Y(4790)$, are observed for the first time. So far, the $Y(4710)$ and $Y(4790)$ were observed in only one process: $Y(4710)$ in $e^+e^- \rightarrow K^+K^-J/\psi$ [49] and $Y(4790)$ in $e^+e^- \rightarrow D_s^{*+}D_s^{*-}$ [50], while the $Y(4500)$ seems to have been observed in three different processes: $e^+e^- \rightarrow K^+K^-J/\psi$ [23], $\pi^+\pi^-J/\psi$ [24], and $D^{*0}D^{*-}\pi^-$ [25]. The $Y(4500)$ was well constructed in the $e^+e^- \rightarrow K^+K^-J/\psi$ with a more than 8σ significance [23], whose mass and width were determined to be $(4484.7 \pm 13.3 \pm 24.1)$ MeV and $(111.1 \pm 30.1 \pm 15.2)$ MeV, respectively. It is noted that the signal of the $Y(4500)$ was also hinted in the dipion process $e^+e^- \rightarrow \pi^+\pi^-J/\psi$, exhibiting similar mass and width, although the significance was reported to be only $\sim 3\sigma$ [24]. The process $e^+e^- \rightarrow D^{*0}D^{*-}\pi^-$ provides possible evidence of the $Y(4500)$, but this channel shows a lower mass of $(4469.1 \pm 26.2 \pm 3.6)$ MeV and larger width of $(246.3 \pm 36.7 \pm 9.4)$ MeV [25]. Very recently, the $e^+e^- \rightarrow \pi^+\pi^-h_c$ cross section measured by the BESIII Collaboration also signals the $Y(4500)$, giving a mass of $(4467.4_{-5.4-2.7}^{+7.2+3.2})$ MeV and a width of $(62.8_{-14.4-7.0}^{+19.2+9.9})$ MeV with the statistical significance greater than 5σ [52].

The nature of these newly observed resonances remains an open question. Not only is the experimental status of the $Y(4500)$, particularly in its mass and width, difficult to reach a unanimous conclusion, but the theoretical interpretations also remain debated. The $Y(4500)$ is compatible with the D -wave charmonium $\psi(4^3D_1)$ having mass of 4484 MeV predicted in the unquenched potential model [5] but also accords with the $\psi(3^3D_1)$ having mass of 4486 (4463) MeV predicted by the quenched (unquenched) quark model [53]. Particularly, in the latter work [53], the total width of the $\psi(3^3D_1)$ is predicted to be about 119 MeV, agreeing well with the experimental results [23,24]. If the $Y(4500)$ is the D -wave charmonium $\psi(3^3D_1)$, it would decay dominantly via the open charmed channels $DD_0(2550)$, $DD_2^*(2460)$, and $D^{(*)}D_1(2420)$. Moreover, it is also consistent with the heavy-antiheavy hadronic molecule $D_s\bar{D}_{s1}(2536)$ with mass of 4503 MeV [54]. The interpretation of $Y(4500)$ as a tetraquark state with quark content $c\bar{c}s\bar{s}$ [30] or $c\bar{c}q\bar{q}$ ($q = u, d$) [55,56] appears feasible as well. In the case of $c\bar{c}q\bar{q}$ components, the QCD sum rules give a total width of about 200 MeV [56], in accordance with the BESIII data [25]. The primary decay modes include the open-charm channels $D\bar{D}$, $D^*\bar{D}^*$, and $DD_1(2420)$ and the hidden ones $\chi_{c0(c1)}\omega$ and $J/\psi f_0(500)$. In Ref. [57], the authors explained the $Y(4500)$ as the $5S-4D$ mixture and refitted the BESIII data, giving a mass of (4504 ± 4) MeV and a width of about 50 MeV. This predicted width is 2–5 times smaller than the BESIII measurements [23–25], which could be attributed to the cross-section enhancement by the charmoniumlike structures around 4.6 GeV [57]. Additionally, the branching fraction of the $Y(4500) \rightarrow K^+K^-J/\psi$ was

predicted to be $(1.3-2.0) \times 10^{-3}$ using the charmed meson loop mechanism, where the K^+K^- were assumed to be produced via the scalar meson $f_0(980)$. So far, experimental and theoretical studies on the $Y(4500)$ are still scarce. To deepen understanding of the $Y(4500)$ nature, more effort is needed.

Although the process $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ signals weakly the $Y(4500)$ resonance [24], the ratio of the Born cross sections for the $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ to $e^+e^- \rightarrow K^+K^-J/\psi$ is approximately 2.5 near the center-of-mass energy of 4.5 GeV [23,24,58]. If the $Y(4500)$ is the only contribution to the $\pi^+\pi^-J/\psi$ and K^+K^-J/ψ , this naive cross section ratio implies that the decay process $Y(4500) \rightarrow \pi^+\pi^-J/\psi$ might occur with a higher rate around 4.5 GeV.

In the present work, we shall study the dipionic transition of the $Y(4500)$ to J/ψ using an effective Lagrangian approach. Following the pioneering theoretical studies [5,57], we also regard the $Y(4500)$ as a mixture of the $\psi(5S)$ and $\psi(4^3D_1)$. QCD multipole expansion can deal well with the hadronic transitions between low-lying heavy quarkonium [59,60]. However, coupled-channel effects become important for higher charmonia [57,61–63]. We assume the decay to occur via the charmed meson loops, including the triangle and box loops. This work contributes to testing the $5S-4D$ mixing scheme of $Y(4500)$ and offers deeper insights into the role of coupled-channel effects in the decay of higher charmonia. In the following, we first give in Sec. II the Lagrangians. Then, in Sec. III, the numerical results and discussion are described in detail. Finally, a summary is given in Sec. IV.

II. THEORETICAL CONSIDERATION

The dipionic processes $Y(4500) \rightarrow \pi^+\pi^-J/\psi$ are assumed to occur possibly via the intermediate meson loops, including the box and triangle ones [64,65] as shown in Fig. 1. In view of the experimental observations that the invariant mass distributions for the K^+K^- [23,58] exhibit clear signals of the mesons $f_0(980)$ and $f_2(1270)$, along with the appreciable decay rate of $Y(4500) \rightarrow J/\psi f_0(500)$ predicted by the tetraquark model [56], we incorporate these intermediate mesons into the triangle loops as depicted in Figs. 1(o)–1(r). The contributions from the different types of loops will be calculated separately in the following, as we did in our previous work [62,63]. The present results show that the $\pi\pi$ invariant mass distributions due to different loops vary significantly. The great difference might be easily recognized by the experimental measurements. Furthermore, the interference effects of the three different types of loop mechanisms are also considered, showing significant influence on the invariant mass distributions of the final states.

Following the treatment in Refs. [5,57], here we also consider the $Y(4500)$ as the mixed state of the $\psi(5S)$ and $\psi(4^3D_1)$. The $\psi(5S)$ has a mass of 4443 MeV and the

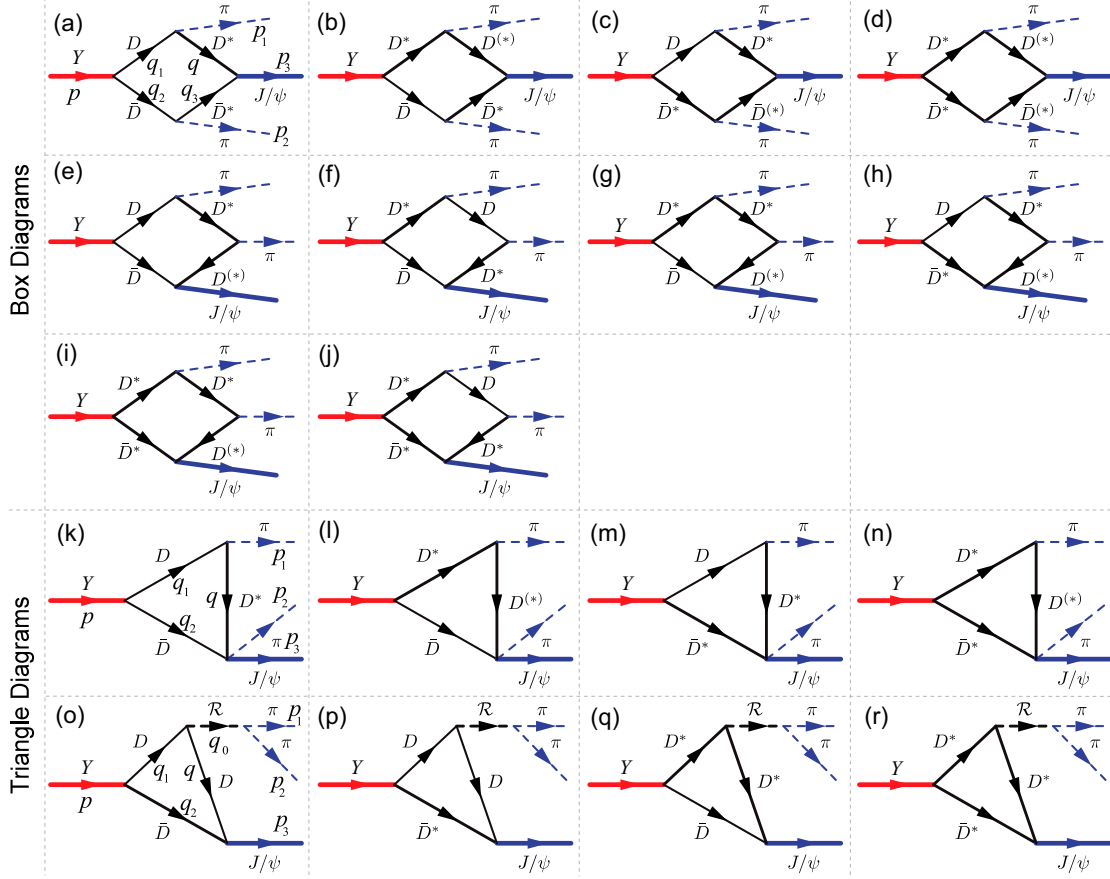


FIG. 1. Feynman diagrams for the $Y(4500) \rightarrow \pi^+ \pi^- J/\psi$ decay process via the intermediate meson loops. As indicated, the diagrams (a)–(j) describe the box loops, while the diagrams (k)–(r) are the triangle loops. The Y represents the $Y(4500)$, and the $\mathcal{R}$ stands for the $f_0(500)$, $f_0(980)$, and $f_2(1270)$. The relevant kinematics $[p, p_{1(2,3)}, q, q_{0(1,2,3)}]$ are explicitly indicated in the graphs. The charge conjugated loops are not shown here but included in the calculations.

$\psi(4^3D_1)4484$ MeV, predicted in the unquenched potential model [5]. Within the $S-D$ mixing framework, the wave function of the $Y(4500)$ has the following form [5,57]:

$$\tilde{Y}(4500) = -\tilde{\psi}(5S) \sin \theta + \tilde{\psi}(4^3D_1) \cos \theta, \quad (1)$$

where the mixing angle θ is determined to be 30° [57], agreeing well with the widely accepted mixing angle for the higher charmonium $S-D$ [66]. The mixing angle can be determined by either the mass-mixing formula or fitting the experimental dielectron width [5]. Here, it should be pointed out that, in some quark models [53,67–70], the masses of $\psi(4S)$ and $\psi(3^3D_1)$ are predicted between 4400 and 4500 MeV. Hence, the $4S-3D$ mixing scenario for the $Y(4500)$ appears also possible, which corresponds to a bit smaller mixing angle of $\lesssim 25^\circ$. In this work, we consider only the $5S-4D$ mixing assumption, regardless of the possibility of the $4S-3D$ mixing. The dipionic transitions of the $\psi(4415)$, another mixed state of the $\psi(5S)$ and $\psi(4^3D_1)$, have been studied in Refs. [47,71,72], in which the $\psi(4415)$ is regarded as the pure S -wave charmonium. Moreover, Ref. [73] shows that, for the S -wave dominant state with a mixing angle around 30° , its hidden-charm

decay deviates mildly from that of the pure S -wave charmonium. Thus, in our calculations, we focus on only the dipionic transition of the $Y(4500)$.

According to the heavy quark effective theory, the interactions of the S -wave charmonium $\psi(nS)$ and D -wave $\psi(n^3D_1)$ with the charmed and anticharmed mesons are described by the Lagrangians [57,74]

$$\begin{aligned} \mathcal{L}_S = & -ig_{\psi DD} \bar{\psi}_\mu D^\dagger \overleftrightarrow{\partial}^\mu D \\ & + g_{\psi DD^*} \varepsilon_{\mu\alpha\beta} \partial^\mu \bar{\psi}^\nu (D^\dagger \overleftrightarrow{\partial}^\alpha D^{*\beta} - D^{*\dagger\beta} \overleftrightarrow{\partial}^\alpha D) \\ & + ig_{\psi D^* D^*} \bar{\psi}_\mu (\partial^\nu D^{*\dagger\mu} D_\nu^* - D_\nu^{*\dagger} \partial^\nu D^{*\mu} \\ & + D^{*\dagger\nu} \overleftrightarrow{\partial}^\mu D_\nu^*), \end{aligned} \quad (2a)$$

$$\begin{aligned} \mathcal{L}_D = & ig_{\psi_1 DD} \bar{\psi}_1 \mu D^\dagger \overleftrightarrow{\partial}^\mu D \\ & - g_{\psi_1 DD^*} \varepsilon_{\mu\alpha\beta} \partial^\mu \bar{\psi}_1^\nu (D^\dagger \overleftrightarrow{\partial}^\alpha D^{*\beta} - D^{*\dagger\beta} \overleftrightarrow{\partial}^\alpha D) \\ & + ig_{\psi_1 D^* D^*} \bar{\psi}_1 \mu (\partial^\nu D^{*\dagger\mu} D_\nu^* - D_\nu^{*\dagger} \partial^\nu D^{*\mu} \\ & + 4D^{*\dagger\nu} \overleftrightarrow{\partial}^\mu D_\nu^*). \end{aligned} \quad (2b)$$

Here, $\mathcal{D}^{(*)\dagger} = (\bar{D}^{(*)0}, D^{(*)-}, D_s^{(*)-})$ is the charmed meson triplet. In the case of the $Y(4500)$ as the $\psi(5S)$ - $\psi(4^3D_1)$ mixture, we need the coupling constants $g_{\psi\mathcal{D}^{(*)}\mathcal{D}^{(*)}}$ and $g_{\psi_1\mathcal{D}^{(*)}\mathcal{D}^{(*)}}$, which are estimated in the Appendix. For the J/ψ , the couplings $g_{J\mathcal{D}\mathcal{D}}$, $g_{J\mathcal{D}\mathcal{D}^*}$, and $g_{J\mathcal{D}^*\mathcal{D}^*}$ are linked to each other by a global coupling strength g_J and the masses of the involved mesons as

$$g_{J\mathcal{D}\mathcal{D}} = 2g_J\sqrt{m_J m_D}, \quad (3a)$$

$$g_{J\mathcal{D}\mathcal{D}^*} = 2g_J\sqrt{m_D m_{D^*}/m_J}, \quad (3b)$$

$$g_{J\mathcal{D}^*\mathcal{D}^*} = 2g_J\sqrt{m_J m_{D^*}}. \quad (3c)$$

In terms of the vector meson dominance [75,76], $g_J = \sqrt{m_J}/(2m_D f_J)$. The J/ψ decay constant f_J can be extracted from $J/\psi \rightarrow e^+e^-$ [77–79]:

$$\Gamma_{ee} = \frac{16\pi\alpha_{\text{EM}}^2}{27m_J} f_J^2, \quad (4)$$

where $\alpha_{\text{EM}} = 1/137$ is the fine-structure constant. Using the newly updated PDG data [3] $m_J = (3096.900 \pm 0.006)$ MeV and $\Gamma_{ee} = (5.53 \pm 0.10)$ keV, we find $f_J = (416 \pm 4)$ MeV and, thereby, $g_J = (1.13 \pm 0.01)$ GeV $^{-3/2}$.

In the heavy quark limit and chiral symmetry, the interactions of the light pseudoscalar mesons with the heavy charmed mesons read [36,80,81]

$$\begin{aligned} \mathcal{L}_\pi = & -ig_{\mathcal{D}\mathcal{D}^*\mathcal{P}}(\mathcal{D}^{i\dagger}\partial^\mu\mathcal{P}_{ij}^\dagger\mathcal{D}_\mu^{*j} - \mathcal{D}_\mu^{*i\dagger}\partial^\mu\mathcal{P}_{ij}^\dagger\mathcal{D}^j) \\ & + \frac{1}{2}g_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}}\epsilon_{\mu\nu\alpha\beta}\mathcal{D}_i^{*\mu\dagger}\partial^\nu\mathcal{P}^{ij\dagger}\partial^\alpha\mathcal{D}_j^{*\beta}, \end{aligned} \quad (5)$$

where the $\mathcal{P}$ represents the pseudoscalar mesons in the matrix form

$$\mathcal{P} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}_0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}. \quad (6)$$

The coupling constants $g_{\mathcal{D}^{(*)}\mathcal{D}^*\mathcal{P}}$ are linked to each other by the following relation:

$$g_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}} = \frac{g_{\mathcal{D}\mathcal{D}^*\mathcal{P}}}{\sqrt{m_D m_{D^*}}} = \frac{2g}{f_\pi} \quad (7)$$

with the pion decay constant $f_\pi = (130.2 \pm 1.2)$ MeV [3]. Using the Lagrangian in Eq. (5), we get

$$\Gamma(D^{*+} \rightarrow D^0\pi^+) = \frac{g^2}{6\pi f_\pi^2 m_{D^*}} |\mathbf{p}_\pi|^3, \quad (8a)$$

$$\Gamma(D^{*+} \rightarrow D^+\pi^0) = \frac{g^2}{12\pi f_\pi^2 m_{D^*}} |\mathbf{p}_\pi|^3. \quad (8b)$$

According to the world average data [3], it yields $g = 0.57 \pm 0.01$, in good agreement with the value $g = 0.59 \pm 0.07 \pm 0.01$ obtained in Ref. [82].

The couplings between the intermediate mesons $\mathcal{R} = \{\mathcal{S}, f_2(1270)\}$ with $\mathcal{S} = \{f_0(500), f_0(980)\}$ and the charmed meson pair can be described by the following Lagrangians [83–86]:

$$\begin{aligned} \mathcal{L}_{\mathcal{R}\mathcal{D}\mathcal{D}} = & g_{\mathcal{S}\mathcal{D}\mathcal{D}}\mathcal{S}\mathcal{D}\mathcal{D}^\dagger - g_{\mathcal{S}\mathcal{D}^*\mathcal{D}^*}\mathcal{S}\mathcal{D}_\mu^{*\dagger}\mathcal{D}^{*\mu\dagger} \\ & + g_{f_2\mathcal{D}^*\mathcal{D}^*}f_2^{\mu\nu}\mathcal{D}_\mu^{*\dagger}\mathcal{D}_\nu^{*\dagger}. \end{aligned} \quad (9)$$

Here, the coupling constants for the $\mathcal{S} = \{f_0(500), f_0(980)\}$ are

$$g_{\mathcal{S}\mathcal{D}\mathcal{D}} = am_D g_\pi, \quad (10a)$$

$$g_{\mathcal{S}\mathcal{D}^*\mathcal{D}^*} = am_{D^*} g_\pi, \quad (10b)$$

where $a = 1/\sqrt{6}$ and $1/\sqrt{3}$ for the $f_0(500)$ and $f_0(980)$, respectively, and $g_\pi = 3.73$ [84]. In this work, we take $M_{f_0(500)} = 449$ MeV, $\Gamma_{\text{tot}}[f_0(500)] = 550$ MeV; $M_{f_0(980)} = 993$ MeV, $\Gamma_{\text{tot}}[f_0(980)] = 61.3$ MeV [84]. It is noted that, for the interaction between the $f_2(1270)$ and the charmed meson pair, we consider only the S -wave coupling while the higher-wave couplings are neglected. The coupling constant of the $f_2(1270)$ to a pair of charmed mesons is unknown. We shall make an estimation later.

The decay of the scalar meson $\mathcal{S}$ into two pions is described by

$$\mathcal{L}_{\mathcal{S}\pi\pi} = g_{\mathcal{S}\pi\pi}\mathcal{S}\pi\pi, \quad (11)$$

where $g_{\mathcal{S}\pi\pi} = 3.25$ GeV for the $f_0(500)$ and $g_{\mathcal{S}\pi\pi} = 1.13$ GeV for the $f_0(980)$ [84]. Furthermore, the interaction of the $f_2(1270)$ with the $\pi^+\pi^-$ reads

$$\mathcal{L}_{f_2\pi\pi} = g_{f_2\pi\pi}f_2^{\mu\nu}\partial_\mu\pi^+\partial_\nu\pi^-. \quad (12)$$

Using the PDG data $\mathcal{B}(f_2 \rightarrow \pi\pi) = 84.3\%$ and $\Gamma_i(f_2) = 186.6$ MeV [3], we find $g_{f_2\pi\pi} = 14.64$ GeV $^{-1}$ under the assumption of $\Gamma(f_2 \rightarrow \pi^+\pi^-) = 2\Gamma(f_2 \rightarrow \pi^0\pi^0)$.

Before proceeding, we provide here the estimation of the coupling constant $g_{f_2\mathcal{D}^*\mathcal{D}^*}$. In the heavy quark limit [$m_c(b) \rightarrow \infty$], the couplings of the f_2 to the charmed mesons and to the bottom ones appear equal. Fitting the Belle data for the process $\Upsilon(5S) \rightarrow \Upsilon(1S)\pi^+\pi^-$ [87] yields $g_{f_2B^*B^*}g_{f_2\pi\pi} = 12.36 \pm 20.11$ [85]. As a consequence, the coupling $g_{f_2\mathcal{D}^*\mathcal{D}^*}$, if the masses of the b and c quarks both approach infinity, is 0.84 GeV. When the flavor SU(4) is an exact symmetry, the $g_{f_2\mathcal{D}^*\mathcal{D}^*}/g_{\chi_{c2}\mathcal{D}^*\mathcal{D}^*} = 1$ accordingly. Using the formulas in Ref. [63], we find

$g_{\chi_{c2}D^*D^*} \approx 11.21$ GeV, and, hence, $g_{f_2D^*D^*} \approx 11.21$ GeV under the SU(4) symmetry. Based on the two estimations above, we take a midpoint value $g_{f_2D^*D^*} = 6.0$ GeV in our calculations.

The effective four-body interactions among the charmonium J/ψ , charmed mesons, and the pions are described by the following Lagrangian [71,88,89]:

$$\begin{aligned} \mathcal{L}_{J\mathcal{D}D\pi} = & -ig_{J\mathcal{D}D\pi}\epsilon_{\mu\nu\alpha\beta}J^\mu\partial^\nu\mathcal{D}\partial^\alpha\pi\partial^\beta\mathcal{D}^\dagger \\ & + g_{J\mathcal{D}D^*\pi}J^\mu(\mathcal{D}\pi\mathcal{D}_\mu^{*\dagger} + \mathcal{D}_\mu^*\pi\mathcal{D}^\dagger) \\ & - ig_{J\mathcal{D}^*D^*\pi}\epsilon_{\mu\nu\alpha\beta}J^\mu\mathcal{D}^{*\nu}\partial^\alpha\pi\mathcal{D}^{*\dagger\beta} \\ & - ih_{J\mathcal{D}^*D^*\pi}\epsilon_{\mu\nu\alpha\beta}\partial^\mu J^\nu\mathcal{D}^{*\alpha}\pi\mathcal{D}^{*\dagger\beta}, \end{aligned} \quad (13)$$

where $\pi = \vec{\tau} \cdot \vec{\pi}$ with τ_i being the Pauli matrices and $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$. Following Ref. [88], we adopt the four-point coupling constants obtained under the framework of SU(4) symmetry:

$$g_{J\mathcal{D}D^*\pi} = 33.92, \quad (14a)$$

$$g_{J\mathcal{D}^*D^*\pi} = h_{J\mathcal{D}^*D^*\pi} = 38.19 \text{ GeV}^{-1}, \quad (14b)$$

$$g_{J\mathcal{D}D\pi} = 16.0 \text{ GeV}^{-3}. \quad (14c)$$

For the box diagrams, the general expression of the amplitudes can be described by

$$\mathcal{M}_{\text{box}} = \int \frac{d^4q}{(2\pi)^4} \frac{V_\psi V_J V_{\pi^+} V_{\pi^-}}{P_1 P_2 P_3 P} \mathcal{F}(q_3, M_3) \mathcal{F}(q, M). \quad (15)$$

For the triangle loops involving four-body interactions (hereafter referred as FB loops), the amplitudes have the following form:

$$\mathcal{M}_{\text{FB}} = \int \frac{d^4q}{(2\pi)^4} \frac{V_\psi V_J V_{\pi^+} V_{\pi^-}}{P_1 P_2 P} \mathcal{F}^2(q, M), \quad (16)$$

while for the other triangle loops with the mesons $f_0(500)$, $f_0(980)$, and $f_2(1270)$ (called $\mathcal{R}$ loops for short) the amplitudes can be written as

$$\mathcal{M}_{\mathcal{R}} = \int \frac{d^4q}{(2\pi)^4} \frac{V_\psi V_J V_{\mathcal{R}} V_{\pi\pi}}{P_1 P_2 P} \frac{\mathcal{N} \mathcal{F}^2(q, M)}{q_0^2 - M_{\mathcal{R}}^2 + iM_{\mathcal{R}}\Gamma_{\mathcal{R}}}. \quad (17)$$

In Eqs. (15)–(17), the V 's are the vertices related to the mesons specified by the index; the P_1 , P_2 , P_3 , and P stand for the propagators of the intermediate charmed mesons with momentum q_1 , q_2 , q_3 , and q , respectively (see Fig. 1); the width effect of the intermediate state $\mathcal{R}$ is considered. The numerator

$$\mathcal{N} = 1 \quad \text{for } f_0(500) \quad \text{and} \quad f_0(980),$$

$$\mathcal{N} = \frac{1}{2}(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta} + \bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}) - \frac{1}{3}\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} \quad \text{for } f_2(1270)$$

with $\bar{g}^{\mu\nu} = (g^{\mu\nu} - q_0^\mu q_0^\nu / M_{f_2}^2)$. Here, it should be pointed out that, for the triangle $\mathcal{R}$ loops, we introduce two parameters ϕ_1 and ϕ_2 to describe the phase difference of the $f_0(980)$ and $f_2(1270)$ with respect to the $f_0(500)$. That is to say, the amplitude $\mathcal{M}_{\mathcal{R}}$ due to the $\mathcal{R}$ loops in Figs. 1(o)–1(r) is of the form

$$\mathcal{M}_{\mathcal{R}} = \mathcal{M}_{f_0(500)} + e^{i\phi_1}\mathcal{M}_{f_0(980)} + e^{i\phi_2}\mathcal{M}_{f_2(1270)}. \quad (18)$$

To describe the off-shell effects of the intermediate charmed mesons and the structure effect of the interactions vertices, a monopole form factor is introduced [90–92]:

$$\mathcal{F}(q, M) = \frac{M^2 - \Lambda^2}{q^2 - \Lambda^2}. \quad (19)$$

Here, $\Lambda = M + \alpha\Lambda_{\text{QCD}}$ with $\Lambda_{\text{QCD}} = 0.22$ GeV. The parameter α depends on the given processes but is usually expected to be around unity [92].

According to the mixed wave function of the $Y(4500)$ in Eq. (1), the total amplitude of the decays for the $Y(4500) \rightarrow \pi^+\pi^-J/\psi$ is written as

$$\mathcal{M}_{\text{tot}} = -\mathcal{M}_S \sin\theta + \mathcal{M}_D \cos\theta, \quad (20)$$

where $\mathcal{M}_S$ and $\mathcal{M}_D$ are the amplitudes due to the pure $\psi(5S)$ and $\psi(4^3D_1)$ contributions, respectively, of which the proportion is described by the mixing angle θ . It is recalled that we adopt $\theta = 30^\circ$ in this work.

Once the amplitude $\mathcal{M}_{\text{tot}}$ is obtained, the differential partial decay width is then evaluated by [3]

$$\frac{d\Gamma}{dm_{\pi\pi}^2 dm_{J/\psi\pi}^2} = \frac{1}{3} \frac{1}{(2\pi)^3} \frac{1}{32m^3} \sum_{\text{spin}} |\mathcal{M}_{\text{tot}}|^2. \quad (21)$$

Here, the symbol $\sum_{\text{spin}}$ means the summation over the spins of the initial $Y(4500)$ and final J/ψ . Additionally, the $m_{\pi\pi}$ and $m_{J/\psi\pi}$ stand for the invariant mass of the two pions and of the J/ψ and π , respectively, while m is the $Y(4500)$ mass. The calculations were conducted with the assistance of the FeynCalc [93,94] and LoopTools [95] packages.

III. NUMERICAL RESULTS AND DISCUSSION

The following calculations are devoted to the analysis of the partial decay widths contributed from different kinds of Feynman diagrams. The calculations were performed with the coupling constants summarized in Table I. To obtain more theoretical information about the decay process $Y(4500) \rightarrow \pi^+\pi^-J/\psi$, we evaluated the invariant mass spectra of the $\pi^+\pi^-$ and π^+J/ψ via those three kinds of

TABLE I. The coupling constants used to evaluate the decay process $Y(4500) \rightarrow \pi^+\pi^- J/\psi$. The couplings for the four-body interactions are taken from Ref. [88], the couplings related to the $f_0(500)$ and $f_0(980)$ are from Ref. [84], and the estimations for the others based on experimental data or theoretical predictions are discussed in the text and the Appendix.

Coupling	Value	Coupling	Value
$g_{\psi DD}$	0.37	$g_{\psi_1 DD}$	-0.72
$g_{\psi DD^*}$	0.041 Ge V^{-1}	$g_{\psi_1 DD^*}$	0.065 Ge V^{-1}
$g_{\psi D^* D^*}$	0.38	$g_{\psi_1 D^* D^*}$	0.20
$g_{J DD}$	7.44	$g_{J DD\pi}$	16.0 Ge V^{-3}
$g_{J DD^*}$	2.49 Ge V^{-1}	$g_{J DD^* \pi}$	33.92
$g_{D^* D^*}$	8.00	$g(h)_{J D^* D^* \pi}$	38.19 Ge V^{-1}
$g_{f_0(500) DD}$	2.84 Ge V	$g_{f_0(980) DD}$	4.02 Ge V
$g_{f_0(500) D^* D^*}$	3.06 Ge V	$g_{f_0(980) D^* D^*}$	4.33 Ge V
$g_{f_0(500) \pi \pi}$	3.25 Ge V	$g_{f_0(980) \pi \pi}$	1.13 Ge V
$g_{f_2 D^* D^*}$	6.0 Ge V	$g_{f_2 \pi \pi}$	14.64 Ge V^{-1}
$g_{D^* D^* \pi}$	17.31	$g_{D^* D^* \pi}$	8.94 Ge V^{-1}

loops depicted in Fig. 1. These spectra can be directly compared with the future BESIII and Belle II experiments. In the absence of experimental data, we are incapable of limiting the superposition among these three kinds of loops. As a consequence, in the following, we first describe their contributions separately.

In Fig. 2, two-dimensional distributions of the invariant mass of the $\pi^+\pi^-$ and $\pi^+ J/\psi$ pairs due to the box loops [Figs. 1(a)–1(j)] are shown for three different model parameters $\alpha = 1.5, 2.5$, and 3.5 . Moreover, the corresponding projected line spectra are also attached marginally. Roughly speaking, the spectral pattern are nearly model (α) independent. The scattered events in the Dalitz plot are concentrated in the kinematic regions $m_{23} = 3.6\text{--}4 \text{ GeV}$ and $m_{12} = 1\text{--}1.4 \text{ GeV}$. As shown in Figs. 1(a)–1(j), the $J/\psi\pi$ final states originate from the $D^{(*)}\bar{D}^{(*)} \rightarrow J/\psi\pi$ scattering processes mediated by the t-channel charmed meson exchange. Therefore, in the range of $m_{23} = 3.6\text{--}4$,

i.e., the mass threshold of $D^{(*)}\bar{D}^{(*)}$, the enhancement of the interaction between $D^{(*)}\bar{D}^{(*)}$ and $J/\psi\pi$ leads to the event clustering. The partial decay width contributed from the box loops shows strong dependence on the α , which increases from 30 to 600 keV as α is varied from 1.5 to 5.0.

Next, in Fig. 3, the results contributed from the FB loops [Figs. 1(k)–1(n)] are compared for the three model parameters $\alpha = 1.5, 2.5$, and 3.5 . Here, the selection of α 's is the same as that for the box loop case so that the calculated data shown before in Fig. 2 can be included in the comparison. It is seen that the invariant mass of the $\pi^+\pi^-$ is nearly independent of the model parameter. However, the $\pi^+ J/\psi$ mass distribution exhibits a noticeable change with the model parameter. In the first graph in Fig. 3, two clear band distributions are shown at $\alpha = 1.5$. The horizontal band is located at $m_{23} = 4.02 \text{ GeV}$ with a significant peak in the m_{23} invariant mass spectra, while the oblique band appears near $m_{23} = 3.6 \text{ GeV}$ as a lower broad bump. Similar to the analysis of $Y(4260) \rightarrow J/\psi\pi\pi$ in Ref. [38], the oblique band and lower bump originate from the interference between the $J/\psi\pi^+$ and $J/\psi\pi^-$ rescattering from $D^*\bar{D}^*$ and are a reflection of the narrow structure at 3.9 GeV . These two structures in the mass spectrum of the $\pi^+ J/\psi$ invariant mass might correspond to the exotic states $Z_c(3900)$ and $Z_c(4020)$ [3]. From Fig. 3, these two structures gradually diminish with increasing α and manifest as two narrow peaks at $\alpha = 3.5$. It is noticed that the shape of the $m_{\pi^+ J/\psi}$ distribution at $\alpha = 1.5$ is similar to the experimental data of the $Y(4230) \rightarrow \pi^+\pi^- J/\psi$ [96]. Additionally, the invariant mass spectrum of the $\pi^+ J/\psi$ obtained by the present model is similar to that through the initial single chiral particle emission mechanism proposed by Chen and Liu [71,72,97]. The partial decay width through this kind of loops is about 65 keV at $\alpha = 1.5$ and can reach up to 960 keV at $\alpha = 5.0$.

Figure 4 shows the invariant mass distributions obtained by means of the $\mathcal{R}$ diagrams [namely, Figs. 1(o)–1(r)]. In this case, the spectra of the $\pi^+\pi^-$ and $\pi^+ J/\psi$ are found to

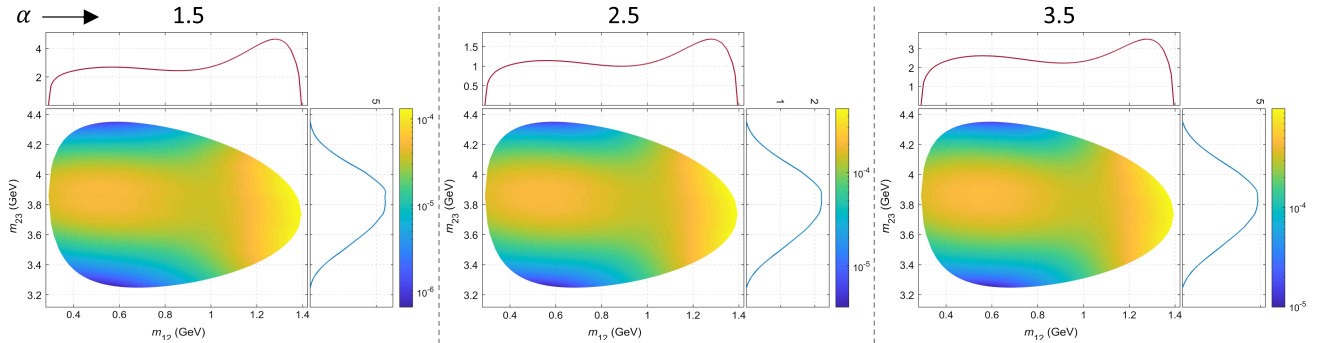


FIG. 2. The Dalitz plots and distributions of the invariant mass of the $\pi^+\pi^-$ (m_{12}) and $\pi^+ J/\psi$ (m_{23}) due to the box diagrams in Figs. 1(a)–1(j) at $\alpha = 1.5, 2.5$, and 3.5 . The solid lines are the corresponding spectra projected onto the m_{12} (red) and m_{23} axis, namely, $(d\Gamma/dm_{12})$ and $(d\Gamma/dm_{23})$. For graphical reasons, the projected spectra for $\alpha = 1.5$ are increased by 10^5 , while for $\alpha = 2.5$ and 3.5 they are increased by 10^4 . The color bars are not the same.

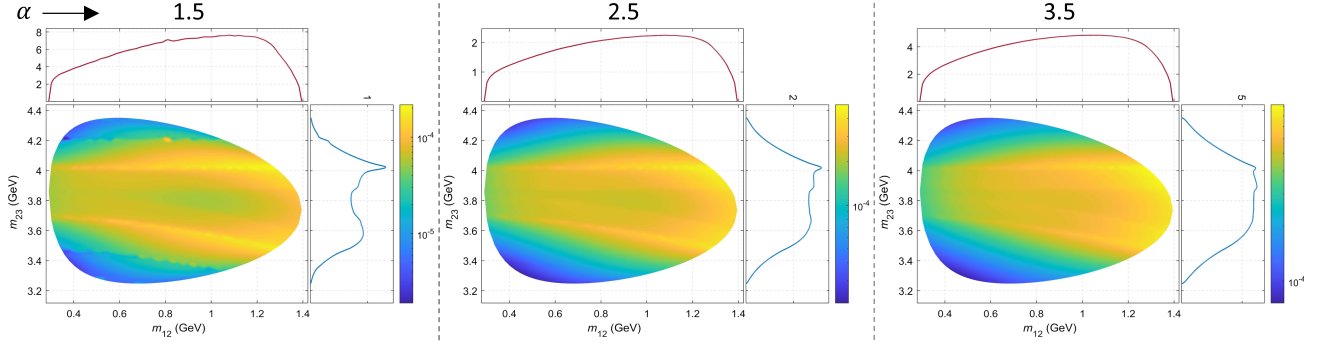


FIG. 3. As in Fig. 2, but these data are obtained using the triangle loops shown in Figs. 1(k)–1(n). For graphical reasons, $(d\Gamma/dm_{12}) \times 10^5$ and $(d\Gamma/dm_{23}) \times 10^4$ for $\alpha = 1.5$, while $(d\Gamma/dm_{12}) \times 10^4$ and $(d\Gamma/dm_{23}) \times 10^4$ for $\alpha = 2.5$ and 3.5 .

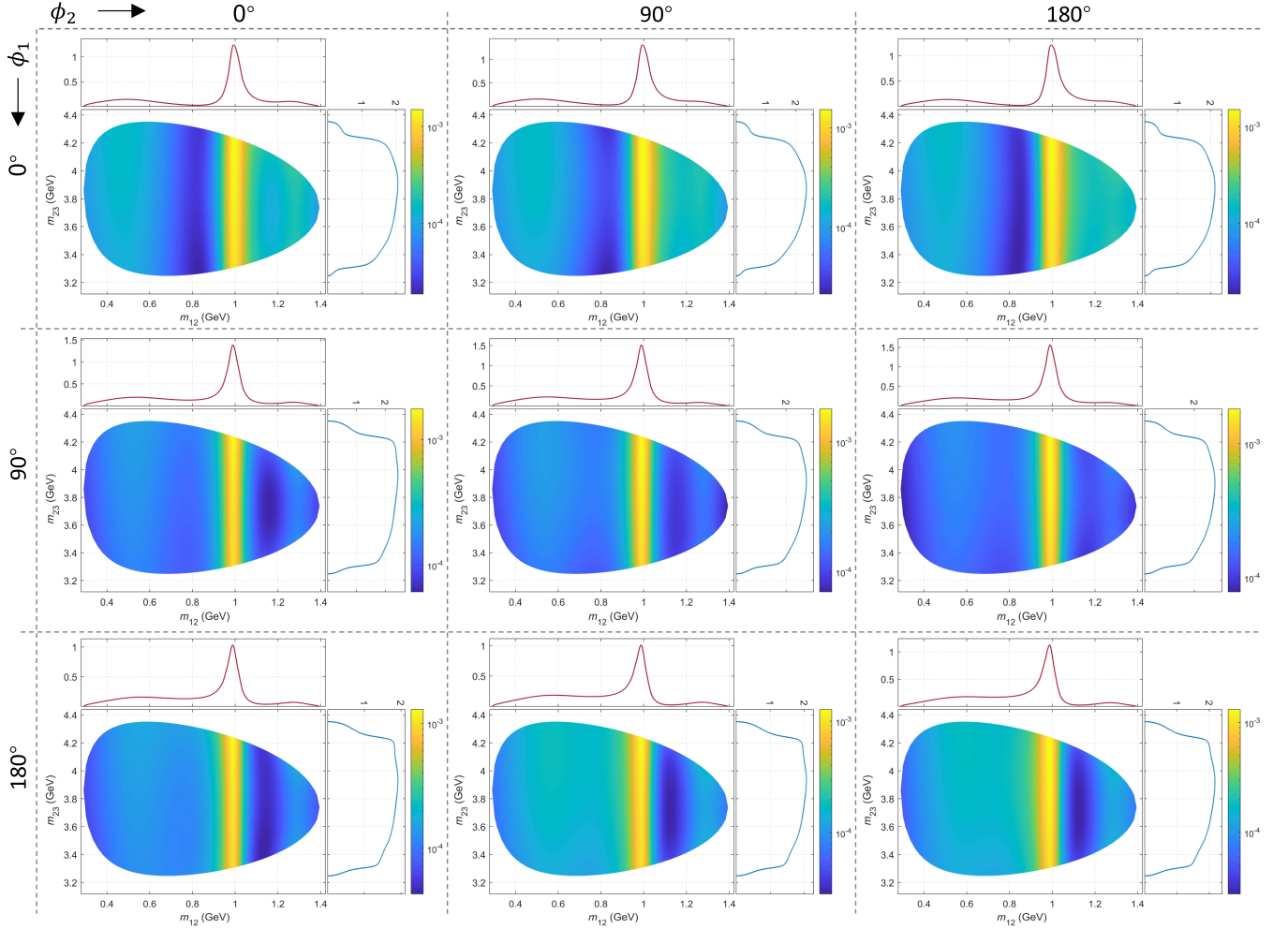


FIG. 4. Distributions of the invariant mass of the $\pi^+\pi^-$ (m_{12}) and π^+J/ψ (m_{23}) for different combinations of the ϕ_1 and ϕ_2 (in units of degree) at $\alpha = 3.5$, obtained using the triangle $\mathcal{R}$ loops (o)–(r) in Fig. 1. The solid lines are the corresponding spectra projected onto the m_{12} (red) and m_{23} axis, namely, $(d\Gamma/dm_{12}) \times 10^3$ and $(d\Gamma/dm_{23}) \times 10^4$. The color bars for different combinations are not the same.

be completely independent of the model parameter α . Hence, we take the results at $\alpha = 3.5$ for illustration. To indicate the influence of the phases ϕ_1 and ϕ_2 [defined in Eq. (18)], we choose three special values of 0° , 90° and

180° . On the whole, the spectra are weakly dependent on the phase angles ϕ_1 and ϕ_2 . This finding indicates that the interference effect among the $f_0(500)$, $f_0(980)$, and $f_2(1270)$ contributions, if their relevant parameters are

TABLE II. Partial decay widths (in units of keV) due to the $\mathcal{R}$ loops for different combinations of phases ϕ_1 and ϕ_2 . Here, the model parameters $\alpha = 1.5, 2.5$, and 3.5 are chosen.

	$\phi_2 = 0^\circ$			$\phi_2 = 90^\circ$			$\phi_2 = 180^\circ$		
	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 3.5$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 3.5$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 3.5$
$\phi_1 = 0^\circ$	10.7	60.7	175.4	12.0	67.7	195.7	11.1	62.3	180.0
$\phi_1 = 90^\circ$	13.6	76.5	221.3	15.3	86.1	249.2	15.3	86.3	250.0
$\phi_1 = 180^\circ$	10.8	60.7	175.6	11.5	64.8	187.6	12.0	67.6	195.6

those we use here, is of minor importance. The Dalitz plots exhibit a distinct band at $m_{12} = 0.98$ GeV, regardless of the ϕ_1 and ϕ_2 values. Meanwhile, a significant peak appears at 0.98 GeV in the $\pi^+\pi^-$ invariant mass spectra. Given that the $Y(4500)$ is regarded as a mixed state of the $\psi(5S)$ and $\psi(4^3D_1)$, with a dominant D -wave component, the charmed meson loop contributions via the $f_0(500/980)$ are strongly suppressed in the heavy quark limit. This suppression arises from the fact that the angular momenta of the $\psi(4^3D_1)$ and J/ψ are $L = 2$ and $L = 0$, respectively. Similarly, the $\psi(5S)$ state in the wave function of the $Y(4500)$ also suppresses the charmed meson loop contributions via the $f_2(1270)$. One should notice that the broad widths of the $f_0(500)/f_2(1270)$ amplify the suppression of associated loop contributions and result in two small bumps around $m_{12} = 0.5$ and 1.27 GeV in the line shape. In addition, for both charmonium [61,98] and bottomonium [99,100] dipion transitions, the dipion systems are predominantly S -wave dominated, highlighting the essential role of scalar resonances in these processes. As a result, only a single peak or band located at $m_{12} = 0.98$ GeV is found for the triangle $\mathcal{R}$ loops, corresponding to the $f_0(980)$. At $\alpha = 3.5$, the $\mathcal{R}$ loop mechanism predicts the partial decay width to be of the order of 200 keV, reflecting the suppression effect discussed above. In Table II, the partial decay widths due to the $\mathcal{R}$ loops for different combinations of phases ϕ_1 and ϕ_2 are listed. Despite the α independence of the invariant mass distributions, the partial width depends strongly on the model parameter α . According to the calculated results, it can be concluded that the phase influence in the case of $\mathcal{R}$ loops is of minor importance.

From the invariant mass distributions in Figs. 2–4, we found that the FB and box diagrams play a role as background terms with respect to the $\mathcal{R}$ loops diagrams for the $\pi\pi$. As for $J/\psi\pi$, the $\mathcal{R}$ loops and box diagrams play a role as background terms with respect to the FB diagrams. The predicted line shapes are helpful for experimentally searching the decay $Y(4500) \rightarrow J/\psi\pi\pi$. In Fig. 5, we plot the partial decay widths contributed from the three kinds of loops as a function of the model parameter α . Although patterns of the invariant mass distributions are hardly dependent on the model parameter α as mentioned above, the partial decay widths increase distinctly with increasing α . It is seen that, under the present conditions we adopted,

the FB loops show the most importance among the three kinds of loops. In the range $\alpha \in [1.5, 5]$, the decay widths through the FB, box, and $\mathcal{R}$ loops are, respectively,

$$\Gamma_{\text{FB}} = (64.6\text{--}960.2) \text{ keV}, \quad (22a)$$

$$\Gamma_{\text{box}} = (32.2\text{--}591.1) \text{ keV}, \quad (22b)$$

$$\Gamma_{\mathcal{R}} = (10.8\text{--}499.4) \text{ keV}. \quad (22c)$$

To characterize the interference effects among the three kinds of loop contributions, we introduce two other phase angles, χ_1 and χ_2 . Consequently, the total contribution to the $Y(4500) \rightarrow \pi^+\pi^-J/\psi$ from the three kinds of loop diagrams shown in Fig. 1 could be expressed as

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_{\text{box}} + e^{i\chi_1}\mathcal{M}_{\text{FB}} + e^{i\chi_2}\mathcal{M}_{\mathcal{R}}. \quad (23)$$

In the absence of experimental results, the χ_1 and χ_2 remain completely free. Notice that the patterns shown in

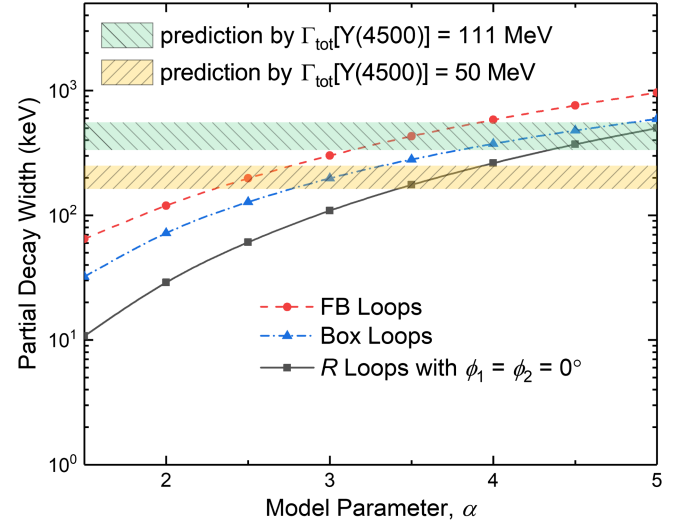


FIG. 5. Partial decay width as a function of the model parameter α . The closed circles, triangles, and squares represent results contributed the FB, box, and $\mathcal{R}$ loops, respectively. The calculations for $\mathcal{R}$ loops were conducted at phases $\phi_1 = \phi_2 = 0^\circ$. The two bands depict the estimations using the BESIII experiments [58] and the theoretical predictions for the $Y(4500) \rightarrow K^+K^-J/\psi$ [57] (see the text).

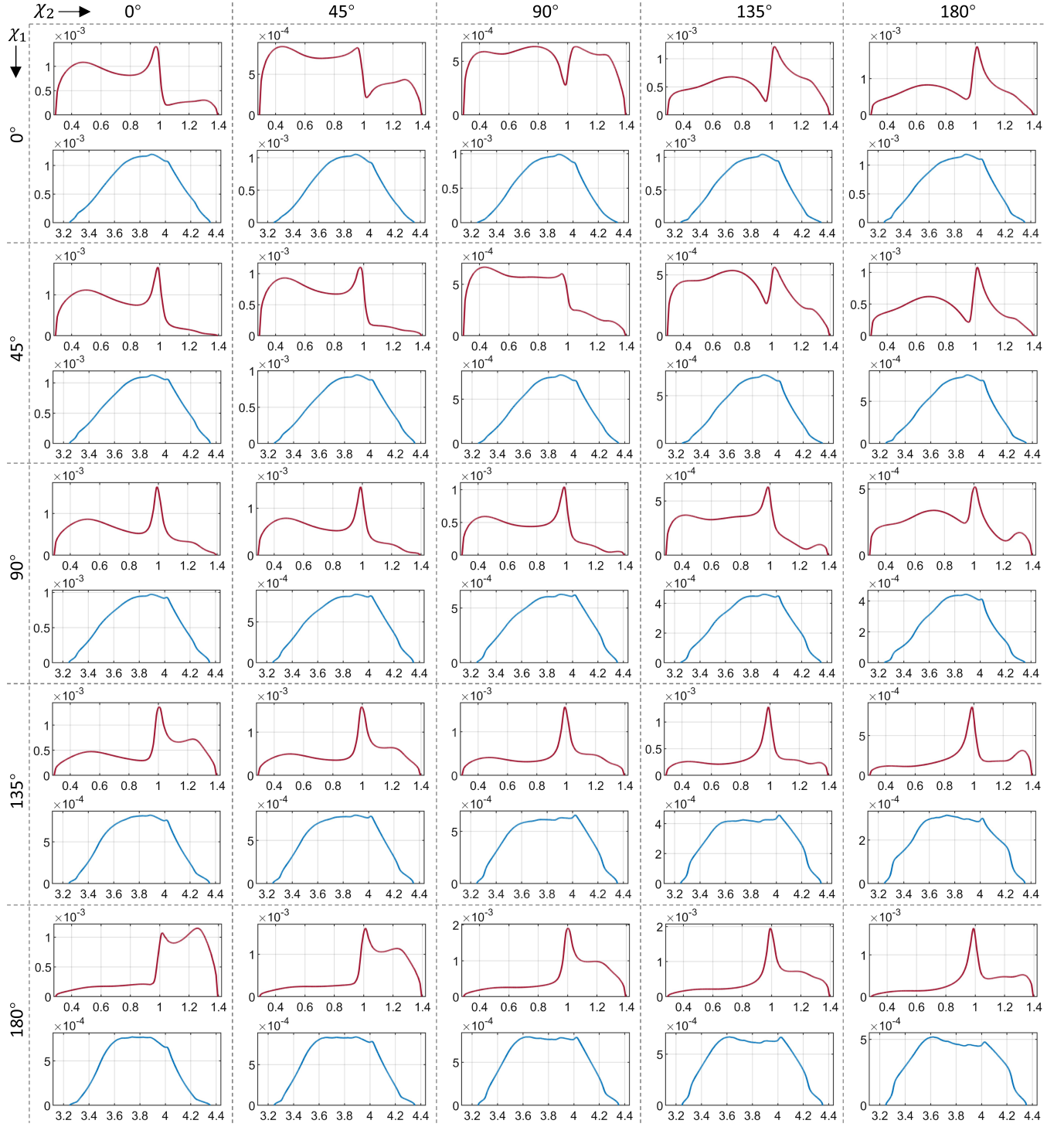


FIG. 6. Distributions of the invariant mass of the $\pi^+\pi^-$ (m_{12}) and π^+J/ψ (m_{23}) due to the three kinds of loop Feynman diagrams depicted in Fig. 1. The calculations were performed using $\alpha = 3.0$ and $\phi_1 = \phi_2 = 0^\circ$. The phase angles χ_1 and χ_2 were varied from 0° to 180° as indicated. The solid red line represents the $d\Gamma/dm_{12}$, and the blue line is the $d\Gamma/dm_{23}$. For graphical reasons, the x - and y -axis labels are omitted.

Figs. 2–4 are nearly independent of the model parameter α . Moreover, the phase angles ϕ_1 and ϕ_2 [see Eq. (18)], describing the interference of the $\mathcal{R}$ loops, also have little effect on the decay process (see Fig. 4 and Table II). Hence,

these calculations were conducted using $\alpha = 3.0$ and $\phi_1 = \phi_2 = 0^\circ$. In Fig. 6, the invariant mass distributions of the $\pi^+\pi^-$ and π^+J/ψ are presented for the χ_1 and χ_2 in the range of 0° – 180° . It is seen that the pattern of the $\pi^+\pi^-$

invariant mass distributions exhibits strong dependence on the χ_1 and χ_2 . In contrast, for the π^+J/ψ system, the invariant-mass-distribution pattern shows a rather small sensitivity to the phase angle χ_2 . Moreover, it varies marginally with the χ_1 when compared to that of the $\pi^+\pi^-$. The significant dependence of the $\pi^+\pi^-$ invariant mass distribution on the angles χ_1 and χ_2 is understandable, because the $\pi^+\pi^-$ invariant mass intensities (namely, the $d\Gamma/dm_{12}$ shown in Figs. 2 and 3) for the box and FB loops have nearly the same order of magnitude as the peak intensity near the $f_0(980)$ for the $\mathcal{R}$ loops, thereby leading to obvious interference effect.

We plot in Fig. 7 the partial decay width of the $Y(4500) \rightarrow \pi^+\pi^-J/\psi$ as a function of the χ_1 and χ_2 . As seen, the width value varies significantly with the χ_1 and χ_2 . Given the dependence of the partial decay widths on the model parameter α , as illustrated in Fig. 5, the foregoing predicted width ranges alone are insufficient to constrain the phase angles χ_1 and χ_2 . A precise measurement of the invariant mass distributions of the $\pi^+\pi^-$ and $\pi J/\psi$ would enable the determination of χ_1 and χ_2 , thereby providing constraints on the model parameter α .

If the $Y(4500)$ is the only contribution to the $\pi^+\pi^-J/\psi$ and K^+K^-J/ψ , the BESIII experiments [58] imply $\Gamma[Y(4500) \rightarrow K^+K^-J/\psi]/\Gamma[Y(4500) \rightarrow \pi^+\pi^-J/\psi] \approx 0.4$. In view of the theoretical prediction in Ref. [57] that the $\Gamma[Y(4500) \rightarrow K^+K^-J/\psi] \approx (65\text{--}100)$ keV when $\Gamma_{\text{tot}}[Y(4500)] = 50$ MeV is adopted, the partial width $\Gamma[Y(4500) \rightarrow \pi^+\pi^-J/\psi] \approx (162.5\text{--}250)$ keV. However, if the experimental center value $\Gamma_{\text{tot}}[Y(4500)] = 111$ MeV is adopted, the corresponding decay width would be increased by approximately a factor of 2. These two estimations are indicated in Fig. 5 as the two bands. Unfortunately, without any experimental constraints, the parameters χ_1 and χ_2 are entirely free. Nevertheless, the

predicted partial width of $Y(4500) \rightarrow \pi^+\pi^-J/\psi$, varying from 250 to 700 keV with $\alpha = 3.0$ and $\phi_1 = \phi_2 = 0^\circ$, is consistent with the estimation obtained by combining the measurement by BESIII Collaboration and theoretical prediction for the $Y(4500) \rightarrow K^+K^-J/\psi$.

IV. SUMMARY

The dipionic transition of the newly observed charmoniumlike resonance $Y(4500)$ by the BESIII Collaboration to the J/ψ was investigated using an effective Lagrangian approach. This work was motivated by BESIII observations that the Born cross section of the $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ exhibits a small enhancement around 4.5 GeV [24], hinting the well-constructed $Y(4500)$ in the $e^+e^- \rightarrow K^+K^-J/\psi$ [23]. In this study, the transition was assumed to occur via charmed meson loops, including box loops and two kinds of triangle loops. Similar to the treatment by Wang and Liu [57], the $Y(4500)$ was considered as a mixture of the $\psi(5S)$ and $\psi(4^3D_1)$.

Our calculations indicate that the patterns of the invariant mass distributions of $\pi^+\pi^-$ and π^+J/ψ are hardly dependent on the model, especially for those contributed from the box and $\mathcal{R}$ loops. Different loop mechanisms give different invariant mass spectra. The interference among the three different kinds of loops is also exhibited. We could determine the decay mechanism of the process $Y(4500) \rightarrow \pi^+\pi^-J/\psi$ by comparing the present calculated spectra with the future experimental measurements by BESIII or Belle II. The partial decay widths are found to be model dependent. Combining the BESIII experiments [23] and the theoretical predictions [57], the width of $Y(4500) \rightarrow \pi^+\pi^-J/\psi$ was roughly estimated to between 160 and 250 keV or from 360 to 550 keV. The different ranges arise from different total widths of the $Y(4500)$. Our results agree with the combined predictions. We hope that our present calculations would be tested by the future BESIII or Belle II experiments.

ACKNOWLEDGMENTS

We thank Xiao-Hai Liu at Tianjin University for some helpful discussion. Particular thanks are devoted to Jun-Zhang Wang at Peking University for some helpful information about the work [57]. This work is partly supported by the National Natural Science Foundation of China under Grants No. 12475081, No. 12105153, and No. 12405093 and by the Natural Science Foundation of Shandong Province under Grants No. ZR2025 MS04, No. ZR2021 MA082, and No. ZR2022ZD26. It is also supported by Taishan Scholar Project of Shandong Province (Grant No. tsqn202103062) and partly by National Key Research and Development Program under Grant No. 2024YFA1610504.

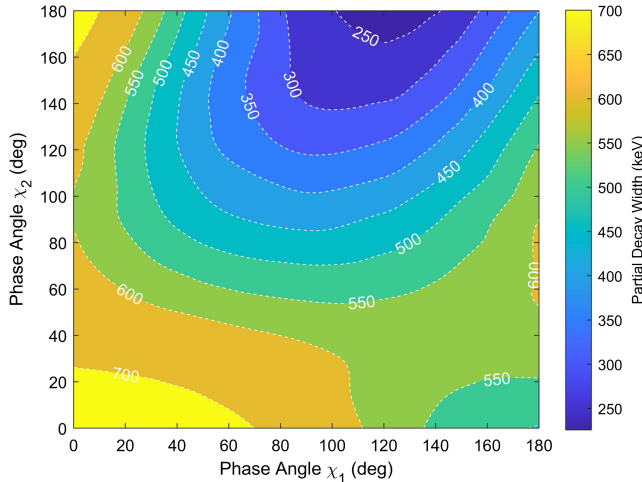


FIG. 7. Partial decay width due to the three kinds of the loops in Fig. 1. The $\alpha = 3.0$ and $\phi_1 = \phi_2 = 0^\circ$ were used.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: ESTIMATE OF COUPLING CONSTANTS

In this appendix, we perform the estimations of the coupling constants for the $Y(4500)$. As noted, we treat the $Y(4500)$ as the mixed state of the $\psi(5S)$ and $\psi(4D)$. According to the mixing scheme [5],

$$\begin{pmatrix} \tilde{\psi}(4415) \\ \tilde{Y}(4500) \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \tilde{\psi}(5S) \\ \tilde{\psi}(4^3D_1) \end{pmatrix}. \quad (\text{A1})$$

To reasonably estimate the couplings, we could fit the partial width of the $Y(4500)$ and $\psi(4415)$ decaying into the open charmed meson pairs by the Lagrangians in Eq. (2). Using the theoretical widths predicted in Ref. [5], we give

$$|g_{\psi DD} \sin \theta + g_{\psi_1 DD} \cos \theta| = 0.437, \quad (\text{A2a})$$

$$|g_{\psi DD} \cos \theta - g_{\psi_1 DD} \sin \theta| = 0.678, \quad (\text{A2b})$$

$$|g_{\psi DD^*} \sin \theta + g_{\psi_1 DD^*} \cos \theta| = 7.63 \times 10^{-2}, \quad (\text{A2c})$$

$$|g_{\psi DD^*} \cos \theta - g_{\psi_1 DD^*} \sin \theta| = 3.13 \times 10^{-3}, \quad (\text{A2d})$$

$$0.162 g_{\psi DD^*}^2 \sin^2 \theta - 0.382 g_{\psi DD^*} g_{\psi_1 DD^*} \sin \theta \cos \theta + g_{\psi_1 DD^*}^2 \cos^2 \theta = 2.38 \times 10^{-2}, \quad (\text{A2e})$$

$$g_{\psi DD^*}^2 \cos^2 \theta + 2.44 g_{\psi DD^*} g_{\psi_1 DD^*} \sin \theta \cos \theta + 6.06 g_{\psi_1 DD^*}^2 \sin^2 \theta = 0.251. \quad (\text{A2f})$$

It is expected that the couplings $g_{\psi D^{(*)} D^{(*)}}$ for the $\psi(5S)$ with the open charmed mesons are subject to the relations in Eq. (3). Moreover, we assume the couplings $g_{\psi D^{(*)} D^{(*)}}$ to be positive. Hence, solving Eq. (A2) with $\theta = 30^\circ$ gives

$$g_{\psi DD} = 0.37, \quad (\text{A3a})$$

$$g_{\psi DD^*} = 4.10 \times 10^{-2} \text{ GeV}^{-1}, \quad (\text{A3b})$$

$$g_{\psi D^* D^*} = 0.38, \quad (\text{A3c})$$

$$g_{\psi_1 DD} = -0.72, \quad (\text{A3d})$$

$$g_{\psi_1 DD^*} = 6.50 \times 10^{-2} \text{ GeV}^{-1}, \quad (\text{A3e})$$

$$g_{\psi_1 D^* D^*} = 0.20. \quad (\text{A3f})$$

-
- [1] Z. Liu and R. E. Mitchell, New hadrons discovered at BESIII, *Sci. Bull.* **68**, 2148 (2023).
 - [2] T. Gershon (LHCb Collaboration), Exotic hadron naming convention, [arXiv:2206.15233](https://arxiv.org/abs/2206.15233).
 - [3] S. Navas *et al.* (Particle Data Group), Review of particle physics, *Phys. Rev. D* **110**, 030001 (2024).
 - [4] X. H. Mo, G. Li, C. Z. Yuan, K. L. He, H. M. Hu, J. H. Hu, P. Wang, and Z. Y. Wang, Determining the upper limit of Γ_{ee} for the $Y(4260)$, *Phys. Lett. B* **640**, 182 (2006).
 - [5] J.-Z. Wang, D.-Y. Chen, X. Liu, and T. Matsuki, Constructing J/ψ family with updated data of charmoniumlike Y states, *Phys. Rev. D* **99**, 114003 (2019).
 - [6] J.-Z. Wang, Z.-F. Sun, X. Liu, and T. Matsuki, Higher bottomonium zoo, *Eur. Phys. J. C* **78**, 915 (2018).
 - [7] M. Ablikim *et al.* (BESIII Collaboration), Precise measurement of the $e^+e^- \rightarrow \pi^+\pi^- J/\psi$ cross section at center-of-mass energies from 3.77 to 4.60 GeV, *Phys. Rev. Lett.* **118**, 092001 (2017).
 - [8] J.-H. Chen, J. Chen, F.-K. Guo, Y.-G. Ma, C.-P. Shen, Q.-Y. Shou, Q. Shou, Q. Wang, J.-J. Wu, and B.-S. Zou, Production of exotic hadrons in pp and nuclear collisions, *Nucl. Sci. Tech.* **36**, 55 (2025).
 - [9] M.-Z. Liu, Y.-W. Pan, Z.-W. Liu, T.-W. Wu, J.-X. Lu, and L.-S. Geng, Three ways to decipher the nature of exotic hadrons: Multiplets, three-body hadronic molecules, and correlation functions, *Phys. Rep.* **1108**, 1 (2025).
 - [10] H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, The hidden-charm pentaquark and tetraquark states, *Phys. Rep.* **639**, 1 (2016).
 - [11] H.-X. Chen, W. Chen, X. Liu, Y.-R. Liu, and S.-L. Zhu, An updated review of the new hadron states, *Rep. Prog. Phys.* **86**, 026201 (2022).
 - [12] L. Meng, B. Wang, G.-J. Wang, and S.-L. Zhu, Chiral perturbation theory for heavy hadrons and chiral effective field theory for heavy hadronic molecules, *Phys. Rep.* **1019**, 1 (2023).
 - [13] M. Ablikim *et al.* (BESIII Collaboration), Future physics programme of BESIII, *Chin. Phys. C* **44**, 040001 (2020).
 - [14] F.-K. Guo, X.-H. Liu, and S. Sakai, Threshold cusps and triangle singularities in hadronic reactions, *Prog. Part. Nucl. Phys.* **112**, 103757 (2020).
 - [15] N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C.-P. Shen, C. E. Thomas, A. Vairo, and C.-Z. Yuan, The xyz states: Experimental and theoretical status and perspectives, *Phys. Rep.* **873**, 1 (2020).

- [16] Y.-R. Liu, H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, Pentaquark and tetraquark states, *Prog. Part. Nucl. Phys.* **107**, 237 (2019).
- [17] F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, and B.-S. Zou, Hadronic molecules, *Rev. Mod. Phys.* **90**, 015004 (2018).
- [18] X. Liu, An overview of XYZ new particles, *Chin. Sci. Bull.* **59**, 3815 (2014).
- [19] Z.-G. Wang, Review of the QCD sum rules for exotic states, *Front. Phys. (Beijing)* **21**, 016300 (2025).
- [20] B. Aubert *et al.* (BABAR Collaboration), Observation of a broad structure in the $\pi^+\pi^-J/\psi$ mass spectrum around 4.26-GeV/ c^2 , *Phys. Rev. Lett.* **95**, 142001 (2005).
- [21] T. Coan, Y. Gao, F. Liu *et al.* (CLEO Collaboration), Charmonium decays of $Y(4260)$, $\psi(4160)$, and $\psi(4040)$, *Phys. Rev. Lett.* **96**, 162003 (2006).
- [22] C. Z. Yuan *et al.* (Belle Collaboration), Measurement of $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ cross-section via initial state radiation at Belle, *Phys. Rev. Lett.* **99**, 182004 (2007).
- [23] M. Ablikim *et al.* (BESIII Collaboration), Observation of the $Y(4230)$ and a new structure in $e^+e^- \rightarrow K^+K^-J/\psi$, *Chin. Phys. C* **46**, 111002 (2022).
- [24] M. Ablikim, M. N. Achasov, P. A. Adlarson *et al.* (BESIII Collaboration), Study of the resonance structures in $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ process, *Phys. Rev. D* **106**, 072001 (2022).
- [25] M. Ablikim, M. Achasov, P. Adlarson *et al.* (BESIII Collaboration), Observation of three charmoniumlike states with $J^{PC} = 1^{--}$ in $e^+e^- \rightarrow D^{*0}D^{*-}\pi^+$, *Phys. Rev. Lett.* **130**, 121901 (2023).
- [26] S.-L. Zhu, The possible interpretations of $Y(4260)$, *Phys. Lett. B* **625**, 212 (2005).
- [27] S. Dubničká, A. Dubničková, A. Issadykov, M. Ivanov, and A. Liptaj, $Y(4260)$ as a four-quark state, *Phys. Rev. D* **101**, 094030 (2020).
- [28] Z.-G. Wang, Analysis of the vector tetraquark states with P-waves between the diquarks and antidiquarks via the QCD sum rules, *Eur. Phys. J. C* **79**, 29 (2019).
- [29] Z.-G. Wang, Lowest vector tetraquark states: $Y(4260/4220)$ or $Z_c(4100)$, *Eur. Phys. J. C* **78**, 933 (2018).
- [30] T.-W. Chiu and T.-H. Hsieh (TWQCD Collaboration), $Y(4260)$ on the lattice, *Phys. Rev. D* **73**, 094510 (2006).
- [31] C. Z. Yuan, P. Wang, and X. H. Mo, The $Y(4260)$ as an $\omega\chi_{c1}$ molecular state, *Phys. Lett. B* **634**, 399 (2006).
- [32] R. M. Albuquerque and M. Nielsen, QCD sum rules study of the $J^{PC} = 1^{--}$ charmonium Y mesons, *Nucl. Phys.* **A815**, 53 (2009).
- [33] G.-J. Ding, Are $Y(4260)$ and $Z_2^+(4250)D_1D$ or D_0D^* hadronic molecules?, *Phys. Rev. D* **79**, 014001 (2009).
- [34] F. Close, C. Downum, and C. E. Thomas, Novel charmonium and bottomonium spectroscopies due to deeply bound hadronic molecules from single pion exchange, *Phys. Rev. D* **81**, 074033 (2010).
- [35] F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, and Q. Zhao, Production of the $X(3872)$ in charmonia radiative decays, *Phys. Lett. B* **725**, 127 (2013).
- [36] G. Li and X.-H. Liu, Investigating possible decay modes of $Y(4260)$ under $D_1(2420)\bar{D} + \text{c.c.}$ molecular state ansatz, *Phys. Rev. D* **88**, 094008 (2013).
- [37] X.-H. Liu and G. Li, Exploring the threshold behavior and implications on the nature of $Y(4260)$ and $Z_c(3900)$, *Phys. Rev. D* **88**, 014013 (2013).
- [38] Q. Wang, C. Hanhart, and Q. Zhao, Decoding the riddle of $Y(4260)$ and $Z_c(3900)$, *Phys. Rev. Lett.* **111**, 132003 (2013).
- [39] M. Cleven, Q. Wang, F.-K. Guo, C. Hanhart, U.-G. Meißner, and Q. Zhao, $Y(4260)$ as the first S -wave open charm vector molecular state?, *Phys. Rev. D* **90**, 074039 (2014).
- [40] F.-Z. Peng, M.-J. Yan, M. S. Sánchez, and M. P. Valderrama, Light- and heavy-quark symmetries and the $Y(4230)$, $Y(4360)$, $Y(4500)$, $Y(4620)$ and $X(4630)$ resonances, *Phys. Rev. D* **107**, 016001 (2023).
- [41] X.-K. Dong, Y.-H. Lin, and B.-S. Zou, Prediction of an exotic state around 4240 MeV with $J^{PC} = 1^{-+}$ as C-parity partner of $Y(4260)$ in molecular picture, *Phys. Rev. D* **101**, 076003 (2020).
- [42] M. Cleven and Q. Zhao, Cross section line shape of $e^+e^- \rightarrow \chi_{c0}\omega$ around the $Y(4260)$ mass region, *Phys. Lett. B* **768**, 52 (2017).
- [43] G. Li, C.-S. An, P.-Y. Li, D. Liu, X. Zhang, and Z. Zhou, Investigations on the charmless decays of $Y(4260)$, *Chin. Phys. C* **39**, 063102 (2015).
- [44] C.-F. Qiao, One explanation for the exotic state $Y(4260)$, *Phys. Lett. B* **639**, 263 (2006).
- [45] Y.-H. Chen, L.-Y. Dai, F.-K. Guo, and B. Kubis, Nature of the $Y(4260)$: A light-quark perspective, *Phys. Rev. D* **99**, 074016 (2019).
- [46] H.-f. Fu and L. Jiang, Coupled-channel-induced $S - D$ mixing of charmonia and testing possible assignments for $Y(4260)$ and $Y(4360)$, *Eur. Phys. J. C* **79**, 460 (2019).
- [47] D.-Y. Chen, J. He, and X. Liu, Nonresonant explanation for the $Y(4260)$ structure observed in the $e^+e^- \rightarrow J/\psi\pi^+\pi^-$ process, *Phys. Rev. D* **83**, 054021 (2011).
- [48] X.-H. Liu, Influence of threshold effects induced by charmed meson rescattering, *Phys. Rev. D* **90**, 074004 (2014).
- [49] M. Ablikim, M. Achasov, P. Adlarson *et al.* (BESIII Collaboration), Observation of a Vector Charmoniumlike State at 4.7 GeV/ c^2 and Search for Z_{cs} in $e^+e^- \rightarrow K^+K^-J/\psi$, *Phys. Rev. Lett.* **131**, 211902 (2023).
- [50] M. Ablikim *et al.* (BESIII Collaboration), Precise measurement of the $e^+e^- \rightarrow D_s^{*+}D_s^{*-}$ cross sections at center-of-mass energies from threshold to 4.95 GeV, *Phys. Rev. Lett.* **131**, 151903 (2023).
- [51] M. Ablikim, M. N. Achasov, P. A. Adlarson *et al.* (BESIII Collaboration), Observation of structures in the processes $e^+e^- \rightarrow \omega\chi_{c1}$ and $\omega\chi_{c2}$, *Phys. Rev. Lett.* **132**, 161901 (2024).
- [52] M. Ablikim *et al.* (BESIII Collaboration), Observation of three resonant structures in the cross section of $e^+e^- \rightarrow \pi^+\pi^-h_c$, *Phys. Rev. Lett.* **135**, 071901 (2025).
- [53] Q. Deng, R.-H. Ni, Q. Li, and X.-H. Zhong, Charmonia in an unquenched quark model, *Phys. Rev. D* **110**, 056034 (2024).
- [54] X.-K. Dong, F.-K. Guo, and B.-S. Zou, A survey of heavy-antiheavy hadronic molecules, *Progr. Phys.* **41**, 65 (2021).
- [55] Z.-G. Wang, Analysis of the decay $Y(4500) \rightarrow D^*D^-\pi$ with the light-cone QCD sum rules, *Nucl. Phys.* **B993**, 116265 (2023).

- [56] Z.-G. Wang, Strong decays of the vector tetraquark states with the masses about 4.5 GeV via the QCD sum rules, *Nucl. Phys.* **B1005**, 116580 (2024).
- [57] J.-Z. Wang and X. Liu, Confirming the existence of a new higher charmonium $\psi(4500)$ by the newly released data of $e^+e^- \rightarrow K^+K^-J/\psi$, *Phys. Rev. D* **107**, 054016 (2023).
- [58] M. Ablikim, M. N. Achasov, S. A. N. Ahmed *et al.* (BESIII Collaboration), Measurement of $e^+e^- \rightarrow K\bar{K}J/\psi$ cross sections at center-of-mass energies from 4.189 to 4.600 GeV, *Phys. Rev. D* **97**, 071101 (2018).
- [59] T.-M. Yan, Hadronic transitions between heavy quark states in quantum chromodynamics, *Phys. Rev. D* **22**, 1652 (1980).
- [60] Y.-P. Kuang and T.-M. Yan, Predictions for hadronic transitions in the $b\bar{b}$ system, *Phys. Rev. D* **24**, 2874 (1981).
- [61] B. Wang, H. Xu, X. Liu, D.-Y. Chen, S. Coito, and E. Eichten, Using $X(3823) \rightarrow J/\psi\pi^+\pi^-$ to identify coupled-channel effects, *Front. Phys. (Beijing)* **11**, 111402 (2016).
- [62] Z.-S. Jia, Z.-H. Zhang, W.-H. Qin, and G. Li, Hunting for X_b via hidden bottomonium decays $X_b \rightarrow \pi\pi\chi_{bJ}$, *Phys. Rev. D* **109**, 034017 (2024).
- [63] S.-D. Liu, F. Wang, Z.-S. Jia, G. Li, X.-H. Liu, and J.-J. Xie, Pionic transitions of the spin-2 partner of $X(3872)$ to χ_{cJ} , *Phys. Rev. D* **110**, 054048 (2024).
- [64] Y.-H. Chen, Predictions of $\Upsilon(4S) \rightarrow h_b(1P, 2P)\pi^+\pi^-$ transitions, *Chin. Phys. C* **44**, 023103 (2020).
- [65] Y.-H. Chen, M. Cleven, J. T. Daub, F.-K. Guo, C. Hanhart, B. Kubis, U.-G. Meißner, and B.-S. Zou, Effects of Z_b states and bottom meson loops on $\Upsilon(4S) \rightarrow \Upsilon(1S, 2S)\pi^+\pi^-$ transitions, *Phys. Rev. D* **95**, 034022 (2017).
- [66] A. M. Badalian, B. L. G. Bakker, and I. V. Danilkin, The $S-D$ mixing and di-electron widths of higher charmonium 1^{--} states, *Phys. At. Nucl.* **72**, 638 (2009).
- [67] T. Barnes, S. Godfrey, and E. S. Swanson, Higher charmonia, *Phys. Rev. D* **72**, 054026 (2005).
- [68] D. Ebert, R. N. Faustov, and V. O. Galkin, Spectroscopy and regge trajectories of heavy quarkonia and bmesons, *Eur. Phys. J. C* **71**, 1825 (2011).
- [69] L.-C. Gui, L.-S. Lu, Q.-F. Lü, X.-H. Zhong, and Q. Zhao, Strong decays of higher charmonium states into open-charm meson pairs, *Phys. Rev. D* **98**, 016010 (2018).
- [70] W.-J. Deng, H. Liu, L.-C. Gui, and X.-H. Zhong, Charmonium spectrum and electromagnetic transitions with higher multipole contributions, *Phys. Rev. D* **95**, 034026 (2017).
- [71] D.-Y. Chen and X. Liu, Predicted charged charmoniumlike structures in the hidden-charm dipion decay of higher charmonia, *Phys. Rev. D* **84**, 034032 (2011).
- [72] D.-Y. Chen, X. Liu, and T. Matsuki, Predictions of charged charmoniumlike structures with hidden-charm and open-strange channels, *Phys. Rev. Lett.* **110**, 232001 (2013).
- [73] M. N. Anwar, Y. Lu, and B.-S. Zou, Modeling charmonium- η decays of $J^{PC} = 1^{--}$ higher charmonia, *Phys. Rev. D* **95**, 114031 (2017).
- [74] R. Casalbuoni, A. Deandrea, N. Di Bartolomeo, R. Gatto, F. Feruglio, and G. Nardulli, Phenomenology of heavy meson chiral Lagrangians, *Phys. Rep.* **281**, 145 (1997).
- [75] P. Colangelo, F. De Fazio, and T. Pham, Nonfactorizable contributions in b decays to charmonium: The case of $B^- \rightarrow K^-h_c$, *Phys. Rev. D* **69**, 054023 (2004).
- [76] A. Deandrea, G. Nardulli, and A. D. Polosa, J/ψ couplings to charmed resonances and to π , *Phys. Rev. D* **68**, 034002 (2003).
- [77] A. M. Badalian, B. L. G. Bakker, and I. V. Danilkin, Dielectron widths of the S -, D -vector bottomonium states, *Phys. At. Nucl.* **73**, 138 (2010).
- [78] G. Li, F.-I. Shao, C.-W. Zhao, and Q. Zhao, $Z_b/Z'_b \rightarrow \Upsilon\pi$ and $h_b\pi$ decays in intermediate meson loops model, *Phys. Rev. D* **87**, 034020 (2013).
- [79] S. Liu, Z. Cai, Z. Jia, G. Li, and J. Xie, Hidden-bottom hadronic transitions of $\Upsilon(10753)$, *Phys. Rev. D* **109**, 014039 (2024).
- [80] Y. Wang, Q. Wu, G. Li, W.-H. Qin, X.-H. Liu, C.-S. An, and J.-J. Xie, Investigations of charmless decays of $X(3872)$ via intermediate meson loops, *Phys. Rev. D* **106**, 074015 (2022).
- [81] Q. Wu, D.-Y. Chen, and T. Matsuki, A phenomenological analysis on isospin-violating decay of $X(3872)$, *Eur. Phys. J. C* **81**, 193 (2021).
- [82] C. Isola, M. Ladisa, G. Nardulli, and P. Santorelli, Charming penguin contributions in $B \rightarrow K^*\pi, K(\rho, \omega, \phi)$ decays, *Phys. Rev. D* **68**, 114001 (2003).
- [83] C. Meng and K.-T. Chao, Peak shifts due to $B^{(*)} - \bar{B}^{(*)}$ rescattering in $\Upsilon(5S)$ dipion transitions, *Phys. Rev. D* **78**, 034022 (2008).
- [84] Z.-Y. Bai, Y.-S. Li, Q. Huang, X. Liu, and T. Matsuki, $\Upsilon(10753) \rightarrow \Upsilon(nS)\pi^+\pi^-$ decays induced by hadronic loop mechanism, *Phys. Rev. D* **105**, 074007 (2022).
- [85] D.-Y. Chen, J. He, X.-Q. Li, and X. Liu, Dipion invariant mass distribution of the anomalous $\Upsilon(1S)\pi^+\pi^-$ and $\Upsilon(2S)\pi^+\pi^-$ production near the peak of $\Upsilon(10860)$, *Phys. Rev. D* **84**, 074006 (2011).
- [86] D.-Y. Chen, X. Liu, and T. Matsuki, Search for missing $\psi(4S)$ in the $e^+e^- \rightarrow \pi^+\pi^-\psi(2S)$ process, *Phys. Rev. D* **93**, 034028 (2016).
- [87] K. F. Chen *et al.* (Belle Collaboration), Observation of anomalous $\Upsilon(1S)\pi^+\pi^-$ and $\Upsilon(2S)\pi^+\pi^-$ production near the $\Upsilon(5S)$ resonance, *Phys. Rev. Lett.* **100**, 112001 (2008).
- [88] Y. Oh, T. Song, and S. H. Lee, J/ψ absorption by π and ρ mesons in a meson exchange model with anomalous parity interactions, *Phys. Rev. C* **63**, 034901 (2001).
- [89] Z. Lin and C. M. Ko, Model for J/ψ absorption in hadronic matter, *Phys. Rev. C* **62**, 034903 (2000).
- [90] M. P. Locher, Y. Lu, and B. S. Zou, Rates for the reactions $\bar{p}p \rightarrow \pi\phi$ and $\gamma\phi$, *Z. Phys. A* **347**, 281 (1994).
- [91] X.-Q. Li, D. V. Bugg, and B.-S. Zou, Possible explanation of the “ $\rho\pi$ puzzle” in $J/\psi, \psi'$ decays, *Phys. Rev. D* **55**, 1421 (1997).
- [92] H.-Y. Cheng, C.-K. Chua, and A. Soni, Final state interactions in hadronic B decays, *Phys. Rev. D* **71**, 014030 (2005).
- [93] R. Mertig, M. Böhm, and A. Denner, FeynCalc—computer-algebraic calculation of Feynman amplitudes, *Comput. Phys. Commun.* **64**, 345 (1991).
- [94] V. Shtabovenko, R. Mertig, and F. Orellana, FeynCalc 9.3: New features and improvements, *Comput. Phys. Commun.* **256**, 107478 (2020).
- [95] T. Hahn and M. Perez-Victoria, Automatized one-loop calculations in 4 and d dimensions, *Comput. Phys. Commun.* **118**, 153 (1999).

- [96] M. Ablikim, M. Achasov, X. Ai *et al.* (BESIII Collaboration), Observation of a charged charmoniumlike structure in $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ at $\sqrt{s} = 4.26$ GeV, *Phys. Rev. Lett.* **110**, 252001 (2013).
- [97] D.-Y. Chen, X. Liu, and T. Matsuki, Reproducing the $Z_c(3900)$ structure through the initial-single-pion-emission mechanism, *Phys. Rev. D* **88**, 036008 (2013).
- [98] M. Ablikim *et al.* (BES Collaboration), Production of sigma in $\psi(2S) \rightarrow \pi^+\pi^-J/\psi$, *Phys. Lett. B* **645**, 19 (2007).
- [99] D. Cronin-Hennessy *et al.* (CLEO Collaboration), Study of di-pion transitions among $\Upsilon(3S)$, $\Upsilon(2S)$, and $\Upsilon(1S)$ states, *Phys. Rev. D* **76**, 072001 (2007).
- [100] C. Meng and K.-T. Chao, Scalar resonance contributions to the dipion transition rates of $\Upsilon(4S, 5S)$ in the re-scattering model, *Phys. Rev. D* **77**, 074003 (2008).