

Canonical ensemble of a self-gravitating matter thin shell in AdS space

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We build the canonical ensemble of a hot self-gravitating matter thin shell in anti-de Sitter (AdS) space by finding its partition function through the Euclidean path integral approach with fixed temperature at the conformal boundary. We obtain the reduced action of the system by restricting the path integral to spherically symmetric metrics with given boundary conditions and with the Hamiltonian constraint satisfied. The stationary conditions, i.e., the mechanical equilibrium and the thermodynamic equilibrium, are obtained from minimizing the reduced action. Evaluating the perturbed reduced action at the stationary points yields the mechanical stability condition and the thermodynamic stability condition. The reduced action calculated at the stationary points gives the partition function in the zero-loop approximation, and from it, the thermodynamic properties of the system are acquired. Within thermodynamics alone, the only stability condition that one can establish is thermodynamic stability, which follows from the computation of the heat capacity. For given specific pressure and temperature equations of state for the shell, we obtain the solutions of the ensemble. There are four different thin shell solutions, one of them is fully stable, i.e., is stable mechanically and thermodynamically. For the equations of state given, we find a first order phase transition from the matter thermodynamic phase to the Hawking-Page black hole phase. Moreover, there is a maximum temperature above which the shell ceases to exist, presumably at these high temperatures the shell inevitably collapses to a black hole.

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I. INTRODUCTION

Lumps of excited matter fields have temperature. Lumps of excited gravitational fields also have temperature, the most notable case being a black hole. The interface of contact between a black hole and the external world is the event horizon, and so it is expected that this horizon characterizes the black hole temperature itself. The sequence of developments that led to the realization that black holes are objects with temperature and have thermodynamic properties was gradual and intriguing. It started with the possibility of energy extraction from a black hole and its consequences, notably the appearance of the irreducible mass concept and the demonstration that the black hole event horizon area always increases. This indicated that, possibly, fundamental phenomena were lurking behind what could be expected. These

anticipations were confirmed when Bekenstein [1], in order to save the second law of thermodynamics, advanced the hypothesis that black holes possess entropy proportional to its area, paving the way to envisage black holes in a new light, namely, as thermodynamic objects. The black hole thermodynamic characterization was pushed forward when it was found that there exist four laws of black hole mechanics [2] in striking analogy to the four laws of thermodynamics. It was then finalized by Hawking [3] with the discovery that black holes interacting with quantum fields emit radiation with a black body spectrum, at the Hawking temperature $T_H = \frac{\kappa}{2\pi}$, where κ is the surface gravity of the horizon. With this expression for the temperature and the first law of black hole mechanics, which is the energy conservation law, the entropy could be determined exactly with the factor of one over four, i.e., a black hole possessed the Bekenstein-Hawking entropy given by $S = \frac{1}{4} \frac{A_+}{l_p^2}$, where A_+ is the area of the event horizon and l_p is the Planck length. The connection between black holes, thermodynamics, and quantum fields put black holes in the center of physics phenomena. It led to an intensive research on quantum fields in curved spacetimes, on the quantum nature of spacetime, and to the study of quantum statistical mechanics through the construction of statistical ensembles in curved spacetimes.

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In relation to statistical ensembles, the development came from the Euclidean path integral approach of Feynman applied to gravitational systems. It is known that, when Euclideanized, the path integral gives statistical mechanics, so when the same procedure is made to black hole spacetimes, the path integral should give statistical mechanics of black holes. The idea is that one can calculate the partition function of stationary black hole spaces through the Euclidean path integral approach to quantum gravity; i.e., the partition function is formally given by $Z = \int Dg_{\alpha\beta} e^{-I[g_{\mu\nu}]}$, where $g_{\alpha\beta}$ is the Euclidean metric and I is the Euclidean action. The path integral in curved spaces possesses a number of difficulties which are still unresolved, but at some level one can proceed. Gibbons and Hawking [4] used the Euclidean Schwarzschild solution to find the corresponding action and study the canonical ensemble of the system, where the temperature is fixed at spatial infinity. In order to deal with the path integral, the saddle point approximation was performed, where the action would be expanded around the Euclidean Einstein solutions and only the zeroth order term would be dealt with, justifying the usage of the action evaluated at the classical solutions to describe the partition function of the canonical ensemble. They further used the Euclidean Reissner-Nordström and Kerr black hole solutions to find the corresponding actions in the zero order approximation, and study the grand canonical ensembles of those systems, where the temperature and the appropriate potential are fixed at spatial infinity. Yet, for these black holes, and in particular, for the Schwarzschild black hole, it was found that the heat capacity was negative, indicating thermodynamic instability and showing that, under small thermal perturbations, these black holes either evaporate or grow indefinitely [5]. Taking into account up to second order terms in the expansion of the action named as one-loop approximation, which amounts to performing quantum perturbations of the background metric, it was found that such unstable Schwarzschild solutions had an unstable mode of perturbations that allowed the tunneling of hot flat space to a black hole state which could then grow indefinitely [6], an indication that hot flat space was not a global minimum of the action. On the other hand, if one examines a Schwarzschild black hole in anti-de Sitter space, i.e., AdS space, there is still a black hole solution analogue to the Gibbons-Hawking unstable solution but another solution arises which is thermodynamically stable, and additionally, hot AdS space can undergo a first order phase transition to the stable black hole [7]. By enclosing the Schwarzschild black hole in a spherical box with fixed radius and fixed temperature at its boundary, it was found that if the box is small enough then the unstable mode in Schwarzschild geometry ceases to exist [8]. Since AdS is a natural box, and an assembled box acts as a stabilizer of the geometry, this motivated York to enclose the pure Schwarzschild black hole inside a finite cavity to study

the corresponding canonical ensemble [9]. York used also the classical solutions of the ensemble to compute the action and so the free energy of the system. However, to analyze the stability and phase transitions, he built a generalized free energy where the relation between the temperature and the horizon radius was relaxed. The York formalism was then formally constructed in [10], where the path integral was constrained to the hypersurface of paths obeying the Hamiltonian and momentum constraints, yielding a reduced action corresponding to the generalized free energy of the spherically symmetric black hole inside a cavity. One can then perform the saddle point approximation by expanding the reduced action around its stationary points and one can obtain the expansion up to second order corrections to analyze stability in the hypersurface of constrained paths. Now, if one constructs only the contributions of the stationary points, one is performing the zero-loop approximation, which is then equivalent to the Gibbons-Hawking approach. For the black hole in a cavity, it was found that two black hole solutions also exist, with the smallest being unstable and the largest being stable, and also that hot flat space could undergo a first order phase transition to the stable black hole. Thus, the York formalism makes the canonical ensemble of asymptotically flat black holes valid, while the canonical ensemble of AdS black holes is valid naturally. Within this formalism one can even obtain quantum gravity corrections by calculating the one-loop terms of the constrained path integral. Further developments were made in [11–25] for several different types of black holes in spaces with negative, zero, or positive cosmological constant.

It is also important to establish in a self-consistent manner the ways that gravity and matter interact and the effects of this interplay on the thermodynamics of gravitating matter systems. An effective useful model is the self-gravitating matter thin shell. In [26], the first law of thermodynamics was applied to these shells, and it was found that the Einstein equations give an equation of state for the pressure and that the temperature equation of state has a precise dependence on the gravitational radius of the system. The case of electrically charged shells was first treated in [27]. The results were extended to arbitrary dimensions in [28,29]. A feature of such a shell is that one can pick the same temperature equation of state of the black hole for the shell and put it very close to its gravitational radius, one then has an object mimicking the thermodynamics of a black hole. For shells made of extremal matter, this can be done smoothly, while for the other cases, issues of stability and diverging pressure may arise. Another instance where the first law of thermodynamics of self-gravitating systems can be used with benefit is in the quasiblack hole approach, where matter systems on the verge of becoming a black hole are analyzed [30–32].

In this work, we analyze a more fundamental way of studying the thermodynamics of self-gravitating matter

systems by building their statistical ensemble. This was sketched first in [11], where a thin shell was considered with a black hole inside, and it was found that the entropy of the system is the sum of the entropies of the shell and the black hole. This was further worked in detail in [23], where it was shown that the entropy and the thermodynamics are given simply as a function of the gravitational radius of the system. We develop these ideas and explore the matter sector of spherically symmetric AdS spaces. For that, we carefully build the canonical ensemble of a hot self-gravitating matter thin shell in AdS using York's path integral formalism. By establishing the reduced action through the constraint equations, we are able to show that the path integral depends only on the gravitational radius of the system and on the radius of the shell. One stationary point gives the mechanical solution which shows that the radius of the shell depends only on the gravitational radius, making the action depending solely on this latter radius. The other stationary point yields the thermodynamic solution which sets the gravitational radius as a function of the temperature of the ensemble. The zero order approximation then gives the partition function as a function of this temperature. Through perturbations of the ensemble, we analyze the corresponding stability criteria of the system. The solutions for the stationary points of the action depend on the choice of the equation of state of the matter. We give a linear equation of state for the pressure and an equation of state for the temperature that allow for stable solutions. From the zero-loop approximation, we obtain the thermodynamics of the system, namely the free energy, the mean energy, the entropy, and the heat capacity. We compare the free energy of the self-gravitating matter thin shell with the free energy of the black hole sector and analyze the existence of first order phase transitions in AdS. In this way, we simulate the interaction of matter and gravitation both treated at zero-loop level with striking results. One can include results at one-loop approximation valid for nonself-gravitating radiation in AdS space and compare with the results at zero-loop approximation found for the self-gravitating hot shell. Our analysis is valid for microscopic configurations not near the Planck scale.

The paper is organized as follows. In Sec. II, we work out the canonical ensemble, characterized by a fixed temperature, of a hot self-gravitating matter thin shell in AdS space. We define the partition function of the system through the Euclidean path integral approach, postulate the general Euclidean action and the matter free energy, impose spherical symmetry and provide the boundary conditions. We then enforce the Hamiltonian and momentum constraints to finally find the reduced action. We show that at this stage the partition function reduces to a sum over metrics with different gravitational radii and different radii of the shell. In Sec. III, we perform the zero-loop approximation; i.e., we investigate the reduced action evaluated at

its stationary points. The mechanical stationary point yields that the radius of the shell is a function of its gravitational radius and so the partition function reduces to a sum over metrics with different gravitational radii alone. The thermodynamic stationary point yields that the gravitational radius is a function of the ensemble temperature, and so finally, we find that the partition function is a function of the temperature alone, which is fixed. We also find the stability conditions, and study mechanical and thermodynamic stability of the shell. In Sec. IV, we derive the thermodynamics from the path integral in the zero-loop approximation and show that positive heat capacity for the system is equivalent to the ensemble thermodynamic stability. In Sec. V, we give specific equations of state for the matter, namely, the matter pressure equation and the matter temperature equation to fully determine the system. We find the radius of the shell as a function of its gravitational radius, the matter entropy as a function of the gravitational radius, and the gravitational radius as a function of the ensemble temperature. In Sec. VI, we compare the results for the hot thin shell in AdS with the results for the Hawking-Page black hole in AdS; notably we find the favorable states for different ensemble temperatures. In Sec. VII, we conclude.

We choose units where the Planck constant is $\hbar = 1$, the Boltzmann constant is $k = 1$ and the speed of light is $c = 1$. The gravitational constant is written as the square of the Planck length $G = l_p^2$ in these units, while mass is understood as the inverse of the reduced Compton wavelength associated to that mass, and temperature is understood as the inverse of a thermal wavelength. We adopt Greek indices for tensors defined on the whole space and Latin indices for tensors defined on hypersurfaces of constant spacelike coordinates.

II. CANONICAL ENSEMBLE OF A HOT SELF-GRAVITATING MATTER THIN SHELL IN ASYMPTOTICALLY ADS SPACE AND THE REDUCED ACTION

A. The partition function given by the Euclidean path integral approach

Generally, in quantum systems represented by fields ψ at temperature T , one can build the statistical mechanics partition function Z with expression $Z = \text{Tr}(e^{-\beta H})$, where $\beta = \frac{1}{T}$ is the inverse temperature and H is the Hamiltonian of the system. The partition function Z gives all the statistical mechanics information about the system, in particular its thermodynamics. Through the Feynman path integral and by performing the Euclideanization of time, one obtains the path integral formula for the partition function $Z = \int D\psi e^{-I[\psi]}$, where I is the Euclidean action and the integration is done over periodic fields in the imaginary time variable in the case the fields ψ are bosonic.

In the Euclidean path integral approach to quantum gravity, one extends this formula to gravitational systems.

The partition function of curved space with matter can then be given by $Z = \int Dg_{\alpha\beta} D\psi e^{-I[g_{\mu\nu}, \psi]}$, where $g_{\alpha\beta}$ represents the Euclidean metric, ψ describes the matter fields, and I is the Euclidean action. The integration is done over metrics and matter fields that are periodic in Euclidean time, with fixed quantities at the boundary of the curved space. When dealing with the canonical ensemble, the data fixed at the boundary are simply the inverse temperature $\beta = \frac{1}{T}$, which is given by the total Euclidean proper time at the boundary. For the case of asymptotically AdS spaces, the Euclidean proper time length at infinity becomes zero, and we instead use the asymptotic behavior of AdS to fix a quantity $\bar{\beta}$, which is proportional to the inverse of the temperature $\bar{T}$ of a conformal boundary. We choose the proportionality constant to be one so that $\bar{\beta} = \frac{1}{\bar{T}}$.

Due to the difficulties in performing the full path integral, we employ the zero-loop approximation of the path integral, but we do it in a different way. We assume that we can put the path integral over the matter fields inside the path integral over metrics in the sense of $Z = \int Dg_{\alpha\beta} e^{-I_g} \int D\psi e^{-I_\psi}$, where $I_g = I_g[g_{\mu\nu}]$ is the Euclidean gravitational action and $I_\psi = I_\psi[g_{\mu\nu}, \psi]$ is the Euclidean matter action of any field ψ . We assume minimal coupling between the field ψ and the metric $g_{\alpha\beta}$. While for the general case one cannot perform the path integral on matter, for the case of a matter thin shell in spherical symmetry one can perform the path integral exactly, if the action is quadratic in the field. This is because the metric components will be constants in the action of the matter thin shell and the path integral becomes an integration over Gaussian functions, yielding $\int D\psi e^{-I_\psi[g_{\mu\nu}, \psi]} = e^{-I_m[g_{\mu\nu}]}$, where I_m is an effective matter action. So, the partition function, after performing the integration over the matter fields, becomes

$$Z = \int Dg_{\alpha\beta} e^{-I[g_{\mu\nu}]}, \quad (1)$$

with $I[g_{\mu\nu}]$ given by

$$I = I_g + I_m, \quad (2)$$

more precisely, $I[g_{\mu\nu}] = I_g[g_{\mu\nu}] + I_m[g_{\mu\nu}]$. This motivates the expression of the action below. With the action, we perform the zero-loop approximation of the path integral by first using the Hamiltonian and momentum constraints to obtain the reduced action I_* . Then, the stationary points of the reduced action are found and stability can be analyzed.

B. Euclidean action

The asymptotically AdS space can be represented by M , which splits into two spaces M_1 and M_2 due to the presence of a matter thin shell N separating them. The outer

boundary of M lies at spatial infinity and is represented by B . The gravitational part of the Euclidean action considered here is the Euclidean Einstein-Hilbert action with the Gibbons-Hawking-York boundary term written as $I_g = -\frac{1}{16\pi l_p^2} \int_{M \setminus \{N\}} (R + \frac{6}{l^2}) \sqrt{g} d^4x + \int_N \frac{[K]}{8\pi l_p^2} \sqrt{h} d^3y - \frac{1}{8\pi l_p^2} \int_B K \sqrt{h} d^3y - I_{\text{subtr}}$, where l_p is the Planck length, R is the Ricci scalar, g is the metric determinant, $l = \sqrt{-\frac{3}{\Lambda}}$ is defined as the AdS length, with Λ being the negative cosmological constant, h_{ab} is the induced metric from the space metric $g_{\alpha\beta}$ on the hypersurface in analysis, h is the determinant of h_{ab} , K_{ab} is the extrinsic curvature of the hypersurface in analysis, with trace K given by $K = n^\alpha{}_{;\alpha}$, n^α being the normal vector to the hypersurface in analysis, i.e., either N or B , the bracket $[K] = K|_{M_2} - K|_{M_1}$ means the difference between K evaluated at M_2 and K evaluated at M_1 , and I_{subtr} is the action of a reference space, in this case AdS. In relation to the matter part of the Euclidean action, we take as Lagrangian density, the matter free energy per unit area $\mathcal{F}_m$. This stems from the fact we are dealing with the canonical ensemble, which is connected to the thermodynamic Helmholtz free energy. Then, $I_m = \int \mathcal{F}_m[h_{ab}] \sqrt{h} d^3x$, where $\mathcal{F}_m$ is a functional of the induced metric h_{ab} on the shell, with h being the determinant of h_{ab} . The Euclidean action of the system $I = I_g + I_m$ is then given by

$$I = -\frac{1}{16\pi l_p^2} \int_{M \setminus \{N\}} \left(R + \frac{6}{l^2} \right) \sqrt{g} d^4x + \int_N \left(\frac{[K]}{8\pi l_p^2} + \mathcal{F}_m[h_{ab}] \right) \sqrt{h} d^3y - \frac{1}{8\pi l_p^2} \int_B K \sqrt{h} d^3y - I_{\text{subtr}}, \quad (3)$$

with all quantities having been properly defined.

C. Spherically symmetric geometry

In the analysis we perform of the path integral, we only consider paths which are spherically symmetric. We also assume that the space is static. We write then the metric for M_1 as

$$ds_{M_1}^2 = b_1^2(y) \frac{b_2^2(y_m)}{b_1^2(y_m)} d\tau^2 + a_1^2(y) dy^2 + r(y)^2 d\Omega^2, \quad 0 \leq y < y_m. \quad (4)$$

For the thin shell N , we write the induced metric as

$$ds_N^2 = b_2^2(y_m) d\tau^2 + r^2(y_m) d\Omega^2, \quad y = y_m. \quad (5)$$

For M_2 , we write

$$ds_{M_2}^2 = b_2^2(y)d\tau^2 + a_2^2(y)dy^2 + r^2(y)d\Omega^2, \quad y_m < y < 1. \quad (6)$$

Here b_1 , b_2 , a_1 , a_2 , and r are functions of the coordinate y . The Euclidean time coordinate τ is chosen to be an angular coordinate in the interval $0 < \tau < 2\pi$ on M , the radial coordinate y takes values as above, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the 2-sphere with surface area $\Omega = 4\pi$, and the coordinates θ and ϕ are the usual spherical coordinates. The points at the thin shell are located at $y = y_m$ and we exhibit y_m as a label that is fixed, while the radius of the shell $r(y_m)$ depends on the arbitrary function $r(y)$.

D. Matter free energy and stress-energy tensor

The matter Lagrangian density can be identified to the thermodynamic Helmholtz free energy density since we are dealing with the canonical ensemble. The Helmholtz free energy potential F is described by $F = E - TS$, where E is the thermodynamic energy, S the entropy, and T the temperature of the reservoir. We must then analyze the associated density quantities, and so the free energy density $\mathcal{F}_m$ can be written as

$$\mathcal{F}_m[h_{ab}] = \epsilon_m[h_{ab}] - T_m[h_{ab}]s_m[h_{ab}], \quad (7)$$

where ϵ_m is the total energy density of the matter, T_m is the local temperature of the shell, and s_m is the entropy density of the matter. All these quantities are functionals of the induced metric h_{ab} . Since from Eq. (5), one has that h_{ab} depends on two arbitrary quantities that are seen as constants at the shell, $b_2(y_m)$ and $r(y_m)$, the matter free energy density $\mathcal{F}_m[h_{ab}]$ depends only locally on these two quantities, and so a dependence on derivatives of h_{ab} is ruled out.

We define the radius α of the shell as

$$\alpha = r(y_m). \quad (8)$$

Although in order to keep a consistent nomenclature, we should have defined the radius of the shell as $r_m = r(y_m)$, we prefer to stick to $\alpha = r(y_m)$ to not overcrowd the symbols ahead. One can define a local temperature at some point y , as $T(y) = \frac{1}{2\pi b_2(y)}$. So the local temperature of the shell is

$$T_m = \frac{1}{2\pi b_2(y_m)}. \quad (9)$$

The rationale for this definition comes from continuity, since in the canonical ensemble, one fixes the Euclidean proper time length at the boundary and assigns to it the meaning of an inverse temperature. One must keep in mind however that this definition does not give information about

the specific expression of the temperature since $b_2(y_m)$ is arbitrary.

The free energy per unit area $\mathcal{F}_m[h_{ab}] = \mathcal{F}_m[b_2(y_m), r(y_m)]$ can then be put in the form,

$$\mathcal{F}_m[h_{ab}] = \mathcal{F}_m[\alpha, T_m], \quad (10)$$

upon using Eqs. (8) and (9). Now, we assume the first law to describe the matter energy density as $d\epsilon_m = T_m ds_m - 2(\epsilon_m - T_m s_m + p_m) \frac{d\alpha}{\alpha}$, where p_m is the matter tangential pressure at the shell. Thus, from Eq. (7), the free energy density has the differential,

$$d\mathcal{F}_m = -s_m dT_m - 2(\mathcal{F}_m + p_m) \frac{d\alpha}{\alpha}. \quad (11)$$

With the known differential of the free energy density regarding its dependence on the metric components, one can compute the surface stress-energy tensor S^{ab} as the functional derivative, $S^{ab} = -\frac{2}{\sqrt{h}} \frac{\delta(\sqrt{h}\mathcal{F}_m)[h_{ab}]}{\delta h_{ab}}$. From Eq. (5), one has $h_{\tau\tau} = b_2^2(y_m)$ and $h_{\theta\theta} = \frac{h_{\phi\phi}}{\sin^2(\theta)} = r^2(y_m)$. Then, from $\alpha = r(y_m)$ and $T_m = \frac{1}{2\pi b_2(y_m)}$, see Eqs. (8) and (9), one finds that the variation yields $S^\tau_\tau = -\mathcal{F}_m + T_m \frac{\partial \mathcal{F}_m}{\partial T_m}$ and $S^\theta_\theta = S^\phi_\phi = -\frac{1}{2}\alpha \frac{\partial \mathcal{F}_m}{\partial \alpha} - \mathcal{F}_m$, where the partial derivatives are done keeping the hidden variable constant, and $\delta h = h h^{ab} \delta h_{ab}$ has been used. We note that some care is due while performing the variational derivative to obtain S^θ_θ and S^ϕ_ϕ , as one must calculate $d\alpha$ in $d\mathcal{F}_m$ as $d\alpha = d\left(\sqrt{\frac{h_{\theta\theta}h_{\phi\phi}}{\sin^2\theta}}\right)$. From Eqs. (7) and (11), one has $\epsilon_m = \mathcal{F}_m - T_m \frac{\partial \mathcal{F}_m}{\partial T_m}$ and $p_m = -\frac{1}{2}\alpha \frac{\partial \mathcal{F}_m}{\partial \alpha} - \mathcal{F}_m$. Thus, the stress-energy tensor S^a_b has components,

$$S^\tau_\tau = -\epsilon_m, \quad S^\theta_\theta = S^\phi_\phi = p_m. \quad (12)$$

The fluid is thus isotropic; more specifically, it is a perfect fluid. Note that $\epsilon_m = \epsilon_m(\alpha, T_m)$ and $p_m = p_m(\alpha, T_m)$. The rest mass m of the shell is important for the analysis below, and we define it as

$$m = 4\pi\alpha^2\epsilon_m. \quad (13)$$

Since $\epsilon_m = \epsilon_m(\alpha, T_m)$, one has $m = m(\alpha, T_m)$. The dependence of the thermodynamic quantities in α and T_m is helpful when one makes variations of the action on the metric components to find the Hamiltonian constraint; however, it is also helpful to invert the first law of thermodynamics to get $ds_m = \frac{1}{T_m} d\epsilon_m + 2(\epsilon_m - T_m s_m + p_m) \frac{d\alpha}{\alpha}$. We can integrate over the area to obtain the first law of thermodynamics in the form,

$$T_m dS_m = dm + p_m dA_m, \quad (14)$$

where

$$A_m = 4\pi\alpha^2, \quad (15)$$

$$S_m = s_m A_m, \quad (16)$$

are the area of the shell and the entropy of the matter in the shell, respectively. Written like this, also the quantities S_m , T_m , and p_m become functions of m and α . We shall explicitly indicate which dependences are used below.

E. Boundary conditions

1. Origin: $y=0$

We must further impose regularity conditions and boundary conditions on the metrics that are being summed in the path integral. In the region M_1 , the interior region, we impose regularity conditions at $y=0$ corresponding to flat conditions at the origin, i.e.,

$$r|_{y=0} = 0, \quad b_1|_{y=0} \text{ finite and positive}, \quad (17)$$

$$\left. \frac{r'}{a_1} \right|_{y=0} = 1, \quad \left. \frac{1}{r'} \left(\frac{r'}{a_1} \right)' \right|_{y=0} = 0, \quad \left. \frac{b_1'}{a_1} \right|_{y=0} = 0, \quad (18)$$

where a primed quantity means derivative with respect to y , e.g., $r' = \frac{dr}{dy}$. The two conditions in Eq. (17) are the result of choosing the topology $S^1 \times R^3$ for the region in M_1 . The three conditions in Eq. (18) are needed so that the Ricci scalar is well-defined to avoid the occurrence of singularities in M_1 .

2. Infinity: $y=1$

In the region M_2 , the exterior region, we impose that the space behaves as asymptotically AdS, when $y \rightarrow 1$. The outer boundary B is defined as the hypersurface with constant y in the limit of $y \rightarrow 1$, with the radius $r(y)$ diverging in this limit, so $y \rightarrow 1$ implies $r(y) \rightarrow \infty$. Since AdS means a constant curvature space, the metric $g_{\alpha\beta}$ should satisfy asymptotically the vacuum Einstein equations with negative cosmological constant, i.e., the Ricci tensor should obey $R_{\alpha\beta} = -\frac{3}{l^2} g_{\alpha\beta}$, and the Ricci scalar should be $R = -\frac{12}{l^2}$. To establish exactly the asymptotic behavior of the metric, one can perform a conformal transformation such that $\bar{g}_{\alpha\beta} = W^2 g_{\alpha\beta}$, where $\bar{g}_{\alpha\beta}$ is the unphysical metric and W is a conformal factor. Following [33], the conformal factor must vanish at the boundary of the conformal space but in such a way that the conformal metric is nonsingular. The only way to satisfy these conditions is if the conformal factor behaves as $W = cr^{-1}(y)$ near the boundary, where c is a constant that we have chosen here as $c=1$. If one makes a coordinate transformation $W = W(y)$, the

conformal metric near the hypersurface B , which is now described by $W=0$, has in the neighborhood $\mathcal{N}(B)$ of B the following corresponding line element, $d\bar{s}^2|_{\mathcal{N}(B)} = \frac{b_2^2(y)}{r^2(y)} d\tau^2 + \left(\frac{a_2(y)r(y)}{r'(y)}\right)^2 dW^2(y) + d\Omega^2$. The asymptotic behavior of the metric is going to be chosen as boundary conditions to be imposed on the elements of $d\bar{s}^2$ [33,34]. The fact that the original metric $g_{\alpha\beta}$ should satisfy asymptotically the vacuum Einstein equations with negative cosmological constant, $R_{\alpha\beta} = -\frac{3}{l^2} g_{\alpha\beta}$, yields two conditions. One condition is that the conformal induced metric on $\mathcal{N}(B)$ can be written as $d\bar{s}^2 = d\bar{\tau}^2 + d\Omega^2$, where $\bar{\tau}$ is a particular Euclidean time coordinate that can be recast into the coordinate τ as $\bar{\tau} = \gamma\tau$ with γ being a constant. The other condition is that $\bar{g}^{\alpha\beta} \nabla_\alpha W \nabla_\beta W = \frac{1}{l^2}$. Imposing these two conditions to the conformal metric in $d\bar{s}^2$, one gets two following boundary conditions that can be cast in terms of the original metric components. The first boundary condition is

$$\left. \frac{b_2(y)}{r(y)} \right|_{y \rightarrow 1} = \frac{\bar{\beta}}{2\pi l}, \quad (19)$$

where $\frac{\bar{\beta}}{2\pi l}$ is meant as the constant γ of the first condition above, and the parameter $\bar{\beta}$ is defined to be the fixed quantity of the ensemble. In some sense, the parameter $\bar{\beta}$ is proportional to the total proper length of the conformal boundary, and we identify it as being equal to the inverse of the local temperature $\bar{T}$ of the conformal boundary, such that $\bar{\beta} = \frac{1}{\bar{T}}$. It must be pointed out that fixing the inverse temperature in this conformal boundary as $\bar{\beta}$ is a choice of the formalism, with the only freedom being the choice of a constant of proportionality. Here, the choice coincides with the usual Euclidean proper time approach formalism, e.g., the way we define the temperature at infinity for a black hole yields the same as the Hawking-Page definition. This can be seen further below. The second boundary condition is

$$\left. \frac{a_2(y)r(y)}{r'(y)} \right|_{y \rightarrow 1} = l, \quad (20)$$

which gives the typical asymptotic behavior of a_2 in asymptotically AdS space.

We note that one could have chosen a different constant of proportionality c in the conformal transformation as long as the asymptotic AdS behavior is imposed. This indeed leads to a $\bar{\beta}$ only differing by a constant multiplication factor which can be thought of as a change of scale for the temperature and energy measured by the conformal observer at the boundary, e.g., a change of units in the measuring tools of the observer. The physical results do not alter from such choice.

F. Hamiltonian and momentum constraint equations

We impose the Hamiltonian and momentum constraint equations so that the path integral is along the constraint paths. In this context, the Hamiltonian constraint arises from the first variation of the action with respect to the metric components $b_1(y)$ and $b_2(y)$, and so the application of the constraint to the paths can be seen as a first step towards zero-loop approximation, establishing the relation between the curvature of space and the energy density of matter, as in general relativity. Moreover, this is powerful enough that allows the analysis of stability. Only afterwards, we perform the full zero-loop approximation. We start with the Hamiltonian constraint, consisting of one equation for each region M_1 and M_2 , and a junction condition on the matter shell N . We then analyze the momentum constraint.

The Hamiltonian constraint in the regions M_1 and M_2 makes use of the Einstein tensor component G^τ_τ , which can be written as $G^\tau_\tau|_{M_i} = \frac{1}{r'^2} (r[(\frac{r'}{a_i})^2 - 1])'$ for each M_i , with $i = \{1, 2\}$. The Hamiltonian constraint is the Einstein equation $G^\tau_\tau = \frac{3}{l_p^2}$, or explicitly, $\frac{1}{r'^2} (r[(\frac{r'}{a_i})^2 - 1])' = \frac{3}{l_p^2}$, for each region M_i . Thus, the Hamiltonian constraint for each region is satisfied if

$$\left(\frac{r'}{a_1}\right)^2 = 1 + \frac{r^2}{l^2} \equiv f_1(r), \quad (21)$$

$$\left(\frac{r'}{a_2}\right)^2 = 1 + \frac{r^2}{l^2} - \frac{\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}}{r} \equiv f_2(r, \tilde{r}_+), \quad (22)$$

where the regularity condition $\frac{r'}{a_1}|_{y=0} = 1$ in Eq. (18) was used, $\tilde{r}_+$ is the gravitational radius of the system, and it is featured as an integration constant obeying $\tilde{r}_+ < \alpha$, and we have defined the functions $f_1(r)$ and $f_2(r, \tilde{r}_+)$ for convenience. Due to the order of the differential equation in the Hamiltonian constraint equation, the regularity condition $\frac{1}{r}(\frac{r'}{a_1})'|_{y=0} = 0$ in Eq. (18) was not used but it is naturally satisfied. The same thing happens for the function $\frac{r'}{a_2}$ which obeys naturally the boundary condition Eq. (20).

The Hamiltonian constraint in the hypersurface N is described by the junction condition $[K^\tau_\tau] - [K] = -8\pi l_p^2 S^\tau_\tau$, where S^τ_τ is the $\tau\tau$ component of the surface stress-energy tensor. The extrinsic curvature components of constant y surfaces are given by $K^\tau_\tau|_{M_i} = \frac{b'_i}{a_i b_i}$, and $K^\theta_\theta|_{M_i} = K^\phi_\phi|_{M_i} = \frac{r'}{a_i r}$, with the trace of the extrinsic curvature being $K = \frac{b'_i}{a_i b_i} + 2 \frac{r'}{a_i r}$. The surface stress-energy tensor is the functional derivative S^{ab} , with $S^\tau_\tau = -\epsilon_m$, see Eq. (12). Then, for the mass $m = 4\pi\alpha^2\epsilon_m$, Eq. (13), we find the Hamiltonian constraint at the shell is $m = \frac{\alpha}{l_p^2} [\frac{r'}{a_1} - \frac{r'}{a_2}]|_{y=y_m}$, i.e.,

$$m = \frac{\alpha}{l_p^2} (\sqrt{f_1(\alpha)} - \sqrt{f_2(\alpha, \tilde{r}_+)}), \quad (23)$$

with $f_1(\alpha) = (\frac{r'}{a_1})^2|_\alpha = 1 + \frac{\alpha^2}{l^2}$ and $f_2(\alpha, \tilde{r}_+) = (\frac{r'}{a_2})^2|_\alpha = 1 + \frac{\alpha^2}{l^2} - \frac{\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}}{\alpha}$; see Eqs. (21) and (22), respectively. While the dependence of m in the metric components is described by $m = m(\alpha, T_m)$, one can invert in order to T_m and get $T_m = T_m(m, \alpha)$. Using now the junction condition Eq. (23), we obtain the temperature of the shell as a function of $\tilde{r}_+$ and α as $T_m = T_m(m(\tilde{r}_+, \alpha), \alpha)$, as long as the equation of state $T_m(m, \alpha)$ is provided.

In relation to the momentum constraints, due to the spherical symmetry of the metrics in Eqs. (4) and (6) and the symmetry on translations in τ , the momentum constraints in the regions M_1 and M_2 are satisfied *a priori*. Moreover, the momentum constraint at the shell is satisfied since the matter shell stress tensor is diagonal as $\mathcal{F}_m$ is a functional only of b_2 and α .

G. Reduced action and the partition function from the constrained path integral

If one investigates the regularity conditions at the origin in Eqs. (17) and (18), the boundary conditions at infinity in Eqs. (19) and (20), the Hamiltonian constraint solutions in Eqs. (21)–(23), and the fact that the Ricci scalar here is given by $R|_{M_i} = -2G^\tau_\tau|_{M_i} - \frac{2}{a_i b_i r^2} (\frac{r^2 b'_i}{a_i})'$ for each region M_i , the action in Eq. (3) with these considerations yields the reduced action I_* with the form $I_* = -\frac{\bar{\beta}}{l_p^4} (r^2 \sqrt{f_2(r)})|_{y \rightarrow 1} - S_m(m(\tilde{r}_+, \alpha), \alpha) - I_{\text{subtr}}$, where an equation of state for the shell is assumed so that $S_m(T_m(m, \alpha), \alpha) = S_m(m, \alpha)$. The first term in I_* is formally divergent, the reason being that we are in asymptotically AdS space. We must therefore pick the reference action I_{subtr} so that pure hot AdS space has vanishing action with the same inverse temperature $\bar{\beta}$ at the conformal boundary, thus regularizing the reduced action. Pure hot AdS space is defined as AdS at fixed $\bar{\beta}$ with nothing in it, it has vanishing action, and could as well be called classical hot AdS space. This notion of pure, or classical, hot AdS space, is very important in the zero-loop approximation of the path integral as it renders the approach self-consistent at zero-loop order. Pure hot AdS space can be obtained by setting $\tilde{r}_+ = 0$ and $S_m = 0$, and so $I_{\text{subtr}} = -\frac{\bar{\beta}}{l_p^4} (r^2 \sqrt{f_1(r)})|_{y \rightarrow 1}$. By inspecting the limit $(r^2 \sqrt{f_1(r)} - r^2 \sqrt{f_2(r)})|_{y \rightarrow 1} = \frac{l}{2} (\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2})$, the reduced action becomes

$$I_*[\bar{\beta}; \tilde{r}_+, \alpha] = \frac{\bar{\beta}}{2l_p^2} \left(\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2} \right) - S_m(m(\tilde{r}_+, \alpha), \alpha), \quad (24)$$

where $m(\tilde{r}_+, \alpha)$ is given by the right-hand side of Eq. (23). The partition function of Eq. (1) with its path integral reduces thus to the following expression:

$$Z[\bar{\beta}] = \int D\tilde{r}_+ D\alpha e^{-I_*[\bar{\beta}; \tilde{r}_+, \alpha]}, \quad (25)$$

as the sum over different metrics with spherical symmetry reduces to the sum over metrics with different $\tilde{r}_+$ and different α . For clarification, the Hamiltonian constraint that was imposed makes the dependence of the action in the metric components $b_1(y)$ and $b_2(y)$ disappear, and moreover makes the components a_1 and a_2 be functional dependent on $\tilde{r}_+$ and on $r(y)$. The integration over α arises due to the sum over metric functions $r(y)$. Although the Hamiltonian constraint ensures that the metric in the bulk has the same form for any arbitrary function $r(y)$, through a coordinate transformation $r = r(y)$, the value $\alpha = r(y_m)$ that separates the regions M_1 and M_2 depends on the specific function $r(y)$, and so one must sum over the possible values of α .

III. THE ZERO-LOOP APPROXIMATION AND STABILITY CRITERIA FROM THE REDUCED ACTION OF A HOT SELF-GRAVITATING THIN SHELL IN ASYMPTOTICALLY ADS SPACE

A. The stationary conditions, the hot thin matter shell solutions in AdS space, and the zero-loop approximation from the reduced action

With the reduced action of the system being given by Eq. (24), we now minimize it to find the action in the zero-loop approximation. To find the minimum of the action, we have to find its stationary conditions which are given by

$$\frac{\partial I_*[\bar{\beta}; \tilde{r}_+, \alpha]}{\partial \alpha} = 0, \quad (26)$$

$$\frac{\partial I_*[\bar{\beta}; \tilde{r}_+, \alpha]}{\partial \tilde{r}_+} = 0. \quad (27)$$

The stationary conditions given in Eqs. (26) and (27) can be understood as the remaining Einstein equations whose solutions minimize the action. Since the reduced action is essentially the Einstein-Hilbert action together with the matter action, having the Hamiltonian constraint being imposed, the minimization of the action in relation to α and to $\tilde{r}_+$ is equivalent to the minimization in relation to the metric components $g_{\theta\theta}$ and to g_{yy} . And so, these conditions yield the Dirac delta terms of $G_{\theta\theta} = 8\pi T_{\theta\theta}$, and the equation $G_{yy} = 0$, where $G_{\theta\theta}$ and G_{yy} are the corresponding components of the Einstein tensor, and $T_{\theta\theta}$ is the corresponding component of the stress-energy tensor. In order to further develop the derivatives of Eqs. (26) and (27), we have to find $\frac{\partial S_m}{\partial \alpha}$ and $\frac{\partial S_m}{\partial \tilde{r}_+}$. We remind that, from the first law of

thermodynamics given in Eq. (14), the matter entropy has the differential form $dS_m = \frac{dm(\tilde{r}_+, \alpha)}{T_m} + \frac{p_m}{T_m} dA_m$, and so to find the derivatives of S_m we have to find $\frac{\partial A_m}{\partial \alpha}$, $\frac{\partial A_m}{\partial \tilde{r}_+}$, $\frac{\partial m}{\partial \alpha}$, and $\frac{\partial m}{\partial \tilde{r}_+}$. From the expression of A_m , Eq. (15), we have

$$\frac{\partial A_m}{\partial \alpha} = 8\pi\alpha, \quad \frac{\partial A_m}{\partial \tilde{r}_+} = 0. \quad (28)$$

From the expression of $m(\tilde{r}_+, \alpha)$, Eq. (23), we have

$$\begin{aligned} \frac{\partial m}{\partial \alpha} &\equiv -8\pi\alpha p_g, & \frac{\partial m}{\partial \tilde{r}_+} &= \frac{1 + \frac{3\tilde{r}_+^2}{l_p^2}}{2l_p^2 \sqrt{f_2(\alpha, \tilde{r}_+)}} \\ p_g &\equiv \frac{1}{8\pi\alpha l_p^2} \left(\frac{1 + 2\frac{\alpha^2}{l_p^2} - \frac{\tilde{r}_+ + \frac{\tilde{r}_+^3}{l_p^2}}{2\alpha}}{\sqrt{f_2(\alpha, \tilde{r}_+)}} - \frac{1 + 2\frac{\alpha^2}{l_p^2}}{\sqrt{f_1(\alpha)}} \right), \end{aligned} \quad (29)$$

where p_g is defined as the gravitational pressure. Using then the chain rule on the first law of thermodynamics, we get the following derivatives for the matter entropy:

$$\frac{\partial S_m}{\partial \alpha} = \frac{8\pi\alpha}{T_m} (p_m - p_g), \quad \frac{\partial S_m}{\partial \tilde{r}_+} = \frac{1 + 3\frac{\tilde{r}_+^2}{l_p^2}}{2l_p^2 T_m \sqrt{f_2(\alpha, \tilde{r}_+)}}. \quad (30)$$

Since the differential of the entropy has been recast in terms of α and $\tilde{r}_+$, the temperature of the shell and the pressure of the shell also have that dependence as $T_m = T_m(m(\alpha, \tilde{r}_+), \alpha) = T_m(\alpha, \tilde{r}_+)$ and $p_m = p_m(m(\tilde{r}_+, \alpha), \alpha) = p_m(\alpha, \tilde{r}_+)$. From now on, we abbreviate this dependence to avoid cluttering; however, the dependence must always be assumed.

Then, using the reduced action given in Eq. (24) and the derivatives of the matter entropy in Eq. (30), one finds the stationary conditions. The stationary condition of Eq. (26) yields

$$p_g = p_m. \quad (31)$$

This equation gives the condition for mechanical equilibrium of the shell and we call Eq. (31) as the balance of pressure equation. The stationary condition of Eq. (27) yields

$$\bar{\beta} = \frac{1}{T_m \sqrt{f_2(\alpha, \tilde{r}_+)}}. \quad (32)$$

This equation gives the condition for thermodynamic equilibrium of the shell, and we call Eq. (32) as the balance of temperature equation.

One can verify that Eq. (31) only depends on $\tilde{r}_+$ and α , which means the solutions to this equation can be expressed as

$$\alpha = \alpha(\tilde{r}_+). \quad (33)$$

Then, one can input such solutions into Eq. (32) to obtain an equation only dependent on $\tilde{r}_+$ and $\tilde{\beta}$, which can be solved by a function $\tilde{r}_+(\tilde{\beta})$, i.e.,

$$\tilde{\beta} = \frac{\iota(\tilde{r}_+)}{1 + 3\frac{\tilde{r}_+^2}{\tilde{l}_p^2}}, \quad \text{implying } \tilde{r}_+ = \tilde{r}_+(\tilde{\beta}), \quad (34)$$

where $\iota(\tilde{r}_+)$ is a function of $\tilde{r}_+$ that appears for convenience and is given for the thin shell by

$$\iota(\tilde{r}_+) = \frac{1 + 3\frac{\tilde{r}_+^2}{\tilde{l}_p^2}}{T_m(\tilde{r}_+, \alpha(\tilde{r}_+))\sqrt{f_2(\alpha(\tilde{r}_+))}}. \quad (35)$$

In comparison, for the Hawking-Page black hole one has $\iota(\tilde{r}_+) = 4\pi r_+$. Of course, the expression of Eq. (35) means we are dealing with a thin shell.

Since, from Eq. (33), one has $\alpha = \alpha(\tilde{r}_+)$, the reduced action $I_*[\tilde{\beta}; \alpha, \tilde{r}_+]$ of Eq. (24) under the mechanical stationarity condition can be written as an effective reduced action of the form $I_*[\tilde{\beta}; \tilde{r}_+]$. This in turn implies that the partition function given in Eq. (25) is now $Z[\tilde{\beta}] = \int D\tilde{r}_+ e^{-I_*[\tilde{\beta}; \tilde{r}_+]}$, as the zero-loop approximation in the path integral over α as been done, i.e., using the reduced action evaluated at the stationary point provided by Eq. (31), or what amounts to the same thing, by Eq. (33). It is interesting to note that this behavior can be deduced from the structure of the reduced action in Eq. (24) together with Eq. (25). In fact, the path integral over α in the partition function, $\int D\alpha e^{S_m}$, corresponds to the partition function of the microcanonical ensemble. Therefore, this indicates that the canonical ensemble of the full system can be described by an effective reduced action determined by the microcanonical ensemble of a hot self-gravitating matter thin shell with fixed $\tilde{r}_+$, $I_*[\tilde{\beta}; \tilde{r}_+, \alpha(\tilde{r}_+)] = I_*[\tilde{\beta}; \tilde{r}_+]$, while the solutions $\alpha(\tilde{r}_+)$ are but a consequence of performing the zero-loop approximation on the path integral over α , i.e., of performing the zero loop approximation on the microcanonical ensemble. Given Eqs. (24), (33), and (34), we see that the solutions $\tilde{r}_+(\tilde{\beta})$ of the canonical ensemble are the solutions of the problem. Having $\tilde{r}_+(\tilde{\beta})$, one finds $\alpha = \alpha(\tilde{r}_+(\tilde{\beta}))$, and then the action $I_*[\tilde{\beta}; \tilde{r}_+(\tilde{\beta})]$, which is the action of the stationary points. This action is the zero-loop approximation action $I_0(\tilde{\beta})$. Indeed,

$$I_0[\tilde{\beta}] \equiv I_*[\tilde{\beta}; \tilde{r}_+(\tilde{\beta})]; \quad (36)$$

i.e., the zero-loop action $I_0[\tilde{\beta}]$ is found by evaluating the reduced action around its stationary points with $\alpha(\tilde{r}_+(\tilde{\beta}))$ and $\tilde{r}_+(\tilde{\beta})$ being found from the stationary conditions. The partition function of the canonical ensemble can then be obtained to be

$$Z(\tilde{\beta}) = e^{-I_0[\tilde{\beta}]}, \quad (37)$$

in the zero loop approximation, and the thermodynamic properties of the system can be extracted.

B. The stability criteria from the reduced action of a hot self-gravitating thin shell in asymptotically AdS space

Going a step further within this formalism, we can apply the zero-loop approximation of the path integral in Eq. (25), and go one order up to first order approximation by evaluating the reduced action around its stationary points up until second order and write

$$I_*[\tilde{\beta}; \tilde{r}_+, \alpha] = I_0[\tilde{\beta}] + \sum_{ij} H_{ij} \delta r^i \delta r^j, \quad (38)$$

where $I_0[\tilde{\beta}] = I_*[\tilde{\beta}; \tilde{r}_+(\tilde{\beta}), \alpha(\tilde{\beta})]$ is the reduced action evaluated at the stationary points given in Eq. (36), with $\tilde{r}_+(\tilde{\beta})$ and $\alpha(\tilde{\beta})$ being found from the stationary conditions of I_* , and $H_{ij} = \frac{\partial^2 I_*}{\partial r^i \partial r^j} \big|_0$ is the Hessian of the reduced action I_* evaluated at the stationary points, with the parameters $r^i = (\alpha, \tilde{r}_+)$, with $i = 1, 2$. The partition function can then be written in the saddle point approximation as

$$Z[\tilde{\beta}] = e^{-I_0[\tilde{\beta}]} \int D r^i e^{-\sum_{jk} H_{jk} \delta r^j \delta r^k}, \quad (39)$$

where the first and second factors are the zero and first-loop contributions. Although we only consider the zero-loop contribution, i.e., the zero-loop approximation, we also take into account the first-loop contribution in the sense that it gives us some information about the stability of the approximation. For the path integral to converge and so for the formalism to be stable, the Hessian H_{ij} must be positive definite, i.e., the stationary points must correspond to a local minimum of the reduced action.

The components of the Hessian are

$$H_{\alpha\alpha} = \frac{8\pi\alpha}{T_m} \left(\left(\frac{\partial p_g}{\partial \alpha} \right)_{\tilde{r}_+} - \left(\frac{\partial p_m}{\partial \alpha} \right)_m + 8\pi\alpha p_m \left(\frac{\partial p_m}{\partial m} \right)_\alpha \right), \quad (40)$$

$$H_{\alpha\tilde{r}_+} = \left(\frac{1 + 3\frac{\tilde{r}_+^2}{\tilde{l}_p^2}}{2T_m\sqrt{f_2}\tilde{l}_p} \right) \left(\frac{\frac{\alpha}{\tilde{l}_p^2} + \frac{\tilde{r}_+^3}{2\alpha^2}}{f_2} - 8\pi\alpha \left(\frac{\partial p_m}{\partial m} \right)_\alpha \right), \quad (41)$$

$$H_{\tilde{r}_+\tilde{r}_+} = \left(\frac{1 + 3\frac{\tilde{r}_+^2}{\tilde{l}_p^2}}{2T_m\sqrt{f_2}\tilde{l}_p} \right)^2 \left[\frac{1}{\tilde{l}_p^2} \left(\frac{\partial T_m}{\partial m} \right)_\alpha - \frac{T_m}{\alpha\sqrt{f_2}} \right], \quad (42)$$

where

$$\left(\frac{\partial p_g}{\partial \alpha}\right)_{\tilde{r}_+} = \frac{1}{8\pi\alpha^2 l_p^2} \left(\frac{3\frac{\tilde{r}_+ + \tilde{r}_+^2}{2\alpha} \left(1 - \frac{\alpha^2}{l^2}\right) - 1 - 3\frac{(\tilde{r}_+ + \frac{\tilde{r}_+^2}{l^2})^2}{4\alpha^2}}{f_2^{3/2}} + \frac{1}{f_1^{3/2}} \right), \quad (43)$$

and since here we are working with three variables, we have explicitly put the variable that is kept constant in the subscript of the parenthesis of the partial derivative. The sufficient conditions for the positive definiteness of the Hessian are chosen to be

$$H_{\alpha\alpha} > 0, \quad (44)$$

$$H_{\tilde{r}_+\tilde{r}_+} - \frac{H_{\alpha\tilde{r}_+}^2}{H_{\alpha\alpha}} > 0. \quad (45)$$

Applying Eqs. (40)–(42) to Eqs. (44) and (45), and including the marginal case, one has

$$\left(\frac{\partial p_g}{\partial \alpha}\right)_{\tilde{r}_+} - \left(\frac{\partial p_m}{\partial \alpha}\right)_m + 8\pi\alpha p_m \left(\frac{\partial p_m}{\partial m}\right)_\alpha \geq 0, \quad (46)$$

$$\frac{d\tilde{r}_+}{d\tilde{T}} \geq 0, \quad (47)$$

respectively. Indeed, one can obtain the derivative of the solution $\tilde{r}_+(\tilde{\beta})$ by applying the derivative of $\tilde{\beta}$ to Eqs. (31) and (32), obtaining $\frac{d\tilde{r}_+}{d\tilde{\beta}} = -\frac{1}{2l_p^2} \left(1 + 3\frac{\tilde{r}_+^2}{l^2}\right) \left(H_{\tilde{r}_+\tilde{r}_+} - \frac{H_{\alpha\tilde{r}_+}^2}{H_{\alpha\alpha}}\right)^{-1}$. And so Eq. (45) implies that $\frac{d\tilde{r}_+}{d\tilde{\beta}} < 0$, or in terms of temperature $\frac{d\tilde{r}_+}{d\tilde{T}} > 0$, leading to Eq. (47), when one includes the marginal case. Regarding the meaning of these stability conditions, one can verify that Eq. (46) is precisely the mechanical stability condition for a static shell in AdS with constant $\tilde{r}_+$. Regarding the other condition in Eq. (47), in some sense, it is a thermal stability condition.

Another comment that we can make is in regard to $\frac{d\alpha}{d\tilde{\beta}}$. One can also obtain the derivative of the solution $\alpha(\tilde{\beta})$ by applying the derivative of $\tilde{\beta}$ to Eqs. (31) and (32), obtaining $\frac{d\alpha}{d\tilde{\beta}} = -\frac{H_{\tilde{r}_+\alpha}}{H_{\alpha\alpha}} \frac{d\tilde{r}_+}{d\tilde{\beta}}$, i.e., $\frac{d\alpha}{d\tilde{T}} = -\frac{H_{\tilde{r}_+\alpha}}{H_{\alpha\alpha}} \frac{d\tilde{r}_+}{d\tilde{T}}$. Thus, if mechanical stability holds, $H_{\alpha\alpha} > 0$, Eq. (44), then the radius of the shell α decreases with ensemble temperature if $H_{\tilde{r}_+\alpha} > 0$, and increases if $H_{\tilde{r}_+\alpha} < 0$. The sign of $H_{\tilde{r}_+\alpha}$ depends on the particular shell one is studying.

IV. THERMODYNAMICS OF THE HOT SELF-GRAVITATING THIN SHELL IN THE ZERO-LOOP APPROXIMATION

In the statistical mechanics formalism of the canonical ensemble, the partition function is given by the free energy F as $Z = e^{-\tilde{\beta}F}$, while the zero-loop approximation gives a partition function $Z = e^{-I_0}$. By connecting both, we have $F = \tilde{T}I_0$, where $\tilde{T}$ is the temperature of the system, $\tilde{T} = \frac{1}{\tilde{\beta}}$. Then, from Eq. (24), the free energy is

$$F = \frac{\tilde{r}_+(l^2 + \tilde{r}_+^2)}{2l_p^2 l^2} - \tilde{T}S_m, \quad (48)$$

with $\tilde{r}_+$ given by the solution $\tilde{r}_+ = \tilde{r}_+(\tilde{T})$ of Eq. (34), and α given by the solution $\alpha = \alpha(\tilde{r}_+(\tilde{T}))$ of Eq. (33) together with Eq. (34).

One can now obtain the thermodynamic quantities for the system from the derivatives of the free energy. In terms of the thermodynamic energy E , the temperature $\tilde{T}$, and the entropy S_m , the free energy of a system and its differential are given by

$$F = E - \tilde{T}S, \quad dF = -Sd\tilde{T}, \quad (49)$$

respectively. From the Eqs. (48) and (49), we obtain that the entropy of the system is

$$S = S_m, \quad (50)$$

and the mean energy is

$$E = \frac{1}{2l_p^2} \tilde{r}_+ \left(1 + \frac{\tilde{r}_+^2}{l^2}\right), \quad (51)$$

with $\tilde{r}_+ = \tilde{r}_+(\tilde{T})$. One can identify the Schwarzschild-AdS mass M as the right-hand side of Eq. (51); i.e., $E = M$.

Regarding thermodynamic stability, one must verify if the heat capacity C is positive. If

$$C \geq 0, \quad (52)$$

the system is thermodynamically stable, where we include the limiting case; otherwise, it is unstable. The heat capacity is defined as $C = \frac{dE}{d\tilde{T}}$. Using Eq. (51) together with Eq. (34), we find

$$C = \frac{1}{l_p^2} \frac{\left(1 + \frac{3\tilde{r}_+^2}{l^2}\right) l^2 (\tilde{r}_+)}{12\tilde{r}_+ \iota(\tilde{r}_+) - 2\left(1 + \frac{3\tilde{r}_+^2}{l^2}\right) \frac{\partial \iota(\tilde{r}_+)}{\partial \tilde{r}_+}}, \quad (53)$$

where $\iota(\tilde{r}_+) = \frac{1+3\frac{\tilde{r}_+^2}{l^2}}{T_m(\tilde{r}_+, \alpha(\tilde{r}_+))\sqrt{f_2(\alpha(\tilde{r}_+))}}$, see Eq. (35), and $\tilde{r}_+ = \tilde{r}_+(\tilde{T})$. We see that the heat capacity is positive if

$$6\frac{\tilde{r}_+}{l^2} - t'(\tilde{r}_+)\tilde{T} \geq 0, \quad (54)$$

where we have included the limiting case. Using Eq. (34), one finds that Eq. (54) is equivalent to $\frac{d\tilde{r}_+}{d\tilde{T}} \geq 0$ which is Eq. (47) of the ensemble theory. We now see that the remaining stability condition Eq. (44) is not present or cannot be accessed by the thermodynamics of the system. It is moreover interesting to see that the thermodynamic stability of the canonical ensemble given by Eqs. (52)–(54) is also given by the saddle point approximation of the effective reduced action $I_*[\tilde{\beta}; \tilde{r}_+]$, only dependent in the parameter $\tilde{r}_+$. We suppose that this is due to the fact that $\tilde{r}_+$ is associated to the quasilocal energy and so the effective reduced action $I_*[\tilde{\beta}; \tilde{r}_+]$ plays the role of the appropriate generalized free energy that when minimized yields indeed the thermodynamic equilibrium and stability of the canonical ensemble. It is important to note also that the thermodynamic stability of the canonical ensemble is different from the intrinsic stability of the system in the sense of Callen, as intrinsic stability requires more conditions on the concavity of the free energy.

V. SPECIFIC CASE OF MATTER THIN SHELL WITH BAROTROPIC EQUATION OF STATE

A. The ensemble quantities

In order to proceed with the analysis of the canonical ensemble of a self-gravitating matter thin shell, we must now give the equations of state for the matter in the shell. Here we give an equation of state for the pressure in the form of a barotropic equation, i.e.,

$$p_m(m, \alpha) = \frac{1}{3} \frac{m}{4\pi\alpha^2}. \quad (55)$$

The equation of state for the temperature of the matter is chosen as

$$T_m = \frac{4}{3C_0} \frac{m^{\frac{1}{2}}}{(4\pi\alpha^2)^{\frac{1}{2}}}, \quad (56)$$

where C_0 is a constant with units. Then, integrating the first law of thermodynamics yields that the matter entropy has the equation,

$$S_m = C_0 m^{\frac{3}{2}} (4\pi\alpha^2)^{\frac{1}{2}}. \quad (57)$$

A more general equation for $p_m(m, \alpha)$ in Eq. (55) could be chosen, e.g., $p_m(m, \alpha) = \lambda \frac{m}{4\pi\alpha^2}$, where λ is a constant, with $\lambda = \frac{1}{2}$ corresponding to the barotropic equation of state of a two-dimensional ultrarelativistic gas. The mechanical stability condition, Eq. (46), requires that $\lambda < \frac{1}{2}$. Since the dominant energy condition requires $-1 < \lambda < 1$, a reasonable range for λ is $0 < \lambda < \frac{1}{2}$, as we require the pressure to

be positive. If we integrate this general expression for the pressure, using from the first law of thermodynamics that $p_m = -(\frac{\partial m}{\partial A_m})_{S_m}$, the partial derivative in relation the area A_m being defined at constant matter entropy, we would obtain the expression for S_m in the form $S_m(m, \alpha) = S_m(m(4\pi\alpha^2)^\lambda)$, i.e., S_m is an arbitrary function of $m(4\pi\alpha^2)^\lambda$. We could pick a power-law expression for the entropy of the form $S_m(m, \alpha) = C_0 m^\delta (4\pi\alpha^2)^{\delta\lambda}$, with C_0 a constant and δ a number. Then, the temperature equation of state could be deduced using the first law of thermodynamics, i.e., $\frac{1}{T_m} = (\frac{\partial S_m}{\partial m})_{A_m}$ giving $T_m = \frac{m^{1-\delta}}{\delta C_0 (4\pi\alpha^2)^{\delta\lambda}}$. We could instead have picked up this equation of state for T_m from the necessity of the equality of the second order cross derivatives to have an exact S_m and then integrate the first law to find S_m itself. We have narrowed the analysis to specific λ and δ , using Eqs. (55)–(57). Namely, we have adopted $\lambda = \frac{1}{3}$ as given in Eq. (55), and which is in the interval, $0 < \lambda < \frac{1}{2}$. We have also singled out $\delta = \frac{3}{4}$, so that the temperature and the entropy have equations given in Eq. (56) and Eq. (57), respectively. For numerical purposes, it is best to write

the pressure as $l_p^2 p_m(m, \alpha) = \frac{1}{3} \frac{m^{\frac{1}{2}}}{4\pi\alpha^2}$, the temperature as $T_m = \frac{1}{3} \frac{4}{3c_0} (\frac{l_c}{l_p})^{\frac{1}{2}} (\frac{l_p}{l_c})^{\frac{1}{2}} (\frac{m l_c^2}{l_p^2})^{\frac{1}{4}} (4\pi\alpha^2)^{-\frac{1}{4}}$, and the entropy as $S_m = c_0 (\frac{l_c}{l_p})^{\frac{1}{2}} (\frac{l_p}{l_c})^{\frac{3}{2}} (\frac{m l_c^2}{l_p^2})^{\frac{3}{4}} (4\pi\alpha^2)^{\frac{1}{4}}$, i.e., $C_0 = c_0 l_c^{\frac{1}{2}}$, where c_0 has no units and l_c can be understood as the Compton wavelength associated to the rest mass of the constituents of the shell. The motivation for the matter equations of state given above is both physical and mathematical. Physically, the equations of state resemble the equations of state of a radiation gas. Namely, the equation of state for the pressure $p_m(m, \alpha)$ can be thought of as the equation of state of a three-dimensional radiation gas confined in a very thin shell of small width l_c . It can also be thought of as a two-dimensional gas of a fundamental field with some Compton wavelength l_c , which is the view we take here. Mathematically, it allows for an analytical treatment of the balance of the pressure which facilitates the search for the solutions of the shell and the analysis of its stability.

With the equations of state described, we can solve numerically Eq. (32), to obtain two solutions for the radius α of the shell which we write as $\alpha_u(\tilde{r}_+)$ and $\alpha_s(\tilde{r}_+)$, see Fig. 1, the meaning of the subscripts u and s will turn up shortly. We see that the solution $\alpha_u(\tilde{r}_+)$ is monotonically increasing, while $\alpha_s(\tilde{r}_+)$ is monotonically decreasing, until both meet a common point. By evaluating the stability condition in Eq. (46), it turns out that the solution $\alpha_u(\tilde{r}_+)$ is mechanically unstable, while $\alpha_s(\tilde{r}_+)$ is mechanically stable, hence the nature of the subscripts for the solutions. This mechanical behavior of the thin shell is rather like the radius-mass behavior of a white dwarf or a neutron star. The two solutions translate into two possible radii for the shell for a given energy, one which is large and another which is

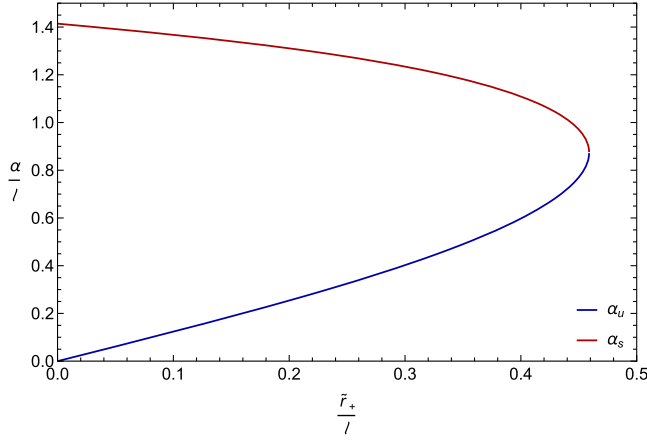


FIG. 1. Solutions of the balance of pressure α_u and α_s as function of $\tilde{r}_+$.

small. The small radius solution has very high pressure and is unstable, the large radius solution has low pressure and is stable. Physically, this is similar to the two solutions appearing in models of astrophysical objects, such as white dwarfs, neutron stars with polytropic-type equations of state, one solution being unstable while the other being stable.

Knowing the solutions $\alpha_u(\tilde{r}_+)$ and $\alpha_s(\tilde{r}_+)$ for the radius of the shell, we can put them into Eq. (32) in order to obtain the solutions for the gravitational radius $\tilde{r}_+(\bar{T})$. One finds that there are four solutions in total, two solutions $\tilde{r}_{+u1}(\bar{T})$ and $\tilde{r}_{+u2}(\bar{T})$ with shell radius α_u , i.e., $\alpha_u(\tilde{r}_{+u1}(\bar{T}))$ and $\alpha_u(\tilde{r}_{+u2}(\bar{T}))$, respectively, and other two solutions $\tilde{r}_{+s1}(\bar{T})$ and $\tilde{r}_{+s2}(\bar{T})$ with shell radius α_s , i.e., with $\alpha_s(\tilde{r}_{+s1}(\bar{T}))$ and $\alpha_s(\tilde{r}_{+s2}(\bar{T}))$, respectively. These solutions are shown in Fig. 2. Regarding stability, the solutions are thermodynamically stable if the gravitational radius increases with

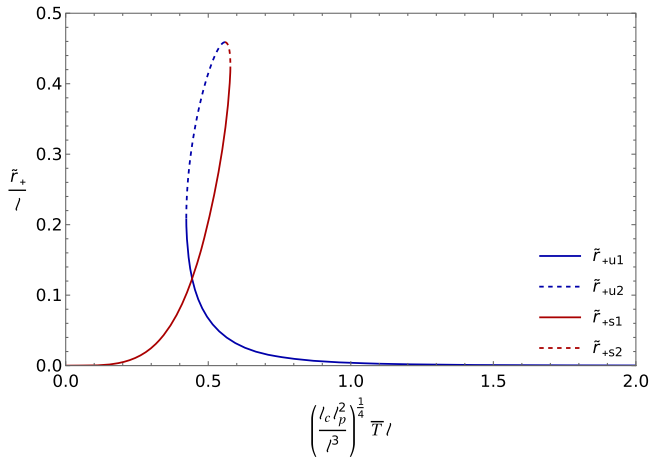


FIG. 2. Solutions of the ensemble $\tilde{r}_{+u1}$, $\tilde{r}_{+u2}$, $\tilde{r}_{+s1}$, and $\tilde{r}_{+s2}$, as functions of $\tilde{T}l(l_c/l_p)^{1/4}(l_p/l)^{1/2}$. Both $\tilde{r}_{+u1}$ and $\tilde{r}_{+u2}$ have shell radius α_u , while both $\tilde{r}_{+s1}$ and $\tilde{r}_{+s2}$ have shell radius α_s .

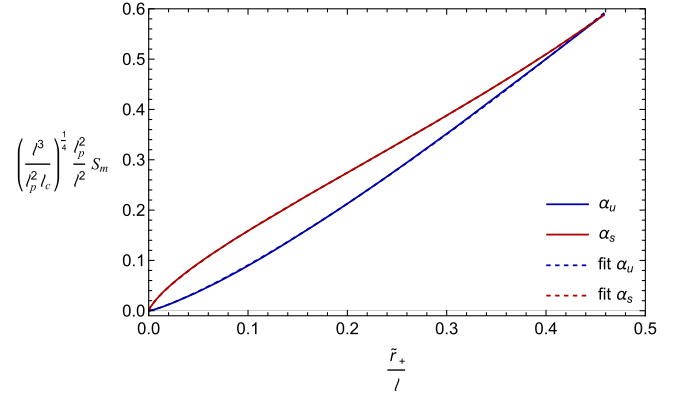


FIG. 3. Matter entropy $(\frac{l_p^3}{l_c^3})^{1/4} \frac{l_p^2}{l^2} S_m$ in function of the gravitational radius $\tilde{r}_+/l$ for the two shell radius solutions $\alpha_u(\tilde{r}_+)$ and $\alpha_s(\tilde{r}_+)$. A fit was performed for each branch, with $(\frac{l_p^3}{l_c^3})^{1/4} \frac{l_p^2}{l^2} S_m = 1.54662(\tilde{r}_+/l)^{1.2323}$ for the case of $\alpha_u(\tilde{r}_+)$ and $(\frac{l_p^3}{l_c^3})^{1/4} \frac{l_p^2}{l^2} S_m = 0.898912(\tilde{r}_+/l)^{0.755675} + 0.867397(\tilde{r}_+/l)^{2.91424}$ for $\alpha_s(\tilde{r}_+)$, with respective coefficients of determination $R^2 = 0.999992$ and $R^2 = 1$, with this last equality being approximate.

the temperature $\bar{T}$. From the figure, we see that $\tilde{r}_{+u1}(\bar{T})$ and $\tilde{r}_{+s2}(\bar{T})$ are thermodynamically unstable, and $\tilde{r}_{+u2}(\bar{T})$ and $\tilde{r}_{+s1}(\bar{T})$ are thermodynamically stable.

The entropy S_m is now analyzed; see Fig. 3. We have performed a polynomial fit to the matter entropy to understand its leading power of $\tilde{r}_+$. The fit for S_m given in Eq. (57) for the solution α_u as a function of $\tilde{r}_{+u}$ is described by the function $(\frac{l_p^3}{l_c^3})^{1/4} \frac{l_p^2}{l^2} S_m = 1.54662(\tilde{r}_{+u}/l)^{1.2323}$ with a coefficient of determination $R^2 = 0.999992$. The fit of S_m given in Eq. (57) for the solution α_s as a function of $\tilde{r}_{+s}$ is described by the function $(\frac{l_p^3}{l_c^3})^{1/4} \frac{l_p^2}{l^2} S_m = 0.898912(\tilde{r}_{+s}/l)^{0.755675} + 0.867397(\tilde{r}_{+s}/l)^{2.91424}$ with a coefficient of determination $R^2 = 1$, with this equality being approximate. Another fit of S_m for the solution α_s was tried with just one power, giving $(\frac{l_p^3}{l_c^3})^{1/4} \frac{l_p^2}{l^2} S_m = 1.12374(\tilde{r}_{+s}/l)^{0.867504}$, with $R^2 = 0.999607$, however the differences of the fit are visible in the plot, and so we have tried the fit with two powers. These obtained fits are very near the numerical results for the S_m , which is surprising, and one could wonder if there might be an analytic solution. But in order to obtain the expression of S_m , one needs to solve a quintic polynomial equation and we were not able to find an analytic solution. A feature that the fits do not capture is the fact that S_m is only defined in the interval $0 < \tilde{r}_+/l < 0.4589$ with this last number being approximate.

Another equivalent indicator of thermal stability is given by the positivity of the heat capacity, see Fig. 4. Yet, it must be noticed that $\tilde{r}_{+u2}(\bar{T})$ has a shell radius $\alpha_u(\tilde{r}_{+u2}(\bar{T}))$,

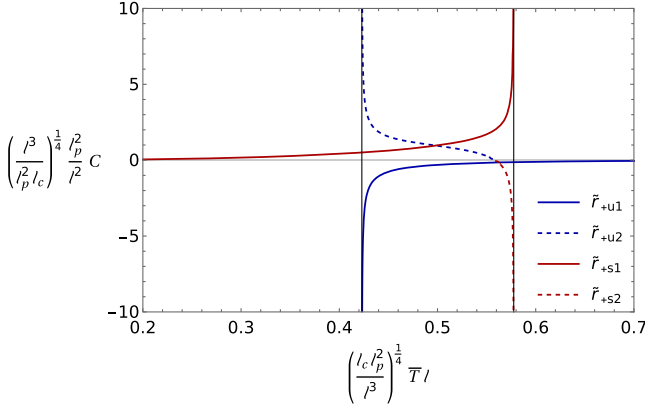


FIG. 4. Adimensional heat capacity for the solutions $\tilde{r}_{+u1}$, $\tilde{r}_{+u2}$, $\tilde{r}_{+s1}$, and $\tilde{r}_{+s2}$ as functions of $\tilde{T}l(\frac{l_c}{l})^{1/4}(\frac{l_p}{l})^{1/2}$, where the solutions α_u and α_s are also assumed. The solutions $\tilde{r}_{+u1}$ and $\tilde{r}_{+s2}$ are thermodynamically unstable, while $\tilde{r}_{+u2}$ and $\tilde{r}_{+s1}$ are thermodynamically stable.

which is mechanically unstable. Therefore, the only fully stable solution is $\tilde{r}_{+s1}(\tilde{T})$ with shell radius $\alpha_s(\tilde{r}_{+s1}(\tilde{T}))$.

VI. HOT THIN SHELL VERSUS BLACK HOLE IN AdS

A. The black hole

For completeness, we now give the relevant quantities of the canonical ensemble of a Schwarzschild-AdS black hole, which are important for the comparison between a hot thin shell and a black hole in AdS. In order to obtain the reduced action and then the zero-loop action, one can carry on in a similar way the calculations above but now with black hole boundary conditions rather than thin shell ones, or one can simply use the Hawking-Page results.

The action of the Hawking-Page black hole solutions is

$$I_{\text{bh}} = \frac{1}{2l_p^2 \tilde{T}} \left(r_+ + \frac{r_+^3}{l^2} \right) - \frac{\pi r_+^2}{l_p^2}, \quad (58)$$

with the radius r_+ being a function of $\tilde{T}$. Indeed here, the gravitational radius r_+ , which is also a horizon radius since we are now dealing with a black hole, is given by the equation $\tilde{\beta} \equiv \frac{1}{\tilde{T}} = \frac{\iota(r_+)}{1+3\frac{r_+^2}{l^2}}$; see also Eq. (34) with

$$\iota(r_+) = 4\pi r_+.$$

Therefore, one has to solve for r_+ the equation $\tilde{T} = \frac{1+3\frac{r_+^2}{l^2}}{4\pi r_+}$. The solutions $\frac{r_+}{l}$ as a function of $\tilde{T}l$ are given by

$$\frac{r_+}{l} = \frac{2\pi \tilde{T}l}{3} \pm \frac{1}{3} \sqrt{(2\pi \tilde{T}l)^2 - 3}. \quad (59)$$

Thus, for $\tilde{T}l \geq \frac{\sqrt{3}}{2\pi}$, there are two black hole solutions, $r_{+1}(\tilde{T})$, the solution with the minus sign, which is

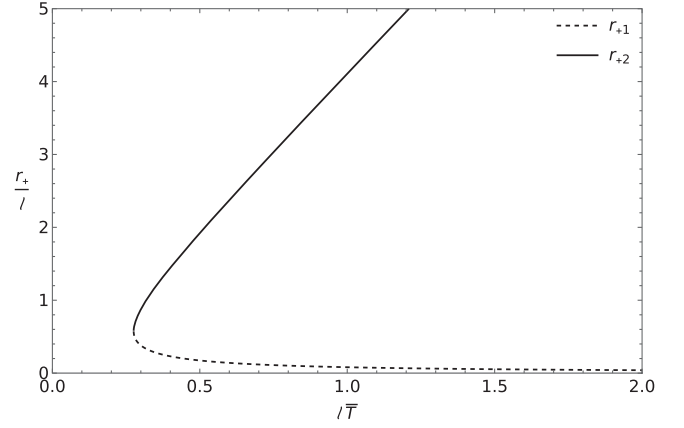


FIG. 5. Solutions of the ensemble $\frac{r_{+1}}{l}$ and $\frac{r_{+2}}{l}$ for the black hole in asymptotically AdS.

thermodynamically unstable, and $r_{+2}(\tilde{T})$, the solution with the plus sign, which is stable. When equality holds, $\tilde{T}l = \frac{\sqrt{3}}{2\pi}$, one has a degenerate solution, $r_{+1}(\tilde{T}) = r_{+2}(\tilde{T}) = \frac{1}{\sqrt{3}}$. For $\tilde{T}l < \frac{\sqrt{3}}{2\pi}$, there are no solutions; see Fig. 5. The numerical value of this critical temperature is $\tilde{T}l = \frac{\sqrt{3}}{2\pi} = 0.276$ approximately, with the corresponding horizon radius $\frac{r_+}{l} = \frac{1}{\sqrt{3}} = 0.577$.

The black hole entropy can be obtained from the action as

$$S_{\text{bh}} = \frac{\pi r_+^2}{l_p^2}, \quad (60)$$

i.e., the Bekenstein-Hawking entropy, with r_+ standing for r_{+1} or r_{+2} . The entropy describes the usual parabola; see Fig. 6.

The heat capacity for the Schwarzschild-AdS black hole is

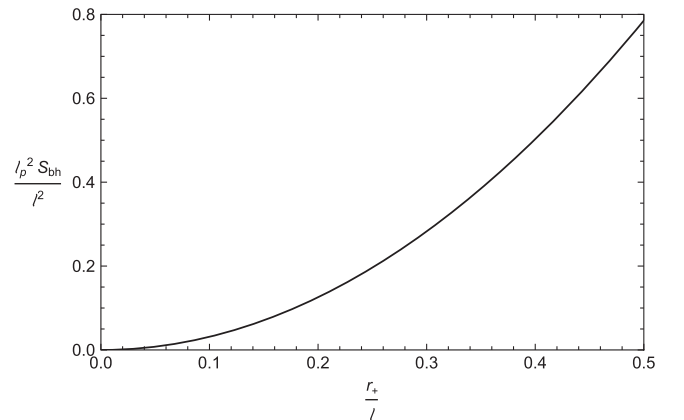


FIG. 6. Black hole entropy $\frac{l_p^2}{l^2} S_{\text{bh}}$ as a function of the horizon radius $\frac{r_+}{l}$, which stands either for $\frac{r_{+1}}{l}$ or for $\frac{r_{+2}}{l}$.

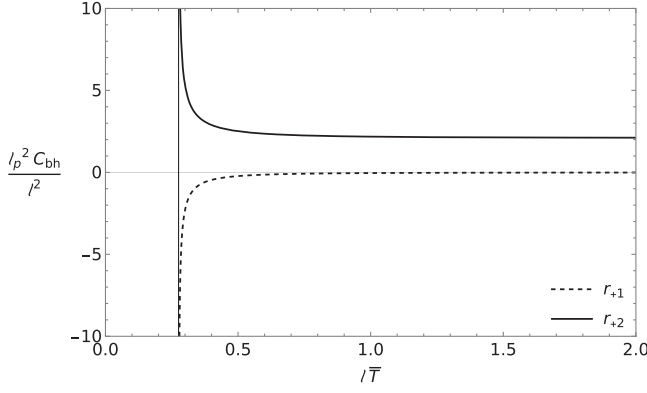


FIG. 7. Adimensional heat capacity $\frac{l_p^2 C_{bh}}{l^2}$ of the black hole solutions r_{+1} and r_{+2} in function of $\bar{T}l$.

$$C_{bh} = \frac{2\pi r_+^2 \left(1 + 3 \frac{\tilde{r}_+^2}{l^2}\right)}{3 \left(\frac{r_+^2}{l^2} - \frac{1}{3}\right)}, \quad (61)$$

for each solution $r_{+1}(\bar{T})$ and $r_{+2}(\bar{T})$; see Fig. 7. The heat capacity is positive for $\frac{r_+}{l} > \frac{\sqrt{3}}{3}$, and so r_{+1} is thermodynamic unstable, and r_{+2} is thermodynamic stable.

It is of relevance to remark that pure hot AdS and black hole in AdS compete to be the prominent thermodynamic phase, with the phase that has the minimum action being the one that is favored. Pure hot AdS has zero action, $I_{\text{PAdS}} = 0$, so if $I_{bh} > 0$ then AdS is favored, if $I_{bh} = 0$ the two phases coexist equally, and if $I_{bh} < 0$ then the black hole is favored. From the black hole action, Eq. (58), one finds that as one increases $\bar{T}l$ from zero, there is a first order phase transition from thermal AdS to the stable black hole state, the transition happening at $\bar{T}l = \frac{1}{\pi} = 0.318$, the latter number being approximate, to the stable black hole with horizon radius $\frac{r_{+2}}{l} = 1$.

B. Hot thin shell versus black hole and favorable states

1. Gravitational radii, entropies, and heat capacities

It is now of interest to compare the properties we have obtained for the hot thin shell in AdS with the properties of the black hole in AdS. More specifically, we can compare the gravitational radii of each thin shell with the gravitational radius of the black hole, examine their entropies, and analyze their heat capacities.

Let us start with the comparison of the two possible gravitational radii of each thin shell with the gravitational radii of the black hole. For the mechanically unstable hot thin shell, which is given by the shell radius α_u , we have found that there are two branches for the gravitational radius. One of the branches, $\tilde{r}_{+u1}$ is thermodynamically unstable, while the other branch $\tilde{r}_{+u2}$ is thermodynamically stable. As well, for the Hawking-Page black hole, the horizon radius r_+ has one branch r_{+1} , which is

thermodynamically unstable, and another branch r_{+2} , which is thermodynamically stable. It is clear from Figs. 1 and 5 that the two solutions for the gravitational radius with thin shell radius α_u , which is mechanically unstable, share similarities with the two solutions for the horizon radius of the black hole. In detail, the thermodynamically unstable branch of the thin shell solution, $\tilde{r}_{+u1}$, follows the same behavior as the thermodynamically unstable black hole solution, r_{+1} . As well, the thermodynamically stable branch of the thin shell solution, $\tilde{r}_{+u2}$, follows the same behavior as the thermodynamically stable black hole solution, r_{+2} . These similarities could perhaps be expected since the mechanically unstable shell is bound to collapse into a black hole and can be understood as a black hole precursor. For the mechanically stable hot thin shell, which is given by the shell radius α_s , we have found that there are also two branches for the gravitational radius. One of the branches, $\tilde{r}_{+s1}$ is thermodynamically unstable, while the other branch $\tilde{r}_{+s2}$ is thermodynamically stable. From Figs. 1 and 5, it can be seen that the two solutions for the gravitational radius with thin shell radius α_s , which is mechanically stable, share no similarities with the two solutions for the horizon radius of the black hole. However, the solutions $\tilde{r}_{+s1}$ and $\tilde{r}_{+s2}$ appear to have similarities with the behavior of the Davies black hole solutions, which correspond to an electrically charged black hole [25] in the canonical ensemble. These black hole solutions have a stable branch that start at zero temperature with a horizon radius given by the electric charge, and then the horizon radius increases with the temperature up until a maximum temperature. The same happens with the mechanically and thermodynamically stable hot matter thin shell, starting at zero temperature with zero gravitational radius instead of a non-zero value. This behavior is expected from a solution of hot self-gravitating matter that models hot AdS space with radiation at the same order of approximation.

Let us now compare the matter thin shell entropy with the black hole entropy. Both entropies depend on their own gravitational radius, which in the black hole case is also a horizon radius. For the mechanical unstable and stable thin shells, we have seen that the matter entropy S_m can be described approximately by a power law. For the unstable shell α_u , we found that $S_m = \xi \tilde{r}_+^{1.2323}$, for some ξ . For the stable shell α_s , we found that $S_m = \xi \tilde{r}_+^{0.867504}$, for some other ξ . For the black hole, the entropy is also given by a power law $S_{bh} = \chi r_+^2$, for some χ . We see that both the mechanical unstable shell α_u and the black hole have exponents γ satisfying $\gamma > 1$, while the stable shell has an exponent $\gamma < 1$. This behavior reinforces the similarity properties between the mechanical unstable shell and the black hole, as advocated above. To understand this, we resort to a Bekenstein argument [1]. It is stated in it, without making calculations, that one should expect that black holes have an entropy with an exponent γ obeying $\gamma > 1$. Suppose that they had an exponent $\gamma < 1$, then one would have that two

isolated black holes that will merge into one should have a final mass lower than the sum of the initial mass, part of the initial mass being lost in gravitational radiation. Concomitantly, the entropy of the final black hole should be greater than the sum of the entropies of the initial black hole, to have the second law of thermodynamics obeyed. The two conditions cannot be met simultaneously when $\gamma < 1$. Suppose for these purposes that the black hole entropy is proportional to a power of the gravitational radius, $S_{\text{bh}} = \chi r_{+}^{\gamma}$, for some χ and γ . For black hole a and black hole b merging into a third black hole c , one has from the first condition $r_{+c} < r_{+a} + r_{+b}$ and from the second condition, $r_{+c}^{\gamma} > r_{+a}^{\gamma} + r_{+b}^{\gamma}$; i.e., one has the range for the horizon radius of the black hole c obeying $(r_{+a}^{\gamma} + r_{+b}^{\gamma})^{\frac{1}{\gamma}} < r_{+c} < r_{+a} + r_{+b}$. This inequality can only be fulfilled for $\gamma > 1$. From thermodynamic arguments then it was chosen $\gamma = 2$, the correct value for black holes. The point here we want to make is that the unstable shell α_u has an exponent $\gamma = 1.2323$ that is indeed greater than one, and following the arguments above it is black hole like; i.e., the shell behaves as the black hole that it can originate from gravitational collapse. The stable shell α_s has an exponent $\gamma = 0.867504$ that is less than one, and thus, it does not behave as a black hole that it could originate upon collapse. Another remark in relation to the entropy is that for the black hole $S_{\text{bh}} = \pi \frac{l_p^2}{l_c^2} (\frac{r_{+}}{l_c})^2$, the exponent is $\gamma = 2$, and it grows much faster than for the matter for large gravitational radius, as expected, since it is known that black holes have the maximum entropy. However, for small gravitational radius, the matter entropy S_m is larger than the black hole entropy, so this might indicate that there are no stable black holes for small gravitational radius; see Fig. 5 noting that r_{+1} is unstable.

Let us continue with the comparison of the heat capacities for the hot thin shell and black hole, C and C_{bh} , respectively, with the help of Figs. 4 and 7. The heat capacity C as a function of the temperature for the mechanically unstable shell $\alpha_u(\tilde{r}_{+})$ behaves in the same manner as the heat capacity of the black hole C_{bh} as a function of temperature. There are parts that are thermodynamically unstable, $\tilde{r}_{+u1}$ for the shell and r_{+1} for the black hole, and parts that are thermodynamically stable $\tilde{r}_{+u2}$ for the shell, and r_{+2} for the black hole. The heat capacity C as a function of the temperature for the mechanically stable shell α_s behaves differently from the heat capacity of the black hole C_{bh} as a function of temperature. However, it shows similarities to the heat capacity of the electrically charged black hole in the canonical ensemble [25]. In particular, the thermodynamically stable $\tilde{r}_{+s1}$ branch of the heat capacity is similar to the heat capacity of the stable branch of an electrically charged black hole. These similarities displayed here for the heat capacity, are the same as the similarities we have found

above when comparing the gravitational radii, and indeed they come from the fact that the heat capacity is related to the first derivative in temperature of the gravitational radius, and so the similarities from the gravitational radius solutions are carried into the heat capacity.

2. Favorable states: Comparison between hot thin shell, black hole, and pure hot AdS thermodynamic states

We want now to identify the favorable states of the ensemble, i.e., given a fixed temperature we want to know if the hot thin shell is favored in relation to the black hole or if the contrary happens.

In order to do this, to identify the favorable states at a fixed temperature of the ensemble, we must compare the action of the stable hot self-gravitating matter thin shell with the action of the stable black hole solution of Schwarzschild-AdS. This is so because the sector with a self-gravitating matter shell may compete with the black hole sector and the pure hot AdS sector in the path integral. From thermodynamics, it is known that the preferred configuration is the one with the least free energy. This also means the one with the least action, since $I_0 = \tilde{\beta}F$. This can be seen because the partition function is $Z = e^{-I_0}$, and thus, the configuration with less I_0 is the more probable one. Now, if at a certain temperature, the configuration with the least action changes, then this marks a first order phase transition as the action is continuous but not differentiable there. In [7], hot thermal AdS, i.e., hot AdS in one-loop approximation, and the stable black hole solution were discussed, where it was discovered the Hawking-Page phase transition from hot thermal AdS to the stable black hole. Here, the stable self-gravitating matter thin shell can be understood as one possible description of hot AdS with thermal self-gravitating matter, i.e., hot curved AdS.

To help in the comparison of the actions, one can write the action of the matter thin shell given in Eq. (25) as $\frac{l_p^2}{l_c^2} I_0 = \frac{\tilde{r}_{+}^2(1+\frac{\tilde{r}_{+}^2}{l_c^2})}{2\tilde{T}l} - c_0(\frac{l_p^2 l_c}{\tilde{r}_{+}})^{\frac{1}{4}}(\frac{m l_p^2}{\tilde{T}})^{\frac{3}{4}}(4\pi \frac{\alpha^2}{l_c^2})^{\frac{1}{4}}$, with $\tilde{r}_{+}$ given by the stable solution $\tilde{r}_{+} = \tilde{r}_{+}(\tilde{T})$ of Eq. (34), i.e., $\tilde{r}_{+s2}$, and α given by the stable solution $\alpha = \alpha(\tilde{r}_{+}(\tilde{T}))$ of Eq. (33) together with Eq. (34), i.e., $\alpha_s = \alpha_s(\tilde{r}_{+s2}(\tilde{T}))$, and the action has been set without units. It is useful to define $z = (\frac{l_p^3}{l_c l_c})^{\frac{1}{4}}$ as the parameter without units that establishes the relevant scale ratios between the Planck length, AdS length, and the Compton length in the case of the hot matter thin shell with the chosen equations of state above. Since the parameter z is given by the ratio between the scales of the matter entropy and the black hole entropy, its value establishes how dominant the matter entropy is compared to the black hole entropy, which affects the phase structure, as we shall see. Then, one can write the unitless action of the matter thin shell as

$$\frac{l_p^2}{l^2} I_0 = \frac{l_p^2}{l^2} \frac{1}{z} \bar{I}_0 \left(\frac{\bar{T}l}{z} \right), \quad z \equiv \left(\frac{l^3}{l_p^2 l_c} \right)^{\frac{1}{4}},$$

$$\frac{l_p^2}{l^2} \bar{I}_0 \left(\frac{\bar{T}l}{z} \right) \equiv \frac{\bar{r}_+ (1 + \frac{\bar{r}_+^2}{l^2})}{2 \frac{\bar{T}l}{z}} - c_0 \left(\frac{m l_p^2}{l} \right)^{\frac{3}{4}} \left(4\pi \frac{\alpha^2}{l^2} \right)^{\frac{1}{4}}. \quad (62)$$

This property of the thin shell action is also useful for numerical purposes, as one can compare actions with a given parameter z . The behavior of the action of the matter thin shell, Eq. (62), evaluated at the solutions of the shell, can be summarized as follows. The action is zero at $\bar{l}\bar{T} = 0$, and it decreases for increasing $\bar{l}\bar{T}$. At a final temperature $\bar{l}\bar{T}_f$, the stable thin shell solution ceases to exist, which can be interpreted as the matter having larger thermal agitation than the permitted to have a shell, implying that the shell can collapse to a black hole or disperse to infinity. This $\bar{l}\bar{T}_f$ depends on the parameter $z = (\frac{l^3}{l_p^2 l_c})^{\frac{1}{4}}$, which itself depends on the natural gravitational scale ratio $\frac{l}{l_p}$ and on the matter scale ratio $\frac{l}{l_c}$.

In relation to the action of the stable black hole, one has the Hawking-Page action given by

$$\frac{l_p^2}{l^2} I_{bh} = \frac{r_+}{2\bar{T}l} \left(1 + \frac{r_+^2}{l^2} \right) - \frac{\pi r_+^2}{l^2}, \quad (63)$$

with r_+ being the stable black hole solution, i.e., r_{+2} , given as a function of $\bar{l}\bar{T}$ by $\frac{r_+}{l} = \frac{2\pi\bar{l}\bar{T}}{3} + \frac{1}{3} \sqrt{(2\pi\bar{l}\bar{T})^2 - 3}$. We have seen that the stable solution only exists for $\bar{l}\bar{T} \geq \frac{\sqrt{3}}{2\pi} = 0.276$, with last equality being approximate.

Now, we must compare the matter action with the black hole action for each possible parameter z and c_0 , and also with pure hot AdS space characterized by $I_{\text{PAdS}} = 0$. Here, we make the choice $c_0 = 1$ since c_0 can be in some sense absorbed by the parameter z , and then make the comparison of the actions only on z . The plot of the actions is shown in Fig. 8, where for the matter shell we put three values of z , namely, $z = 0.1$, $z = 0.581$, and $z = 1$. The action of matter thin shell starts from zero at $\bar{l}\bar{T} = 0$ and decreases until a minimum negative value for a maximum temperature $\bar{l}\bar{T}_f$. Note that the maximum temperature depends on the parameter z as $\bar{l}\bar{T}_f = 0.577z$, where 0.577 is approximate. Above this temperature $\bar{l}\bar{T}_f$ the shell stops to exist and probably collapses. With respect to the black hole case, the action I_{bh} only starts to exist for $\bar{l}\bar{T} \geq \frac{\sqrt{3}}{2\pi} = 0.276$, and decreases with increasing temperature. The black hole action is positive in the range $\frac{\sqrt{3}}{2\pi} \leq \bar{l}\bar{T} \leq 0.318$ where 0.318 is approximate, is zero at $\bar{l}\bar{T} = 0.318$, and is negative for $\bar{l}\bar{T} > 0.318$. The point of intersection between the two actions is given by the equality $I_{bh}(\bar{l}\bar{T}) = \frac{1}{z} \bar{I}_0(\frac{\bar{l}\bar{T}}{z})$ for each z . For example, in the case of $z = 1$, one has that the actions

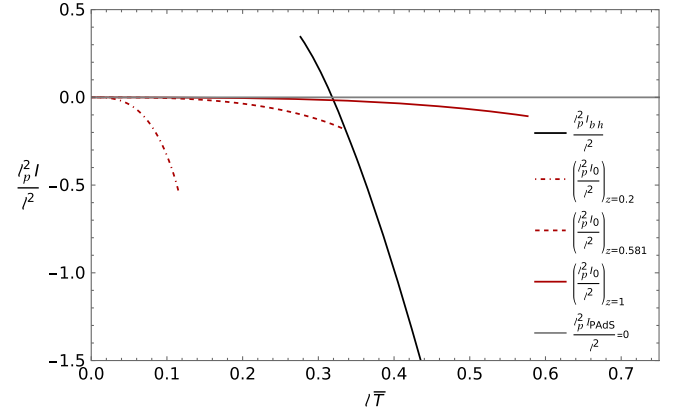


FIG. 8. Plot of the actions I_{bh} , I_0 , and I_{PAdS} as functions of the temperature $\bar{l}\bar{T}$. The solution that has lower action between stable black hole, hot shell, and pure hot AdS is the one that is favored. It is chosen $z = (\frac{l^3}{l_p^2 l_c})^{\frac{1}{4}} = 0.2, 0.581, 1$ to compare the actions. For $z = 0.2$, the hot shell ceases to exist at temperature $\bar{l}\bar{T}_f = 0.115$, for $z = 0.581$ at temperature $\bar{l}\bar{T}_f = 0.335$, and for $z = 1$, at temperature $\bar{l}\bar{T}_f = 0.577$.

intersect at $\bar{l}\bar{T} = 0.320$, and so as one increases the temperature around this point, there is a first order phase transition from the hot matter thin shell to the stable black hole. This is analogous to the case of the Hawking-Page phase transition, where the matter is treated in AdS space at one-loop approximation, rather than in zero loop. It can be found that the intersection between the matter thin shell action and the black hole action only happens for a range of z . As one decreases z , the maximum temperature of the thin shell also decreases, while the black hole action is unaltered. And so, there must be a minimum value of z for which the intersection occurs. The minimum value can be found by considering that the two actions intersect exactly at the maximum temperature of the shell, i.e., $I_{bh}(0.577z) = \frac{1}{z} \bar{I}_0(0.577)$, which numerically can be solved and gives $z = 0.581$ approximately and the first order phase transition occurs for this case at $\bar{l}\bar{T} = 0.336$, approximately. As a consequence, the action of the matter thin shell intersects the action of the black hole only if $0.581 \leq z < \infty$, with first number being approximate. If $0 < z < 0.581$, the matter shell solution ceases to exist before it intersects the curve of the action of the black hole. Therefore, there is only a first order phase transition from the matter thin shell to the black hole when $0.581 \leq z < \infty$.

A comment can be made regarding pure hot AdS space, i.e., AdS space in zero loop, in comparison with the hot thin shell and black hole configurations. Since the matter action is always negative, the matter thin shell is always more favorable than pure hot AdS space as long as the thin shell solution exists. However, we must also consider the existence of the stable black hole. For $0.581 < z < \infty$, the matter thin shell action always intersects the black hole action, and so hot pure AdS is never the favorable phase.

For $0.547 \leq z \leq 0.581$, the matter thin shell and the black hole actions do not intersect but at the point where the matter thin shell ceases to exist, the action of the black hole is negative, and so these configurations are more favorable than pure hot AdS space. For the range $0 \leq z \leq 0.547$, the matter thin shell ceases to exist at a temperature at which either there is no black hole solution or the action of the black hole solution is positive, so there is an interval of temperatures where the pure hot AdS space is favorable.

It is worth stressing that the parameter z can be restricted from validity arguments of the zero-loop approximation. The zero-loop approximation should be valid for the cases $l \gg l_p$ but with l not that large and since α is comparable to l , one must have $l \gg l_c$ also so that matter at the thin shell can be judged thermodynamic. Moreover, all scales must be much greater than the Planck scale l_p , as we are using the zero-loop approximation. Therefore, we must have $l \gg l_c \gg l_p$, which means a large value of $z = (\frac{l^3}{l_p^3 l_c})^{\frac{1}{4}}$. In this regime, the first order phase transition can always occur, and both the matter thin shell and the black hole solutions are more favorable than pure hot AdS space $I_{\text{PAdS}} = 0$. This strengthens the interpretation that the matter thin shell with the chosen equation of state models hot AdS space with self-gravitating radiation matter at low temperatures.

3. Favorable states: Comparison between thin shell, black hole, and hot thermal AdS thermodynamic states

We have just seen how the action of a stable self-gravitating matter system in AdS, which is a realization of hot curved AdS, compares with the action of the stable AdS black hole solution, and the action of pure AdS, describing classical AdS space devoid of any matter.

It is also interesting to substitute pure AdS for hot thermal AdS, i.e., AdS space with nonself-gravitating radiation obtained from the one-loop approximation, and compare with the action for black hole and the thin shell. The action I_{TAdS} for hot thermal AdS is given by $I_{\text{TAdS}} = -\frac{\pi^4 (l\bar{T})^3}{45}$, where we have considered that hot thermal AdS is made of particles, each with effective number of spin states equal to 2, such as gravitons do. For the purpose of comparison with the other actions we write I_{TAdS} as

$$\frac{l_p^2}{l^2} I_{\text{TAdS}} = -\frac{l_p^2}{l^2} \frac{\pi^4 (l\bar{T})^3}{45}. \quad (64)$$

We can then compare the first order phase transition treated above with the Hawking-Page phase transition, which is a transition between hot thermal AdS with action given in Eq. (64) and the black hole action given in Eq. (58) [7]. Moreover, when the temperature of the radiation is sufficiently high, the radiation forms a singularity of the Buchdahl type in the center and then it presumably

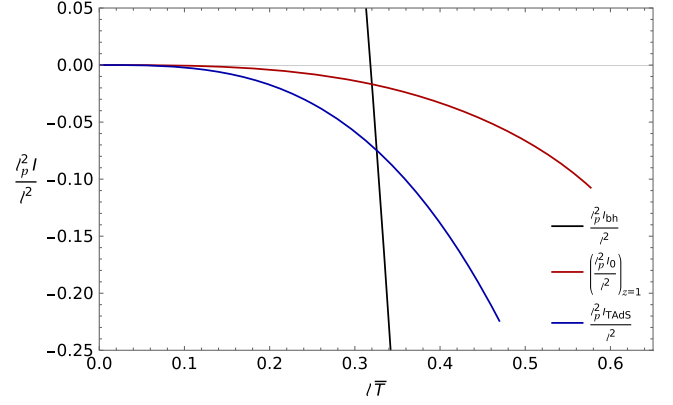


FIG. 9. Plot of the actions I_{bh} , I_0 , and I_{TAdS} as functions of the temperature $l\bar{T}$. The solution that has lower action between stable black hole, hot shell, and thermal hot AdS, i.e., AdS with nonself-gravitating radiation, is the one that is favored. It is chosen $z = (\frac{l}{l_c})^{\frac{1}{4}}(\frac{l}{l_p})^{\frac{1}{2}} = 1$ and $\frac{l_p}{l} = 1$ to compare the actions. For $z = 1$, the hot shell ceases to exist at temperature $l\bar{T}_f = 0.577$. Thermal hot AdS ceases to exist at temperature $l\bar{T}_{\text{Buch}} = 0.4701$.

collapses to a black hole. One finds that this maximum Buchdahl radiation temperature for which radiation ceases to exist is given by $l\bar{T}_{\text{Buch}} = 0.4234(\frac{2\pi^2}{30})^{-\frac{1}{4}}(\frac{l}{l_p})^{\frac{1}{2}}$ [7]; see also [35]. In Fig. 9, the actions are plotted for the stable black hole, the hot matter thin shell, and hot thermal AdS. In the figure, we have used the numbers $z = (\frac{l}{l_c})^{\frac{1}{4}}(\frac{l}{l_p})^{\frac{1}{2}} = 1$ and $\frac{l_p}{l} = 1$ to compare the actions. First, we comment on the transitions between stable black hole and hot thermal AdS thermodynamic states, and second, we compare the results for the hot matter thin shell and hot thermal AdS. In relation to the first point, we have seen that there are no black holes, and therefore, no stable black hole, in the range of temperatures $0 \leq l\bar{T} < \frac{\sqrt{3}}{2\pi} = 0.276$, the last number being approximate. From $\frac{\sqrt{3}}{2\pi} \leq l\bar{T} < 0.325$, hot thermal AdS is favored in relation to a black hole state. At $l\bar{T} = 0.325$ hot thermal AdS and black hole coexist equally, and this is the temperature at which a first order phase transition occurs. This is the Hawking-Page phase transition. For $0.325 < l\bar{T} < l\bar{T}_{\text{Buch}}$, where $l\bar{T}_{\text{Buch}} = 0.4701$, the black hole is favored over hot thermal AdS, meaning that it is more probable to find the system in a black hole state. For $l\bar{T}_{\text{Buch}} < l\bar{T} < \infty$, the system is in a collapsed black hole state in AdS, meaning that at these temperatures, it is not possible to find the system in a hot thermal AdS state. In relation to the second point, we see from the figure that the first order phase transition between the matter thin shell and the black hole is analogous to the Hawking-Page phase transition between hot thermal AdS and the black hole. However, it seems that hot thermal AdS, i.e., AdS with nonself-gravitating radiation, is more favorable than the matter thin shell, with the differences being very small as

one increases z and $\frac{l}{l_p}$. Since the action for hot thermal AdS does not include gravitation, as it corresponds to nonself-gravitating radiation, it is clear that the hot thin shell should mimic self-gravitating radiation due to its analogous behavior around the phase transition. In addition, we see they share the feature of a maximum temperature, $l\tilde{T}_f = 0.577$ for the hot thin shell and $lT_{\text{Buch}} = 0.4701$ for hot thermal AdS, the values being for $z = 1$ and $l = l_p$.

VII. CONCLUSIONS

In this work, we have studied the canonical ensemble of a hot self-gravitating matter thin shell in AdS by finding the partition function of the system via the Euclidean path integral approach to quantum gravity. We have restricted the study to spherically symmetric metrics and have established the boundary conditions. Imposing the Hamiltonian constraint, we obtained the reduced action of the matter thin shell in AdS, and the stationary and stability conditions. There are two equations for the stationary condition, i.e., the balance of pressure and the balance of temperature, and two stability conditions, i.e., the mechanical stability condition and the thermodynamic stability condition. The fact that one obtains the two stability conditions shows the power of the reduced action in the Euclidean path integral approach, in that it gives not only information about thermodynamics but also of mechanics.

We showed that for the case of the matter thin shell in AdS, one can obtain the canonical ensemble of the thin shell by establishing an effective reduced action only dependent on the gravitational radius of the thin shell. This eases the analysis of the canonical ensemble and further shows that one can build an effective reduced action dependent only on the gravitational radius for this case, hiding the description of matter in the form of the effective entropy. Here, we established the link between the effective entropy and the specific description of the shell with an equation of state, in the zero-loop approximation.

The thermodynamics of the system follows directly from zero-loop approximation consisting of the reduced action evaluated at the stationary points, which is equivalent to finding the action for the specific solution of Einstein equation. In this approximation, one obtains directly the relevant thermodynamic quantities, namely, the mean free

energy, the entropy, the mean energy, and the heat capacity. There is a correspondence between thermodynamic stability in the ensemble theory and positive heat capacity in the derived thermodynamics, as it should. On the other hand, within thermodynamics itself, one cannot determine mechanical stability by varying the thermodynamic quantity fixed at the conformal boundary, i.e., the temperature. This fact seems to be a consequence of applying the zero-loop approximation to the internal degree of freedom, the radius of the shell. It also means that the zero-loop approximation of the effective reduced action yields the expected thermodynamic stability condition, since it can be seen as a generalized free energy function.

We introduced an equation of state and obtained the solutions of the canonical ensemble for the matter thin shell. There are in total four solutions, with only one of them being stable both mechanically and thermodynamically. We also have compared the action of the stable self-gravitating matter shell solution with the Hawking-Page AdS black hole stable solution and verified the existence of a first order phase transition in a physically reasonable range of scale lengths. We showed that this first order phase transition is analogous to the Hawking-Page phase transition, and that the hot matter thin shell can mimic nonself-gravitating radiation in AdS. We note also that the phase structure is dependent on the choice of the equation of state and further study of this configuration for different families of equations of state should be done to uncover other possible phase structures. It will also be interesting to uncover the sector with a black hole and a shell together to fully understand the space of configurations and respective phase transitions.

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DATA AVAILABILITY

The data that support the findings of this article are openly available in [36].

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