

Landau theory for lattice higher-form gauge theories and the Kramers-Wannier duality

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We derive a Landau field theory for a lattice higher gauge theory defined on p -dimensional open cells (i.e., sites, links, faces, cubes, etc.), and study its continuum-limit and phases. In this approach, the p -dimensional Wilson-surface operator of the higher gauge theory is promoted to a fundamental functional field that is charged under the p -form global symmetry. By explicitly solving the functional equation of motion, we show that the classical solution exhibits the area (perimeter) law in the strong (weak) gauge coupling limit. In the deconfined phase, we also construct topological defects for both $U(1)$ and $\mathbb{Z}_N$ higher-form gauge theories, as analogs of vortex and domain-wall solutions in conventional field theories with 0-form global symmetries. Besides, we examine low-energy effective theory by identifying the phase modulations of the functional field as low-energy modes, and discuss the Coleman-Mermin-Wagner theorem for higher-form global symmetries. Finally, we discuss infrared duality among Landau field theories, which originates from Kramers-Wannier duality in lattice higher-form gauge theories.

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I. INTRODUCTION

In modern physics, the concept of symmetry has been extended and generalized in various directions [1–17]. The awareness of these generalized symmetries has broadened our understanding of quantum phases of matter and motivated us to explore new paradigm beyond conventional Landau theory for particles, i.e., field theory with 0-form global symmetries.

In this paper, we derive a Landau field theory for a lattice higher gauge theory defined on p -dimensional open cells in a D -dimensional Euclidean hypercubic lattice. Here, p -dimensional open cells include sites ($p = 0$), links ($p = 1$), faces ($p = 2$), cubes ($p = 3$), and their higher-dimensional ones. Hereafter, we refer to the p -dimensional open cell as a p -link as in ordinary lattice gauge theories. The higher gauge theory defined on p -links possesses a p -form global symmetry, whose order parameter is the vacuum expectation value of the p -dimensional Wilson-surface operator $W[C_p]$. By performing the Hubbard-Stratonovich (HS) transformation to the higher gauge theory, we derive a scalar field theory defined on the space of p -dimensional (self-avoiding) closed surfaces. This is an extension of the previous works [18,19], where the Landau

theory of $SU(N)$ (lattice) gauge theory was formulated in the same manner.

We then examine the classical continuum-limit of the Landau theory and study phases of the higher gauge theory at the mean-field level. One of the crucial findings is that the kinetic term of the scalar functional takes the form of a d'Alembert operator constructed from the so-called area derivatives [20–27]. By solving the functional equation of motion, we show that the classical solution in the Landau theory exhibits the area (perimeter) law in the strong (weak) gauge coupling limit, corresponding to the confined (deconfined) phase of the original higher gauge theory, as expected. In the broken or deconfined phase, the phase modulations in the scalar field are identified as low-energy fluctuation modes, and we examine their low-energy effective theory. In the case of compact $U(1)$ p -form gauge theory, a naive effective theory is the p -form Maxwell theory describing a gapless p -form field A_p , but the existence of topological defects (or monopoles for $D = p + 2$) can render these fluctuation modes massive and prevent the spontaneous breaking of the p -form global symmetry for $p \geq D - 2$. This is the Coleman-Mermin-Wagner theorem for compact $U(1)$ higher-form gauge theories [1,28]. In the case of finite p -form gauge theory, on the other hand, the low-energy effective theory is found to be a BF-type topological field theory coupled to topological defects, exhibiting topological order in general.

Following the mean-field analysis, we further discuss Kramers-Wannier (KW) duality [29–33] in finite higher-form gauge theories and its implication for infrared (IR) duality in continuum Landau field theories. In general, the

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KW duality asserts that the higher gauge theory with an Abelian group G defined on p -links is dual (or isomorphic) to the “gauged” higher gauge theory defined on $(D - p - 2)$ links with the Pontrjagin dual $\hat{G} \simeq G$ as its gauge group. For example, for $D = 2$ and $p = 0$, it corresponds to the famous KW duality between the two-dimensional $\mathbb{Z}_2$ Ising model and the two-dimensional gauged $\mathbb{Z}_2$ Ising model. We derive this KW duality in the present lattice formulation and elucidate how it can lead to an IR duality between Landau field theories in the continuum limit. Here, IR duality means that two theories describe the same critical phenomena or flow to the same IR critical point by renormalization group. It should be noted that we do not intend to provide a rigorous proof of such an IR duality here; rather, we argue that the HS transformation on dual higher-form gauge theories leads to an exact duality between two lattice Landau theories, which in turn yields an IR duality between two continuum Landau theories at the mean-field level, after taking the classical continuum-limit. In Fig. 1, we summarize these duality relations. (See the main text for the concrete definitions of partition functions.)

From the perspective of quantum many-body theory of higher dimensional objects, such an IR duality implies that closed extended objects with different dimensions can belong to the same universality class provided that they share the same higher-form global symmetry. These observations may offer valuable insights into the quantum nature of strings and branes.

The organization of this paper is as follows. In Sec. II, we formulate higher-form gauge theories in a Euclidean D -dimensional hypercubic lattice and explain higher-form global symmetries. In Sec. III, we derive a Landau field theory for the lattice higher gauge theory by performing the Hubbard-Stratonovich transformations and discuss its continuum limit. We then study the phases of higher gauge theory, including the explicit constructions of topological defects in the Landau theory in Sec. IV. Here, the Coleman-Mermin-Wagner theorem for higher-form global symmetries is also discussed in the present framework. In

Sec. V, we discuss Kramers-Wannier duality in higher-form gauge theories and how it can lead to an IR duality in the corresponding continuum Landau theories. Section VI is devoted to summary. Appendixes provide supplementary details.

II. LATTICE HIGHER GAUGE THEORY

We consider a D -dimensional Euclidean hypercubic lattice Λ_D with a lattice spacing a . In this discretized space, a p -dimensional closed surface C_p is represented as a union of elementary p -dimensional minimum hypercubes, as illustrated in the right panel of Fig. 2. Following the same convention as conventional lattice gauge theories, we refer to such a minimum p -dimensional hypercube as p -link, and represent it as L_p in general. In particular, a p -link extending in the p -dimensional subspace $(X^{\mu_1}, \dots, X^{\mu_p})$ is explicitly denoted by $L_{\mu_1 \dots \mu_p}(\hat{i})$, where $\hat{i}$ represents the center-of-mass position of it.

We focus on *self-avoiding* p -dimensional closed surfaces C_p with orientations, where “self-avoiding” means that C_p is not a union of multiple p -dimensional surfaces that share lower dimensional q -links ($q \leq p - 1$), as depicted in Fig. 2. In this figure, a two-dimensional surface C_2 is given by a union of two self-avoiding surfaces sharing a single one-link. The set of all self-avoiding p -dimensional closed surfaces is denoted by Γ_p . Besides, the surface with the opposite orientation to C_p is represented by $C_p^{-1} \in \Gamma_p$.

A minimum closed p -dimensional surface, called a p -plaquette, is defined by the boundary of a $(p + 1)$ -link L_{p+1} , and is denoted by P_p in general. See the left figures in Fig. 2 for $p = 1$ and 2. Moreover, we represent the number of p -links on a p -dimensional surface $C_p \in \Gamma_p$ as $|C_p|$. For example, $|P_p| = 2(p + 1)$ for $\forall$ p -plaquettes P_p . In particular, the continuum limit of a surface C_p is obtained by

$$a \rightarrow 0, \quad |C_p| \rightarrow \infty, \quad \text{with } \text{Vol}[C_p] := a^p |C_p| = \text{fixed}. \quad (1)$$

Finally, we identify 0-plaquette as a 1-link, which allows us to discuss lattice field theories with 0-form global symmetries as well.

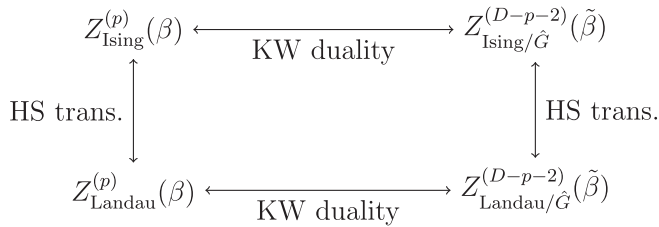


FIG. 1. KW duality in lattice higher-form gauge theories and corresponding Landau field theories. The upper KW duality in higher-form gauge theories is already well established, while the lower one is obtained in this paper by performing the HS transformation. In the continuum limit, this leads to an IR duality between continuum Landau field theories.

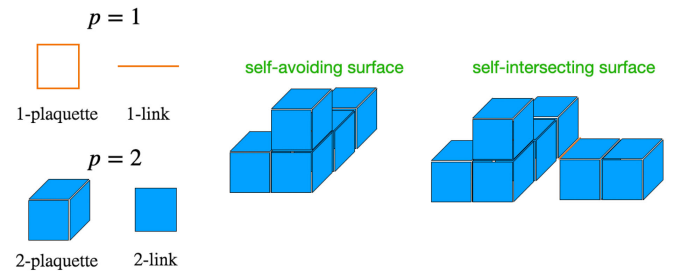


FIG. 2. Left: p -plaquettes and p -links. Here we show $p = 1$ and $p = 2$ cases. Right: Examples of closed two-dimensional surfaces. The left (right) one is a self-avoiding (intersecting) surface.

A. Higher gauge theory on the lattice

In the following, we denote a general Abelian group as G . As in ordinary lattice gauge theories, we assign a group variable for each p -link as

$$U_{L_p}(\hat{i}) = \begin{cases} e^{\frac{2\pi i}{N}m} (m = 0, 1, \dots, N-1) & \text{for } G = \mathbb{Z}_N \\ e^{i\theta} (\theta \in [-\pi, \pi]) & \text{for } G = \text{U}(1) \end{cases}. \quad (2)$$

As usual, we also define a p -link variable corresponding to L_p^{-1} as

$$U_{L_p^{-1}}(\hat{i}) := U_{L_p}^\dagger(\hat{i}). \quad (3)$$

In addition, the gauge transformation of a link variable is defined by

$$U_{L_p}(\hat{i}) \rightarrow U_{L_p}(\hat{i}) \prod_{L_{p-1} \subset \mathbb{P}_{p-1} = \partial L_p} g(L_{p-1}), \quad g(L_{p-1}) \in G, \quad (4)$$

where $\prod_{L_{p-1} \subset \mathbb{P}_{p-1} = \partial L_p}$ means the product over all $(p-1)$ -links on the boundary $\mathbb{P}_{p-1} = \partial L_p$. For example, it reproduces the usual gauge transformation

$$U_{L_\mu}(\hat{i}) \rightarrow g(i) U_{L_\mu}(\hat{i}) g^\dagger(i + \hat{\mu}) \quad (5)$$

when $p = 1$. For a given closed surface $C_p \in \Gamma_p$, we define a Wilson-surface operator by

$$W[C_p] := \prod_{L_p \subset C_p} U_{L_p}(\hat{i}), \quad (6)$$

which is invariant under the gauge transformation (4) by definition.

The partition function of higher gauge theory defined on p -links is

$$Z_{\text{Ising}}^{(p)}(\beta) = \int [dU] \exp(-S_E), \quad (7)$$

$$S_E = \beta \sum_{\mathbb{P}_p} \left[1 - \frac{1}{2} (W[\mathbb{P}_p] + W[\mathbb{P}_p^{-1}]) \right], \quad (8)$$

where $\beta = 1/g^2$ is the inverse gauge coupling, $\sum_{\mathbb{P}_p}$ is the summation over all positive oriented p -plaquettes, and

$$\int [dU] := \begin{cases} \prod_{L_p} \left(\frac{1}{|G|} \sum_{g(L_p) \in G} \right) & \text{for a finite Abelian group } G \\ \prod_{L_p} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta_{L_p} & \text{for } G = \text{U}(1) \end{cases} \quad (9)$$

is the path-integral measure of link variables. For $p = 0$,

Eqs. (7) and (8) reduce to an Ising model because a 0-plaquette is identified as a 1-link. In this sense, the lattice gauge theory (7) can be viewed as a higher-link version of Ising models. This is the reason why we put the subscript “Ising” in the partition function (7).

B. Higher form global symmetry

The lattice gauge theory (7) is invariant under the following p -form global transformation

$$W[C_p] \rightarrow g(C_p) W[C_p], \quad g(C_p) \in G, \quad (10)$$

with the condition $g(C_p) = 1$ for all $C_p \in \Gamma_p$ that shrink to a point in the continuum-limit (1). This condition ensures the local flatness of gauge parameters in the continuum limit. Besides, $g(C_p)$ satisfies

$$g(C_p^1 + C_p^2) = g(C_p^1) g(C_p^2) \quad (11)$$

for a combined surface $C_p^1 + C_p^2 \in \Gamma_p$ because $W[C_p^1 + C_p^2] = W[C_p^1] W[C_p^2]$. More explicitly, the p -form transformation is expressed by

$$g(C_p) = \begin{cases} e^{i\theta \int_{C_p} \Lambda_p}, \theta \in \mathbb{R} & \text{for } G = \text{U}(1), d\Lambda_p = 0, \\ e^{\frac{i}{N} \int_{C_p} \Lambda_p} & \text{for } G = \mathbb{Z}_N \end{cases}, \quad \int_{C_p} \Lambda_p \in 2\pi\mathbb{Z}, \quad (12)$$

in the continuum-limit expression, where Λ_p is a differential p form. The corresponding symmetry/topological operator is a codimension $(p+1)$ operator $U[C_{D-p-1}]$ satisfying

$$U[C_{D-p-1}] W[C_p] U^{-1}[C_{D-p-1}] = g(C_p)^{I[C_p, C_{D-p-1}]} W[C_p], \quad (13)$$

as an (equal-time) operator relation, where C_{D-p-1} is a $(D-p-1)$ -dimensional subspace and $I[C_p, C_{D-p-1}]$ is the intersection number between C_p and C_{D-p-1} . See Fig. 3 for a graphical representation of this relation.

For $G = \text{U}(1)$, the symmetry operator is explicitly given by

$$U_\theta[C_{D-p-1}] = \exp \left(i\theta \int_{C_{D-p-1}} \star F_{p+1} \right), \quad F_{p+1} = dA_p \quad (14)$$

in the continuum-limit expression. When G is a finite Abelian group, it can be realized as a locally flat configuration of the background $(p+1)$ -form gauge field coupled

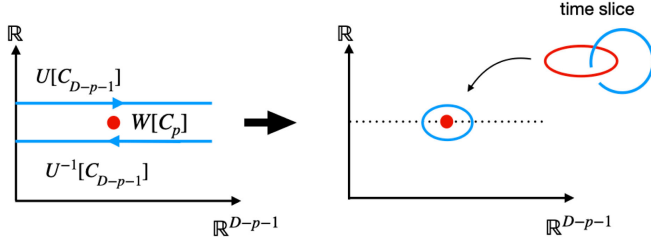


FIG. 3. Graphical representation of the operator relation (13). Here, the red point corresponds to the Wilson-surface operator $W[C_p]$, and the blue lines correspond to the symmetry operators $U[C_{D-p-1}]$, $U[C_{D-p-1}]^{-1}$.

to Eq. (7) [1]. See also Appendix A for the explicit check of Eq. (13) in the continuum higher-form gauge theories.

III. LANDAU THEORY FOR HIGHER GAUGE THEORY

We derive a Landau field theory for the lattice higher gauge theory (7) and examine its (classical) continuum limit. The key step in the derivation is to rewrite the lattice action (8) into a quadratic form by introducing an infinitesimal deformation operator of closed surfaces. Such a quadratic form enables us to perform the HS transformation, and we obtain a scalar field theory defined on the space of self-avoiding closed surfaces Γ_p .

A. Hubbard Stratonovich transformation

Given a self-avoiding surface $C_p \in \Gamma_p$ and a p -plaquette P_p , we define a combined surface $C_p + P_p$ by erasing the common p -links (with opposite orientations) between C_p and P_p , and attaching the remaining p -links of P_p that are not common with C_p , as illustrated in Fig. 4. An trivial but important property is

$$W[C_p]^* W[C_p + P_p] = W[P_p] \quad (15)$$

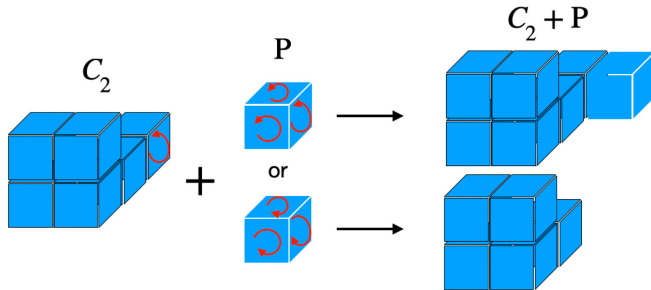


FIG. 4. Combination of a two-dimensional surface C_2 and two-plaquette P_2 . In the upper case, a single two-link (face) is erased and the remaining five two-links are attached to C_2 . In the lower case, on the other hand, five two-links are erased and the remaining one two-link is attached.

when $C_p + P_p \in \Gamma_p$. When $C_p + P_p \notin \Gamma_p$, Eq. (15) does not necessarily hold.

We introduce a translation operator along a p -plaquette P_p by [18,34]

$$\Pi_{P_p} W[C_p] := \begin{cases} W[C_p + P_p] & \text{for } C_p + P_p \in \Gamma_p \\ 0 & \text{for } C_p + P_p \notin \Gamma_p \end{cases} \quad (16)$$

Using this operator, we can express the lattice action (8) as

$$S_E = \text{constant} - \frac{\beta}{2} \sum_{C_p \in \Gamma_p} w[C_p] W[C_p]^* (k_0 + \hat{H}) w[C_p]^* W[C_p], \quad (17)$$

where the operator $\hat{H}$ is defined by

$$\hat{H} W[C_p] := \frac{1}{2(D-p)|C_p|} \sum_{\substack{P_p \\ P_p \cap C_p \neq \emptyset}} \Pi_{P_p} W[C_p]. \quad (18)$$

We will soon prove the equivalence below. Here, $\sum_{\substack{P_p \\ P_p \cap C_p \neq \emptyset}}$ implies the summation over all p -plaquettes having common p -links with C_p , and k_0 is a sufficiently large positive constant to make $k_0 + \hat{H}$ positive definite. The introduction of k_0 does not change any observables because it simply corresponds to a constant shift of the action. Note also that the normalization of $\hat{H}$ is chosen to guarantee $\hat{H} \rightarrow 1$ in the continuum limit $a \rightarrow 0$. See the next subsection for more details.

Moreover, $w[C_p]$ is an arbitrary weight functional that is invariant under spacetime translation and satisfies

$$\frac{1}{2(D-p)} \sum_{\substack{C_p \in \Gamma_p \\ C_p \cap P_p \neq \emptyset}} \frac{1}{|C_p|} w[C_p] w[C_p + P_p]^* = 1, \quad (19)$$

where $\sum_{\substack{C_p \in \Gamma_p \\ C_p \cap P_p \neq \emptyset}}$ means the summation over all closed surfaces $C_p \in \Gamma_p$ such that (i) they have common p -links with a given p -plaquette P_p and (ii) $C_p + P_p$ also belongs to Γ_p . The simplest choice satisfying these conditions is

$$w[C_p] = b \times \exp \left(-\frac{\alpha}{2} \text{Vol}[C_p] + i \int_{C_p} \lambda_p \right), \quad (20)$$

where λ_p is a differential closed p -form $d\lambda_p = 0$ with $\int_{C_p} \lambda_p \in 2\pi\mathbb{Z}$, and $\alpha > 0$ is a brane tension with mass dimension p . The normalization factor b is determined by Eq. (19) as

$$\begin{aligned}
 \frac{b^2}{2(D-p)} \sum_{\substack{C_p \in \Gamma_p \\ C_p \cap P_p \neq \emptyset}} \frac{1}{|C_p|} e^{-\frac{g}{2}(\text{Vol}[C_p] + \text{Vol}[C_p + P_p]) - i \int_{P_p} \lambda_p} &\approx \frac{b^2}{2(D-p)} \sum_{\substack{C_p \in \Gamma_p \\ C_p \cap P_p \neq \emptyset}} \frac{1}{|C_p|} e^{-\alpha \text{Vol}[C_p]} = 1 \\
 \therefore b^2 &\approx \frac{2(D-p)}{\sum_{\substack{C_p \in \Gamma_p \\ C_p \cap P_p \neq \emptyset}} e^{-\alpha \text{Vol}[C_p]} / |C_p|}, \tag{21}
 \end{aligned}$$

where we have used the Stokes theorem in the first line. The volume-suppression factor in Eq. (20) was introduced in order to make the summation over p -dimensional surfaces well defined.

Now we can prove the equivalence between Eqs. (8) and (17) as follows:

$$\begin{aligned}
 \sum_{C_p \in \Gamma_p} w[C_p] W^*[C_p] \hat{H} w[C_p]^* W[C_p] &= \sum_{C_p \in \Gamma_p} w[C_p] W^*[C_p] \frac{1}{2(D-p)|C_p|} \sum_{\substack{P_p \\ P_p \cap C_p \neq \emptyset}} w[C_p + P_p]^* W[C_p + P_p] \\
 &= \sum_{P_p, P_p^{-1}} W[P_p] \frac{1}{2(D-p)} \sum_{\substack{C_p \in \Gamma_p \\ C_p \cap P_p \neq \emptyset}} \frac{1}{|C_p|} w[C_p] w[C_p + P_p]^* \\
 &= \sum_{P_p} (W[P_p] + W[P_p^{-1}]), \tag{22}
 \end{aligned}$$

where we have used Eq. (15) [Eq. (19)] from the first (second) line to the second (third) line. Note also that $\hat{H}$ is self-adjoint as

$$\begin{aligned}
 \sum_{C_p \in \Gamma_p} \phi_1^*[C_p] \hat{H} \phi_2[C_p] &= \sum_{C_p \in \Gamma_p} \frac{1}{2(D-p)|C_p|} \sum_{\substack{P_p \\ P_p \cap C_p \neq \emptyset}} \phi_1^*[C_p] \phi_2[C_p + P_p] \\
 &= \sum_{P_p} \sum_{\substack{C_p \in \Gamma_p \\ C_p \cap P_p \neq \emptyset}} \frac{1}{2(D-p)|C_p|} \phi_1^*[C_p] \phi_2[C_p + P_p]. \tag{23}
 \end{aligned}$$

Since $C'_p := C_p + P$ belongs to Γ_p by the definition of Π_p , Eq. (23) can be also written as

$$= \sum_{P_p} \sum_{\substack{C_p \in \Gamma_p \\ C_p \cap P_p \neq \emptyset}} \frac{1}{2(D-p)|C_p|} \phi_1^*[C'_p + P_p^{-1}] \phi_2[C'_p] = \sum_{C_p \in \Gamma_p} \{ \hat{H} \phi_1^\dagger[C_p] \} \phi_2[C_p], \tag{24}$$

which confirms $\hat{H}^\dagger = \hat{H}$.

Since Eq. (17) is a quadratic form, we can perform the HS transformation and obtain

$$Z_{\text{Ising}}^{(p)}(\beta) \propto \int [dU] \int [d\phi] \exp \left\{ - \sum_{C_p \in \Gamma_p} \left(\frac{2}{\beta} \phi[C_p]^* (k_0 + \hat{H})^{-1} \phi[C_p] + w[C_p]^* W[C_p] \phi[C_p] + \text{H.c.} \right) \right\} \tag{25}$$

$$= \int [d\phi] \exp \left\{ - \sum_{C_p \in \Gamma_p} \left(\frac{2}{\beta} \phi[C_p]^* (k_0 + \hat{H})^{-1} \phi[C_p] + V(\phi, \phi^*) \right) \right\}, \tag{26}$$

where $\phi[C_p] \in \mathbb{C}$ is a functional field on the space of self-avoiding p -dimensional closed surfaces Γ_p , and

$$\sum_{C_p \in \Gamma_p} V(\phi, \phi^*) = -\log \left[\int [dU] \exp \left(- \sum_{C_p \in \Gamma_p} w[C_p] W[C_p]^* \phi[C_p] + \text{H.c.} \right) \right] \tag{27}$$

is the potential of $\phi[C_p]$. One can see that the lattice higher gauge theory is now recast as a scalar field theory defined on Γ_p .

In this functional formulation, the p -form global transformation (10) is expressed by

$$\phi[C_p] \rightarrow g(C_p)\phi[C_p], \quad g(C_p) \in G. \quad (28)$$

Although the invariance of the potential (27) under this transformation is trivial, the invariance of the kinetic term may be nontrivial. By the definition of $\hat{H}$ (18), we can see

$$\begin{aligned} g(C_p)^* \hat{H} g(C_p) &= \frac{1}{2(D-p)|C_p|} \sum_{\substack{P_p \\ \text{Pr } C_p \neq \emptyset}} g(C_p)^* g(C_p + P_p) \\ &= \frac{1}{2(D-p)|C_p|} \sum_{\substack{P_p \\ \text{Pr } C_p \neq \emptyset}} g(P_p) = 1, \end{aligned} \quad (29)$$

where we have used Eq. (11) and $g(P_p) = 1$.¹ This proves the invariance of the kinetic term under Eq. (28).

In general, the invariance under the p -form transformation (28) restricts the functional form of $V(\phi, \phi^*)$. For example, one can generally expand it as

$$\begin{aligned} \sum_{C_p \in \Gamma_p} V(\phi, \phi^*) &= \sum_{C_p \in \Gamma_p} \{ \mu_2(C_p) \phi[C_p]^* \phi[C_p] + \mu_4(C_p) (\phi[C_p]^* \phi[C_p])^2 + \dots \} \\ &+ \sum_{C_p^1 \in \Gamma_p} \sum_{C_p^2 \in \Gamma_p} \sum_{C_p^3 \in \Gamma_p} \delta(C_p^1 - C_p^2 - C_p^3) \mu_3(C_p^1, C_p^2, C_p^3) \phi[C_p^1]^* \phi[C_p^2] \phi[C_p^3] + \text{H.c.} + \dots, \end{aligned} \quad (30)$$

where $\delta(C_p)$ is the delta function on Γ_p , i.e.,

$$\sum_{C'_p \in \Gamma_p} \delta(C_p - C'_p) \phi[C'_p] = \phi[C_p], \quad (31)$$

for an arbitrary functional $\phi[C_p]$. The existence of this delta function in the second line in Eq. (30) ensures the invariance under the p -form global transformation as

$$\begin{aligned} &\delta(C_p^1 - C_p^2 - C_p^3) g(C_p^1)^* g(C_p^2) g(C_p^3) \\ &= \delta(C_p^1 - C_p^2 - C_p^3) g(C_p^1 - C_p^2 - C_p^3) \\ &= \delta(C_p^1 - C_p^2 - C_p^3), \end{aligned} \quad (32)$$

where we have used Eq. (11). The same argument applies to

higher-order nonlocal interactions in general. We should note that the coupling constants $\{\mu_i(C_p)\}$ in the expansion (30) can depend on C_p for a general weight functional $w[C_p]$. For the simplest example (20), however, they become constant as

$$\begin{cases} \mu_2(C_p) & \propto w[C_p]^* w[C_p] = b^2 \\ \mu_3(C_p^1, C_p^2, C_p^3) & \propto w[C_p^1]^* w[C_p^2] w[C_p^3] = b^3 \end{cases} \quad (33)$$

by absorbing the volume suppression factor $e^{-\alpha \text{Vol}[C_p]}$ into the Wilson-surface operator as $e^{-\alpha \text{Vol}[C_p]} W[C_p] \rightarrow W[C_p]$ in Eq. (27). Furthermore, when G is a finite Abelian group, the potential can also contain explicit U(1) breaking terms. For example, for $G = \mathbb{Z}_N$, there can be

$$\sum_{C_p \in \Gamma_p} V(\phi, \phi^*) \ni \sum_{C_p \in \Gamma_p} \{ \alpha^{(1)}(C_p) \phi[C_p]^N + \alpha^{(2)}(C_p) (\phi[C_p]^N)^2 + \dots \} + \text{H.c.}, \quad (34)$$

where the couplings $\{\alpha^{(i)}(C_p)\}$ transform trivially under the $\mathbb{Z}_N$ transformation (28). For a general finite Abelian group G , the structure of the potential becomes more complicated.

One more comment is in order. In Eq. (26), the canonically normalized kinetic term can be realized by the field redefinition $g\phi[C_p] \rightarrow \phi[C_p]$, which in turn leads to $g^{-n} \phi[C_p]^n$ in a general n th order term in the potential $V(\phi, \phi^*)$. This implies that the weak-coupling limit in the gauge theory $g \rightarrow 0$ corresponds to the strong coupling regime $g^{-n} \rightarrow \infty$ in the Landau field theory, and vice versa. In Sec. IV, we will see that this correspondence plays an important role to demonstrate the area/perimeter

law in the mean-field analysis within the Landau theory.

B. Classical continuum limit

Let us discuss classical continuum limit of the Landau theory (26). We represent a D -dimensional Euclidean curved space with a metric $g_{\mu\nu}(X)$ by Σ_D . In the continuum limit $a \rightarrow 0$, a self-avoiding surface $C_p \in \Gamma_p$ is represented

¹For a given p -link $L_p \subset C_p$, there are $2(D-p)$ p -plaquettes that contain L_p , which leads to $\sum_{\substack{P_p \\ \text{Pr } C_p \neq \emptyset}} 1 = 2(D-p)|C_p|$.

by a set of embedding functions

$$X^\mu: S^p \rightarrow \Sigma_D, \quad (\mu = 1, 2, \dots, D), \quad (35)$$

where S^p is the p -dimensional sphere whose general intrinsic coordinates are denoted by $\{\xi^i\}_{i=1}^p$. Then, the induced metric on C_p is

$$\begin{aligned} ds^2 &= g_{\mu\nu}(X) dX^\mu dX^\nu \\ &= g_{\mu\nu}(X(\xi)) \frac{\partial X^\mu}{\partial \xi^i} \frac{\partial X^\nu}{\partial \xi^j} d\xi^i d\xi^j = h_{ij}(\xi) d\xi^i d\xi^j, \end{aligned} \quad (36)$$

and we denote its determinant as $h(\xi) := \det(h_{ij}(\xi)) \geq 0$. Then, the volume of C_p is expressed as

$$\text{Vol}[C_p] = a^p |C_p| \rightarrow \int_{S^p} d^p \xi \sqrt{h(\xi)} \quad (37)$$

in the continuum limit (1). This is invariant under the reparametrization $\xi^i \rightarrow \xi'^i = f^i(\xi)$ on C_p . This can be also expressed in a differential form as

$$\text{Vol}[C_p] = \int_{C_p} E_p, \quad (38)$$

with

$$\begin{aligned} E_p &= \frac{1}{p!} (E_p(\xi))^{\mu_1 \dots \mu_p} dX_{\mu_1} \wedge \dots \wedge dX_{\mu_p} \\ &:= \frac{1}{p! \sqrt{h(\xi)}} \{X^{\mu_1}, \dots, X^{\mu_p}\} dX_{\mu_1} \wedge \dots \wedge dX_{\mu_p}, \end{aligned} \quad (39)$$

where

$$\{X^{\mu_1}, \dots, X^{\mu_p}\} = \epsilon^{i_1 \dots i_p} \frac{\partial X^{\mu_1}}{\partial \xi^{i_1}} \dots \frac{\partial X^{\mu_p}}{\partial \xi^{i_p}} \quad (40)$$

is the Nambu-bracket [35]. Geometrically, E_p corresponds to the normalized infinitesimal volume element on C_p , which implies that the discretized version of the component $(E_p(\xi))^{\mu_1 \dots \mu_p}$ is

$$(E_p(\hat{i}))^{\mu_1 \dots \mu_p} = \frac{1}{p!} \sum_{\sigma \in S_p} \delta_{\nu_1}^{\sigma(\mu_1)} \dots \delta_{\nu_p}^{\sigma(\mu_p)} \quad (41)$$

for $L_{\nu_1 \dots \nu_p}(\hat{i}) \subset C_p$. Here, S_p is the symmetric group of degree p . For the later convenience, we also represent the volume of the $(p+1)$ -dimensional minimal surface M_{p+1} bounded by C_p as

$$\text{Vol}[M_{p+1}] := \int_{M_{p+1}} E_{p+1}. \quad (42)$$

With these preparations, we can study the classical continuum-limit of Eq. (26). The most crucial and non-trivial point is the treatment of the kinetic term $(k_0 + \hat{H})^{-1}$, which is constructed by the plaquette shifting operator (16). This fact motivates us to generalize the notion of “derivative” from the ordinary one $\partial/\partial x^\mu$ to a functional one acting on functional fields $\phi[C_p]$. In fact, such a generalization has been investigated for the construction of effective Landau field theories with higher-form global symmetries [19,25–27,36]. In the present lattice formulation, it can be explicitly defined by

$$\frac{\delta \psi[C_p]}{\delta \sigma^{\mu_1 \dots \mu_{p+1}}(\hat{i})} := \frac{\psi[C_p + \mathbf{P}_{\mu_1 \dots \mu_{p+1}}(\hat{i})] - \psi[C_p]}{a^{p+1}} \quad (43)$$

for a general functional field $\psi[C_p]$ and a general p -dimensional surface C_p . (That is, it does not need to be a self-avoiding surface.) Here, $\mathbf{P}_{\mu_1 \dots \mu_{p+1}}(\hat{i})$ is a p -plaquette sitting in the $(p+1)$ -dimensional subspace $(X^{\mu_1}, \dots, X^{\mu_{p+1}})$ with a center-of-mass position $\hat{i}$. Since a permutation of the target-space coordinates corresponds to a change of orientation of C_p , we can naturally define

$$\mathbf{P}_{\sigma(\mu_1) \dots \sigma(\mu_{p+1})}(\hat{i}) = \begin{cases} \mathbf{P}_{\mu_1 \dots \mu_{p+1}}(\hat{i}) & \text{for } \text{sgn}(\sigma) = 1 \\ \mathbf{P}_{\mu_1 \dots \mu_{p+1}}^{-1}(\hat{i}) & \text{for } \text{sgn}(\sigma) = -1 \end{cases}, \quad (44)$$

where σ is an element of the symmetric group S_{p+1} and $\text{sgn}(\sigma)$ is the sign of the permutation. This leads to the antisymmetric property of the area derivatives [19,25–27,36],

$$\frac{\delta}{\delta \sigma^{\mu_1 \dots \mu_{p+1}}(\hat{i})} = \text{sgn}(\sigma) \frac{\delta}{\delta \sigma^{\mu_1 \dots \mu_{p+1}}(\hat{i})}. \quad (45)$$

Note that the center-of-mass position $\hat{i}$ corresponds to a point $\xi = (\xi^1, \dots, \xi^p)$ on C_p in the continuum limit $a \rightarrow 0$.

One of the important properties of the area derivatives is [25,26]

$$\frac{\delta}{\delta \sigma^{\mu_1 \dots \mu_{p+1}}(\xi)} \left(\int_{C_p} B_p \right) = (dB_p)_{\mu_1 \dots \mu_{p+1}}(X(\xi)) \quad (46)$$

for an arbitrary differential p -form $B_p(X)$, which then allows us to calculate the area derivative on an arbitrary function of $\int_{C_p} B_p$ as

$$\frac{\delta}{\delta \sigma^{\mu_1 \dots \mu_{p+1}}(\xi)} f \left(\int_{C_p} B_p \right) = (dB_p)_{\mu_1 \dots \mu_{p+1}}(X(\xi)) f'(x) \Big|_{x=\int_{C_p} B_p}. \quad (47)$$

This can be interpreted as a generalization of the chain rule of ordinary derivative ∂_μ .

In the Landau field theory (26), we must take into account the fact that the translation operator Π_{p_p} is nonzero only for p -plaquettes that share common p -links with C_p . This corresponds to projecting the area derivative (43) onto the direction of a shared p -link as

$$\frac{\delta}{\delta S^\mu(\hat{i})} := (E_p(\hat{i}))^{\nu_1 \cdots \nu_p} \frac{\delta}{\delta \sigma^{\mu \nu_1 \cdots \nu_p}(\hat{i})}, \quad (48)$$

where the projection tensor $(E_p(\hat{i}))^{\nu_1 \cdots \nu_p}$ is given by Eq. (41).

Now we can evaluate $\hat{H}$ for $a \rightarrow 0$ as

$$\begin{aligned} \hat{H}\phi[C_p] &= \frac{1}{2(D-p)|C_p|} \sum_{\substack{p_p \\ \text{Pr } C_p \neq \emptyset}} \Pi_{p_p} \phi[C_p] = \phi[C_p] + \frac{a^{p+1}}{2(D-p)|C_p|} \sum_{i \in C_p} \sum_{\mu=1}^D \sum_{s=\pm 1} \text{sgn}(s) \frac{\delta \phi[C_p]}{\delta S^\mu(\hat{i})} \\ &\quad + \frac{2(a^{p+1})^2}{4(D-p)|C_p|} \sum_{i \in C_p} \sum_{\mu=1}^D \frac{\delta \phi[C_p]}{\delta S^\mu(\hat{i}) \delta S^\mu(\hat{i})} + \mathcal{O}(a^{3(p+1)}) \end{aligned} \quad (49)$$

$$= \phi[C_p] + \frac{(a^{p+1})^2}{2(D-p)|C_p|} \sum_{i \in C_p} \sum_{\mu=1}^D \frac{\delta^2 \phi[C_p]}{\delta S^\mu(\hat{i}) \delta S^\mu(\hat{i})} + \mathcal{O}(a^{3(p+1)}), \quad (50)$$

where the linear terms in a^{p+1} vanish due to the contributions with opposite orientations. Then, using the Einstein notation, we obtain

$$\begin{aligned} \lim_{a \rightarrow 0} \phi[C_p]^* \frac{\hat{H} - 1}{(a^{p+1})^2} \phi[C_p] &= \frac{1}{2(D-p)|C_p|} \lim_{a \rightarrow 0} \sum_{i \in C_p} \phi[C_p]^* \frac{\delta^2}{\delta S^\mu(\hat{i}) \delta S_\mu(\hat{i})} \phi[C_p] \\ &= \frac{1}{2(D-p)\text{Vol}[C_p]} \int_{S^p} d^p \xi \sqrt{h(\xi)} \phi[C_p]^* \frac{\delta^2}{\delta S^\mu(\xi) \delta S_\mu(\xi)} \phi[C_p] \end{aligned} \quad (51)$$

in the continuum limit. In addition, the sum over all self-avoiding surfaces becomes the path integral of embedding functions $\{X^\mu(\xi)\}_{\mu=1}^D$ as

$$a^D \sum_{C_p \in \Gamma_p} \rightarrow \mathcal{N} \int \mathcal{D}X, \quad (52)$$

where $\mathcal{N}$ is a normalization factor (that should be determined appropriately) and $\mathcal{D}X$ is the path-integral measure induced by the diffeomorphism and reparametrization invariant norm

$$||\delta X||^2 := \int_{S^p} d^p \xi \sqrt{h(\xi)} g_{\mu\nu}(X(\xi)) \delta X^\mu(\xi) \delta X^\nu(\xi). \quad (53)$$

It should be noted that Eq. (52) contains the integral over the center-of-mass position $\{x^\mu\}_{\mu=1}^D$ of C_p as

$$\int \mathcal{D}X = \int_{\Sigma_D} \star 1 \cdots, \quad (54)$$

because a spacetime translation of a surface C_p results in another surface C'_p . This is the reason why we have included the factor a^D in Eq. (52).

Expanding $(k_0 + \hat{H})^{-1}$ in powers of $\hat{H} - 1$ and performing a field redefinition (without including the inverse gauge coupling β), we finally obtain the continuum limit of Eq. (26) as

$$Z_{\text{Landau}}^{(p)}(\beta) := \int \mathcal{D}\phi \mathcal{D}\phi^* e^{-S_E}, \quad (55)$$

$$\begin{aligned} S_E &= \mathcal{N} \int \mathcal{D}X \left[U(\phi, \phi^*) - \frac{1}{\beta \text{Vol}[C_p]} \int_{S^p} d^p \xi \sqrt{h(\xi)} \phi[C_p]^* \frac{\delta^2}{\delta S^\mu(\xi) \delta S_\mu(\xi)} \phi[C_p] \right] \\ &\quad + (\text{higher area-derivative terms}), \end{aligned} \quad (56)$$

where

$$U(\phi, \phi^*) := g^2 T_p^2 \phi^* \phi + V(\phi, \phi^*) \quad (57)$$

with $T_p^2 \propto a^{-2(p+1)}/(k_0 + 1)$, which corresponds to a *bare* p -brane tension.

In the continuum theory (55), the p -form global transformation (28) becomes

$$\phi[C_p] \rightarrow \begin{cases} e^{i\theta \int_{C_p} \Lambda_p} \phi[C_p] & \text{for } G = \text{U}(1) \\ e^{\frac{i}{N} \int_{C_p} \Lambda_p} \phi[C_p] & \text{for } G = \mathbb{Z}_N \end{cases}, \quad (58)$$

where $\theta \in \mathbb{R}$ and Λ_p is a closed differential p -form satisfying $\int_{C_p} \Lambda_p \in 2\pi\mathbb{Z}$. In fact, one can check that the kinetic term in Eq. (56) is invariant under these transformations due to Eq. (46) and the closeness of Λ_p . For $G = \text{U}(1)$, we can calculate the corresponding conserved current by the Nöther method as [26,27,36]

$$\begin{aligned} J_{p+1}(X) = \mathcal{N} \int \mathcal{D}X \frac{i}{\text{Vol}[C_p](p+1)!} \int_{S^p} d^p \xi \sqrt{h(\xi)} \\ \times \left(\frac{\delta \phi^*[C_p]}{\delta S^{[\mu_1]}(\xi)} \phi[C_p] - \phi^*[C_p] \frac{\delta \phi[C_p]}{\delta S^{[\mu_1]}(\xi)} \right) \\ \times E_{\mu_2 \dots \mu_{p+1}}(X(\xi)) \\ \times \frac{\delta^{(D)}(X - X(\xi))}{\sqrt{g(X(\xi))}} dX^{\mu_1} \wedge \dots \wedge dX^{\mu_{p+1}}, \quad (59) \end{aligned}$$

which satisfies $d\star J_{p+1}(X) = 0$ when $\phi[C_p]$ obeys the equation of motion. Then, the conserved charge is given by

$$\mathcal{Q}_p[C_{D-p-1}] = \int_{C_{D-p-1}} \star J_{p+1}, \quad (60)$$

where C_{D-p-1} is a $(D-p-1)$ -dimensional closed subspace. It is a good exercise to check the transformation law

$$\mathcal{Q}_p[C_{D-p-1}] \phi[C_p] = \text{Link}[C_{D-p-1}, C_p] \phi[C_p] \quad (61)$$

in the path-integral formulation [26].

C. Mass dimensions and upper critical dimension

Readers who are primarily interested in the mean-field analysis can skip this part and proceed directly to the next section.

As in ordinary quantum field theory, mass dimensions of operators and coupling constants are important for dimensional analysis. In the following, we represent the mass dimension of a general quantity $\mathcal{O}$ by $[\mathcal{O}]$. In the continuum action (56), we can see

$$\left[\frac{\delta}{\delta S^{\mu}(\xi)} \right] = \left[\frac{\delta}{\delta \sigma^{\mu_1 \dots \mu_{p+1}}(\xi)} \right] = p+1, \quad [\mathcal{N} \mathcal{D}X] = -D \quad (62)$$

by the definitions (43) and (52). Then, the mass dimension of the scalar field is found to be

$$[\phi[C_p]] = \frac{D-2(p+1)}{2}, \quad (63)$$

which is consistent with the ordinary mass dimension of scalar field $[\phi(X)] = (D-2)/2$ for $p=0$ and the previous results [25–27] for $p=1$ as well.

Once the mass dimension of the scalar field is known, mass dimensions of various coupling constants can be also determined. For example, focusing on local potential terms such as

$$U(\phi, \phi^*) = \mu \phi[C_p]^* \phi[C_p] + \lambda (\phi[C_p]^* \phi[C_p])^2 + \dots, \quad (64)$$

we have

$$[\mu] = 2(p+1), \quad [\lambda] = 4(p+1) - D. \quad (65)$$

In the next section, we will see that $T_p := \sqrt{|\mu|}$ corresponds to a world-volume tension of the p -brane. The mass dimension of self-coupling λ suggests that a naive upper-critical dimension of the Landau theory is $D_c = 4(p+1)$, as predicted by the Parisi's heuristic argument based on Hausdorff dimension of random surfaces [37] as follows.

The dimension of $A \cap B$ for general two objects A and B is

$$d(A \cap B) = d(A) + d(B) - D. \quad (66)$$

For example, for $D=3$ and $d(A)=d(B)=2$, we have $d(A \cap B) = 2+2-3=1$, i.e., the intersection of two surfaces in three-dimensional space is a line. By applying this formula to two random surfaces S_A and S_B , we obtain

$$d(S_A \cap S_B) = 2D_H - D, \quad (67)$$

where D_H is the Hausdorff dimension of random surfaces. This implies that for $D > 2D_H$ there is essentially no intersection between two surfaces, and critical behaviors of the system shall be described by the mean-field dynamics around the Gaussian fixed point. For $D < 2D_H$, however, two random surfaces intersect in general, suggesting that their interactions become relevant for critical behaviors of the system. Thus, we can naively conclude that the upper critical dimension of random surfaces is $D_c = 2D_H$ by this heuristic argument.

Then, the remaining question is the determination of Hausdorff dimension D_H , and here come some subtleties: The definition of the Hausdorff dimension is subject to ambiguities of defining a characteristic size of random surfaces. For example, the Hausdorff dimension of a noncritical closed string (i.e., $p = 1$ and $D < 1$) has been examined in much literature [38–42], yielding various predictions due to a variety of choices of characteristic size. Let us here provide a reasonable argument based on universality or the central-limit theorem. Regardless of a microscopic theory of random closed p -dimensional surface C_p , the average world-volume $\langle \text{Vol}[M_{p+1}] \rangle$ of n -step random p -dimensional surfaces would behave as [19]

$$\langle \text{Vol}[M_{p+1}] \rangle_n \sim a^{p+1} n^{1/2} \quad \text{for } n \rightarrow \infty \quad (68)$$

as long as the central-limit theorem holds. This then implies that a typical length $L := \langle \text{Vol}[M_{p+1}] \rangle_n^{\frac{1}{p+1}}$ and n are related as

$$n(L) \sim \left(\frac{L}{a} \right)^{2(p+1)}, \quad (69)$$

leading to the Hausdorff dimension of random surfaces as $D_H = 2(p+1)$. Combined with the above argument, the upper critical dimension can be estimated as

$$D_c = 4(p+1), \quad (70)$$

which in fact coincides with the critical dimension determined by $[\lambda] = D - 4(p+1) = 0$. However, as already mentioned in Refs. [19,25,36], this estimation is too naive and the existence of explicit $U(1)$ breaking terms and nonlocal interactions would provably alter critical behaviors and result in a different upper-critical dimension. More dedicated studies are necessary for its precise determination.

IV. PHASES OF HIGHER GAUGE THEORY

In this section, we study phases of higher-form gauge theories in the Landau theory framework. See also Refs. [25–27] for similar mean-field analyses.

By taking the variation of the action (56) with respect to $\phi[C_p]$, we obtain the equation of motion as

$$\frac{1}{\text{Vol}[C_p]} \int_{S^p} d^p \xi \sqrt{h(\xi)} \frac{\delta^2 \phi[C_p]}{\delta S^\mu(\xi) \delta S_\mu(\xi)} - \frac{1}{g^2} \frac{\delta U(\phi, \phi^*)}{\delta \phi^*[C_p]} = 0. \quad (71)$$

Let us investigate solutions of this equation in the strong and weak coupling limits, respectively.

A. Strong coupling limit

In the strong coupling limit $g \rightarrow \infty$, we can neglect the interaction terms in the potential (57) as

$$\begin{aligned} \frac{1}{g^2} U(\phi, \phi^*) &= T_p^2 \phi^* \phi + \frac{1}{g^2} V(\phi, \phi^*) \\ &\approx T_p^2 \phi^* \phi \quad (\text{for } g \rightarrow \infty) \end{aligned} \quad (72)$$

with $T_p^2 > 0$. In this case, Eq. (71) has a trivial solution $\langle \phi[C_p] \rangle \approx 0$, which means that the system is in the unbroken phase of the p -form global symmetry. Compared to ordinary quantum field theories, however, this trivial solution should be interpreted as an asymptotic value of $\langle \phi[C_p] \rangle$ in the large volume limit $\text{Vol}[C_p] \rightarrow \infty$ as follows.²

As a nontrivial solution of Eq. (71), we consider the following ansatz:

$$\phi[C_p] = f(\text{Vol}[M_{p+1}])/\sqrt{2}, \quad f(z) \in \mathbb{C}, \quad (73)$$

with a boundary condition $f(\infty) = 0$, where M_{p+1} is the $(p+1)$ -dimensional minimal surface surrounded by C_p , i.e., $\partial M_{p+1} = C_p$. Putting this ansatz into Eq. (71), we obtain

$$p(z) f''(z) + q(z) f'(z) - T_p^2 f(z) \approx 0, \quad z = \text{Vol}[M_{p+1}], \quad (74)$$

where

$$p(z) = \frac{1}{\text{Vol}[C_p]} \int_{S^p} d^p \xi \sqrt{h(\xi)} \left[\frac{1}{p!} (E_{p+1})^{\mu\nu_1 \dots \nu_p} (E_p)_{\nu_1 \dots \nu_p} \right]^2, \quad (75)$$

$$q(z) = \frac{1}{\text{Vol}[C_p]} \int_{S^p} d^p \xi \sqrt{h(\xi)} \frac{1}{p!} \frac{\delta[(E_{p+1})^{\mu\nu_1 \dots \nu_p} (E_p)_{\nu_1 \dots \nu_p}]}{\delta S^\mu(\xi)}. \quad (76)$$

For a concrete example, let us consider a spherical configuration $C_p \simeq S^p$ specified by a fixed radius $r = R$ in the $(p+1)$ -dimensional subspace with the polar coordinates $(r, \xi_1, \dots, \xi_p)$. In this case, the projection tensors are explicitly calculated as

$$(E_p)^{\nu_1 \dots \nu_p} = \frac{e^{\nu_1 \dots \nu_p}}{R^p \sqrt{h(\xi)}}, \quad \nu_i \in \{\xi_1, \dots, \xi_p\}, \quad (77)$$

²Although the invariance under spacetime translation forbids $\langle \phi[C_p] \rangle$ to be dependent on the center-of-mass position of C_p , it can still depend on the relative coordinates $\{Y^\mu(\xi) = X^\mu(\xi) - x^\mu\}_{\mu=1}^D$.

$$(E_{p+1})^{\nu_1 \dots \nu_{p+1}} = \frac{\epsilon^{\nu_1 \dots \nu_{p+1}}}{R^p \sqrt{h(\xi)}}, \quad \nu_i \in \{r, \xi_1, \dots, \xi_p\}, \quad (78)$$

by which we obtain

$$p(z) = 1, \quad q(z) = 0. \quad (79)$$

Then, the equation of motion (74) takes a very simple form

$$f''(z) - T_p^2 f(z) \approx 0, \quad (80)$$

which allows the area-law solution

$$f(z) \approx c \times e^{-T_p \text{Vol}[M_{p+1}]}, \quad (81)$$

where c is a constant that is determined by a boundary condition at $\text{Vol}[C_p] = 0$ in principle. For more general configurations, $p(z)$ and $q(z)$ would become more complicated functions of $z = \text{Vol}[M_{p+1}]$, but we can still argue that the large-volume behavior of the classical solution remains unchanged on dimensional ground. In fact, one can check $[p(z)] = 0$ and $[q(z)] = p + 1$, which implies that $p(z) \sim \text{constant}$ and $q(z) \sim z^{-1}$ for $z \rightarrow \infty$ as long as C_p is a nonsingular surface characterized by a single length $R \sim \text{Vol}[C_p]^{\frac{1}{p}} \sim \text{Vol}[M_{p+1}]^{\frac{1}{p+1}}$. These functional behaviors of $p(z)$ and $q(z)$ lead to the same equation of motion as Eq. (80) and result in the area law in the large-volume limit.

The area law of the charged operator $\phi[C_p]$ characterizes the unbroken p -form global symmetry and implies that the original p -form gauge theory is in the confined phase. Of course, this result had been anticipated from the strong coupling expansion in the gauge theory, but we have rederived it in the Landau theory framework.

B. Weak coupling limit

Let us study the weak coupling limit $g \rightarrow 0$ next. In this case, the interaction terms in $U(\phi, \phi^*)/g^2$ cannot be neglected in general. Although the complete functional form of $U(\phi, \phi^*)$ is unknown, we can still argue that the p -form global symmetry must be spontaneously broken in this weak coupling limit by the following argument: By Eq. (30), the potential $U(\phi, \phi^*)$ is generally expanded as

$$\begin{aligned} \frac{1}{g^2} U(\phi, \phi^*) &= \left(T_p^2 - \frac{1}{g^2} |\omega[C_p]|^2 \right) \phi^* \phi \\ &+ (\text{higher-order terms}) \end{aligned} \quad (82)$$

in the vicinity of $\phi = 0$, which shows that the quadratic term eventually becomes negative for a sufficiently small $|g|$, developing a nonzero vacuum expectation value (VEV) $\langle \phi[C_p] \rangle = v/\sqrt{2} \neq 0$, regardless of the detailed structure of higher-order self interactions (as long as it is bounded below). Hence, the system is in the broken phase of the

p -form global symmetry. Note that this mean-field result can be significantly altered by the proliferation of topological defects (i.e., monopoles for $p = D - 2$), as we will see later.

In the Landau theory, the low-energy effective modes are identified by the phase modulations

$$\phi[C_p] = \frac{v}{\sqrt{2}} \exp \left(i \int_{C_p} A_p \right), \quad (83)$$

where A_p is a differential real-valued p -form field. By putting this into Eq. (56), we can calculate the effective action of A_p , which must respect the original p -form symmetry. For example, the effective action must be invariant under

$$A_p \rightarrow \begin{cases} A_p + \theta \Lambda_p & \text{for } G = \text{U}(1) \\ A_p + \frac{1}{N} \Lambda_p & \text{for } G = \mathbb{Z}_N \end{cases}, \quad (84)$$

where $\theta \in \mathbb{R}$ and Λ_p is a differential closed p -form satisfying $\int_{C_p} \Lambda_p \in 2\pi\mathbb{Z}$.

Let us see each case below.

1. U(1) higher gauge theory

In this case, the effective action in the absence of topological defects is the simple p -form Maxwell theory [26]

$$S_{\text{eff}}[A_p] = \frac{v^2}{2\beta} \int_{\Sigma_D} F_{p+1} \wedge \star F_{p+1}, \quad F_{p+1} = dA_p, \quad (85)$$

where we have used the differential property (46) of the area derivatives. See Appendix B for the details. This theory describes a gapless higher-form field A_p with a gauge symmetry $A_p \rightarrow A_p + d\Lambda_{p-1}$, where Λ_{p-1} is a differential $(p-1)$ -form gauge parameter. In the subsection IV D, however, we will see that the presence of topological defects renders A_p (or its dual field $\tilde{A}_{D-p-2}$) massive and prevents the spontaneous breaking of compact U(1) higher-form symmetry for $p \geq D - 2$.

In the case of noncompact U(1) higher-form gauge theories, on the other hand, strong IR fluctuations of these gapless modes can spoil the spontaneous breaking of higher-form global symmetries [28]. In fact, they contribute to the expectation value of the Wilson-surface $W[C_p]$ as

$$\begin{aligned} \langle W[C_p] \rangle &= \left\langle e^{i \int_{C_p} A_p} \right\rangle \\ &= \exp \left(-\frac{g^2}{2} \int_{\Sigma_D} \delta_{D-p}(C_p) \wedge \star \hat{D} \delta_{D-p}(C_p) \right) \end{aligned} \quad (86)$$

$$= \exp \left(-\frac{\text{Vol}[C_p]}{2} \int \frac{d^{D-p}k}{(2\pi)^{D-p}} \frac{1}{k^2} \right) \quad (87)$$

where δ_{D-p} is the Poincare dual form of C_p , i.e.,

$$\int_{\Sigma_D} f_p \wedge \delta_{D-p}(C_p) = \int_{C_p} f_p \quad (88)$$

for $\forall$ p -form f_p , and $\hat{D} = d^\dagger d + \alpha^{-1} d d^\dagger$, where α is the gauge-fixing parameter. The UV divergence in the exponent in Eq. (87) can be always removed by the redefinition of $W[C_p]$. For $p \geq D-2$, however, we also encounter an IR divergence, which cannot be (simultaneously) removed by the field redefinition. As a result, it leads to $\langle W[C_p] \rangle \rightarrow 0$, and the p -form global symmetry cannot be spontaneously broken. This is the Coleman-Mermin-

Wagner theorem for noncompact $U(1)$ higher-form global symmetries [1,28].

2. Finite higher gauge theory

When G is a finite Abelian group, the phase modulation (83) is no longer gapless because the scalar potential $U(\phi, \phi^*)$ produces a mass for A_p . Regardless of a detailed functional form of $U(\phi, \phi^*)$, the effective potential $V(A_p)$ must respect the original discrete p -form (shift) symmetry. For example, when $G = \mathbb{Z}_N$, it satisfies

$$V\left(A_p + \frac{1}{N}\Lambda_p\right) = V(A_p), \quad (89)$$

where Λ_p is a differential closed p -form with $\int_{C_p} \Lambda_p \in 2\pi\mathbb{Z}$. In the low-energy limit, we can use the Villain formula [43] as

$$\exp\left(-\int_{\Sigma_D} V(A_p) \star 1\right) \approx \sum_{n \in \mathbb{Z}} \exp\left(\mathcal{N} \int \mathcal{D}X \left\{ 1 - \frac{1}{2} \left(N \oint_{C_p} A_p - 2\pi n \right)^2 \right\}\right), \quad (90)$$

which can be further rewritten in a path-integral form by introducing two auxiliary higher-form fields as

$$= \int \mathcal{D}f_p \int \mathcal{D}B_{D-p-1} \exp\left(\mathcal{N} \int \mathcal{D}X \left\{ 1 - \frac{1}{2} \left(\oint_{C_p} (NA_p - f_p) \right)^2 \right\} - \frac{i}{2\pi} \int_{\Sigma_D} B_{D-p-1} \wedge df_p \right) \quad (91)$$

with $\int_{C_p} f_p \in 2\pi\mathbb{Z}$. Then, by repeating similar calculations as Appendix B, we obtain

$$\int \mathcal{D}f_p \int \mathcal{D}B_{D-p-1} \exp\left(-\int_{\Sigma_D} \frac{\lambda_2}{2} \left(A_p - \frac{f_p}{N}\right) \wedge \star \left(A_p - \frac{f_p}{N}\right) - \frac{i}{2\pi} \int_{\Sigma_D} B_{D-p-1} \wedge df_p\right), \quad (92)$$

where λ_2 is a coupling constant. We can eliminate f_p by the equation of motion as

$$\frac{f_p}{N} = A_p + i \frac{N(-1)^{(D-p)(p+1)}}{2\pi\lambda_2} \star dB_{D-p-1}, \quad (93)$$

and the effective action becomes

$$\begin{aligned} S_{\text{eff}}[A_p, B_{D-p-1}] &= \frac{v^2}{2\beta} \int_{\Sigma_D} F_{p+1} \wedge \star F_{p+1} + \int_{\Sigma_D} \left[\frac{N^2(-1)^{D-p}}{8\pi^2\lambda_2} dB_{D-p-1} \wedge \star dB_{D-p-1} \right. \\ &\quad \left. - \frac{N^2(-1)^{(D-p)(p+1)}}{4\pi^2\lambda_2} d(B_{D-p-1} \wedge \star dB_{D-p-1}) - i \frac{N}{2\pi} B_{D-p-1} \wedge dA_p \right]. \end{aligned} \quad (94)$$

The low-energy physics of the system is captured by neglecting all kinetic terms, resulting in

$$S_{\text{eff}}[A_p, B_{D-p-1}] \approx -i \frac{N}{2\pi} \int_{\Sigma_D} B_{D-p-1} \wedge dA_p. \quad (95)$$

which is a BF-type topological field theory. In addition to the original $\mathbb{Z}_N$ p -form global symmetry (84), Eq. (95) has an emergent $\mathbb{Z}_N$ $(D-p-1)$ -form global symmetry,

$$B_{D-p-1} \rightarrow B_{D-p-1} + \frac{1}{N} \Lambda_{D-p-1}, \quad d\Lambda_{D-p-1} = 0, \quad \int_{C_{D-p-1}} \Lambda_{D-p-1} \in 2\pi\mathbb{Z}, \quad (96)$$

where Λ_{D-p-1} is a $(D-p-1)$ -form differential form and C_{D-p-1} is a $(D-p-1)$ -dimensional closed subspace in Σ_D . The charged objects are the Wilson surfaces

$$W[C_p] = \exp \left(i \int_{C_p} A_p \right), \quad U[C_{D-p-1}] = \exp \left(i \int_{C_{D-p-1}} B_{D-p-1} \right). \quad (97)$$

The topological theory (95) exhibits topological order when the topology of spacial manifold Σ_{D-1} is nontrivial. See Refs. [12,14,26,27] and references therein for more details about topological order.

C. Topological defects

As in ordinary quantum field theories with 0-form global symmetries, the equation of motion (71) in the Landau theory admits topologically nontrivial solutions, i.e., topological defects. In the following, we explicitly construct such a topological defect for $G = \text{U}(1)$ and $\mathbb{Z}_N$, respectively.

1. Compact U(1) higher gauge theory

In the previous subsection, we have seen that the effective theory of A_p is the p -form Maxwell theory (85) in the absence of topological defects. In addition to the original $\text{U}(1)$ (electric) p -form global symmetry $d\star F_{p+1} = 0$, this effective theory has an emergent (magnetic) $\text{U}(1)$ $(D-p-2)$ -form symmetry $dF_{p+1} = 0$, under which the 't Hooft-surface operator

$$U[M_{D-p-2}] = e^{i \int_{M_{D-p-2}} \tilde{A}_{D-p-2}} \quad (98)$$

is charged, where $\tilde{A}_{D-p-2}$ is the dual field of A_p and M_{D-p-2} is a $(D-p-2)$ -dimensional topological defect coupled to $\tilde{A}_{D-p-2}$. Below, we explicitly construct such a topological defect as a static solution of Eq. (71). This is a higher-dimensional version of the conventional global vortex solution for $\text{U}(1)$ 0-form global symmetry.

For simplicity, we consider the flat space $\Sigma_{D-1} = \mathbb{R}^{D-1}$ and represent it as

$$\mathbb{R}^{D-1} = \mathbb{R}^{D-p-3} \times \Sigma_{p+2} := \mathbb{R}^{D-p-3} \times S^{p+1} \times [0, \infty), \quad (99)$$

where the coordinate r of $[0, \infty)$ corresponds to the radius of the $(p+2)$ -dimensional subspace Σ_{p+2} . See Fig. 7 as an illustration. Besides, we represent the volume-form of the q -dimensional sphere S^q as Ω_q , i.e.,

$$\int_{S^q} \Omega_q = \text{Vol}[S^q]. \quad (100)$$

For $q = 0$, we define $\text{Vol}[S^0] = 1$.

We consider a spherical p -brane configuration $C_p \simeq S^p$ embedded in S^{p+1} and specified by (r, θ_1) , where θ_1 is one

of the polar coordinates $\theta_1 \in [0, \pi]$ of S^{p+1} . This is schematically shown by a red point in Fig. 5. This spherical configuration corresponds to reducing the (path-)integral measure of C_p as

$$\mathcal{N} \int \mathcal{D}X \rightarrow \int_0^\infty dr r^{p+1} \int_0^\pi d\theta_1 (\sin \theta_1)^p. \quad (101)$$

Besides, the volume of C_p is expressed as

$$\text{Vol}[C_p] = (r \sin \theta_1)^p \int_{S^p} \Omega_p = (r \sin \theta_1)^p \times \text{Vol}[S^p]. \quad (102)$$

We then consider the following ansatz

$$\phi[C_p] = \frac{1}{\sqrt{2}} \left(\int_{C_p} \chi_p \right) \exp \left(i \int_{C_p} A_p^{\text{defect}} \right) \quad (103)$$

with

$$\chi_p = \chi(r) (\sin \theta_1)^p \Omega_p, \quad \chi(r) \in \mathbb{R}, \quad (104)$$

and $F_{p+1}^{\text{defect}} := dA_p^{\text{defect}}$ is the normalized volume-form on $S^{p+1} = C_p \times [0, \pi]$ satisfying

$$\int_{S^{p+1}} F_{p+1}^{\text{defect}} = 2\pi q, \quad q \in \mathbb{Z}. \quad (105)$$

This normalization guarantees the single-valuedness of the scalar field $\phi[C_p]$ in the volumeless limit as

$$\phi[C_p]|_{\theta_1=0} = \phi[C_p]|_{\theta_1=\pi}. \quad (106)$$

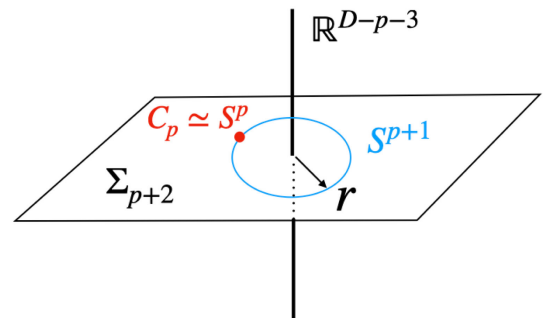


FIG. 5. Topological defect in the compact $\text{U}(1)$ p -form gauge theory. The red point corresponds to the embedded p -brane $C_p \simeq S^p$ that winds along S^{p+1} .

Note that Eq. (105) is nothing but the Dirac quantization condition for the field strength of A_p .

Denoting the world-manifold of the defect as

$$M_{D-p-2} := \mathbb{R} \times \mathbb{R}^{D-p-3}, \quad (107)$$

we can also relate F_{p+1}^{defect} to the Poincare dual-form of the

world-manifold as follows³:

$$\delta_{p+2}(M_{D-p-2}) = \frac{1}{2\pi q} dF_{p+1}^{\text{defect}}. \quad (108)$$

In fact, one can check

$$\int_{\Sigma_D} \delta_{p+2}(M_{D-p-2}) \wedge \delta_{D-p-2}(\Sigma_{p+2}) = \int_{\Sigma_{p+2}} \delta_{p+2}(M_{D-p-2}) = 1, \quad (109)$$

which represents the linking between M_{D-p-2} and S^{p+1} .

The area derivative on the ansatz (103) is evaluated as

$$\begin{aligned} \frac{\delta\phi[C_p]}{\delta S^\mu} &= \frac{1}{p!} (E_p)^{\nu_1 \dots \nu_p} \left(d\chi_p + i \left(\int_{C_p} \chi_p \right) dA_p^{\text{defect}} \right)_{\mu\nu_1 \dots \nu_p} e^{i \int_{C_p} A_p^{\text{defect}}} \\ &= \frac{1}{r^p} \left[\delta_\mu^r \frac{d\chi}{dr} + \delta_\mu^{\theta_1} \left(\frac{p}{\sin \theta_1} + i \frac{2\pi q \text{Vol}[S^p] (\sin \theta_1)^p}{\text{Vol}[S^{p+1}]} \right) \chi \right] e^{i \int_{C_p} A_p^{\text{defect}}}, \end{aligned} \quad (110)$$

which leads to the effective action (energy) of $\chi(r)$ as

$$\begin{aligned} &\int_0^\infty dr \int_0^\pi d\theta_1 \frac{(\sin \theta_1)^p}{r^{p-1}} \left[\frac{1}{2} \left(\frac{d\chi}{dr} \right)^2 + \frac{1}{2r^2} \left\{ \frac{p^2}{(\sin \theta_1)^2} + \left(\frac{2\pi q \text{Vol}[S^p] (\sin \theta_1)^p}{\text{Vol}[S^{p+1}]} \right)^2 \right\} \chi^2 + r^{2p} U(\phi) \right] \\ &\propto \int_0^\infty dr \frac{1}{r^{p-1}} \left[\frac{1}{2} \left(\frac{d\chi}{dr} \right)^2 + \frac{c_2}{2r^2} \chi^2 + r^{2p} \tilde{U}(\chi) \right], \end{aligned} \quad (111)$$

where c_2 is a numerical constant specified by p and q , and $\tilde{U}(\chi)$ is the effective potential of χ after integrating out θ_1 . Equation (111) reduces to the conventional vortex energy for $p = 0$. By introducing dimensionless quantities

$$f = \frac{\chi}{v}, \quad y = vr, \quad \bar{U}(f) = \frac{1}{v^4} \tilde{U}(\chi), \quad (112)$$

we obtain the dimensionless equation of motion,

$$\frac{1}{y^{p+1}} \frac{d}{dy} \left(\frac{1}{y^{p-1}} \frac{df}{dy} \right) - \frac{c_2}{y^{2(p+1)}} f - \frac{\delta \bar{U}(f)}{\delta f} = 0 \quad (113)$$

with boundary conditions $f(0) = 0$ and $f(\infty) = 1$. For simplicity, we consider a local potential

$$\bar{U}(f) = \frac{\lambda}{4} (f^2 - 1)^2, \quad \lambda > 0, \quad (114)$$

for numerical calculations. In Fig. 6, we show numerical plots of $f(y)$ for $p = 0$ (blue), $p = 1$ (orange), $p = 2$ (green), and $p = 3$ (red) with $\bar{\lambda} = 1$ and $q = 1$. The blue one corresponds to the conventional global vortex profile. One can see that the variation of p corresponds to the change of the slope of $f(y)$ in the vicinity of the origin, as expected from the equation of motion (113).

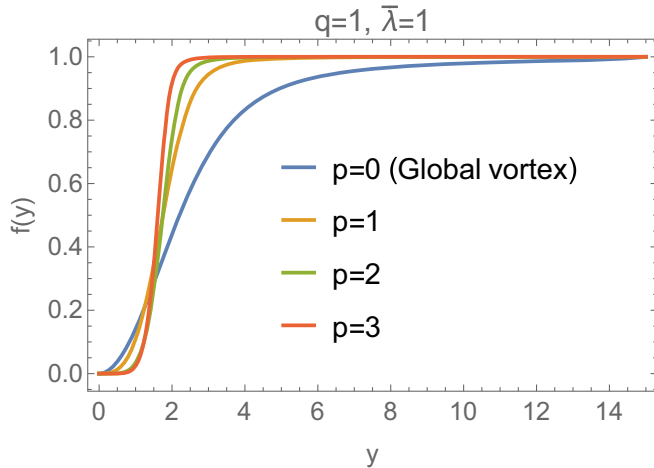


FIG. 6. Field profiles of the topological defect for $G = U(1)$. Each color corresponds to a different choice of p .

³This expression of the Poincare dual is valid only for spherically symmetric functions. Otherwise, we must include the delta functions for angular variables.

2. Finite higher-form gauge theories

Next let us consider the discrete case $G = \mathbb{Z}_N$. The emergent $\mathbb{Z}_N$ $(D-p-1)$ -form symmetry (96) in the effective theory implies that there must exist a $(D-p-1)$ -dimensional defect M_{D-p-1} coupled to the dual field B_{D-p-1} , and we will construct it explicitly below.

In this case, we represent the space as

$$\mathbb{R}^{D-1} = \mathbb{R}^{D-p-2} \times \Sigma_{p+1} := \mathbb{R}^{D-p-2} \times S^p \times [0, \infty), \quad (115)$$

where $r \in [0, \infty)$ denotes the radius of S^p . Note that the boundary of Σ_{p+1} is $\partial\Sigma_{p+1} = \Sigma_{p+1}|_{r=\infty} \simeq S^p$. A p -dimensional surface C_p is embedded as $C_p \simeq S^p$ in Σ_{p+1} , as depicted by a green circle in the left panel in Fig. 7. This corresponds to the reduction of the path-integral measure as

$$\mathcal{N} \int \mathcal{D}X \rightarrow \mathcal{N} \int_0^\infty dr r^p. \quad (116)$$

We then consider the following ansatz:

$$\phi[C_p] = \frac{v}{\sqrt{2}} \exp\left(i \int_{S^p} A_p^{\text{defect}}\right), \quad A_p^{\text{defect}} = f(r) \frac{\Omega_p}{\text{Vol}[S^p]}, \quad (117)$$

with the boundary conditions

$$f(0) = 0, \quad f(\infty) = \frac{2\pi}{N} n, \quad n \in \mathbb{Z}. \quad (118)$$

Namely, Eq. (117) represents a topological configuration interpolating two degenerate minima. The corresponding

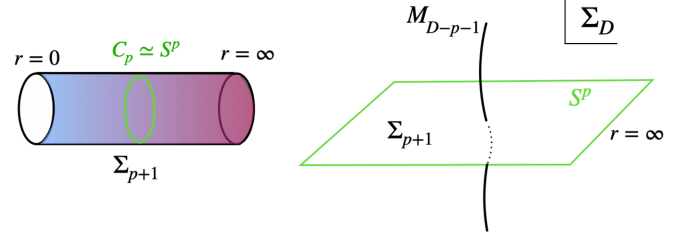


FIG. 7. Left: Topologically nontrivial static configuration in $\mathbb{Z}_N$ higher-form gauge theories. The green circle represents the embedded surface $C_p \simeq S^p$. Right: Linking between the world-manifold M_{D-p-1} of topological defect and S^p .

topological charge is

$$Q_{D-p-1} = \frac{N}{2\pi} \int_{\Sigma_{p+1}} dA_p^{\text{defect}} = \frac{N}{2\pi} \int_{S^p} A_p^{\text{defect}} \Big|_{r=\infty} = n \in \mathbb{Z}, \quad (119)$$

where we have used the Stokes theorem in the second equality.

Denoting the world-manifold of the defect as

$$M_{D-p-1} := \mathbb{R} \times \mathbb{R}^{D-p-2}, \quad (120)$$

we can relate A_p^{defect} to the Poincare dual of M_{D-p-1} as

$$\delta_{p+1}(M_{D-p-1}) = \frac{N}{2\pi Q_{D-p-1}} dA_p^{\text{defect}}. \quad (121)$$

In fact, we can check

$$\int_{\Sigma_D} \delta_{p+1}(M_{D-p-1}) \wedge \delta_{D-p-1}(\Sigma_{p+1}) = \int_{\Sigma_{p+1}} \delta_{p+1}(M_{D-p-1}) = \frac{N}{2\pi Q_{D-p-1}} \int_{\Sigma_{p+1}} dA_p^{\text{defect}} = 1, \quad (122)$$

which represents the linking between M_{D-p-1} and $C_p \simeq S^p$, as illustrated in the right panel in Fig. 7.

The functional derivative on the ansatz (117) is evaluated as

$$\frac{\delta\phi[C_p]}{\delta S^\mu(\xi)} = i \frac{1}{p!} E^{\nu_1 \dots \nu_p} (dA_p^{\text{defect}})_{\mu\nu_1 \dots \nu_p} \phi[C_p] = -i \frac{1}{\text{Vol}[S^p] r^p} \frac{df}{dr} \delta_\mu^r \phi[C_p], \quad (123)$$

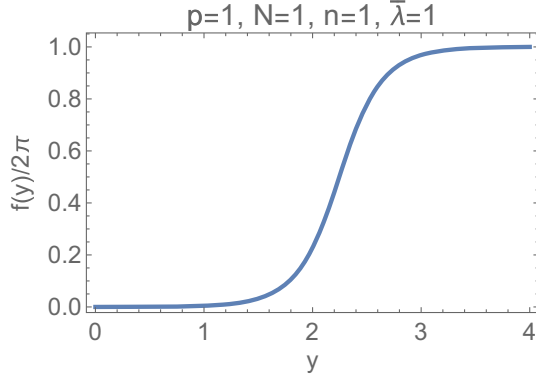
by which we obtain the effective action (energy) of $f(r)$ as

$$\int_0^\infty dr r^p \left[\frac{v^2}{2\text{Vol}[S^p]^2} \left(\frac{1}{r^p} \frac{df}{dr} \right)^2 + U(f) \right] \propto v^{p+3} \int_0^\infty dy y^p \left[\frac{1}{2} \left(\frac{1}{y^p} \frac{df}{dy} \right)^2 + \bar{U}(f) \right], \quad (124)$$

where the dimensionless radius $y := (\text{Vol}[S^p])^{\frac{1}{p+1}} v r$ and potential $\bar{U} := \text{Vol}[S^p]^2 U / v^{2p+4}$ are introduced. By taking the variation of Eq. (124) with respect to $f(y)$, we obtain the dimensionless equation of motion⁴:

$$\frac{1}{y^p} \frac{d}{dy} \left(\frac{1}{y^p} \frac{df}{dy} \right) - \frac{\delta \bar{U}}{\delta f} = 0. \quad (125)$$

⁴This can also be derived from the general equation of motion (71) by putting the ansatz (117) and using (46).

FIG. 8. Field profile of the topological defect for $G = \mathbb{Z}_N$.

In general, the periodic potential $\bar{U}(f) = \bar{U}(f + \frac{2\pi}{N})$ is expressed by a Fourier series of $\cos(Nkf)$ and $\sin(Nkf)$ with $k \in \mathbb{N}$. For a concrete example, let us consider the simplest one

$$\bar{U}(f) = \bar{U}_A(f) := \bar{\lambda}(1 - \cos(Nf)), \quad (126)$$

where $\bar{\lambda}$ is a dimensionless coupling. This potential appears from the first term in Eq. (34) under the ansatz (117). In this case, N dependence is essentially irrelevant because it can be eliminated in the equation of motion (125) by $Nf \rightarrow f$ and $N^{\frac{1}{2}}y \rightarrow y$. Thus, it suffices to calculate the profile $f(y)$ for $N = 1$. In Fig. 8, we show the numerical plot of $f(y)/2\pi$ for $N = 1$, $p = 1$, $n = 1$, and $\bar{\lambda} = 1$.

We can also examine the low-energy effective theory in the presence of topological defect. Instead of Eq. (83), the phase modulation A_p is introduced around the defect solution as

$$\phi[C_p] = \frac{v}{\sqrt{2}} \exp \left(i \int_{C_p} (A_p + A_p^{\text{defect}}) \right), \quad (127)$$

and the low-energy effective action is obtained by replacing A_p in Eq. (95) by $A_p + A_p^{\text{defect}}$ as

$$S_{\text{eff}}[A_p, B_{D-p-1}] \approx i \frac{N}{2\pi} \int_{\Sigma_D} B_{D-p-1} \wedge dA_p + iQ_{D-p-1} \int_{M_{D-p-1}} B_{D-p-1}, \quad (128)$$

where we have used Eq. (121). The second term corresponds to the interaction between the dual field B_{D-p-1} and the topological defect, as expected.

D. Coleman-Mermin-Wagner theorem

We discuss the Coleman-Mermin-Wagner theorem [1,28] for higher-form global symmetries in the present Landau theory framework.

1. Compact $U(1)$ higher gauge theory

In subsection IV B, we saw that noncompact $U(1)$ p -form global symmetry cannot be spontaneously broken for $p \geq D - 2$ due to the IR divergence. The same conclusion holds in the compact $U(1)$ case as well with a different underlying mechanism; the gas of topological defects renders A_p massive, thereby preventing spontaneous symmetry breaking.

In the presence of a single topological defect, low-energy fluctuations are introduced by

$$\phi[C_p] = \frac{v}{\sqrt{2}} \exp \left(i \int_{C_p} [A_p^{\text{defect}} + A_p] \right), \quad (129)$$

where we have taken the thin limit of the defect for simplicity, i.e., $\int_{C_p} \chi_p \rightarrow v$. By putting this into the action (56) and performing similar calculations to Eq. (85), we have

$$S_{\text{eff}}[A_p] = \frac{v^2}{2\beta} \int_{\Sigma_D} (F_{p+1} + F_{p+1}^{\text{defect}}) \wedge \star (F_{p+1} + F_{p+1}^{\text{defect}}), \quad (130)$$

where F_{p+1}^{defect} corresponds to the magnetic field sourced by the defect and satisfies Eq. (108). Since Eq. (130) is a quadratic form, we can dualize it by introducing a $(D - p - 1)$ -form $\tilde{F}_{D-p-1}$ as

$$\frac{\beta}{2v^2(2\pi)^2} \int_{\Sigma_D} \tilde{F}_{D-p-1} \wedge \star \tilde{F}_{D-p-1} + \frac{i}{2\pi} \int_{\Sigma_D} \tilde{F}_{D-p-1} \wedge (F_{p+1} + F_{p+1}^{\text{defect}}), \quad (131)$$

and the path-integral of the original field A_p leads to the flat condition $d\tilde{F}_{D-p-1} = 0$, that is, $\tilde{F}_{D-p-1} = d\tilde{A}_{D-p-2}$, where $\tilde{A}_{D-p-2}$ is interpreted as a dual field of A_p . Using Eq. (108) and performing a partial integration, we obtain the dual effective theory in the presence of a single topological defect M_{D-p-2} as

$$\int \mathcal{D}\tilde{A}_{D-p-2} e^{-S_{\text{eff}}^{\text{dual}}[\tilde{A}_{D-p-2}]}, \quad (132)$$

$$S_{\text{eff}}^{\text{dual}}[\tilde{A}_{D-p-2}] = \frac{\beta}{2v^2(2\pi)^2} \int_{\Sigma_D} \tilde{F}_{D-p-1} \wedge \star \tilde{F}_{D-p-1} + iq \int_{M_{D-p-2}} \tilde{A}_{D-p-2}, \quad (133)$$

where the second term in Eq. (133) is the interaction between the topological defect M_{D-p-2} and the dual field $\tilde{A}_{D-p-2}$. For $D - p - 2 > 0$, M_{D-p-2} is an extended object, and its proliferation is energetically disfavored.

On the other hand, when $D - p - 2 = 0$, M_{D-p-2} is a pointlike object, i.e., monopole, and the summation over monopoles and their locations can lead to an effective potential of the dual scalar field $\varphi := \tilde{A}_{D-p-2}$. In fact, the monopole summation gives⁵

$$\begin{aligned} & \sum_{N=0}^{\infty} \frac{(\xi/2)^N}{N!} \left(\prod_{k=1}^N \sum_{q_k \in \pm 1} \int d^D x_k \right) e^{iq_k \varphi(x_i)} \\ &= \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \left(\int d^D x \, 2 \cos(\varphi(x)) \right)^N \\ &= \exp \left[2\xi \int_{\Sigma_D} \cos(\varphi(x)) \star 1 \right], \end{aligned} \quad (134)$$

where ξ is the fugacity of a single monopole. We then obtain the following effective theory in the weak coupling limit,

$$\langle \phi[C_p] \rangle \approx \frac{1}{Z} \int \mathcal{D}\varphi \exp \left(- \int_{\Sigma_D} \left[\frac{\beta^2}{2v^2(2\pi)^2} d\varphi \wedge \star d\varphi - \xi \cos(\varphi) \star 1 \right] \right) \times \delta(d\tilde{F}_1 - 2\pi\delta_2(C_p)), \quad (137)$$

where the scalar field φ is nonsingular on $\Sigma_D n C_p$, and its field strength $\tilde{F}_1 = d\varphi$ satisfies

$$\begin{aligned} \int_{\Sigma_D} d\tilde{F}_1 \wedge \delta_{D-2}(M_2) &= \int_{\Sigma_D} \tilde{F}_1 \wedge \delta_{D-1}(C_1) = 2\pi \\ \therefore \int_{C_1} d\varphi &= 2\pi \end{aligned} \quad (138)$$

for any loop C_1 that links C_p . Here, we have used the Stokes theorem in the first equality. To understand the qualitative behavior of Eq. (137) in the large volume limit, we do not need detailed information of an analytic saddle. See Fig. 9 for example, where the winding between C_p and C_1 is illustrated in a $D = (p + 2)$ -dimensional space. Since the classical field φ satisfies $d\star d\varphi - \xi \sin(\varphi) \star 1 = 0$ on $\Sigma_D n C_p$, this winding is localized on the intersecting point between C_p and C_1 , that is, the minimal surface M_{p+1} bounded by C_p , costing an energy $\sigma \propto \xi$ per unit volume on M_{p+1} . As a result, the saddle-point Euclidean action is qualitatively given by

$$S_{\text{saddle}} \sim \sigma \text{Vol}[M_{p+1}], \quad (139)$$

which then corresponds to the area-law

$$\langle \phi[C_p] \rangle \sim e^{-\sigma \text{Vol}[M_{p+1}]}. \quad (140)$$

⁵Here, we assume that monopoles with charges $n = \pm 1$ contribute to the partition function because they are quite heavy in the weak coupling limit.

$$\int \mathcal{D}\varphi \exp \left(- \int_{\Sigma_D} \left[\frac{\beta^2}{2v^2(2\pi)^2} d\varphi \wedge \star d\varphi - \xi \cos(\varphi) \star 1 \right] \right), \quad (135)$$

which is a massive scalar theory and indicates that the U(1) p -form global symmetry must be preserved. In fact, we can explicitly check it by calculating the VEV $\langle \phi[C_p] \rangle$ in the effective theory (135). The presence of $\phi[C_p]$ corresponds to adding

$$i \int_{\Sigma_D} \delta_2(C_p) \wedge A_p \quad (136)$$

to Eq. (131), which then leads to the constraint $d\tilde{F}_1 = 2\pi\delta_2(C_p)$ by integrating out A_p . Namely, we have

This implies that compact U(1) p -form global symmetry cannot be spontaneously broken for $p \geq D - 2$ as in the noncompact U(1) case. This is the higher-form version of the Polyakov's argument for the confinement of compact U(1) gauge theory in three dimensions [22,44–46].

2. Finite higher gauge theory

The derivation of the Coleman-Mermin-Wagner theorem for finite higher-form gauge theories is completely parallel to the above compact U(1) case just by replacing the dimensionality of the topological defect from $(D - p - 2)$ to $(D - p - 1)$. In fact, we have already derived the low-energy effective theory (128) in the presence of a single topological defect, and it is equivalent to Eq. (133) by identifying B_{D-p-1} with $\tilde{A}_{D-p-2}$ and including the kinetic term of B_{D-p-1} . Thus, repeating the same discussion, we conclude that discrete p -form global symmetry is not spontaneously broken for $p \geq D - 1$.

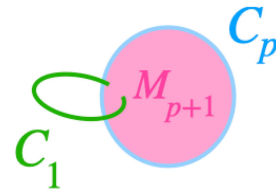


FIG. 9. Winding of the dual scalar field φ along a loop C_1 that links C_p .

V. KRAMERS-WANNIER DUALITY

We discuss KW duality in lattice higher-form gauge theories and its implication for IR duality in continuum Landau field theories.

A. Duality in higher-form gauge theories

In this subsection, we derive the KW duality between the p -form lattice higher gauge theory and the $(D - p - 2)$ -form *gauged* higher gauge theory in our lattice setup. The meaning of *gauged* will be clarified soon below. For an

earlier derivation, see also Ref. [47]. Readers familiar with this subject can skip this subsection.

In the lattice setup introduced in subsection II A, we define a p -chain by the abstract sum

$$\sum_{L_p} \theta_{L_p} L_p, \quad \theta_{L_p} := i^{-1} \log(U_{L_p}), \quad (141)$$

where the summation runs over all positive p -links. The boundary and coboundary operators are defined by

$$\Delta L_p := \sum_{L_{p-1}} \text{Inc}(L_p, L_{p-1}) L_{p-1}, \quad \nabla L_p := \sum_{L_{p+1}} \text{Inc}(L_{p+1}, L_p) L_{p+1}, \quad (142)$$

where $\text{Inc}(L_p, L_{p-1})$ is the incident number defined by

$$\text{Inc}(L_p, L_{p-1}) = \begin{cases} 0 & \text{if } L_{p-1} \not\subset \partial L_p \\ 1 & \text{if } L_{p-1} \subset \partial L_p \text{ and is positively oriented} \\ -1 & \text{if } L_{p-1} \subset \partial L_p \text{ and is negatively oriented} \end{cases}. \quad (143)$$

Then, a general gauge-invariant theory can be written as⁶

$$Z_G^{(p)}(\beta) = \int [dU] e^{-S^{(p)}}, \quad S^{(p)} = \beta \sum_{L_{p+1}} F\left(\sum_{L_p} \text{Inc}(L_{p+1}, L_p) \theta_{L_p}\right), \quad (144)$$

where $F(\theta)$ is an arbitrary real function and the path-integral measure is defined by Eq. (9). For example, the Wilson action (8) corresponds to $F(\theta) = \beta(1 - \cos \theta)$. Moreover, the subscript G in the partition function denotes the p -form global symmetry G in the theory. The integrand in Eq. (144) can be expressed as a Fourier transform as

$$\begin{aligned} e^{-S^{(p)}} &= \prod_{L_{p+1}} e^{-\beta F\left(\sum_{L_p} \text{Inc}(L_{p+1}, L_p) \theta_{L_p}\right)} \\ &= \prod_{L_{p+1}} \left\{ \sum_{\alpha_{L_{p+1}} \in \hat{G}} f(\alpha_{L_{p+1}}) \chi_{\alpha_{L_{p+1}}}\left(\sum_{L_p} \text{Inc}(L_{p+1}, L_p) \theta_{L_p}\right) \right\}, \end{aligned} \quad (145)$$

where $\alpha_{L_{p+1}} \in \hat{G}$ is a dual group element attached to L_{p+1} , $\chi_\alpha(\theta) = e^{i\alpha\theta}$ is the character of the representation $\alpha \in \hat{G}$, and

$$f(\alpha) = \begin{cases} \frac{1}{|\hat{G}|} \sum_{\theta \in G} \chi_\alpha^*(\theta) e^{-\beta F(\theta)} & \text{for a finite abelian group } G \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \chi_\alpha^*(\theta) e^{-\beta F(\theta)} & \text{for } G = \text{U}(1) \end{cases}. \quad (146)$$

Using $\chi_\alpha(\theta) \chi_\beta(\theta) = \chi_{\alpha+\beta}(\theta)$, we can group all terms depending on θ_{L_p} in Eq. (145) as

$$\prod_{L_{p+1}} \chi_{\alpha_{L_{p+1}}}(\text{Inc}(L_{p+1}, L_p) \theta_{L_p}) = \chi_{\sum_{L_{p+1}} \text{Inc}(L_{p+1}, L_p) \alpha_{L_{p+1}}}(\theta_{L_p}), \quad (147)$$

and the summation over θ_{L_p} provides

$$\begin{aligned} &\frac{1}{|\hat{G}|} \sum_{\theta_{L_p} \in G} \chi_{\sum_{L_{p+1}} \text{Inc}(L_{p+1}, L_p) \alpha_{L_{p+1}}}(\theta_{L_p}) \\ &= \delta_{0, \sum_{L_{p+1}} \text{Inc}(L_{p+1}, L_p) \alpha_{L_{p+1}}}. \end{aligned} \quad (148)$$

⁶If the gauge invariance is not imposed, we can also add a potential such as $\sum_{L_p} V(\theta_{L_p})$.

The same result holds for $G = \text{U}(1)$. Now, the partition function (144) is written as

$$Z_G^{(p)}(\beta) = \int [d\hat{U}] e^{-\hat{S}} \times \left(\prod_{L_p} \delta_{0, \sum_{L_{p+1}} \text{Inc}(L_{p+1}, L_p) \alpha_{L_{p+1}}} \right), \quad (149)$$

where

$$\hat{S} = - \sum_{L_{p+1}} \log f(\alpha_{L_{p+1}}), \quad \int [d\hat{U}] = \prod_{L_{p+1}} \sum_{\alpha_{L_{p+1}} \in \hat{G}}. \quad (150)$$

Equation (149) is similar to the original theory (144), but each link variable is an element of $\hat{G}$ and attached to L_{p+1} instead of L_p . We can further express it in a dual form by regarding that these link variables are assigned to dual links $\hat{L}_{D-p-1}$ as

$$Z_G^{(p)}(\beta) = \int [d\hat{U}] e^{-\hat{S}} \times \left(\prod_{\hat{L}_{D-p}} \delta_{0, d_{\hat{L}_{D-p}} \alpha} \right), \quad (151)$$

where

$$d_{\hat{L}_{D-p}} \alpha = \sum_{\hat{L}_{D-p-1}} \text{Inc}(\hat{L}_{D-p}, \hat{L}_{D-p-1}) \alpha_{\hat{L}_{D-p-1}} \quad (152)$$

is the boundary operator and

$$\hat{S} = - \sum_{\hat{L}_{D-p-1}} \log f(\alpha_{\hat{L}_{D-p-1}}), \quad \int [d\hat{U}] = \prod_{\hat{L}_{D-p-1}} \sum_{\alpha_{\hat{L}_{D-p-1}} \in \hat{G}}. \quad (153)$$

In the following, we denote every dual quantity with a hat. From the definition of the boundary operator, one can see that the delta functions in Eq. (151) impose the local flatness conditions for dual link variables. Although Eq. (153) is not gauge invariant, it can be further rewritten in a gauge invariant way by introducing additional $(D-p-2)$ -link variables $\hat{M}_{\hat{L}_{D-p-2}} \in \hat{G}$ as

$$Z_{/\hat{G}}^{(D-p-2)}(\hat{\beta}) := \frac{1}{\text{Vol}[\text{Gauge}]} \int [d\hat{M}] \int [d\hat{U}] e^{-\hat{S}^{(D-p-2)}} \times \prod_{\hat{P}_{D-p-1}} \delta_{1, W[\hat{P}_{D-p-1}]}, \quad (154)$$

where

$$\hat{S}^{(D-p-2)} := - \sum_{\hat{P}_{D-p-2}} K(\hat{M}_{\hat{L}_{D-p-2}}) \log f(\hat{U}_{\hat{L}_{D-p-1}}), \quad W[\hat{P}_{D-p-1}] = \prod_{\hat{L}_{D-p-1} \subset \hat{P}_{D-p-1}} \hat{U}_{\hat{L}_{D-p-1}}. \quad (155)$$

Here, $/\hat{G}$ denotes the gauging of $\hat{G}$, $\hat{\beta}$ is the dual inverse gauge coupling (determined by β), and the argument of $f(\alpha)$ is changed from α to $\hat{U} = e^{i\alpha} \in \hat{G}$. For $G = \text{U}(1)$, we

simply identify $\alpha = \hat{U} \in \mathbb{Z}$. Moreover, $K(x)$ is a real function satisfying $K(1) = 1$ and

$$K(\hat{M}'_{\hat{L}_{D-p-2}}) \log f(\hat{U}'_{\hat{L}_{D-p-1}}) = K(\hat{M}_{\hat{L}_{D-p-2}}) \log f(\hat{U}_{\hat{L}_{D-p-1}}) \quad (156)$$

under the dual gauge transformation

$$\hat{M}_{\hat{L}_{D-p-2}} \rightarrow \hat{M}'_{\hat{L}_{D-p-2}} = g(\hat{L}_{D-p-2}) \hat{M}_{\hat{L}_{D-p-2}}, \quad g(\hat{L}_{D-p-2}) \in \hat{G}, \quad (157)$$

$$\hat{U}_{\hat{L}_{D-p-1}} \rightarrow \left(\prod_{\hat{L}_{D-p-2} \subset \hat{L}_{D-p-1}} g(\hat{L}_{D-p-2})^{-1} \right) \hat{U}_{\hat{L}_{D-p-1}}. \quad (158)$$

The explicit functional form of $K(x)$ depends on a choice of $F(\theta)$. As long as Eq. (156) is satisfied, we can eliminate all $(D-p-2)$ -link variables as $\hat{M}_{\hat{L}_{D-p-2}} = 1$ by this gauge transformation and obtain

$$Z_G^{(p)}(\beta) = Z_{/\hat{G}}^{(D-p-2)}(\hat{\beta}) \quad (159)$$

up to a normalization factor. This is the KW duality between the two lattice higher-form gauge theories and relates the high-temperature phase of one theory to the low-temperature phase of the other in general. In Appendix C, we discuss several examples.

The local flatness condition of the $(D-p-1)$ -form gauge field in the dual theory $Z_{/\hat{G}}^{(D-p-2)}(\hat{\beta})$ implies that the Wilson-surface operator

$$W[C_{D-p-1}] = \prod_{\hat{L}_{D-p-1} \subset \hat{C}_{D-p-1}} \hat{U}_{\hat{L}_{D-p-1}} \quad (160)$$

is a topological operator, corresponding to a p -form global symmetry $\hat{\hat{G}} = G$ as expected from the KW duality (159). In Fig. 10, we show the phase diagram, assuming that there exists a single phase transition as a function of the gauge coupling.⁷

B. Duality in Landau field theories

The KW duality (159) also implies an exact duality between two Landau field theories through the Hubbard-Stratonovich transformation as discussed so far. In the continuum limit, this leads to an IR duality, that is, two theories describe the same critical phenomena or flow to the same IR critical point, similar to the $2+1$ dimensional

⁷This assumption is nontrivial. Many numerical studies support the existence of a Coulomb phase in the intermediate coupling region for $N \geq 5$ [48]. It is quite interesting to discuss such a Coulomb phase in the Landau theory framework.

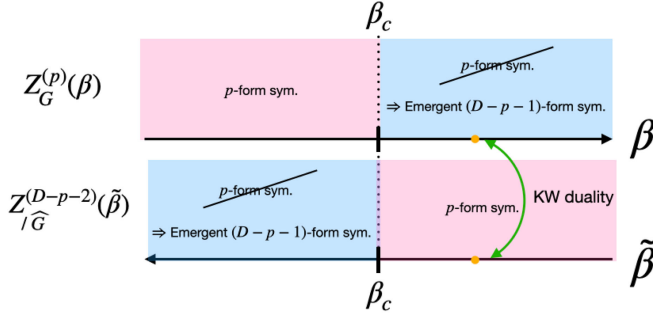


FIG. 10. The Kramers-Wannier duality in finite higher-form gauge theories. The pink (blue) regions correspond to high (low) temperature regions.

particle/vortex duality [27,49,50]. In the following, we elucidate such an IR duality in Landau field theories for a finite higher-form symmetry. To this end, it is necessary to understand the gauging procedure in the Landau theory context. For concreteness, we focus on $G = \mathbb{Z}_N$ in the following.

In order to gauge the $(D-p-2)$ -form global symmetry $\hat{G}$ in $Z_{\text{Landau}}^{(D-p-2)}(\hat{\beta})$, we introduce a $(D-p-1)$ -form gauge field A_{D-p-1} with a locally flat condition

$$NA_{D-p-1} = d\Phi_{D-p-2}, \quad (161)$$

where Φ_{D-p-2} is a differential $(D-p-2)$ -form satisfying $\int_{C_{D-p-1}} d\Phi_{D-p-2} \in 2\pi\mathbb{Z}$, where C_{D-p-1} is a $(D-p-1)$ -dimensional closed subspace. Equation (161) can be realized by introducing a $(p+1)$ -form Lagrangian multiplier B_{p+1} as

$$ic \int_{\Sigma_D} (NA_{D-p-1} - d\Phi_{D-p-2}) \wedge B_{p+1}, \quad (162)$$

where c is a real constant that should be determined

consistently. This action is invariant under the following gauge transformation:

$$\begin{aligned} A_{D-p-1} &\rightarrow A_{D-p-1} + d\Lambda_{D-p-2}, \\ \Phi_{D-p-2} &\rightarrow \Phi_{D-p-2} + N\Lambda_{D-p-2} + d\Lambda_{D-p-3}, \end{aligned} \quad (163)$$

where $\Lambda_{D-p-2}(\Lambda_{D-p-3})$ is a $(D-p-2) - ((D-p-3)-)$ form gauge parameter field. Equation (162) should respect a $\mathbb{Z}_N$ $(D-p-1)$ -form global symmetry, whose charged object is the Wilson-surface operator $W[C_{D-p-1}] = \exp(i \int_{C_{D-p-1}} A_{D-p-1})$. Namely, the coefficient c can be determined by demanding that Eq. (162) is invariant up to $2\pi i n$, $n \in \mathbb{Z}$ under the $\mathbb{Z}_N$ $(D-p-1)$ -form transformation

$$\begin{aligned} A_{D-p-1} &\rightarrow A_{D-p-1} + \frac{1}{N} \Lambda_{D-p-1}, \quad d\Lambda_{D-p-1} = 0, \\ \int_{C_{D-p-1}} \Lambda_{D-p-1} &\in 2\pi\mathbb{Z}. \end{aligned} \quad (164)$$

Then, we find $c = (2\pi)^{-1}$ along with the Dirac quantization condition $\int_{C_{p+2}} dB_{p+1} \in 2\pi\mathbb{Z}$, where C_{p+2} is a $(D-q)$ -dimensional closed subspace. Besides, the variation with respect to Φ_{D-p-2} leads to the condition $dB_{p+1} = 0$, i.e., $B_{p+1} = dB_p$ locally, and we obtain

$$i \frac{N}{2\pi} \int_{\Sigma_D} A_{D-p-1} \wedge dB_p, \quad (165)$$

which is nothing but a BF-type topological field theory, as expected.

Next, let us consider the matter part in the Landau field theory $Z_{\text{Landau}}^{(D-p-2)}(\hat{\beta})$. The key observation is that the kinetic term is expressed in terms of the area derivative (48), which is a $(D-p-1)$ -form tensor in spacetime. Thus, we can naturally find a covariant kinetic term as

$$\mathcal{N} \int \mathcal{D}X \frac{1}{\beta \text{Vol}[C_{D-p-2}]} \int_{S^{D-p-2}} d^{D-p-2} \xi \sqrt{h(\xi)} \phi[C_{D-p-2}]^* \frac{\delta^2}{\delta D^\mu(\xi) \delta D_\mu(\xi)} \phi[C_{D-p-2}], \quad (166)$$

where the covariant derivative D_μ is defined by

$$D_\mu \phi[C_{D-p-2}] := \{X^{\nu_1}, \dots, X^{\nu_{D-p-2}}\} \left(\frac{\delta}{\delta \sigma^{\mu\nu_1 \dots \nu_{D-p-2}}(\xi)} - i A_{\mu\nu_1 \dots \nu_{D-p-2}}(X(\xi)) \right) \phi[C_{D-p-2}]. \quad (167)$$

One can check that this transforms as

$$D_\mu \phi[C_{D-p-2}] \rightarrow e^{i \int_{C_{D-p-2}} \Lambda_{D-p-2}} D_\mu \phi[C_{D-p-2}] \quad (168)$$

under the gauge transformation

$$\phi[C_{D-p-2}] \rightarrow e^{i \int_{C_{D-p-2}} \Lambda_{D-p-2}} \phi[C_{D-p-2}] \quad (169)$$

along with Eq. (163). Besides, the potential $U(\phi, \phi^*)$ can be coupled to the gauge field in a gauge invariant manner as

$$U(\phi[C_{D-p-2}], \phi[C_{D-p-2}]^*) \rightarrow U\left(e^{i \int_{M_{D-p-1}} A_{D-p-1}} \phi[C_{D-p-2}], e^{-i \int_{M_{D-p-1}} A_{D-p-1}} \phi[C_{D-p-2}]^*\right), \quad (170)$$

where M_{D-p-1} is the $(D-p-1)$ -dimensional minimal surface bounded by C_{D-p-2} . One can easily check that the combination $e^{i \int_{M_{D-p-1}} A_{D-p-1}} \phi[C_{D-p-2}]$ is gauge invariant.

To summarize, the gauged version of $Z_{\text{Landau}}^{(D-p-2)}(\hat{\beta})$ is

$$Z_{\text{Landau}/\mathbb{Z}_N}^{(D-p-2)}(\hat{\beta}) := \int \mathcal{D}\phi \mathcal{D}\phi^* \int \mathcal{D}A_{D-p-1} \mathcal{D}B_p e^{-S_{\text{gauged}}^{(D-p-2)}}, \quad (171)$$

where

$$S_{\text{gauged}}^{(D-p-2)} = -i \frac{N}{2\pi} \int_{\Sigma_D} A_{D-p-1} \wedge dB_p + \mathcal{N} \int \mathcal{D}X \left[U\left(e^{i \int_{M_{D-p-1}} A_{D-p-1}} \phi[C_{D-p-2}], e^{-i \int_{M_{D-p-1}} A_{D-p-1}} \phi[C_{D-p-2}]^*\right) \right. \\ \left. - \frac{1}{\beta \text{Vol}[C_{D-p-2}]} \int_{S^{D-p-2}} d^{D-p-2} \xi \sqrt{h(\xi)} \phi[C_{D-p-2}]^* \frac{\delta^2}{\delta D^\mu(\xi) \delta D_\mu(\xi)} \phi[C_{D-p-2}] \right]. \quad (172)$$

Although we have derived Eq. (171) heuristically above, it can be directly derived from Eq. (154) by performing the Hubbard-Stratonovich transformation and taking the continuum limit. This is shown in Appendix D.

Finally, the KW duality (159) suggests that the corresponding Landau field theory is IR dual as

$$Z_{\text{Landau}}^{(p)}(\beta) \sim Z_{\text{Landau}/\mathbb{Z}_N}^{(D-p-2)}(\hat{\beta}), \quad (173)$$

where $\sim$ means that this is valid only near the critical point $\beta \sim \hat{\beta} \sim \beta_c$. Both theories have a p -form global symmetry, and the correspondence of charged and symmetry operators is given as

$$\begin{cases} \phi[C_p] \leftrightarrow e^{i \int_{C_p} B_p} & \text{(charged operators)} \\ U[C_{D-p-1}] \leftrightarrow e^{i \int_{C_{D-p-1}} A_{D-p-1}} & \text{(symmetry operators)} \end{cases}. \quad (174)$$

Since the duality (173) involves closed objects with different dimensions, this can be interpreted as an extension of the particle/vortex duality for U(1) higher-form global symmetry [27,49].

In particular, one can see that field theory of closed p -branes ($p \geq 1$) is can be dual to field theory of particles when

$$D - p - 2 = 0 \quad \therefore \quad p = D - 2. \quad (175)$$

For example, when $D = 3 \Leftrightarrow p = 1$, $Z_{\text{Landau}}^{(1)}(\beta)$ is a closed string field theory with a $\mathbb{Z}_N$ 1-form global symmetry, while the dual theory $Z_{\text{Landau}/\mathbb{Z}_N}^{(0)}(\hat{\beta})$ is a Higgs theory coupled to the $\mathbb{Z}_N$ gauge theory. This indicates that the confinement/deconfinement transition of the three-dimensional $\mathbb{Z}_N$ lattice gauge theory is actually a particlelike transition rather than a stringlike one, as already confirmed by numerical studies [51].

In Tables I and II, we summarize the IR duality. Although we have not fully investigated quantum phases

TABLE I. IR duality in the unbroken phase in $Z_{\text{Ising}}^{(p)}(\beta)$. This corresponds to the unbroken phase in $Z_{\text{Ising}/\mathbb{Z}_N}^{(D-p-2)}(\hat{\beta})$.

	Order parameter	Massive modes
$Z_{\text{Ising}}^{(p)}(\beta)$ (unbroken phase)	Area law of $\phi[C_p]$	$\phi[C_p]$ excitations
$Z_{\text{Ising}/\mathbb{Z}_N}^{(D-p-2)}(\hat{\beta})$ (broken phase)	Perimeter law of $e^{i \int_{C_p} B_p}$	p -dim static topological solitons

TABLE II. IR duality in the broken phase in $Z_{\text{Ising}}^{(p)}(\beta)$. This corresponds to the broken phase in $Z_{\text{Ising}/\mathbb{Z}_N}^{(D-p-2)}(\hat{\beta})$.

	Order parameter	Massive modes
$Z_{\text{Ising}}^{(p)}(\beta)$ (broken phase)	Perimeter law of $\phi[C_p]$	$(D-p-2)$ -dim static topological solitons
$Z_{\text{Ising}/\mathbb{Z}_N}^{(D-p-2)}(\hat{\beta})$ (unbroken phase)	Area law of $e^{i \int_{C_p} B_p}$	$\phi[C_{D-p-2}]$ excitations

of these theories, the correspondence of symmetry, order parameters, and excitation modes provides compelling evidence to support this IR duality from a universality point of view. More dedicated studies are left for future investigations.

VI. SUMMARY

We have derived the continuum Landau field theory for the lattice higher gauge theory and studied the phases of the gauge theory at the mean-field level. By solving the functional equation of motion, we have shown that the classical solution exhibits the area (perimeter) law in the strong (weak) gauge coupling limit. Besides, we have presented explicit constructions of topological defects, which are higher-dimensional analogs of vortex/domain-wall solutions in ordinary field theories with 0-form global symmetries. The existence of topological defects has a critical impact on the phase structure in higher-form gauge theories, and we have derived the Coleman-Mermin-Wagner theorem for higher-form global symmetries in the Landau theory framework.

Moreover, we have formulated the Kramers-Wannier duality in our lattice setup and elucidated how it can lead IR duality in the corresponding Landau field theories. This IR duality relates a field theory of p -dimensional closed objects with p -form global symmetry to a field theory of $(D-p-2)$ -dimensional closed objects with a gauged higher-form symmetry, and can be viewed as a discrete-symmetry analog of the particle-vortex duality [27,49]. Although a complete understanding of quantum theory of extended objects remains elusive, these results suggest that higher-form symmetries plays a crucial role for quantum many-body physics of extended objects.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: HIGHER-FORM GAUGE THEORIES

In this appendix, we check the operator relation (13) in the continuum higher-form gauge theories. In this appendix, we employ the Lorentzian signature $(-, +, \dots, +)$.

The $U(1)$ p -form Maxwell theory is

$$S_p[A_p] = -\frac{1}{2g^2} \int_{\Sigma_D} F_{p+1} \wedge \star F_{p+1}, \quad F_{p+1} = dA_p. \quad (\text{A1})$$

In the path-integral formalism, the VEV of the Wilson-surface operator in the presence of the symmetry operator is

$$\langle U_\theta[C_{D-p-1}]W[C_p] \rangle := \frac{1}{Z} \int \mathcal{D}A_p e^{iS_p[A_p]} U_\theta[C_{D-p-1}]W[C_p] \quad (\text{A2})$$

$$= \frac{1}{Z} \int \mathcal{D}A_p e^{iS'_p[A_p]} W[C_p], \quad (\text{A3})$$

where

$$\begin{aligned} S'_p[A_p] &= S_p[A_p] + \frac{\theta}{g^2} \int_{\Sigma_D} \delta_{p+1}(C_{D-p-1}) \wedge \star F_{p+1} \\ &= -\frac{1}{2g^2} \int_{\Sigma_D} F'_{p+1} \wedge \star F'_{p+1}, \\ F'_{p+1} &= F_{p+1} - \theta \delta_{p+1}(C_{D-p-1}). \end{aligned} \quad (\text{A4})$$

Here, $\delta_{p+1}(C_{D-p-1})$ is the Poincare dual form of C_{D-p-1} . Denoting a $(D-p)$ -dimensional open manifold bounded by C_{D-p-1} as M_{D-p} , we have

$$\delta_{p+1}(C_{D-p-1}) = d\delta_p(M_{D-p}), \quad (\text{A5})$$

which leads to $F'_{p+1} = dA'_p$ with $A'_p = A_p + \theta \delta_p(M_{D-p})$. Then, changing the path-integral variable as $A'_p \rightarrow A_p$, we obtain

$$\begin{aligned} \langle U_\theta[C_{D-p-1}]W[C_p] \rangle &= \frac{1}{Z} \int \mathcal{D}A_p e^{iS_p[A_p]} W[C_p] \\ &\quad \times \exp\left(i\theta \int_{C_p} \delta_p(M_{D-p})\right) \\ &= e^{i\theta \text{Link}[C_p, C_{D-p-1}]} \langle W[C_p] \rangle, \end{aligned} \quad (\text{A6})$$

which proves Eq. (13) in the path-integral formalism.

Next, the $\mathbb{Z}_N$ p -form gauge theory is

$$S_{\text{BF}}[A_p, B_{D-p-1}] = \frac{N}{2\pi} \int_{\Sigma_D} dA_p \wedge B_{D-p-1}. \quad (\text{A7})$$

In this case, the symmetry operator is

$$U_m[C_{D-p-1}] = e^{i\frac{2\pi m}{N} \int_{C_{D-p-1}} B_{D-p-1}}, \quad m \in \mathbb{Z}. \quad (\text{A8})$$

Repeating the same calculation as above, we have

$$\begin{aligned}
 S'_{\text{BF}}[A_p, B_{D-p-1}] &= S_{\text{BF}}[A_p, B_{D-p-1}] + \frac{2\pi m}{N} \int_{\Sigma_D} \delta_{p+1}(C_{D-p-1}) \wedge B_{D-p-1} \\
 &= S_{\text{BF}} \left[A_p + \frac{2\pi m}{N} \delta_p(M_{D-p}), B_{D-p-1} \right],
 \end{aligned} \tag{A9}$$

$$\therefore \langle U_\theta[C_{D-p-1}] W[C_p] \rangle = e^{i \frac{2\pi m}{N} \text{Link}[C_p, C_{D-p-1}]} \langle W[C_p] \rangle, \tag{A10}$$

which is a $\mathbb{Z}_N$ transformation of the Wilson surface operator.

APPENDIX B: EFFECTIVE ACTION IN THE BROKEN PHASE

Here we present the detailed calculation of the effective action of the phase modulation A_p . In the following, we consider the flat space $\Sigma_D = \mathbb{R}^D$ for simplicity.

By putting Eq. (83) into the kinetic term in the continuum action (56), we have

$$\frac{v^2}{2\beta} \mathcal{N} \int \mathcal{D}X \frac{1}{\text{Vol}[C_p](p!)^2} \int_{S^p} d^p \xi \sqrt{h(\xi)} \{ F_{\mu\nu_1 \dots \nu_p}(X(\xi)) E^{\nu_1 \dots \nu_p} \} \{ F^{\mu}_{\rho_1 \dots \rho_p}(X(\xi)) E^{\rho_1 \dots \rho_p} \}, \tag{B1}$$

where $F_{p+1} = dA_p = \frac{1}{(p+1)!} F_{\mu_1 \dots \mu_{p+1}} dX^{\mu_1} \wedge \dots \wedge dX^{\mu_{p+1}}$. Introducing the center-of-mass coordinates and relative coordinates as

$$x^\mu := \frac{1}{\text{Vol}[C_p]} \int_{S^p} d^p \xi \sqrt{h} X^\mu(\xi), \quad Y^\mu(\xi) := X^\mu(\xi) - x^\mu, \tag{B2}$$

we have

$$F_{\mu\nu_1 \dots \nu_p}(X(\xi)) F^{\mu}_{\rho_1 \dots \rho_p}(X(\xi)) = \int \frac{d^D k}{(2\pi)^D} e^{ik \cdot (x+Y)} \tilde{F}_{\mu\nu_1 \dots \nu_p}(k) \tilde{F}^{\mu}_{\rho_1 \dots \rho_p}(k), \tag{B3}$$

where $\tilde{F}_{\mu\nu_1 \dots \nu_p}(k)$ is the Fourier mode of $F_{\mu\nu_1 \dots \nu_p}(x)$. By putting this into Eq. (B1), we obtain

$$\begin{aligned}
 &\int d^D k \left(\int \frac{d^D x}{(2\pi)^D} e^{ik \cdot x} \right) \tilde{F}_{\mu\nu_1 \dots \nu_p}(k) \tilde{F}^{\mu}_{\rho_1 \dots \rho_p}(k) \mathcal{N} \int \mathcal{D}Y \frac{1}{\text{Vol}[C_p]} \int_{S^p} d^p \xi \sqrt{h} e^{ik \cdot Y(\xi)} E^{\nu_1 \dots \nu_p}(Y(\xi)) E^{\rho_1 \dots \rho_p}(Y(\xi)) \\
 &= \tilde{F}_{\mu\nu_1 \dots \nu_p}(0) \tilde{F}^{\mu}_{\rho_1 \dots \rho_p}(0) \frac{\mathcal{N}}{\text{Vol}[C_p]} \int_{S^p} d^p \xi \sqrt{h} \langle E^{\nu_1 \dots \nu_p} E^{\rho_1 \dots \rho_p} \rangle.
 \end{aligned} \tag{B4}$$

Note that $E^{\nu_1 \dots \nu_p}(Y(\xi))$ depends only on the relative coordinates $\{Y^\mu(\xi)\}_{\mu=1}^D$. Assuming that spacetime symmetry is not spontaneously broken, the expectation value can be expanded as a function of momentums $\{k_\mu\}_{\mu=1}^D$ as

$$\begin{aligned}
 \langle E^{\nu_1 \dots \nu_p} E^{\rho_1 \dots \rho_p} \rangle &= \frac{1}{(p!)^3} \sum_{\sigma, \sigma' \in S_p} \text{sgn}(\sigma) \text{sgn}(\sigma') [c_0(k^2) \eta^{\nu_{\sigma(1)} \rho_{\sigma'(1)}} \dots \eta^{\nu_{\sigma(p)} \rho_{\sigma'(p)}} \\
 &\quad + c_1(k^2) k^{\nu_{\sigma(1)}} k^{\rho_{\sigma'(1)}} \eta^{\nu_{\sigma(2)} \rho_{\sigma'(2)}} \dots \eta^{\nu_{\sigma(p)} \rho_{\sigma'(p)}} + \dots],
 \end{aligned} \tag{B5}$$

where $c_0(k^2)$ and $c_1(k^2)$ are functions of k^2 . In the low-energy limit, we can further neglect the k dependence in these functions, and the leading action is

$$\mathcal{N} c_0(0) \frac{v^2}{2\beta(p!)^3} \tilde{F}_{\mu\nu_1 \dots \nu_p}(0) \tilde{F}^{\mu}_{\rho_1 \dots \rho_p}(0) \eta^{\nu_{\sigma(1)} \rho_{\sigma'(1)}} \dots \eta^{\nu_{\sigma(p)} \rho_{\sigma'(p)}} = \mathcal{N} c_0(0) \frac{v^2}{2\beta} \int_{\Sigma_D} F_{p+1} \wedge \star F_{p+1}, \tag{B6}$$

which corresponds to the p -form Maxwell theory (85) by choosing the normalization factor as $\mathcal{N} c_0(0) = 1$.

APPENDIX C: EXAMPLES OF KRAMERS-WANNIER DUALITY

1. Two-dimensional $\mathbb{Z}_2$ Ising model

The two-dimensional $\mathbb{Z}_2$ Ising model corresponds to $D = 2$, $p = 0$, $G = \mathbb{Z}_2$, and

$$F(\theta) = \begin{cases} -1 & \text{for } \theta \equiv 0 \pmod{2} \\ 1 & \text{for } \theta \equiv 1 \pmod{2} \end{cases}. \quad (\text{C1})$$

In this case, the Fourier mode is

$$f(\alpha) = \frac{1}{2} \sum_{\theta=0,1} \chi_\alpha(\theta) e^{-\beta F(\theta)} = \begin{cases} \cosh(\beta) & \text{for } \alpha \equiv 0 \pmod{2} \\ \sinh(\beta) & \text{for } \alpha \equiv 1 \pmod{2} \end{cases}, \quad (\text{C2})$$

or

$$\sqrt{\frac{2}{\sinh(2\beta)}} f(\alpha) = \begin{cases} (\tanh(\beta))^{-\frac{1}{2}} & \text{for } \alpha \equiv 0 \pmod{2} \\ (\tanh(\beta))^{\frac{1}{2}} & \text{for } \alpha \equiv 1 \pmod{2} \end{cases}, \quad (\text{C3})$$

by which we can identify the dual coupling as

$$(\coth(\beta))^{\frac{1}{2}} = e^{\hat{\beta}} \leftrightarrow \sinh(2\beta) \sinh(2\hat{\beta}) = 1. \quad (\text{C4})$$

A critical point is determined by $\beta = \hat{\beta} = \beta_c$, i.e.,

$$\sinh(2\beta_c) = 1 \quad \therefore \beta_c = \frac{1}{2} \log(1 + \sqrt{2}) = 0.440. \quad (\text{C5})$$

2. Four-dimensional $\mathbb{Z}_2$ pure gauge theory

The four-dimensional $\mathbb{Z}_2$ pure lattice gauge theory corresponds to $D = 4$, $p = 1$, $G = \mathbb{Z}_2$, and $F(\theta)$ is the same as Eq. (C1). This system is also self dual, and we obtain the same coupling relation as Eq. (C4) and the critical coupling (C5), assuming that there is a single phase transition. This applies to any $\mathbb{Z}_2$ self-dual model.

3. Three-dimensional pure compact U(1) gauge theory

The three-dimensional pure compact U(1) lattice gauge theory corresponds to $D = 3$, $p = 1$, $G = \text{U}(1)$, and $F(\theta) = 1 - \cos(\theta)$, $\theta \in [-\pi, \pi]$. In the low temperature limit $\beta \rightarrow \infty$, we can use the Villain formula as

$$\begin{aligned} \exp\left(-\beta \sum_{P_1} (1 - \cos(d\theta))\right) &\approx \left(\prod_{P_1} \sum_{n_{P_1} \in \mathbb{Z}}\right) \exp\left(-\sum_{P_1} (d\theta - n_{P_1})^2\right) \\ &= \left(\prod_{P_1} \sqrt{\frac{2\pi}{\beta}} \sum_{m_{P_1} \in \mathbb{Z}}\right) \exp\left(-\sum_{P_1} \left[\frac{2\pi^2}{\beta} m_{P_1}^2 - 2\pi i m_{P_1} d\theta\right]\right), \end{aligned} \quad (\text{C6})$$

where $d\theta = \sum_{L_1 \subset P_1} \text{sgn}(L_1) \theta_{L_1}$ is the discretized exterior derivative, and we have used the mathematical formula

$$\sum_{n \in \mathbb{Z}} e^{-a(y-n)^2} = \sum_{k \in \mathbb{Z}} \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2}{a} k^2 - 2\pi i k y}, \quad (\text{C7})$$

in the second line, which can be derived from the Poisson resummation formula. Note that m_{P_1} is a $\mathbb{Z}$ -valued 2-form field defined on 1-plaquettes. The integration of θ_{L_1} leads to

$$d \star m_{P_1} = 0 \therefore \star m_{P_1} = d\Phi_{\hat{i}}, \quad (\text{C8})$$

where $\Phi_{\hat{i}}$ is a $\mathbb{Z}$ -valued scalar defined on dual sites $\hat{i} \in \hat{\Lambda}_3$. Now, the partition function is written as

$$\begin{aligned} Z_{\text{Ising}}^{(1)}(\beta) &\approx \left(\prod_{\hat{i} \in \hat{\Lambda}_3} \sqrt{\frac{2\pi}{\beta}} \sum_{\Phi_{\hat{i}} \in \mathbb{Z}}\right) \exp\left(-\frac{2\pi^2}{\beta} \sum_{\hat{j} \in \hat{\Lambda}_3} d\Phi_{\hat{j}} \wedge \star d\Phi_{\hat{j}}\right) \\ &= \left(\prod_{\hat{i} \in \hat{\Lambda}_3} \sqrt{\frac{2\pi}{\beta}} \int_{-\infty}^{\infty} d\phi_{\hat{i}} \sum_{n_{\hat{i}} \in \mathbb{Z}}\right) \exp\left(-\frac{2\pi^2}{\beta} \sum_{\hat{i} \in \hat{\Lambda}_3} d\phi_{\hat{i}} \wedge \star d\phi_{\hat{i}} + 2\pi i \sum_{\hat{i} \in \hat{\Lambda}_3} n_{\hat{i}} \phi_{\hat{i}}\right), \end{aligned} \quad (\text{C9})$$

where we have used

$$\sum_{n \in \mathbb{Z}} g(n) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} dx g(x) e^{-2\pi i x k}, \quad (\text{C10})$$

which can be also derived from the Poisson resummation formula. The Lagrangian can be further rewritten as

$$\frac{2\pi^2}{\beta} \phi_i \square \phi_i + 2\pi i n_i \phi_i = \frac{2\pi^2}{\beta} \left(\phi + \frac{1}{2\pi} i \beta_i \square^{-1} \right) \square \left(\phi_i + \frac{1}{2\pi} i \beta \square^{-1} n_i \right) + \frac{\beta}{2} n_i \square^{-1} n_i, \quad (\text{C11})$$

where $\square$ is the three-dimensional d'Alembert operator. The integration of $\{\phi_i\}$ leads to

$$Z(\beta) \approx \left(\prod_{\hat{i} \in \hat{\Lambda}_3} \sum_{n_i \in \mathbb{Z}} \right) \exp \left(\frac{\beta}{2} \sum_{\hat{i} \in \hat{\Lambda}_3} n_i \square^{-1} n_i \right). \quad (\text{C12})$$

In the continuum limit, the summation of monopole gas can be also written as

$$\left(\prod_{\hat{i} \in \hat{\Lambda}_3} \sum_{n_i \in \mathbb{Z}} \right) = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^N \left(\int d^3 x_i \sum_{q_i \in \mathbb{Z}} \right) \quad (\text{C13})$$

with $n_i \rightarrow n(x) = \sum_{i=1}^N q_i \delta^{(3)}(x - x_i)$. In the three-dimensional flat Euclidean space, the propagator is $\square^{-1} = -\frac{1}{4\pi|x|}$, and we obtain

$$Z_{\text{Ising}}^{(1)}(\beta) \approx \sum_{N=0}^{\infty} \frac{e^{\beta\mu N}}{N!} \left(\prod_{i=1}^N \int d^3 x_i \sum_{q_i \in \mathbb{Z}} \right) \exp \left(-\frac{\pi\beta}{4} \sum_{i < j} \frac{q_i q_j}{|x_i - x_j|} \right), \quad (\text{C14})$$

where the self-energy was renormalized into the chemical potential μ . This represents the partition function of monopole gas. Namely, the compact U(1) gauge theory is dual to the monopole-gas system, and it is equivalent to a massive scalar theory as discussed in subsection IV D.

APPENDIX D: LANDAU THEORY FOR GAUGED HIGHER GAUGE THEORY

We derive the Landau theory for the gauged higher gauge theory (154). The flow of the derivation is parallel to the ungauged case by extending the definition of the translation operator (16). In the following, we simply denote $(D - p - 1)$ as q and take the Wilson action (8) as *ungauged* q -form higher gauge theory. Besides, the

minimal $(q + 1)$ -dimensional surface bounded by a q -dimensional closed surface $\hat{C}_q$ is expressed by $\hat{D}_{q+1}$. Although every dual quantity is denoted with a hat below, one can also omit it by regarding the dual lattice $\hat{\Lambda}_D$ as an original lattice.

For a given q -dimensional surface $\hat{C}_q$, we define a Wilson open-surface operator of the $(q + 1)$ -form gauge field by

$$\tilde{W}[\hat{D}_{q+1}] := \prod_{\hat{L}_{q+1} \in \hat{M}_{q+1}} \hat{M}_{\hat{L}_{q+1}}, \quad \hat{M}_{\hat{L}_{q+1}} \in \hat{G}. \quad (\text{D1})$$

Then, we can extend the definition of the translation operator $\Pi_{\hat{p}_q}$ as

$$\Pi_{\hat{p}_q} \tilde{W}[\hat{D}_{q+1}] W[\hat{C}_q] := \begin{cases} \tilde{W}[\hat{D}_{q+1} + \hat{L}_{q+1}] W[\hat{C}_q + \hat{p}_q] & \text{for } \hat{C}_q + \hat{p}_q \in \hat{\Gamma}_q \\ 0 & \text{for } \hat{C}_q + \hat{p}_q \notin \hat{\Gamma}_q \end{cases}, \quad (\text{D2})$$

where $\hat{L}_{q+1}$ is the dual $(q + 1)$ -link satisfying $\partial \hat{L}_{q+1} = \hat{p}_q$. By repeating the same calculation as Eq. (22), we can express the gauged lattice action in a quadratic form as

$$\hat{S}^{(q)} = \text{constant} - \frac{\beta}{2} \sum_{\hat{C}_q \in \hat{\Gamma}_q} w[\hat{C}_q] (\tilde{W}[\hat{D}_{q+1}] W[\hat{C}_q])^* (k_0 + \hat{H}) w[\hat{C}_q]^* (\tilde{W}[\hat{D}_{q+1}] W[\hat{C}_q]). \quad (\text{D3})$$

Performing the HS transformation, we obtain

$$Z_{/\hat{G}}^{(q)} = \int [d\hat{M}] \int [d\phi] \exp \left\{ - \sum_{\hat{C}_q \in \hat{\Gamma}_q} \left(\frac{2}{\beta} \phi[\hat{C}_q]^* (k_0 + \hat{H})^{-1} \phi[\hat{C}_q] + V(\tilde{W}^* \phi, \tilde{W} \phi^*) \right) \right\} \\ \times \prod_{\text{all } \hat{P}_{q+1}} \delta_{\tilde{W}[\hat{P}_{q+1}], 1}, \quad (\text{D4})$$

where

$$V(\tilde{W}^* \phi, \tilde{W} \phi^*) = -\log \left[\frac{1}{\text{Vol}[\text{Gauge}]} \int [d\hat{U}] \exp \left(- \sum_{\hat{C}_q \in \hat{\Gamma}_q} w[\hat{C}_q] (\tilde{W}[\hat{D}_{q+1}] W[\hat{C}_q])^* \phi[\hat{C}_q] + \text{H.c.} \right) \right]. \quad (\text{D5})$$

In particular, the covariant generalization of the area derivative (43) is

$$\frac{\delta \psi[\hat{C}_q]}{\delta \Sigma^{\mu_1 \dots \mu_{p+1}}(\hat{i})} := \frac{\tilde{W}[\hat{L}_{\mu_1 \dots \mu_{q+1}}(\hat{i})] \psi[\hat{C}_q + \hat{P}_{\mu_1 \dots \mu_{q+1}}(\hat{i})] - \psi[\hat{C}_q]}{a^{q+1}}, \quad (\text{D6})$$

where $\hat{L}_{\mu_1 \dots \mu_{q+1}}(\hat{i})$ is the dual $(p+1)$ -link satisfying $\partial \hat{L}_{\mu_1 \dots \mu_{q+1}}(\hat{i}) = \hat{P}_{\mu_1 \dots \mu_{q+1}}(\hat{i})$. In the continuum limit, this becomes

$$= \frac{e^{-ia^{q+1}(A_{q+1})_{\mu_1 \dots \mu_{q+1}}} \psi[\hat{C}_q + \hat{P}_{\mu_1 \dots \mu_{q+1}}(\hat{i})] - \psi[\hat{C}_q]}{a^{p+1}} \\ = \frac{\delta \psi[\hat{C}_q]}{\delta \sigma^{\mu_1 \dots \mu_{q+1}}(\xi)} - i(A_{q+1})_{\mu_1 \dots \mu_{q+1}} \psi[\hat{C}_q] + \mathcal{O}(a^{q+1}), \quad (\text{D7})$$

which coincides with Eq. (167). This proves that the continuum Landau theory of Eq. (D4) is Eq. (172).

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