

Unraveling $K(1690)$ as a pseudoscalar $u\bar{d}\bar{s}$ tetraquark state

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The recently observed $K(1690)$ has been identified as a supernumerary pseudoscalar resonance signal in the strange-meson spectrum predicted by quark model calculations. It is the best candidate of a strange cryptoexotic state. In this work, we systematically study the hadron masses of $u\bar{d}\bar{s}$ tetraquark states with $J^P = 0^-$ in the method of QCD sum rules. For ten interpolating currents, we calculate the correlation functions up to dimension-8 nonperturbative condensates. To calculate the trigluon condensate, we comprehensively consider the contributions from different operators with and without covariant derivatives. The infrared safety can be guaranteed for the completely calculated trigluon condensate by properly addressing the IR divergences in Feynman diagrams. It is demonstrated that the trigluon condensate provides significant contributions to the sum-rule analyses in these light tetraquark systems. Our results support the interpretation of $K(1690)$ resonance to be a pseudoscalar $u\bar{d}\bar{s}$ tetraquark state.

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I. INTRODUCTION

Since the establishment of the quark model [1,2], the search for multiquark states outside of the quark model has always been a research hot spot in both experimental and theoretical aspects. To date, there have been many candidates of multiquark states, such as the hidden-charm pentaquarks, the charged Z_c states, the doubly charged $T_{c\bar{s}0}(2900)^{++}$ state, the doubly charmed T_{cc}^+ state, etc. [3–12]. While most of these confirmed multiquark candidates contain heavy quarks, there also have been some notable advancements in the light hadron sector.

Recently, the COMPASS Collaboration observed three resonance structures in the P -wave $\rho(770)K$ decay channel with quantum numbers $J^P = 0^-$, which lie around 1.4, 1.6, and 1.8 GeV, respectively [13–15]. However, there exist only two excited pseudoscalar strange mesons based on the quark model predictions in the mass region below 2.5 GeV [16–20]. The lightest and heaviest structures of COMPASS are consistent with the established $K(1460)$ and $K(1830)$ states in PDG [21], respectively, and roughly match the two states predicted by quark model calculations.

The mass and decay width of the middle structure in COMPASS's observation are $1687 \pm 10^{+2}_{-67}$ and $140 \pm 20^{+50}_{-50}$ MeV, respectively, and thus dubbed $K(1690)$. This structure is not considered as the known $K(1630)$ in PDG, since the decay width is much smaller for the latter state. Moreover, the spin parity of $K(1630)$ is still undetermined to date. Therefore, $K(1690)$ is a supernumerary signal in the strange-meson spectrum, which can be considered as the clear candidate for a cryptoexotic state with $J^P = 0^-$.

In this work, we explore the possibility of $K(1690)$ as a compact tetraquark state by systematically calculating the mass spectra of $qq\bar{q}\bar{s}$ tetraquarks ($q = u/d$) with $J^P = 0^-$. In Refs. [22,23], the masses of fully light hadron molecules and tetraquark states with $J^{PC} = 0^{--}$ were predicted to be around 1.36 and 2.1 GeV, respectively, in the QCD Coulomb gauge approach. Using the Cornell potential-based phenomenology, the mass spectra of the S -wave and P -wave $s\bar{q}\bar{q}\bar{q}$ tetraquark states were calculated in Ref. [24], giving quite different results for the $\bar{3}_c \otimes 3_c$ and $6_c \otimes \bar{6}_c$ states. In Refs. [25,26], the $qq\bar{q}\bar{q}$ light tetraquark states with $J^{PC} = 0^{--}$ have been studied in QCD sum rules (QCDSR). However, the calculations of trigluon condensate $\langle g^3 f G^3 \rangle$ were incomplete and the IR divergence problem was not properly addressed in these studies. As discussed later, the contribution of trigluon condensate $\langle g^3 f G^3 \rangle$ is very important for the light tetraquark QCDSR analyses. In this work, we shall conduct a detailed study of the $\langle g^3 f G^3 \rangle$ term by including all involved operators and addressing the IR divergence in the calculations.

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The paper is organized as follows. In Sec. II, we briefly introduce the method of QCDSR and construct the pseudoscalar interpolating currents for the $ud\bar{d}\bar{s}$ tetraquark states. We calculate the correlation functions and collect the results in Appendix B. The details for handling the IR divergence of $\langle g^3 f G^3 \rangle$ condensate are shown in Appendix A. In Sec. III, we perform numerical analyses and extract hadron masses of $ud\bar{d}\bar{s}$ tetraquark states. The last section is a brief summary.

II. QCD SUM RULES FOR LIGHT TETRAQUARK

In QCDSR [27,28], we consider the two-point correlation function

$$\Pi(q^2) = i \int d^4x e^{iqx} \langle 0 | T[J(x)J^\dagger(0)] | 0 \rangle, \quad (1)$$

where $J(x)$ is the interpolating current with the same quantum numbers to the interested physical state. Following Ref. [25], we construct the pseudoscalar tetraquark currents with $J^P = 0^-$ and quark component $ud\bar{d}\bar{s}$:

$$\begin{aligned} 6_F \otimes \bar{6}_F(S) & \begin{cases} S_6 = u_a^T C d_b [\bar{d}_a \gamma_5 C \bar{s}_b^T + (a \leftrightarrow b)], \\ V_6 = u_a^T C \gamma_5 d_b [\bar{d}_a C \bar{s}_b^T + (a \leftrightarrow b)], \\ T_3 = u_a^T C \sigma_{\mu\nu} d_b [\bar{d}_a \sigma_{\mu\nu} \gamma_5 C \bar{s}_b^T - (a \leftrightarrow b)], \end{cases} \\ \bar{3}_F \otimes 3_F(A) & \begin{cases} S_3 = u_a^T C d_b [\bar{d}_a \gamma_5 C \bar{s}_b^T - (a \leftrightarrow b)], \\ V_3 = u_a^T C \gamma_5 d_b [\bar{d}_a C \bar{s}_b^T - (a \leftrightarrow b)], \\ T_6 = u_a^T C \sigma_{\mu\nu} d_b [\bar{d}_a \sigma_{\mu\nu} \gamma_5 C \bar{s}_b^T + (a \leftrightarrow b)], \end{cases} \\ \bar{3}_F \otimes \bar{6}_F(M) & \begin{cases} A_6 = u_a^T C \gamma_\mu d_b [\bar{d}_a \gamma_\mu \gamma_5 C \bar{s}_b^T + (a \leftrightarrow b)], \\ P_3 = u_a^T C \gamma_\mu \gamma_5 d_b [\bar{d}_a \gamma_\mu C \bar{s}_b^T - (a \leftrightarrow b)], \end{cases} \\ 6_F \otimes 3_F(M) & \begin{cases} P_6 = u_a^T C \gamma_\mu \gamma_5 d_b [\bar{d}_a \gamma_\mu C \bar{s}_b^T + (a \leftrightarrow b)], \\ A_3 = u_a^T C \gamma_\mu d_b [\bar{d}_a \gamma_\mu \gamma_5 C \bar{s}_b^T - (a \leftrightarrow b)], \end{cases} \end{aligned} \quad (2)$$

in which a and b are color indices and $C = i\gamma_2\gamma_0$ is the charge conjugation operator. The flavor structures of the diquark and antidiquark operators are $6_F \otimes \bar{6}_F(S)$ for the currents S_6, V_6 , and $T_3, \bar{3}_F \otimes 3_F(A)$ for the currents S_3, V_3 , and $T_6, \bar{3}_F \otimes \bar{6}_F(M)$ for the currents A_6 and P_3 , and $6_F \otimes 3_F(M)$ for the currents P_6 and A_3 . The subscripts 6 and 3 of the currents denote the color structures $6 \otimes \bar{6}$ and $\bar{3} \otimes 3$, respectively. According to the flavor structures, the isospin for currents S_6, V_6, T_3, P_6 , and A_3 can be $I = 3/2$ or $1/2$, while the isospin for currents S_3, V_3, T_6, A_6 , and P_3 is $I = 1/2$.

At the hadron level, the correlation function can be described by the dispersion relation [29]

$$\Pi(q^2) = \frac{(q^2)^n}{\pi} \int_0^\infty \frac{\text{Im} \Pi(s)}{s^n(s - q^2)} ds + \sum_{k=0}^{n-1} a_n(q^2)^k, \quad (3)$$

where the a_n is the subtraction constant. The spectral function is usually defined as the imaginary part of correlation function

$$\rho(s) = \frac{1}{\pi} \text{Im} \Pi(s) = f^2 \delta(s - m_X^2) + \dots, \quad (4)$$

where the “narrow resonance” assumption is adopted in the last step and “...” contains contributions from the higher states and continuum. f and m_X are the coupling constant and mass of the lowest-lying hadron state, respectively.

At the quark-gluon level, the correlation function can be computed via the method of operator product expansion (OPE). In our calculation, we use the d -dimensional full quark propagator, taking into account various quark and gluon condensates [30–32]:

$$\begin{aligned} S^{ij}(x) = & \frac{i\Gamma(\frac{d}{2})\mathcal{X}}{2\pi^{d/2}(-x^2)^{d/2}} \delta^{ij} + \frac{m\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}(-x^2)^{d/2-1}} \delta^{ij} - \frac{\delta^{ij}}{12} \langle \bar{\psi}\psi \rangle + \frac{im\delta^{ij}}{12d} \langle \bar{\psi}\psi \rangle \mathcal{X} - \frac{\delta^{ij}}{48d} \langle g\bar{\psi}\sigma G\psi \rangle x^2 \\ & - \frac{i\delta^{ij}x^2\mathcal{X}}{2^4 3^4 (d+2)} g^2 \langle \bar{\psi}\psi \rangle^2 + \frac{i\delta^{ij}mx^2\mathcal{X}}{2^4 3d(d+2)} \langle g\bar{\psi}\sigma G\psi \rangle - \frac{\delta^{ij}x^4 \langle \bar{\psi}\psi \rangle \langle g^2 G^2 \rangle}{2^6 3^2 d(d+2)} - \frac{\delta^{ij}x^4}{2^4 3^4 d(d+2)} g^2 m \langle \bar{\psi}\psi \rangle^2 \\ & - \frac{i\delta^{ij}\Gamma(\frac{d}{2}-1)\mathcal{X}\langle g^3 f G^3 \rangle}{2^8 3^3 d(d+2)\pi^{d/2}(-x^2)^{d/2-3}} + \frac{\Gamma(\frac{d}{2}-1)\gamma^\mu \mathcal{X} \gamma^\nu}{16\pi^{d/2}(-x^2)^{d/2-1}} g G_{\mu\nu}^a T_{ij}^a \\ & + \left[\frac{\Gamma(\frac{d}{2}-2)(\gamma^{\mu\rho\nu} + \gamma^{\rho\mu\nu} - 4g^{\mu\rho}\gamma^\nu)}{96\pi^{d/2}(-x^2)^{d/2-2}} + \frac{\Gamma(\frac{d}{2}-1)(x^\mu \gamma^\rho \mathcal{X} \gamma^\nu + x^\rho \gamma^\mu \mathcal{X} \gamma^\nu)}{48\pi^{d/2}(-x^2)^{d/2-1}} \right] g G_{\mu\nu;\rho}^a T_{ij}^a \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{-\Gamma(\frac{d}{2}-2)(2g^{\{\mu\rho}x^\sigma\} + g^{\{\mu\rho}\gamma^\sigma\}x)}{2^8 \times 3\pi^{d/2}(-x^2)^{d/2-2}} + \frac{\Gamma(\frac{d}{2}-1)x^{\{\mu}x^\rho\gamma^\sigma\}x}{192\pi^{d/2}(-x^2)^{d/2-1}} \right) \gamma_\nu g G_{\mu\nu;\rho\sigma}^a T_{ij}^a \\
& + g^2 G_{\mu\nu}^a G_{\rho\sigma}^b (T^a T^b)_{ij} \left[\frac{-i\Gamma(\frac{d}{2}-1)x^\nu x^\sigma \gamma^\mu \chi \gamma^\rho}{96\pi^{d/2}(-x^2)^{d/2-1}} + \frac{-i\Gamma(\frac{d}{2}-2)g^{\nu\sigma} \gamma^\mu \chi \gamma^\rho}{192\pi^{d/2}(-x^2)^{d/2-2}} \right. \\
& \left. + \frac{i\Gamma(\frac{d}{2}-2)}{2^8 \times 3\pi^{d/2}(-x^2)^{d/2-2}} (-6g^{\nu\sigma} \chi \gamma^{\mu\rho} - 4x^\sigma \gamma^{\mu\nu\rho} + 6x^\mu \gamma^{\nu\rho\sigma} - 4x^\nu \gamma^{\mu\rho\sigma} + 3\gamma^\mu \chi \gamma^{\nu\rho\sigma}) \right], \quad (5)
\end{aligned}$$

where m is the light quark mass, $i, j = 1, 2, 3$ are color indices, and T^a ($a = 1, \dots, 8$) is the Gell-Mann matrix. We also use the following abbreviations:

$$\begin{aligned}
\gamma^{\mu\nu\rho} &:= \gamma^\mu \gamma^\nu \gamma^\rho, & g^{\{\mu\rho}x^\sigma\} &:= g^{\mu\rho}x^\sigma + g^{\mu\sigma}x^\rho + g^{\rho\sigma}x^\mu, & g^{\{\mu\rho}\gamma^\sigma\} &:= g^{\mu\rho}\gamma^\sigma + g^{\mu\sigma}\gamma^\rho + g^{\rho\sigma}\gamma^\mu, \\
x^{\{\mu}x^\rho\gamma^\sigma\} &:= x^\mu x^\rho \gamma^\sigma + x^\mu x^\sigma \gamma^\rho + x^\rho x^\sigma \gamma^\mu, & G_{\mu\nu;\rho}^a &:= \tilde{D}_\rho^{ab} G_{\mu\nu}^b, & G_{\mu\nu;\rho\sigma}^a &:= \tilde{D}_\sigma^{ab} \tilde{D}_\rho^{bc} G_{\mu\nu}^c, \quad (6)
\end{aligned}$$

in which $G_{\mu\nu}$ is the gluon field strength, $\tilde{D}_\mu^{ab} = \delta^{ab}\partial_\mu - gf^{abc}A_\mu^c$ is the covariant derivative operator in the adjoint representation, and A_μ^c is the external gauge field. The origin of the full quark propagator in Eq. (5) includes both the background field and quantum field. In our calculation, we keep track of terms in the fifth order of the background quark field expansion $\frac{x^\mu x^\nu x^\rho x^\sigma}{4!} \langle \psi^i D_\mu D_\nu D_\rho D_\sigma \bar{\psi}^j \rangle$. For the part of quantum fields, we include terms up to dimension 6 to fully compute the result of $\langle g^3 f G^3 \rangle$.

In QCDSR studies for multi-quark systems, the contribution of trigluon condensate $\langle g^3 f G^3 \rangle$ is usually neglected due to the computational complexity and infrared divergence (IR) in fully calculations. However, such contributions may be significant in fully calculations at leading order (LO) by considering $\langle (DG)(DG) \rangle$ and $\langle G(DDG) \rangle$ terms as the following [30,33]:

$$\begin{aligned}
\langle G_{\mu\nu;\rho}^a G_{\alpha\beta;\sigma}^b \rangle &= \delta^{ab} 2Z g_{\rho\sigma} (g_{\mu\alpha} g_{\beta\nu} - g_{\mu\nu} g_{\beta\alpha}) \\
&+ \delta^{ab} Z [g_{\nu\sigma} g_{\mu\alpha} g_{\beta\rho} + g_{\mu\sigma} g_{\nu\beta} g_{\alpha\rho} - (\mu \leftrightarrow \nu)] \\
&+ \delta^{ab} Y [(g_{\nu\rho} g_{\mu\alpha} g_{\beta\sigma} + g_{\mu\rho} g_{\nu\beta} g_{\alpha\sigma}) - (\mu \leftrightarrow \nu)], \\
Z &= -\frac{D(D-2)g^2 \langle \bar{\psi}\psi \rangle^2 + 9(D-1)\langle gfG^3 \rangle}{72(D-2)(D-1)D(D+2)}, \\
Y &= \frac{-D(D-2)g^2 \langle \bar{\psi}\psi \rangle^2 + 27\langle gfG^3 \rangle}{72(D-2)(D-1)D(D+2)}, \quad (7)
\end{aligned}$$

$$\langle G_{\mu\nu;\rho}^a G_{\alpha\beta;\sigma}^b \rangle = -\langle G_{\mu\nu;\rho\sigma}^a G_{\alpha\beta}^b \rangle = -\langle G_{\mu\nu}^a G_{\alpha\beta;\sigma\rho}^b \rangle, \quad (8)$$

in which we adopt the D -dimensional condensates to distinguish the dimension d of the full quark propagator in Eq. (5). D and d are identical, but such symbols are useful for addressing IR divergences, as shown in Appendix A. In Ref. [34], the authors also address the IR divergence of $\langle g^3 f G^3 \rangle$; they just did not summarize the corresponding cancellation tricks. One should note that the IR safety for the trigluon condensate $\langle g^3 f G^3 \rangle$ cannot be guaranteed for calculating the $\langle (DG)(DG) \rangle$ terms by using propagators up to dimension 6, as adopted in Ref. [30].

For the full quark propagator in Eq. (5), the IR divergences appear for the terms involving $\Gamma(d/2-2)$ when $d=4$. Fortunately, the operator mixing is unnecessary for $\langle g^3 f G^3 \rangle$ to address the IR divergence if we neglect the $g^2 \langle \bar{\psi}\psi \rangle^2$ parts in the expressions of $\langle (DG)(DG) \rangle$ and $\langle G(DDG) \rangle$ [35], which are actually of high order in α_s and, thus, numerically negligible. Therefore, its contribution can be neglected. We summarize the handling of the IR divergence for $\langle g^3 f G^3 \rangle$ as follows.

- (i) As shown in Fig. 1, both the $\langle G(DDG) \rangle$ and $\langle G(GG) \rangle$ condensates formed by two different quark propagators are divergent. However, the

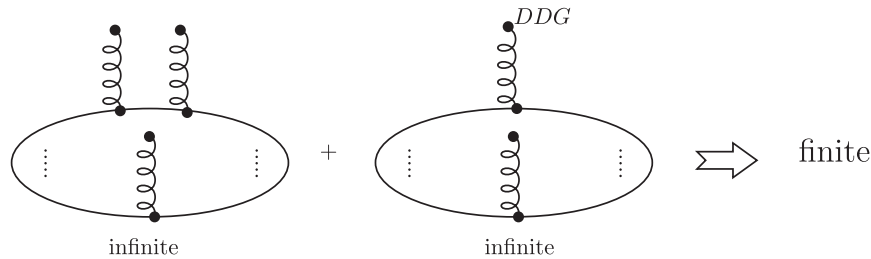


FIG. 1. Cancellation of $\langle (GG)G \rangle$ and $\langle G(DDG) \rangle$ condensates formed by two different quark propagators.

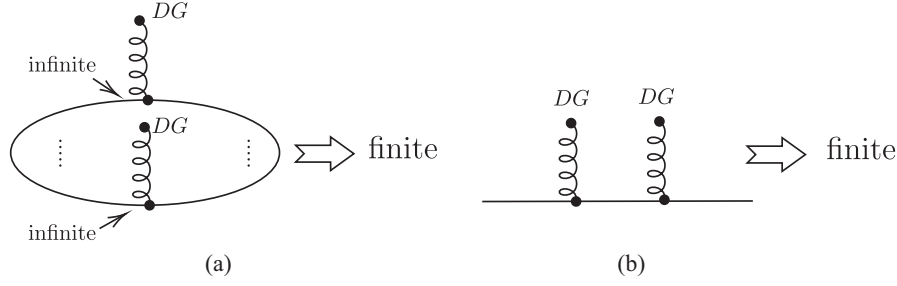


FIG. 2. IR safety for $\langle (DG)(DG) \rangle$ condensate: (a) cancellation of $\langle (DG)(DG) \rangle$ by any two quark propagators; (b) the propagator containing $\langle (DG)(DG) \rangle$ has no IR divergence.

divergences from these two diagrams cancel each other out exactly.

- (ii) Although the propagator including DG is divergent, the $\langle (DG)(DG) \rangle$ condensate in any diagram has no IR divergence, as shown in Fig. 2.
- (iii) As shown in Fig. 3, both the $\langle G(DDG) \rangle$ and $\langle GGG \rangle$ condensates formed by the single quark propagator are divergent. However, the divergences from these two diagrams cancel each other out exactly.

The corresponding propagators are as follows:

$$S_{\langle (DG)(DG) \rangle}^{ij}(x) = \frac{-i\delta^{ij}\langle g^3 f G^3 \rangle \Gamma(\frac{d}{2}-1)\chi}{1728d(d+2)\pi^{d/2}(-x^2)^{d/2-3}}, \quad (9)$$

$$S_{\langle GGG \rangle}^{ij}(x) = \frac{-i\delta^{ij}\langle g^3 f G^3 \rangle \Gamma(\frac{d}{2}-2)\chi}{9216d\pi^{d/2}(-x^2)^{d/2-3}}, \quad (10)$$

$$S_{\langle G(DDG) \rangle}^{ij}(x) = \frac{i\delta^{ij}(d-2)\langle g^3 f G^3 \rangle \Gamma(\frac{d}{2}-2)\chi}{3072d(d+2)\pi^{d/2}(-x^2)^{d/2-3}}, \quad (11)$$

in which the last two are divergent while the sum of them are finite. The detailed discussions can be found in Appendix A. After considering all the above points, we can completely calculate the contribution of trigluon condensate $\langle g^3 f G^3 \rangle$ at the LO of α_s .

We calculate the correlation functions up to dimension 8 by considering the Feynman diagrams depicted in Fig. 4, including the complete contribution of $\langle g^3 f G^3 \rangle$ at LO. For simplicity, we use a quark line with dots to represent the terms with the color factor δ_{ij} in the quark propagator $S_{ij}(x)$ in Eq. (5), as they can be calculated collectively. Gluon lines labeled DG and DDG correspond to terms

involving $G_{\mu\nu;\rho}$ and $G_{\mu\nu;\rho\sigma}$ in Eq. (5), respectively. The IR divergences in the diagrams Figs. 4(d)–4(f) contributing to $\langle g^3 f G^3 \rangle$ can be systematically addressed by applying the cancellation tricks presented in Figs. 1–3. The diagram Fig. 4(h) incorporates contributions from two types of condensates: $\langle \bar{q}G_{\mu\nu}D_\rho q \rangle$ and $\langle \bar{q}\tilde{D}_\rho G_{\mu\nu}q \rangle$. The diagram Fig. 4(i) contains contribution from condensate $\langle \bar{q}G_{\mu\nu;\rho}q \rangle$. The results of the correlation functions are listed in Appendix B.

The correlation function obtained at the quark-gluon level is expected to be equivalent to the one at the hadron level due to the principle of quark-hadron duality. In QCDSR, one can pick out the lowest-lying resonance by performing the Borel transform to the correlation functions at both levels, in order to suppress the contributions from high excited states in Eq. (4) and eliminate the unknown subtraction terms in Eq. (3). Subsequently, the QCD sum rules are obtained as

$$\Pi(M_B^2, s_0) = f^2 e^{-m_X^2/M_B^2} = \int_0^{s_0} \rho(s) e^{-s/M_B^2} ds, \quad (12)$$

in which s_0 is the continuum threshold parameter and M_B is the Borel parameter. Then the hadron mass m_X of the lowest-lying resonance can be extracted to be

$$m_X = \sqrt{\frac{L_1(s_0, M_B^2)}{L_0(s_0, M_B^2)}}, \quad (13)$$

where

$$L_k(s_0, M_B^2) = \int_0^{s_0} \rho(s) s^k e^{-s/M_B^2} ds. \quad (14)$$

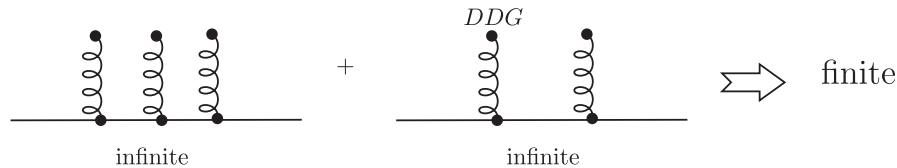


FIG. 3. Cancellation of $\langle GGG \rangle$ and $\langle G(DDG) \rangle$ condensates formed by the single quark propagator.

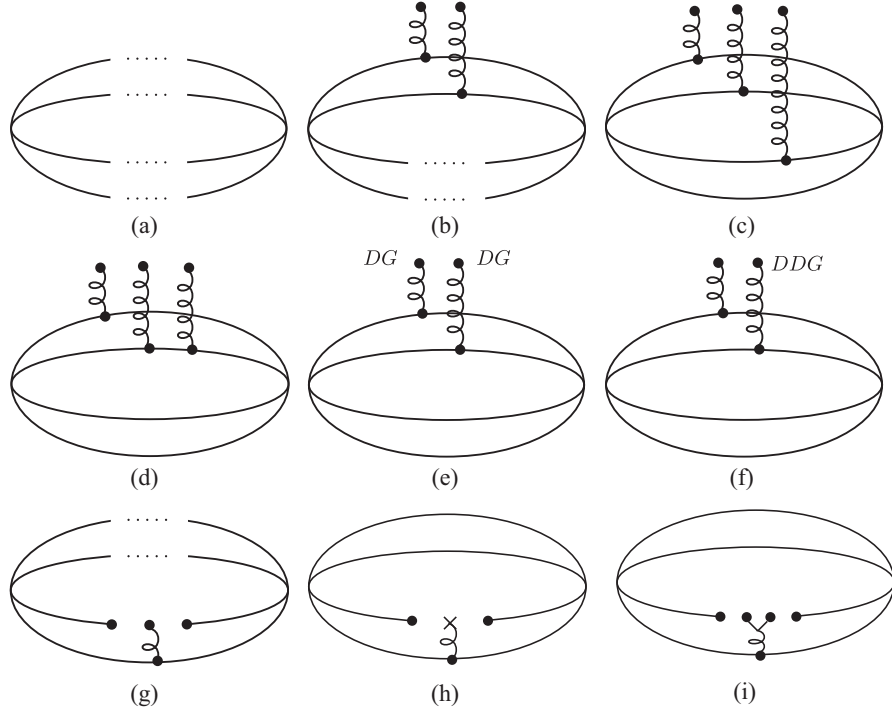


FIG. 4. The Feynman diagrams involved in our calculations for the $ud\bar{d}\bar{s}$ tetraquark systems. A quark line with dots contains terms with the color factor δ^{ij} in $S^{ij}(x)$.

III. NUMERICAL ANALYSIS

To perform the QCD sum-rule numerical analyses, we use the following values of the quark masses, strong coupling, and various QCD condensates [21,36–42]:

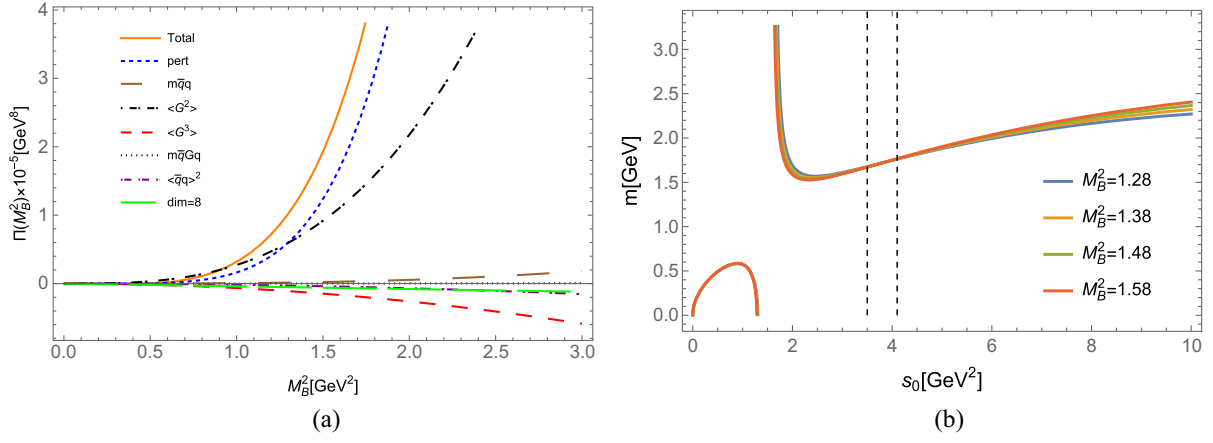
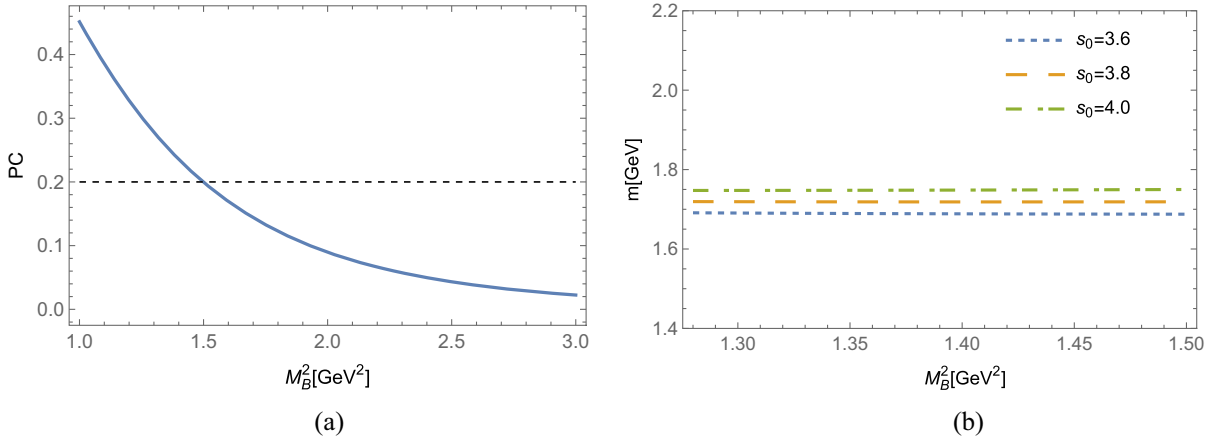
$$\begin{aligned}
 m_u = m_d = 0, \quad m_s &= 93.5 \pm 0.8 \text{ MeV}, \\
 \langle \bar{q}q \rangle &= -(0.24 \pm 0.01)^3 \text{ GeV}^3, \\
 \langle \bar{s}s \rangle &= (0.8 \pm 0.1) \langle \bar{q}q \rangle, \\
 \langle g\bar{q}\sigma Gq \rangle &= (0.8 \pm 0.2) \langle \bar{q}q \rangle \text{ GeV}^2, \\
 \langle g\bar{s}\sigma Gs \rangle &= (0.8 \pm 0.2) \langle g\bar{q}\sigma Gq \rangle, \\
 \langle \alpha_s G^2 \rangle &= (6.39 \pm 0.35) \times 10^{-2} \text{ GeV}^4, \\
 \langle g^3 f G^3 \rangle &= 1.2 \langle \alpha_s G^2 \rangle \text{ GeV}^2, \\
 m_s(\mu) &= m_s(2 \text{ GeV}) \left[\frac{\alpha_s(\mu)}{\alpha_s(2 \text{ GeV})} \right]^{\frac{12}{33-2n_f}}, \\
 \alpha_s &= \frac{1}{b_0 t} \left[1 - \frac{b_1 \log t}{b_0^2 t} + \frac{b_1^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right],
 \end{aligned} \tag{15}$$

where $t = \log \frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33-2n_f}{12\pi}$, $b_1 = \frac{153-19n_f}{24\pi^2}$, $b_2 = \frac{2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2}{128\pi^3}$, and $\Lambda = 332 \text{ MeV}$ for the flavors $n_f = 3$. The s -quark mass m_s is taken as the $\overline{\text{MS}}$ mass at the scale $\mu = 2 \text{ GeV}$. Given that the condensate values are evaluated at the renormalization scale $\mu = 1 \text{ GeV}$, we accordingly adopt m_s at the same scale. To date, there is no

reliable method to determine the value of $\langle g^3 f G^3 \rangle$ from experimental data. In our analyses, we adopt the value from the early review papers of QCDSR [27,28,40], which was taken from estimations of instanton models [40] and lattice calculation [39].

We first study the behaviors of correlation function to show the importance of the trigluon condensate $\langle g^3 f G^3 \rangle$, by taking the interpolating current P_3 as an example in Fig. 5(a). The gluon condensates $\langle g^2 G^2 \rangle$ and $\langle g^3 f G^3 \rangle$ are shown to be the most important nonperturbative terms, while the quark condensates $m_s \langle \bar{q}q \rangle$ and $m_s \langle \bar{q}Gq \rangle$ are much smaller due to the light quark mass. However, the four-quark condensate $\langle \bar{q}q \rangle^2$ also gives a significant contribution to the correlation function [25,27]. Similar situations occur for other tetraquark currents in Eq. (2). In other words, the well-calculated trigluon condensate $\langle g^3 f G^3 \rangle$ may be important to the fully light tetraquark sum-rule analyses and, thus, cannot be neglected. This is very different from the heavy tetraquark systems, in which the quark condensates are usually much larger than gluon condensates due to the existence of heavy quarks [3,43]. The contributions of trigluon condensates have been proven to be negligible for the conventional heavy quarkonium [44,45] and fully heavy baryons [46].

In QCDSR, the study of OPE convergence leads to the lower limit $M_{B \min}^2$ of the Borel parameter. To ensure a good OPE convergence, we require that the $D = 6$ and $D = 8$ condensates give small contributions to the correlation functions

FIG. 5. The OPE convergence (a) and variation of hadron mass with s_0 (b) for the interpolating current P_3 .FIG. 6. The pole contribution (a) and variation of hadron mass with M_B^2 (b) for the interpolating current P_3 .

$$\left| \frac{\Pi_{D=6}(M_B^2)}{\Pi_{\text{pert}}(M_B^2)} \right| \leq 25\%, \quad \left| \frac{\Pi_{D=8}(M_B^2)}{\Pi_{\text{pert}}(M_B^2)} \right| \leq 10\%. \quad (16)$$

Meanwhile, the guarantee of pole contribution (PC) will provide an upper limit for M_B^2 :

$$\text{PC} = \frac{L_0(s_0, M_B^2)}{L_0(\infty, M_B^2)} \geq 10\%. \quad (17)$$

According to Eq. (16), the behavior of OPE series in Fig. 5(a) exhibits very good convergence for $M_B^2 \geq 1.28 \text{ GeV}^2$. As shown in Fig. 5(b), we further study the dependence of the hadron mass on the continuum threshold parameter s_0 with different values of M_B^2 , from which one immediately realizes an optimal value of $s_0 = 3.8 \pm 0.2 \text{ GeV}^2$. Within this choice, the hadron mass is almost independent of the unphysical parameter M_B^2 . The uncertainty of the threshold value s_0 is estimated to be $\pm 0.2 \text{ GeV}^2$, within which the stability of mass sum rules can be guaranteed. In Fig. 6(a), we show the pole

contribution to set the upper limit as $M_{B\text{max}}^2 = 1.5 \text{ GeV}^2$ by requiring $\text{PC} \geq 20\%$ for this current. Within these parameter working region, the mass curves can be plotted in Fig. 6(b). The hadron mass is reading as

$$m_X = 1.72 \pm 0.10 \text{ GeV}, \quad (18)$$

TABLE I. Numerical results for the $ud\bar{d}\bar{s}$ tetraquark states.

Currents	$s_0 (\pm 0.2 \text{ GeV}^2)$	$M_B^2 (\text{GeV}^2)$	$m_X (\text{GeV})$	PC (%)
P_6	3.3	1.14–1.35	1.62 ± 0.10	>15
P_3	3.8	1.28–1.50	1.72 ± 0.10	>20
S_6	6.7	1.40–1.80	2.34 ± 0.11	>25
S_3	...	...	...	...
A_6	1.0	0.54–0.62	0.90 ± 0.50	>10
A_3	2.8	0.95–1.20	1.48 ± 0.20	>10
V_6	8.8	1.34–1.94	2.60 ± 0.20	>40
V_3	3.3	1.42–1.62	1.60 ± 0.16	>10
T_6	2.4	0.75–1.10	1.36 ± 0.05	>15
T_3	4.7	0.90–1.35	1.92 ± 0.04	>25

TABLE II. Numerical results for P_6 and P_3 currents without contribution of $\langle g^3 f G^3 \rangle$. The value R in the last column represents the mass correction comparing to the results in Table I.

Currents	s_0 (± 0.2 GeV ²)	M_B^2 (GeV ²)	m_X (GeV)	PC (%)	R (%)
P_6	2.1	0.88–0.97	1.29 ± 0.12	>15	25.6
P_3	2.7	1.25–1.33	1.45 ± 0.11	>15	18.6

in which the error is from the uncertainties of s_0 , the strange quark mass m_s , and various QCD condensates. After performing numerical analyses for all interpolating currents in Eq. (2), we present the numerical results for these $u\bar{d}\bar{d}\bar{s}$ tetraquarks in Table I. To show the importance of trigluon condensate $\langle g^3 f G^3 \rangle$, we reanalyze the mass sum rules without considering the contributions from this term for the currents P_6 and P_3 . As shown in Table II, the predicted masses are about 20% lower than those in Table I.

There have been some other discussions regarding the value of $\langle g^3 f G^3 \rangle$. For example, QCDSR provided a quite different estimation from the studies of heavy quarkonium systems [41,42]:

$$\langle g^3 f G^3 \rangle = (8.2 \pm 1.0) \text{ GeV}^2 \times \langle \alpha_s G^2 \rangle, \quad (19)$$

which is much larger than that in Eq. (15). Since the trigluon condensate will contribute significantly to our sum-rule analyses, we further investigate the dependence of extracted hadron mass to the coefficient $k = \langle g^3 f G^3 \rangle / \langle \alpha_s G^2 \rangle$ in Fig. 7, by using the interpolating current P_6 as an example. It is clear that the hadron mass is very sensitive to the change of the trigluon condensate.

In these results, the hadron masses extracted from the currents P_6 , P_3 , and V_3 are consistent with the mass of $K(1690)$ observed by the COMPASS Collaboration [13–15], suggesting that $K(1690)$ could be a $u\bar{d}\bar{d}\bar{s}$ tetraquark state. For the currents S_6 and V_6 , the dominant nonperturbative contributions are negative, leading to poor positivity behavior of the spectral functions, and, thus, the mass predictions for these two currents are actually

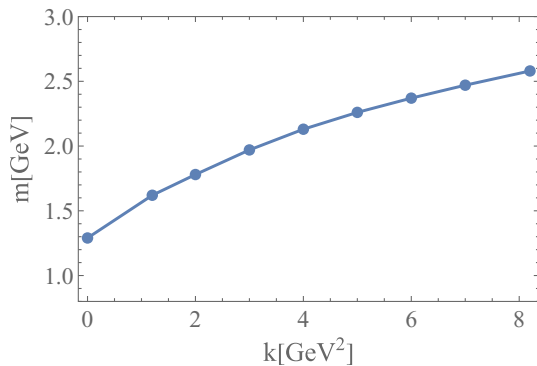


FIG. 7. The sensitivity of extracted hadron mass to coefficient $k = \langle g^3 f G^3 \rangle / \langle \alpha_s G^2 \rangle$ for the interpolating current P_6 .

unreliable. The analysis for current S_3 fails to produce stable mass sum rules. Moreover, the hadron masses extracted from the currents A_3 and T_3 are close to the masses of $K(1460)$ and $K(1830)$, respectively, implying possible tetraquark components in these two strange mesons. More properties of the $u\bar{d}\bar{d}\bar{s}$ tetraquarks should be investigated to further study the inner structures of these resonance states.

IV. SUMMARY

To study the inner structure of the recent observed strange meson $K(1690)$, we investigate the mass spectrum of the light tetraquark $u\bar{d}\bar{d}\bar{s}$ states with quantum numbers $J^P = 0^-$ in the method of QCDSR. We calculate the two-point correlation functions up to dimension-8 nonperturbative condensates at the leading order of α_s . We utilize the quark propagators incorporating DG , DDG , and $\langle g^3 f G^3 \rangle$ terms to comprehensively account for the contributions of trigluon condensate $\langle g^3 f G^3 \rangle$. After properly addressing the IR divergences in OPE, our calculations demonstrate that the trigluon condensate $\langle g^3 f G^3 \rangle$ provides a significant contribution to the mass sum-rule analyses for light tetraquark systems, potentially surpassing the contributions of four-quark condensate $\langle \bar{q}q \rangle^2$. In the case of P_6 and P_3 , the inclusion of $\langle g^3 f G^3 \rangle$ can increase about 20% of mass predictions.

Using the interpolating currents (P_6, P_3, V_3), we predict the hadron mass of pseudoscalar $u\bar{d}\bar{d}\bar{s}$ tetraquark state to be 1.6–1.7 GeV. This result is consistent with the mass of $K(1690)$ observed by COMPASS, supporting the compact tetraquark interpretation of the structure. However, one should note that other possibilities cannot be excluded, such as a strange hybrid meson or a structure of kinematic effect [47,48]. The decay properties of $u\bar{d}\bar{d}\bar{s}$ tetraquark state should be studied to further investigate the inner structure of $K(1690)$, such as the decay processes $u\bar{d}\bar{d}\bar{s} \rightarrow a_0 K, f_0 K, K_0^* \pi, K^* \rho, K \rho, K^* \pi$, etc. Especially in the isospin quartet ($I = 3/2$), the doubly charged $u\bar{u}\bar{d}\bar{s}$ tetraquark can be considered as a characteristic signal in the future.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: IR SAFETY FOR $\langle g^3 f G^3 \rangle$

In this calculation, the IR divergences can appear in the Feynman diagrams Figs. 4(d)–4(f), in which the involved

condensates $\langle(GG)G\rangle$, $\langle G(DDG)\rangle$, and $\langle(DG)(DG)\rangle$ come from any two propagators. This can be expressed as

$$(\dots) * S^{ij} * (\dots) * S^{mn} * (\dots), \quad (\text{A1})$$

where the ellipsis $(\dots)$ represents irrelevant part to divergences. For convenience, we define the following propagators:

$$S_2^{ij}(x) := X^{\mu\nu}(x) g G_{\mu\nu}^a T_{ij}^a = \frac{\Gamma(\frac{d}{2}-1) \gamma^\mu \chi^\nu}{16\pi^{d/2}(-x^2)^{d/2-1}} g G_{\mu\nu}^a T_{ij}^a, \quad (\text{A2})$$

$$S_{4b}^{ij}(x) := Z^{\mu\nu\rho\sigma}(x) g G_{\mu\nu;\rho\sigma}^a T_{ij}^a = \frac{-\Gamma(\frac{d}{2}-2)(2g^{\{\mu\rho}x^{\sigma\}} + g^{\{\mu\rho}\gamma^\sigma\}x^\nu)}{2^8 \times 3\pi^{d/2}(-x^2)^{d/2-2}} g G_{\mu\nu;\rho\sigma}^a T_{ij}^a, \quad (\text{A3})$$

$$\begin{aligned} & (\dots) * S_{4a}^{ij}(x) * (\dots) S_2^{mn} * (\dots) + (\dots) * S_{4b}^{ij}(x) * (\dots) S_2^{mn} * (\dots) \\ & = (\dots) * [(T^b T^c)_{ij} g^2 G_{\mu_1\nu_1}^b G_{\rho_1\sigma_1}^c Y^{\mu_1\nu_1\rho_1\sigma_1}(x) + T_{ij}^a g G_{\mu_2\nu_2;\rho_2\sigma_2}^a Z^{\mu_2\nu_2\rho_2\sigma_2}(x)] * (\dots) * T_{mn}^d g G_{\mu\nu}^d X^{\mu\nu} * (\dots). \end{aligned} \quad (\text{A5})$$

We define the following structures for the dimension-6 condensates:

$$\begin{aligned} \langle G_{\mu\nu}^a G_{\alpha\beta}^b G_{\rho\sigma}^c \rangle &:= f^{abc} W_{\mu\nu;\alpha\beta;\rho\sigma}, \\ \langle G_{\mu\nu;\rho\sigma}^a G_{\alpha\beta}^b \rangle &:= \delta^{ab} U_{\mu\nu\alpha\beta;\rho\sigma}, \end{aligned} \quad (\text{A6})$$

where

$$\begin{aligned} & g^3 W_{\mu\nu;\alpha\beta;\rho\sigma} \\ &= \frac{-\langle g^3 f G^3 \rangle}{24(D-2)(D-1)D} \\ & \times [g_{\beta\nu}(g_{\alpha\rho}g_{\mu\sigma} - g_{\alpha\sigma}g_{\mu\rho}) + g_{\alpha\nu}(g_{\beta\sigma}g_{\mu\rho} - g_{\beta\rho}g_{\mu\sigma}) \\ & + g_{\beta\mu}(g_{\alpha\sigma}g_{\nu\rho} - g_{\alpha\rho}g_{\nu\sigma}) + g_{\alpha\mu}(g_{\beta\rho}g_{\nu\sigma} - g_{\beta\sigma}g_{\nu\rho})]. \end{aligned} \quad (\text{A7})$$

The specific expression of $U_{\mu\nu\alpha\beta;\rho\sigma}$ is given by Eqs. (7) and (8). Then we have

$$\begin{aligned} & (T^b T^c)_{ij} T_{mn}^d \langle G_{\mu_1\nu_1}^b G_{\rho_1\sigma_1}^c G_{\mu\nu}^d \rangle \\ &= (T^b T^c)_{ij} T_{mn}^d f^{bcd} W_{\mu_1\nu_1;\rho_1\sigma_1;\mu\nu}, \end{aligned} \quad (\text{A8})$$

$$T_{ij}^a T_{mn}^d \langle G_{\mu_2\nu_2;\rho_2\sigma_2}^a G_{\mu\nu}^d \rangle = T_{ij}^a T_{mn}^d U_{\mu_2\nu_2\rho_2\sigma_2;\mu\nu}, \quad (\text{A9})$$

and

$$(T^b T^c)_{ij} T_{mn}^d f^{bcd} = \frac{3i}{2} T_{mn}^d T_{ij}^d, \quad (\text{A10})$$

where f^{bce} and d^{bce} are antisymmetric structure and symmetric constants of the SU(3) group, respectively. Substituting Eqs. (A8)–(A10) into Eq. (A5),

$$\begin{aligned} S_{4a}^{ij}(x) &:= g^2 G_{\mu\nu}^a G_{\rho\sigma}^b (T^a T^b)_{ij} Y^{\mu\nu\rho\sigma}(x) \\ &= g^2 G_{\mu\nu}^a G_{\rho\sigma}^b (T^a T^b)_{ij} \left[\frac{-i\Gamma(\frac{d}{2}-2) g^{\nu\sigma} \gamma^\mu \chi^\rho}{192\pi^{d/2}(-x^2)^{d/2-2}} \right. \\ & \quad + \frac{i\Gamma(\frac{d}{2}-2)(-6g^{\nu\sigma} \chi^\rho \gamma^\mu - 4x^\sigma \gamma^{\mu\nu\rho})}{2^8 \times 3\pi^{d/2}(-x^2)^{d/2-2}} \\ & \quad \left. + \frac{i\Gamma(\frac{d}{2}-2)(6x^\mu \gamma^{\nu\rho\sigma} - 4x^\nu \gamma^{\mu\sigma\rho} + 3\gamma^\mu \chi^\nu \gamma^{\rho\sigma})}{2^8 \times 3\pi^{d/2}(-x^2)^{d/2-2}} \right], \end{aligned} \quad (\text{A4})$$

in which the subscript 2/4 represents the dimension of $S^{ij}(x)$. It is very clear that $S_{4a}^{ij}(x)$ and $S_{4b}^{ij}(x)$ are divergent with the factor $\Gamma(d/2-2)$ when $d=4$, which will appear in the Feynman diagrams Figs. 4(d) and 4(f), respectively. The sum of these two diagrams is shown in Fig. 1 with the expression

$$\begin{aligned} & (\dots) * T_{mn}^d T_{ij}^d \left[\frac{3i}{2} W_{\mu_1\nu_1;\rho_1\sigma_1;\mu\nu} g^3 Y^{\mu_1\nu_1\rho_1\sigma_1}(x) \right. \\ & \quad \left. + U_{\mu_2\nu_2\rho_2\sigma_2;\mu\nu} g^2 Z^{\mu_2\nu_2\rho_2\sigma_2}(x) \right] * (\dots) * X^{\mu\nu}, \end{aligned} \quad (\text{A11})$$

in which the result in the square brackets can be calculated as

$$\begin{aligned} & \frac{\langle g^3 f G^3 \rangle \Gamma(\frac{d}{2}-2)(d-4)(\gamma_\mu \chi_\nu - \gamma_\nu \chi_\mu)}{3072(d-2)(d-1)d\pi^{d/2}(-x^2)^{d/2-2}} \\ &= \frac{\langle g^3 f G^3 \rangle \Gamma(\frac{d}{2}-1)(\gamma_\mu \chi_\nu - \gamma_\nu \chi_\mu)}{1536(d-2)(d-1)d\pi^{d/2}(-x^2)^{d/2-2}}. \end{aligned} \quad (\text{A12})$$

It is fantastic that this result is finite, showing that the divergences in Figs. 4(d) and 4(f) exactly cancel out for each other.

To ensure such a cancellation for the calculation in Fig. 4(e), it is important to use the d -dimensional propagator and D -dimensional condensates but not any one in the four-dimensional. The propagator containing $G_{\mu\nu;\rho}^a$ can be written as

$$S_3^{ij}(x) = S_{3a}^{ij}(x) + S_{3b}^{ij}(x) := J_0^{\mu\nu\rho} g G_{\mu\nu;\rho}^a T_{ij}^a, \quad (\text{A13})$$

$$\begin{aligned} S_{3a}^{ij}(x) &:= J_1^{\mu\nu\rho} g G_{\mu\nu;\rho}^a T_{ij}^a \\ &= \frac{\Gamma(\frac{d}{2}-1) g G_{\mu\nu;\rho}^a T_{ij}^a}{48\pi^{d/2}(-x^2)^{d/2-1}} (x_\mu \gamma_\rho \chi_\nu + x_\rho \gamma_\mu \chi_\nu), \end{aligned} \quad (\text{A14})$$

$$S_{3b}^{ij}(x) := J_2^{\mu\nu\rho} g G_{\mu\nu;\rho}^a T_{ij}^a(x) \\ = \frac{\Gamma(\frac{d}{2}-2) g G_{\mu\nu;\rho}^a T_{ij}^a}{96\pi^{d/2}(-x^2)^{d/2-2}} (\gamma_{\mu\rho\nu} + \gamma_{\rho\mu\nu} - 4g_{\mu\rho}\gamma_\nu), \quad (\text{A15})$$

where $J_0^{\mu\nu\rho}$, $J_1^{\mu\nu\rho}$, and $J_2^{\mu\nu\rho}$ are corresponding tensor structures and $J_0^{\mu\nu\rho} = J_1^{\mu\nu\rho} + J_2^{\mu\nu\rho}$. It is clear that only S_{3b}^{ij} is divergent while S_{3a}^{ij} remains finite. The most divergent part of Fig. 4(e) can be written as

$$(\dots) * S_{3b}^{ij} * (\dots) S_{3b}^{mn} * (\dots) = (\dots) * J_2^{\mu_1\nu_1\rho_1} T_{ij}^a g G_{\mu_1\nu_1;\rho_1}^a * (\dots) * J_2^{\mu\nu\rho} T_{mn}^b g G_{\mu\nu;\rho}^b * (\dots) \\ = (\dots) * J_2^{\mu_1\nu_1\rho_1} T_{ij}^a T_{mn}^b g^2 \langle G_{\mu_1\nu_1;\rho_1}^a G_{\mu\nu;\rho}^b \rangle * (\dots) * J_2^{\mu\nu\rho} * (\dots), \quad (\text{A16})$$

in which both $J_2^{\mu_1\nu_1\rho_1}$ and $J_2^{\mu\nu\rho}$ contain the divergent factor $\Gamma(\frac{d}{2}-2)$. However, such a divergence is trivial since the other part in the calculation is zero:

$$J_2^{\mu_1\nu_1\rho_1} T_{ij}^a T_{mn}^b g^2 \langle G_{\mu_1\nu_1;\rho_1}^a G_{\mu\nu;\rho}^b \rangle = T_{ij}^a T_{mn}^b \frac{\langle g^3 f G^3 \rangle \Gamma(\frac{d}{2}-2) (D-d) (g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu)}{128(D-1)D(D^2-4)\pi^{d/2}(-x^2)^{d/2-2}} \\ = 0, \quad (\text{A17})$$

where we use $D = d$ in the last step.

APPENDIX B: TWO-POINT CORRELATION FUNCTIONS FOR ALL INTERPOLATING CURRENTS

In this appendix, we present the correlation functions after Borel transformation up to dimension 8 for the interpolating currents in Eq. (2):

$$\begin{aligned} \Pi_{T_6} &= \frac{3M_B^{10}}{160\pi^6} + \left(\frac{11\langle g^2 G^2 \rangle}{384\pi^6} + \frac{m\langle \bar{s}s \rangle}{4\pi^4} \right) M_B^6 + \left(\frac{7m\langle g\bar{s}\sigma Gs \rangle}{32\pi^4} - \frac{\langle g^3 f G^3 \rangle}{16\pi^6} + \frac{7g^2\langle \bar{q}q \rangle^2}{36\pi^4} + \frac{7g^2\langle \bar{s}s \rangle^2}{108\pi^4} \right) M_B^4 + \frac{11m\langle g^2 G^2 \rangle \langle \bar{s}s \rangle}{192\pi^4} M_B^2, \\ \Pi_{T_3} &= \frac{3M_B^{10}}{320\pi^6} + \left(\frac{\langle g^2 G^2 \rangle}{384\pi^6} + \frac{m\langle \bar{s}s \rangle}{8\pi^4} \right) M_B^6 - \left(\frac{\langle g^3 f G^3 \rangle}{64\pi^6} + \frac{m\langle g\bar{s}\sigma Gs \rangle}{32\pi^4} + \frac{g^2\langle \bar{q}q \rangle^2}{36\pi^4} + \frac{g^2\langle \bar{s}s \rangle^2}{108\pi^4} \right) M_B^4 + \frac{m\langle g^2 G^2 \rangle \langle \bar{s}s \rangle}{192\pi^4} M_B^2, \\ \Pi_{S_6} &= \frac{M_B^{10}}{1280\pi^6} + \left(\frac{m\langle \bar{s}s \rangle}{96\pi^4} - \frac{m\langle \bar{q}q \rangle}{48\pi^4} - \frac{\langle g^2 G^2 \rangle}{3072\pi^6} \right) M_B^6 \\ &\quad + \left(\frac{m\langle g\bar{q}G\sigma q \rangle}{64\pi^4} - \frac{5\langle g^3 f G^3 \rangle}{13824\pi^6} - \frac{\langle \bar{q}q \rangle^2}{12\pi^2} + \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle}{12\pi^2} - \frac{7m\langle g\bar{s}\sigma Gs \rangle}{768\pi^4} - \frac{7g^2\langle \bar{q}q \rangle^2}{864\pi^4} - \frac{7g^2\langle \bar{s}s \rangle^2}{2592\pi^4} \right) M_B^4 \\ &\quad + \left(\frac{\langle \bar{q}q \rangle \langle g\bar{q}G\sigma q \rangle}{12\pi^2} - \frac{m\langle g^2 G^2 \rangle \langle \bar{q}q \rangle}{2304\pi^4} - \frac{\langle g\bar{s}\sigma Gs \rangle \langle \bar{q}q \rangle}{24\pi^2} - \frac{m\langle g^2 G^2 \rangle \langle \bar{s}s \rangle}{1536\pi^4} - \frac{\langle g\bar{q}G\sigma q \rangle \langle \bar{s}s \rangle}{24\pi^2} \right) M_B^2, \\ \Pi_{S_3} &= \frac{M_B^{10}}{2560\pi^6} + \left(\frac{m\langle \bar{s}s \rangle}{192\pi^4} - \frac{m\langle \bar{q}q \rangle}{96\pi^4} + \frac{\langle g^2 G^2 \rangle}{3072\pi^6} \right) M_B^6 \\ &\quad + \left(\frac{\langle g^3 f G^3 \rangle}{6912\pi^6} + \frac{m\langle g\bar{q}G\sigma q \rangle}{128\pi^4} + \frac{m\langle g\bar{s}\sigma Gs \rangle}{768\pi^4} - \frac{\langle \bar{q}q \rangle^2}{24\pi^2} + \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle}{24\pi^2} + \frac{g^2\langle \bar{q}q \rangle^2}{864\pi^4} + \frac{g^2\langle \bar{s}s \rangle^2}{2592\pi^4} \right) M_B^4 \\ &\quad + \left(-\frac{5m\langle g^2 G^2 \rangle \langle \bar{q}q \rangle}{2304\pi^4} + \frac{\langle \bar{q}q \rangle \langle g\bar{q}G\sigma q \rangle}{24\pi^2} - \frac{\langle g\bar{q}G\sigma q \rangle \langle \bar{s}s \rangle}{48\pi^2} - \frac{\langle g\bar{s}\sigma Gs \rangle \langle \bar{q}q \rangle}{48\pi^2} + \frac{m\langle g^2 G^2 \rangle \langle \bar{s}s \rangle}{1536\pi^4} \right) M_B^2, \\ \Pi_{V_6} &= \frac{M_B^{10}}{1280\pi^6} + \left(\frac{m\langle \bar{s}s \rangle}{96\pi^4} + \frac{m\langle \bar{q}q \rangle}{48\pi^4} - \frac{\langle g^2 G^2 \rangle}{3072\pi^6} \right) M_B^6 \\ &\quad + \left(\frac{-5\langle g^3 f G^3 \rangle}{13824\pi^6} - \frac{m\langle g\bar{q}G\sigma q \rangle}{64\pi^4} - \frac{7m\langle g\bar{s}\sigma Gs \rangle}{768\pi^4} + \frac{\langle \bar{q}q \rangle^2}{12\pi^2} - \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle}{12\pi^2} - \frac{7g^2\langle \bar{q}q \rangle^2}{864\pi^4} - \frac{7g^2\langle \bar{s}s \rangle^2}{2592\pi^4} \right) M_B^4 \\ &\quad + \left(\frac{m\langle g^2 G^2 \rangle \langle \bar{q}q \rangle}{2304\pi^4} - \frac{\langle \bar{q}q \rangle \langle g\bar{q}G\sigma q \rangle}{12\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle \langle \bar{q}q \rangle}{24\pi^2} + \frac{\langle g\bar{q}G\sigma q \rangle \langle \bar{s}s \rangle}{24\pi^2} - \frac{m\langle g^2 G^2 \rangle \langle \bar{s}s \rangle}{1536\pi^4} \right) M_B^2, \end{aligned}$$

$$\begin{aligned}
\Pi_{V_3} &= \frac{M_B^{10}}{2560\pi^6} + \left(\frac{m\langle\bar{s}s\rangle}{192\pi^4} + \frac{m\langle\bar{q}q\rangle}{96\pi^4} + \frac{\langle g^2 G^2 \rangle}{3072\pi^6} \right) M_B^6 \\
&+ \left(\frac{\langle g^3 f G^3 \rangle}{6912\pi^6} - \frac{m\langle g\bar{q}G\sigma q \rangle}{128\pi^4} + \frac{m\langle g\bar{s}\sigma Gs \rangle}{768\pi^4} + \frac{\langle\bar{q}q\rangle^2}{24\pi^2} - \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{24\pi^2} + \frac{g^2\langle\bar{q}q\rangle^2}{864\pi^4} + \frac{g^2\langle\bar{s}s\rangle^2}{2592\pi^4} \right) M_B^4 \\
&+ \left(\frac{5m\langle g^2 G^2 \rangle\langle\bar{q}q\rangle}{2304\pi^4} - \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq \rangle}{24\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle\langle\bar{q}q\rangle}{48\pi^2} + \frac{\langle g\bar{q}\sigma Gq \rangle\langle\bar{s}s\rangle}{48\pi^2} + \frac{m\langle g^2 G^2 \rangle\langle\bar{s}s\rangle}{1536\pi^4} \right) M_B^2, \\
\Pi_{P_6} &= \frac{M_B^{10}}{320\pi^6} + \left(\frac{m\langle\bar{s}s\rangle}{24\pi^4} - \frac{m\langle\bar{q}q\rangle}{24\pi^4} + \frac{5\langle g^2 G^2 \rangle}{1536\pi^6} \right) M_B^6 \\
&+ \left(-\frac{113\langle g^3 f G^3 \rangle}{13824\pi^6} - \frac{m\langle g\bar{q}G\sigma q \rangle}{128\pi^4} + \frac{7m\langle g\bar{s}\sigma Gs \rangle}{384\pi^4} - \frac{\langle\bar{q}q\rangle^2}{6\pi^2} + \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{6\pi^2} + \frac{7g^2\langle\bar{q}q\rangle^2}{432\pi^4} + \frac{7g^2\langle\bar{s}s\rangle^2}{1296\pi^4} \right) M_B^4 \\
&+ \left(-\frac{7m\langle g^2 G^2 \rangle\langle\bar{q}q\rangle}{1152\pi^4} - \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq \rangle}{24\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle\langle\bar{q}q\rangle}{48\pi^2} + \frac{\langle g\bar{q}\sigma Gq \rangle\langle\bar{s}s\rangle}{48\pi^2} + \frac{5m\langle g^2 G^2 \rangle\langle\bar{s}s\rangle}{768\pi^4} \right) M_B^2, \\
\Pi_{P_3} &= \frac{M_B^{10}}{640\pi^6} + \left(\frac{m\langle\bar{s}s\rangle}{48\pi^4} - \frac{m\langle\bar{q}q\rangle}{48\pi^4} + \frac{5\langle g^2 G^2 \rangle}{1536\pi^6} \right) M_B^6 \\
&+ \left(-\frac{113\langle g^3 f G^3 \rangle}{13824\pi^6} - \frac{3m\langle g\bar{q}G\sigma q \rangle}{128\pi^4} + \frac{11m\langle g\bar{s}\sigma Gs \rangle}{384\pi^4} - \frac{\langle\bar{q}q\rangle^2}{12\pi^2} + \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{12\pi^2} + \frac{5g^2\langle\bar{q}q\rangle^2}{432\pi^4} + \frac{5g^2\langle\bar{s}s\rangle^2}{1296\pi^4} \right) M_B^4 \\
&+ \left(-\frac{5m\langle g^2 G^2 \rangle\langle\bar{q}q\rangle}{1152\pi^4} - \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq \rangle}{8\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle\langle\bar{q}q\rangle}{16\pi^2} + \frac{\langle g\bar{q}\sigma Gq \rangle\langle\bar{s}s\rangle}{16\pi^2} + \frac{5m\langle g^2 G^2 \rangle\langle\bar{s}s\rangle}{768\pi^4} \right) M_B^2, \\
\Pi_{A_6} &= \frac{M_B^{10}}{320\pi^6} + \left(\frac{m\langle\bar{s}s\rangle}{24\pi^4} + \frac{m\langle\bar{q}q\rangle}{24\pi^4} + \frac{5\langle g^2 G^2 \rangle}{1536\pi^6} \right) M_B^6 \\
&+ \left(-\frac{113\langle g^3 f G^3 \rangle}{13824\pi^6} + \frac{m\langle g\bar{q}G\sigma q \rangle}{128\pi^4} + \frac{7m\langle g\bar{s}\sigma Gs \rangle}{384\pi^4} + \frac{\langle\bar{q}q\rangle^2}{6\pi^2} - \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{6\pi^2} + \frac{7g^2\langle\bar{q}q\rangle^2}{432\pi^4} + \frac{7g^2\langle\bar{s}s\rangle^2}{1296\pi^4} \right) M_B^4 \\
&+ \left(\frac{7m\langle g^2 G^2 \rangle\langle\bar{q}q\rangle}{1152\pi^4} + \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq \rangle}{24\pi^2} - \frac{\langle g\bar{s}\sigma Gs \rangle\langle\bar{q}q\rangle}{48\pi^2} - \frac{\langle g\bar{q}\sigma Gq \rangle\langle\bar{s}s\rangle}{48\pi^2} + \frac{5m\langle g^2 G^2 \rangle\langle\bar{s}s\rangle}{768\pi^4} \right) M_B^2, \\
\Pi_{A_3} &= \frac{M_B^{10}}{640\pi^6} + \left(\frac{m\langle\bar{s}s\rangle}{48\pi^4} + \frac{m\langle\bar{q}q\rangle}{48\pi^4} + \frac{\langle g^2 G^2 \rangle}{1536\pi^6} \right) M_B^6 \\
&+ \left(-\frac{25\langle g^3 f G^3 \rangle}{13824\pi^6} - \frac{m\langle g\bar{q}G\sigma q \rangle}{128\pi^4} - \frac{m\langle g\bar{s}\sigma Gs \rangle}{384\pi^4} + \frac{\langle\bar{q}q\rangle^2}{12\pi^2} - \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{12\pi^2} - \frac{g^2\langle\bar{q}q\rangle^2}{432\pi^4} - \frac{g^2\langle\bar{s}s\rangle^2}{1296\pi^4} \right) M_B^4 \\
&+ \left(-\frac{m\langle g^2 G^2 \rangle\langle\bar{q}q\rangle}{1152\pi^4} - \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq \rangle}{24\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle\langle\bar{q}q\rangle}{48\pi^2} + \frac{\langle g\bar{q}\sigma Gq \rangle\langle\bar{s}s\rangle}{48\pi^2} + \frac{m\langle g^2 G^2 \rangle\langle\bar{s}s\rangle}{768\pi^4} \right) M_B^2.
\end{aligned}$$

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