Nonmaximal symmetry breaking patterns in the supersymmetric $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory

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We study the nonmaximal symmetry breaking patterns of $\mathfrak{su}(8) \to \mathfrak{g}_{531}/\mathfrak{g}_{351}$ allowed in a supersymmetric theory with the affine Lie algebra of $\widehat{\mathfrak{su}}(8)_{k_U=1}$, which was recently proposed to explain the three-generational standard model quark/lepton mass hierarchies as well as the Cabbibo-Kobayashi-Maskawa mixing pattern. Three gauge couplings are found to achieve the unification below the Planck scales, according to the unification relations determined by the affine Lie algebra at a level of 1. We also prove that a separate nonmaximal symmetry breaking pattern of $\mathfrak{su}(8) \to \mathfrak{g}_{621}$ is unrealistic due to the additional massless vectorlike quarks in the spectrum.

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I. INTRODUCTION

Grand unified theories (GUTs) were proposed to not only unify all three Standard Model (SM) symmetries into one simple Lie algebra of either $\mathfrak{su}(5)$ [1] or $\mathfrak{so}(10)$ [2], but to also unify all SM fermions into some anomaly-free irreducible representations (IRs) of some simple Lie algebra $\mathfrak{g}_U$. The Georgi-Glashow $\mathfrak{su}(5)$ GUT [1] and the Fritzsch-Minkowski $\mathfrak{so}(10)$ GUT [2] contain the chiral fermions of $3 \times [\overline{\mathbf{5_F}} \oplus \mathbf{10_F}]$ and $3 \times \mathbf{16_F}$, respectively. Neither framework can fully explain the three-generational SM fermion mass hierarchies and the Cabibbo-Kobayashi-Maskawa (CKM) mixing pattern [3,4] due to three repetitive chiral irreducible anomaly-free fermion sets (IRAFFSs)[1] in the flavor sector.

In a seminal paper by Georgi [7], he first proposed to avoid the repetitive structure in the flavor sector in a unified $\mathfrak{su}(11)$ theory. This was recently relaxed with two following chiral IRAFFSs at the GUT scale:

$$\{f_L\}^{n_g=3}_{\mathfrak{su}(8)} = [\overline{\mathbf{8_F}}^{\omega} \oplus \mathbf{28_F}] \bigoplus [\overline{\mathbf{8_F}}^{\dot{\omega}} \oplus \mathbf{56_F}], \qquad \dim_{\mathbf{F}} = 156,$$
$$\Omega \equiv (\omega, \dot{\omega}), \qquad \omega = (3, \mathrm{IV}, \mathrm{V}, \mathrm{VI}),$$
$$\dot{\omega} = (\dot{1}, \dot{2}, \dot{\mathrm{V}}\mathrm{II}, \dot{\mathrm{V}}\mathrm{III}, \dot{\mathrm{I}}\mathrm{X}), \tag{1.1}$$

starting from a nonsupersymmetric $\mathfrak{su}(8)$ theory [5,6,8]. Here, the undotted/dotted indices represent the $\overline{\mathbf{8_F}}$'s in the rank-two chiral IRAFFS and the rank-three chiral IRAFFS,[2] respectively. We also distinguish the heavy partner fermions and the SM fermions in terms of the Roman numbers and the Arabic numbers. The global symmetries of the chiral fermions in Eq. (1.1) are

$$\tilde{\mathcal{G}}_{\text{global}}[\mathfrak{su}(8), n_g = 3] = [\widetilde{\mathrm{SU}}(4)_{\omega} \otimes \tilde{\mathrm{U}}(1)_{T_2} \otimes \tilde{\mathrm{U}}(1)_{\mathrm{PQ}_2}] \otimes [\widetilde{\mathrm{SU}}(5)_{\dot{\omega}} \otimes \tilde{\mathrm{U}}(1)_{T_3} \otimes \tilde{\mathrm{U}}(1)_{\mathrm{PQ}_3}], \tag{1.2}$$

with the $\tilde{\mathrm{U}}(1)_{T_i}$ being the nonanomalous global symmetries, and the $\tilde{\mathrm{U}}(1)_{\mathrm{PQ}_i}$ being the anomalous global Peccei-Quinn symmetries [9]. To count the SM generations by decomposing the $\mathfrak{su}(8)$ fermion IRs into the $\mathfrak{su}(5)$ IRs [7], one finds three identical SM $\overline{\mathbf{5_F}}$'s and three distinctive $\mathbf{10_F}$'s [8]. It is sufficient to obtain three distinctive SM generations in the UV setup of the $\mathfrak{su}(8)$ theory.

The nonsupersymmetric $\mathfrak{su}(8)$ theory was analyzed according to its maximal symmetry breaking pattern of $\mathfrak{su}(8) \to \mathfrak{g}_{441} \equiv \mathfrak{su}(4)_s \oplus \mathfrak{su}(4)_W \oplus \mathfrak{u}(1)_{X_0}$ [6,10], which was favored by the minimization condition of the adjoint Higgs potential [11]. It was also found that the renormalization group equations (RGEs) of the gauge couplings in the nonsupersymmetric $\mathfrak{su}(8)$ theory cannot achieve the unification in the field theoretical interpretation of $\alpha_{4s}(v_U) = \alpha_{4W}(v_U) = \alpha_1(v_U)$ [8], even if the one-loop threshold effects [12–14] and the higher-dimension

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[1]The definition of the chiral IRAFFS was previously given in Refs. [5,6].

[2]The rank-two and the rank-three chiral IRAFFSs are named after the $\mathfrak{su}(8)$ rank-two and rank-three antisymmetric fermions of $\mathbf{28_F}$ and $\mathbf{56_F}$, respectively.

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operators were taken into account. The main difficulties of achieving the gauge coupling unification are (i) the Cartan discontinuities of different Abelian gauge couplings through the symmetry breaking stage of $\mathfrak{su}(N) \oplus \mathfrak{u}(1)_X \to \mathfrak{su}(N-1) \oplus \mathfrak{u}(1)_{X'}$ given by

$$\alpha_{X'}^{-1} = \frac{2}{N(N-1)}\alpha_N^{-1} + \alpha_X^{-1}, \tag{1.3}$$

where $(\alpha_N, \alpha_{X'}, \alpha_X)$ are the gauge couplings of the $(\mathfrak{su}(N), \mathfrak{u}(1)_X, \mathfrak{u}(1)_{X'})$ symmetries, and (ii) the suggested intermediate symmetry breaking scales, which were previously determined according to the observed SM quark/lepton masses and the CKM mixing patterns [6]. In the context of string theory, it was shown by Ginsparg [15] that the gauge coupling unification can be achieved as

$$k_s\alpha_s(v_U) = k_W\alpha_W(v_U) = k_1\alpha_1(v_U), \tag{1.4}$$

with (k_s, k_W, k_1) being the affine levels of the corresponding strong, weak, and the Abelian gauge couplings in the physical basis.[3] In a recent paper [18], we have proved that the reasonable conformal embedding based on the affine Lie algebra of

$$\widehat{\mathfrak{su}}(4)_{s,k_s=1} \oplus \widehat{\mathfrak{su}}(4)_{W,k_W=1} \oplus \hat{\mathfrak{u}}(1)_{1,k_{1,\text{phys}}=\frac{1}{4}} \subset \widehat{\mathfrak{su}}(8)_{k_U=1} \tag{1.5}$$

can achieve the gauge coupling unification with the $\mathcal{N}=1$ supersymmetric (SUSY) extension between the scale of $v_{441} \le \mu \le v_U$. All chiral superfields are tabulated in Table I so that (i) the $[\mathfrak{su}(8)]^3$ gauge anomaly is vanishing, and (ii) a nonanomalous global symmetry of $[\mathfrak{su}(8)]^2 \cdot \tilde{\mathrm{U}}(1)_T = 0$ is well defined.[4] Correspondingly, we expect the following Yukawa coupling terms from the holomorphic superpotential:

$$W_Y = Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\omega}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\omega} + Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega} + Y_{\mathcal{T}}\mathbf{28_F}\mathbf{28_F}\mathbf{70_H} + \frac{c_4}{M_{\text{pl}}}\mathbf{56_F}\mathbf{56_F}\mathbf{63_H}\mathbf{28_H}^{\dot\omega}, \tag{1.6}$$

with the reduced Planck scale of $M_{\text{pl}} = (8\pi G_N)^{-\frac{1}{2}} = 2.4 \times 10^{18}$ GeV.

It turns out the Yukawa coupling terms of $\overline{\mathbf{8_F}}^{\omega}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\omega}$, $\overline{\mathbf{8_F}}^{\dot\omega}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega}$, and $\mathbf{56_F}\mathbf{56_F}\mathbf{63_H}\mathbf{28_H}^{\dot\omega}$ give vectorlike fermion masses through the sequential stages after the breaking of $\mathfrak{su}(8) \to \mathfrak{g}_{441}$ [5], leaving three generational massless SM quarks/leptons plus twenty-three left-handed sterile neutrinos before the electroweak symmetry breaking (EWSB) stage according to the 't Hooft anomaly matching conditions [19]. Furthermore, the previous analyses of the maximal symmetry breaking patterns [5,6,20] reveal the origins of three-generational SM quark/lepton masses and the observed CKM mixing pattern [3,4] in the quark sector through the Yukawa coupling to one unique SM Higgs boson within the $\mathbf{70_H}$ in the spectrum. Specifically, the Yukawa coupling of $\mathbf{28_F}\mathbf{28_F}\mathbf{70_H}$ in Eq. (1.6) leads to the top quark mass, and other SM up-type quark masses are due to two $d=5$ operators of

$$c_4\mathcal{O}_{\mathcal{F}}^{(4,1)} = c_4\mathbf{56_F}\mathbf{56_F}\cdot\overline{\mathbf{28_H}}_{,\dot\omega}\cdot\mathbf{70_H}, \tag{1.7a}$$

$$c_5\mathcal{O}_{\mathcal{F}}^{(5,1)} = c_5\mathbf{28_F}\mathbf{56_F}\cdot\overline{\mathbf{8_H}}_{,\omega}\cdot\mathbf{70_H}. \tag{1.7b}$$

To generate the masses for all down-type quarks and charged leptons, we conjecture two sets of Higgs mixing terms,

$$d_{\mathscr{A}}\mathcal{O}_{\mathscr{A}}^{d=5} \equiv d_{\mathscr{A}}\epsilon_{\omega_1\omega_2\omega_3\omega_4}\overline{\mathbf{8_H}}^{\dagger}_{,\omega_1}\overline{\mathbf{8_H}}^{\dagger}_{,\omega_2}\overline{\mathbf{8_H}}^{\dagger}_{,\omega_3}\overline{\mathbf{8_H}}^{\dagger}_{,\omega_4}\mathbf{70_H}^{\dagger}, \quad \mathcal{PQ} = 2(2p+3q_2) \neq 0, \tag{1.8a}$$

$$d_{\mathscr{B}}\mathcal{O}_{\mathscr{B}}^{d=5} \equiv d_{\mathscr{B}}(\overline{\mathbf{28_H}}^{\dagger}_{,\dot\kappa_1}\overline{\mathbf{28_H}}_{,\dot\kappa_2})\cdot\overline{\mathbf{28_H}}^{\dagger}_{,\dot\omega_1}\overline{\mathbf{28_H}}^{\dagger}_{,\dot\omega_2}\mathbf{70_H}^{\dagger}, \quad \mathcal{PQ} = 2(p+q_2+q_3), \quad \text{with} \quad \dot\kappa_2 \neq (\dot\kappa_1, \dot\omega_1, \dot\omega_2), \tag{1.8b}$$

where one of the Higgs fields will mix with two renormalizable Yukawa couplings of $Y_{\mathcal{B}}\overline{\mathbf{8_F}}^{\omega}\mathbf{28_F}\overline{\mathbf{8_H}}_{,\omega} + Y_{\mathcal{D}}\overline{\mathbf{8_F}}^{\dot\omega}\mathbf{56_F}\overline{\mathbf{28_H}}_{,\dot\omega} + \text{H.c.}$ in Eq. (1.6). Both $d=5$ Yukawa couplings in Eq. (1.7) and Higgs mixing terms in Eq. (1.8) violate the global symmetries in Eq. (1.2) explicitly, and they were conjectured to be suppressed by the reduced Planck scale of $M_{\text{pl}} = (8\pi G_N)^{-\frac{1}{2}} = 2.4 \times 10^{18}$ GeV [6]. Three intermediate symmetry breaking scales beyond the EW scale enter into the SM quark/lepton mass matrices, and hence are set by requiring a reasonable matching with the experimental measurements in the flavor sector.

TABLE I. The representations and charges of all $\widehat{\mathfrak{su}}(8)_{k_U=1}$ chiral superfields under the global symmetries in Eq. (1.2).

$\mathfrak{su}(8)$	$\widetilde{\mathrm{SU}}(4)_{\omega}$	$\widetilde{\mathrm{SU}}(5)_{\dot\omega}$	$\tilde{\mathrm{U}}(1)_{\text{PQ}}$	$\tilde{\mathrm{U}}(1)_T$
$\overline{\mathbf{8_F}}^{\omega}$	**4**	**1**	p	$-3t$
$\overline{\mathbf{8_F}}^{\dot\omega}$	**1**	**5**	p	$-3t$
$\mathbf{28_F}$	**1**	**1**	q_2	$+2t$
$\mathbf{56_F}$	**1**	**1**	q_3	$+t$
$\overline{\mathbf{8_H}}_{,\omega}/\mathbf{8_H}^{\omega}$	$\bar{\mathbf{4}}/\mathbf{4}$	**1**	$-(p+q_2)/p+q_2$	$+t/-t$
$\overline{\mathbf{28_H}}_{,\dot\omega}/\mathbf{28_H}^{\dot\omega}$	**1**	$\bar{\mathbf{5}}/\mathbf{5}$	$-(p+q_3)/p+q_3$	$+2t/-2t$
$\mathbf{70_H}/\overline{\mathbf{70_H}}$	**1**	**1**	$-2q_2/+2q_2$	$-4t/+4t$
$\mathbf{63_H}$	**1**	**1**	0	0

[3]Some early studies of the higher affine levels in the string-inspired unified theories include Refs. [16,17].

[4]The global $\tilde{\mathrm{U}}(1)_T$ will evolve to the global $\tilde{\mathrm{U}}(1)_{B-L}$ symmetry in the SM [5].

It was previously pointed out by Witten that other nonmaximal symmetry breaking patterns are possible in a SUSY theory [21]. The superpotential for the adjoint field $\mathbf{63_H}$ takes the form of

$$W \supset \frac{1}{2}\mu_U \mathrm{Tr}(\mathbf{63_H})^2 + \frac{1}{3}\lambda_U \mathrm{Tr}(\mathbf{63_H})^3, \tag{1.9}$$

and the corresponding field equation for the unbroken SUSY reads

$$\lambda_U\left(\mathbf{63_H}^2 - \frac{1}{8}\mathbb{I}_8 \mathrm{Tr}(\mathbf{63_H})^2\right) + \mu_U \mathbf{63_H} = 0. \tag{1.10}$$

The vacuum expectation values (VEVs) of the adjoint Higgs field and the corresponding symmetry breaking patterns are

$$\mathfrak{su}(8) \to \mathfrak{g}_{441}: \langle \mathbf{63_H} \rangle = \frac{1}{\sqrt{2}} v_U \mathrm{diag}\left(\frac{1}{4}\mathbb{I}_4, -\frac{1}{4}\mathbb{I}_4\right), \tag{1.11a}$$

$$\mathfrak{su}(8) \to \mathfrak{g}_{531}/\mathfrak{g}_{351}: \langle \mathbf{63_H} \rangle = \frac{3\mu_U}{2\lambda_U}\mathrm{diag}\left(\mathbb{I}_5, -\frac{5}{3}\mathbb{I}_3\right)/\langle \mathbf{63_H} \rangle = \frac{5\mu_U}{2\lambda_U}\mathrm{diag}\left(\mathbb{I}_3, -\frac{3}{5}\mathbb{I}_5\right), \tag{1.11b}$$

$$\mathfrak{su}(8) \to \mathfrak{g}_{621}: \langle \mathbf{63_H} \rangle = \frac{\mu_U}{2\lambda_U}\mathrm{diag}(\mathbb{I}_6, -3\mathbb{I}_2). \tag{1.11c}$$

The maximal symmetry breaking pattern of $\mathfrak{su}(8) \to \mathfrak{g}_{441}$ can be achieved with the vanishing mass term of μ_U in Eq. (1.9), and have been recently studied in Refs. [6,10,18,20]. As for the nonmaximal symmetry breaking patterns in Eqs. (1.11b) and (1.11c), we have also proved in Ref. [18] that the corresponding conformal embeddings are

$$\widehat{\mathfrak{su}}(5)_{k_{s/W}=1} \oplus \widehat{\mathfrak{su}}(3)_{k_{W/s}=1} \oplus \hat{\mathfrak{u}}(1)_{k_{1,\mathrm{phys}}=\frac{1}{4}} \subset \widehat{\mathfrak{su}}(8)_{k_U=1}, \tag{1.12a}$$

$$\widehat{\mathfrak{su}}(6)_{k_s=1} \oplus \widehat{\mathfrak{su}}(2)_{k_W=1} \oplus \hat{\mathfrak{u}}(1)_{k_{1,\mathrm{phys}}=\frac{1}{4}} \subset \widehat{\mathfrak{su}}(8)_{k_U=1}. \tag{1.12b}$$

Given the possible nonmaximal symmetry breaking patterns in the SUSY extension, as well as the conformal embeddings in Eqs. (1.12a) and (1.12b), it is a central question to check whether the gauge coupling unification can be achieved according to Eq. (1.4).

If one considers the nonmaximal symmetry breaking patterns of $\mathfrak{su}(8) \to \mathfrak{g}_{531}$ or $\mathfrak{su}(8) \to \mathfrak{g}_{351}$ at the GUT scale, there can be four possible sequential symmetry breaking patterns,[5] namely,

$$\begin{aligned}&\mathrm{SSW}: \mathfrak{su}(8) \to \mathfrak{g}_{531} \to \mathfrak{g}_{431} \to \mathfrak{g}_{331} \to \mathfrak{g}_{\mathrm{SM}},\\ &\mathfrak{g}_{531} \equiv \mathfrak{su}(5)_s \oplus \mathfrak{su}(3)_W \oplus \mathfrak{u}(1)_{X_0},\\ &\mathfrak{g}_{331} \equiv \mathfrak{su}(3)_c \oplus \mathfrak{su}(3)_W \oplus \mathfrak{u}(1)_{X_2},\end{aligned} \tag{1.13a}$$

$$\begin{aligned}&\mathrm{SWS}: \mathfrak{su}(8) \to \mathfrak{g}_{531} \to \mathfrak{g}_{431} \to \mathfrak{g}_{421} \to \mathfrak{g}_{\mathrm{SM}},\\ &\mathfrak{g}_{431} \equiv \mathfrak{su}(4)_s \oplus \mathfrak{su}(3)_W \oplus \mathfrak{u}(1)_{X_1},\\ &\mathfrak{g}_{421} \equiv \mathfrak{su}(4)_s \oplus \mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_{X_2},\end{aligned} \tag{1.13b}$$

$$\begin{aligned}&\mathrm{WSS}: \mathfrak{su}(8) \to \mathfrak{g}_{531} \to \mathfrak{g}_{521} \to \mathfrak{g}_{421} \to \mathfrak{g}_{\mathrm{SM}},\\ &\mathfrak{g}_{521} \equiv \mathfrak{su}(5)_s \oplus \mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_{X_1},\end{aligned} \tag{1.13c}$$

$$\begin{aligned}&\mathrm{WWW}: \mathfrak{su}(8) \to \mathfrak{g}_{351} \to \mathfrak{g}_{341} \to \mathfrak{g}_{331} \to \mathfrak{g}_{\mathrm{SM}},\\ &\mathfrak{g}_{351} \equiv \mathfrak{su}(3)_c \oplus \mathfrak{su}(5)_W \oplus \mathfrak{u}(1)_{X_0},\\ &\mathfrak{g}_{341} \equiv \mathfrak{su}(3)_c \oplus \mathfrak{su}(4)_W \oplus \mathfrak{u}(1)_{X_1},\end{aligned} \tag{1.13d}$$

with $\mathfrak{g}_{\mathrm{SM}} \equiv \mathfrak{su}(3)_c \oplus \mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_Y$. In contrast to the maximal symmetry breaking pattern of $\mathfrak{su}(8) \to \mathfrak{g}_{441}$, the $\mathfrak{u}(1)_1$ gauge coupling is normalized as $\alpha_1 = \frac{16}{15}\alpha_{X_0}$ in the above nonmaximal symmetry breaking patterns of $\mathfrak{su}(8) \to \mathfrak{g}_{531}/\mathfrak{g}_{351}$. Additionally, there can also be a possible symmetry breaking pattern of $\mathfrak{su}(8) \to \mathfrak{g}_{621} \equiv \mathfrak{su}(6)_s \oplus \mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_{X_0}$, with the sequential breaking of $\mathfrak{su}(6)_s \oplus \mathfrak{u}(1)_{X_0} \to \ldots \to \mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_Y$ in the extended strong sector. However, this turns out to be a no-go pattern, since a pair of vectorlike quarks of $(\mathfrak{u}, \mathfrak{d})$ in the spectrum remains massless through this symmetry breaking pattern.

The possible $d = 5$ gravity-induced operator of

$$\mathcal{O}_{\mathrm{HSW}} = -\frac{1}{2}\frac{c_{\mathrm{HSW}}}{M_{\mathrm{pl}}}\mathrm{Tr}[\mathbf{63_H} U_{\mu\nu} U^{\mu\nu}] \tag{1.14}$$

suggested by Hill-Shafi-Wetterich (HSW) [22,23] could have effect on the gauge coupling unification. Here, the $U_{\mu\nu}$ represents the $\mathfrak{su}(8)$ field strength tensor. After the GUT-scale symmetry breaking with the VEVs in Eq. (1.11), this HSW operator shifts the $\mathfrak{su}(N_s)_s$, $\mathfrak{su}(N_W)_W$, and $\mathfrak{u}(1)_{X_0}$ field strengths for different symmetry breaking patterns as follows:

[5]The acronyms stand for the strong-strong-weak (SSW), strong-weak-strong (SWS), weak-strong-strong (WSS), and weak-weak-weak (WWW) symmetry breaking patterns, respectively.

$$\mathfrak{su}(8) \to \mathfrak{g}_{531}: \ -\frac{1}{2}\left(1-\frac{3}{4}\epsilon\right)\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] - \frac{1}{2}\left(1+\frac{5}{4}\epsilon\right)\mathrm{Tr}[W_{\mu\nu}W^{\mu\nu}] - \frac{1}{4}\left(1+\frac{1}{2}\epsilon\right)\frac{15}{16}X_{0\mu\nu}X_0^{\mu\nu}, \tag{1.15a}$$

$$\mathfrak{su}(8) \to \mathfrak{g}_{351}: \ -\frac{1}{2}\left(1-\frac{5}{4}\epsilon\right)\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] - \frac{1}{2}\left(1+\frac{3}{4}\epsilon\right)\mathrm{Tr}[W_{\mu\nu}W^{\mu\nu}] - \frac{1}{4}\left(1-\frac{1}{2}\epsilon\right)\frac{15}{16}X_{0\mu\nu}X_0^{\mu\nu}, \tag{1.15b}$$

where $\epsilon \equiv \frac{1}{\sqrt{15}}c_{\mathrm{HSW}}\zeta_0$ with $\zeta_0 \equiv \frac{v_U}{M_{\mathrm{pl}}}$. Consequently, the gauge coupling unification is modified into

$$\mathfrak{su}(8) \to \mathfrak{g}_{531}: \ \left(1-\frac{3}{4}\epsilon\right)\alpha_{5s}(v_U) = \left(1+\frac{5}{4}\epsilon\right)\alpha_{3W}(v_U) = \left(1+\frac{1}{2}\epsilon\right)\frac{1}{4}\alpha_1(v_U) = \alpha_U(v_U), \tag{1.16a}$$

$$\mathfrak{su}(8) \to \mathfrak{g}_{351}: \ \left(1-\frac{5}{4}\epsilon\right)\alpha_{3s}(v_U) = \left(1+\frac{3}{4}\epsilon\right)\alpha_{5W}(v_U) = \left(1-\frac{1}{2}\epsilon\right)\frac{1}{4}\alpha_1(v_U) = \alpha_U(v_U), \tag{1.16b}$$

with the affine levels in Eq. (1.12a) and the gravity-induced operator in Eq. (1.14) taken into account.

The rest of the paper will be devoted to analyzing the symmetry breaking patterns in Eqs. (1.11b) and (1.11c), with the focus on the gauge coupling evolutions in terms of the corresponding RGEs. In Secs. II–V, we analyze the sequential symmetry breaking patterns following the non-minimal breaking patterns of $\mathfrak{su}(8) \to \mathfrak{g}_{531}$ and $\mathfrak{su}(8) \to \mathfrak{g}_{351}$ at the GUT scale. Particularly, we derive the RGEs along each symmetry breaking pattern and find that the gauge coupling unification can be achieved in the context of the affine Lie algebra of $\widehat{\mathfrak{su}}(8)_{k_U=1}$. In Sec. VI, we prove that the symmetry breaking pattern in Eq. (1.11c) is unrealistic, since the Yukawa coupling of $\mathbf{56_F56_F63_H28_H}^{\dot{\omega}}$ in Eq. (1.6) leads to vanishing mass terms for vectorlike fermions of $(\mathfrak{u}, \mathfrak{d})$ within the $\mathbf{56_F}$. We summarize the results in Sec. VII. In the Appendix, we list the generic results of the one- and two-loop β coefficients. We present the details of all possible symmetry breaking patterns after the GUT-scale symmetry breaking of $\mathfrak{su}(8) \to \mathfrak{g}_{531}/\mathfrak{g}_{351}$ in Ref. [24]. By reproducing the SM quark/lepton mass hierarchies and the CKM mixing pattern in Ref. [6], we identify the SM flavors as well as determine the intermediate symmetry breaking scales for each pattern. Accordingly, the intermediate symmetry breaking scales are used for the RGEs in Secs. II–V.

II. THE SSW SYMMETRY BREAKING PATTERN

A. Decompositions of the $\mathfrak{su}(8)$ fermions

By following the SSW symmetry breaking pattern in Eq. (1.13), we tabulate the fermion representations at various stages in Tables II–IV. Along this pattern, the global $\tilde{\mathrm{U}}(1)_T$ symmetry evolves to the global $\tilde{\mathrm{U}}(1)_{B-L}$ at the EW scale according to the following sequence:

$$\begin{aligned} &\mathfrak{g}_{531}: \ \mathcal{T}' \equiv \mathcal{T} - 5t\mathcal{X}_0, \qquad \mathfrak{g}_{431}: \ \mathcal{T}'' \equiv \mathcal{T}', \\ &\mathfrak{g}_{331}: \ \mathcal{T}''' \equiv \mathcal{T}'' + 8t\mathcal{X}_2, \qquad \mathfrak{g}_{\mathrm{SM}}: \ \mathcal{B}-\mathcal{L} \equiv \mathcal{T}'''. \end{aligned} \tag{2.1}$$

B. Decompositions of the $\mathfrak{su}(8)$ Higgs fields

We decompose the Higgs fields in Eq. (1.6) into components that can develop VEVs for the sequential symmetry breaking stages in Eq. (1.13a). For Higgs fields of $\overline{\mathbf{8_H}}_{,\omega}$, they read

$$\begin{aligned} \overline{\mathbf{8_H}}_{,\omega} &\supset \underline{\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega}} \oplus \left\langle \left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\omega} \right\rangle \\ &\supset \underline{\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega}} \oplus \left\langle \left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\omega} \right\rangle \\ &\supset \left\langle \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega} \right\rangle. \end{aligned} \tag{2.2}$$

TABLE II. The $\mathfrak{su}(8)$ fermion representation of $\overline{\mathbf{8_F}}^{\Omega}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{431}, \mathfrak{g}_{331}, \mathfrak{g}_{\mathrm{SM}}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{431}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\mathrm{SM}}$
$\overline{\mathbf{8_F}}^{\Omega}$	$(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},+\frac{1}{3})^{\Omega}_{\mathbf{F}}$: $\mathcal{D}_R^{\Omega c}$	
			$(\mathbf{1},\mathbf{1},0)^{\Omega}_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)^{\Omega}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega}$
		$(\mathbf{1},\mathbf{1},0)^{\Omega'}_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)^{\Omega'}_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)^{\Omega'}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega'}$
	$(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2})^{\Omega}_{\mathbf{F}}$: $\mathcal{L}_L^{\Omega} = (\mathcal{E}_L^{\Omega}, -\mathcal{N}_L^{\Omega})^T$
				$(\mathbf{1},\mathbf{1},0)^{\Omega''}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega''}$

TABLE III. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{28_F}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{431}, \mathfrak{g}_{331}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{431}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{28_F}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})_{\mathbf{F}}$: $(\mathfrak{n}_R{}^c,-\mathfrak{e}_R{}^c)^T$
				$(\mathbf{1},\mathbf{1},+1)_{\mathbf{F}}$: $\tau_R{}^c$
	$(\mathbf{5},\mathbf{3},+\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})'_{\mathbf{F}}$: $(\mathfrak{e}'^c_R,\mathfrak{n}'^c_R)^T$
				$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}$: $\check{\mathfrak{n}}^c_R$
		$(\mathbf{4},\mathbf{3},+\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})''_{\mathbf{F}}$: $(\mathfrak{e}''^c_R,\mathfrak{n}''^c_R)^T$
				$(\mathbf{1},\mathbf{1},0)'_{\mathbf{F}}$: $\check{\mathfrak{n}}'^c_R$
			$(\mathbf{3},\mathbf{3},0)_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})_{\mathbf{F}}$: $\mathfrak{D}_L$
				$(\mathbf{3},\mathbf{2},+\frac{1}{6})_{\mathbf{F}}$: $(t_L,b_L)^T$
	$(\mathbf{10},\mathbf{1},-\frac{2}{5})_{\mathbf{F}}$	$(\mathbf{4},\mathbf{1},-\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'_{\mathbf{F}}$: $\mathfrak{D}'_L$
			$(\mathbf{1},\mathbf{1},0)''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)''_{\mathbf{F}}$: $\check{\mathfrak{n}}''^c_R$
		$(\mathbf{6},\mathbf{1},-\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''_{\mathbf{F}}$: $\mathfrak{D}''_L$
			$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})_{\mathbf{F}}$: $t_R{}^c$

TABLE IV. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{56_F}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{431}, \mathfrak{g}_{331}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{431}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{56_F}$	$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$: $\mathfrak{E}_R{}^c$
	$(\mathbf{5},\bar{\mathbf{3}},+\frac{7}{15})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})'''_{\mathbf{F}}$: $(\mathfrak{n}'''^c_R,-\mathfrak{e}'''^c_R)^T$
				$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$: $e_R{}^c$
		$(\mathbf{4},\bar{\mathbf{3}},+\frac{5}{12})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})''_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})''''_{\mathbf{F}}$: $(\mathfrak{n}''''^c_R,-\mathfrak{e}''''^c_R)^T$
				$(\mathbf{1},\mathbf{1},+1)'''_{\mathbf{F}}$: $\mu_R{}^c$
			$(\mathbf{3},\bar{\mathbf{3}},+\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},+\frac{2}{3})_{\mathbf{F}}$: $\mathfrak{U}_L$
				$(\mathbf{3},\bar{\mathbf{2}},+\frac{1}{6})_{\mathbf{F}}$: $(\mathfrak{d}_L,-\mathfrak{u}_L)^T$
	$(\overline{\mathbf{10}},\mathbf{1},-\frac{3}{5})_{\mathbf{F}}$	$(\bar{\mathbf{4}},\mathbf{1},-\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$: $\mathfrak{E}_L$
			$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$: $c_R{}^c$
		$(\mathbf{6},\mathbf{1},-\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''_{\mathbf{F}}$: $\mathfrak{D}'''_L$
			$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})''_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})''_{\mathbf{F}}$: $u_R{}^c$
	$(\mathbf{10},\mathbf{3},-\frac{1}{15})_{\mathbf{F}}$	$(\mathbf{4},\mathbf{3},+\frac{1}{12})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})'''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})''''_{\mathbf{F}}$: $(\mathfrak{e}''''^c_R,\mathfrak{n}''''^c_R)^T$
				$(\mathbf{1},\mathbf{1},0)'''_{\mathbf{F}}$: $\check{\mathfrak{n}}'''^c_R$
			$(\mathbf{3},\mathbf{3},0)'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
				$(\mathbf{3},\mathbf{2},+\frac{1}{6})'_{\mathbf{F}}$: $(u_L,d_L)^T$
		$(\mathbf{6},\mathbf{3},-\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{3},0)''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
				$(\mathbf{3},\mathbf{2},+\frac{1}{6})''_{\mathbf{F}}$: $(c_L,s_L)^T$
			$(\bar{\mathbf{3}},\mathbf{3},-\frac{1}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'''_{\mathbf{F}}$: $\mathfrak{U}_R{}^c$
				$(\bar{\mathbf{3}},\mathbf{2},-\frac{1}{6})_{\mathbf{F}}$: $(\mathfrak{d}_R{}^c,\mathfrak{u}_R{}^c)^T$

For Higgs fields of $\overline{\mathbf{28}_{\mathbf{H},\dot{\omega}}}$, they read

$$\begin{aligned}\overline{\mathbf{28}_{\mathbf{H},\dot{\omega}}} &\supset \underline{\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}}} \oplus \underline{\left(\bar{\mathbf{5}},\bar{\mathbf{3}},-\frac{2}{15}\right)_{\mathbf{H},\dot{\omega}}} \oplus \underline{\left(\overline{\mathbf{10}},\mathbf{1},+\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}}} \\ &\supset \underline{\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left[\underline{\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}}} \oplus \underline{\left(\bar{\mathbf{4}},\bar{\mathbf{3}},-\frac{1}{12}\right)_{\mathbf{H},\dot{\omega}}}\right] \oplus \left\langle\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}}\right\rangle \\ &\supset \underline{\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left[\left\langle\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}}\right\rangle \oplus \left\langle\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)'_{\mathbf{H},\dot{\omega}}\right\rangle\right]. \end{aligned} \tag{2.3}$$

For Higgs field of $\mathbf{70_H}$, it reads

$$\begin{aligned}\mathbf{70_H} \supset & \left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \underline{\left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}} \\ & \supset \underline{\left(\mathbf{4},\bar{\mathbf{3}},+\frac{5}{12}\right)_{\mathbf{H}}} \supset \underline{\left(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3}\right)_{\mathbf{H}}} \supset \left\langle \left(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2}\right)_{\mathbf{H}} \right\rangle. \end{aligned} \tag{2.4}$$

The decompositions of the Higgs fields of $\mathbf{8_H}^{\omega}$ and $\mathbf{28_H}^{\dot{\omega}}$ can be made by taking the conjugate IRs in Eqs. (2.2) and (2.3), and we will neglect the corresponding expressions throughout the text.

C. RGEs of the SSW symmetry breaking pattern

Between the $v_{531} \le \mu \le v_U$, the massless Higgs fields are summarized as follows:

$$\begin{aligned}&\left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\omega} \oplus \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega} \subset \overline{\mathbf{8_{H},\omega}},\\
&\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{5}},\bar{\mathbf{3}},-\frac{2}{15}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\overline{\mathbf{10}},\mathbf{1},+\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}} \subset \overline{\mathbf{28_{H},\dot{\omega}}},\\
&\left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}} \subset \mathbf{70_H},\\
&\left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}} \subset \overline{\mathbf{70_H}},\\
&\left(\mathbf{5},\mathbf{1},-\frac{1}{5}\right)^{\omega}_{\mathbf{H}} \oplus \left(\mathbf{1},\mathbf{3},+\frac{1}{3}\right)^{\omega}_{\mathbf{H}} \subset \mathbf{8_H}^{\omega},\\
&\left(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3}\right)^{\dot{\omega}}_{\mathbf{H}} \oplus \left(\mathbf{5},\mathbf{3},+\frac{2}{15}\right)^{\dot{\omega}}_{\mathbf{H}} \oplus \left(\mathbf{10},\mathbf{1},-\frac{2}{5}\right)^{\dot{\omega}}_{\mathbf{H}} \subset \mathbf{28_H}^{\dot{\omega}}. \end{aligned} \tag{2.5}$$

All fermionic components of $\overline{\mathbf{8_F}}^{\Omega} \oplus \mathbf{28_F} \oplus \mathbf{56_F}$ remain massless after the decomposition into the $\mathfrak{g}_{531}$ IRs. Consequently, we have the two-loop $\mathfrak{g}_{531}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(5)_s}, b^{(1)}_{\mathfrak{su}(3)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_0}}) = \left(+54,+60,+\frac{368}{5}\right), \qquad b^{(2)}_{\mathfrak{g}_{531}} = \begin{pmatrix} 6984/5 & 216 & 672/25 \\ 648 & 728 & 352/15 \\ 16128/25 & 2816/15 & 98816/1125 \end{pmatrix}. \tag{2.6}$$

Between the $v_{431} \le \mu \le v_{531}$, the massless Higgs fields are the following:

$$\begin{aligned}&\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},3,\mathrm{V},\mathrm{VI}} \oplus \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},3,\mathrm{V},\mathrm{VI}} \subset \ldots \subset \overline{\mathbf{8_{H},3,V,VI}},\\
&\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left[\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{4}},\bar{\mathbf{3}},-\frac{1}{12}\right)_{\mathbf{H},\dot{\omega}}\right]\\
&\oplus\left[\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{6},\mathbf{1},+\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}\right] \subset \ldots \subset \overline{\mathbf{28_{H},\dot{\omega}}}, \dot{\omega} = (\dot{\mathrm{I}},\dot{2},\dot{\mathrm{V}}\mathrm{II},\dot{\mathrm{V}}\mathrm{III},\dot{\mathrm{I}}\mathrm{X}),\\
&\left(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3}\right)^{\dot{\mathrm{V}}\mathrm{II}}_{\mathbf{H}} \oplus \left[\left(\mathbf{1},\mathbf{3},+\frac{1}{3}\right)^{\dot{\mathrm{V}}\mathrm{II}}_{\mathbf{H}} \oplus \left(\mathbf{4},\mathbf{3},+\frac{1}{12}\right)^{\dot{\mathrm{V}}\mathrm{II}}_{\mathbf{H}}\right]\\
&\oplus\left[\left(\mathbf{4},\mathbf{1},-\frac{1}{4}\right)^{\dot{\mathrm{V}}\mathrm{II}}_{\mathbf{H}} \oplus \left(\mathbf{6},\mathbf{1},-\frac{1}{2}\right)^{\dot{\mathrm{V}}\mathrm{II}}_{\mathbf{H}}\right] \subset \ldots \subset \mathbf{28_H}^{\dot{\mathrm{V}}\mathrm{II}},\\
&\left(\mathbf{4},\bar{\mathbf{3}},+\frac{5}{12}\right)_{\mathbf{H}} \subset \ldots \subset \mathbf{70_H}. \end{aligned} \tag{2.7}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{431}$ IRs can be found in Ref. [24]. Consequently, we have the two-loop $\mathfrak{g}_{431}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(4)_s}, b^{(1)}_{\mathfrak{su}(3)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_1}}) = \left(+\frac{5}{3}, +\frac{11}{2}, +\frac{115}{6}\right), \qquad b^{(2)}_{\mathfrak{g}_{431}} = \begin{pmatrix} 3511/12 & 76 & 269/24 \\ 285/2 & 233 & 47/4 \\ 1345/8 & 94 & 5267/144 \end{pmatrix}. \tag{2.8}$$

The $\mathfrak{su}(5)_s \oplus \mathfrak{u}(1)_{X_0}$ gauge couplings match with the $\mathfrak{su}(4)_s \oplus \mathfrak{u}(1)_{X_1}$ gauge couplings according to the following conditions:

$$\alpha^{-1}_{5s}(v_{531}) = \alpha^{-1}_{4s}(v_{531}), \qquad \alpha^{-1}_{X_1}(v_{531}) = \frac{1}{10}\alpha^{-1}_{5s}(v_{531}) + \alpha^{-1}_{X_0}(v_{531}). \tag{2.9}$$

Between the $v_{331} \leq \mu \leq v_{431}$, the massless Higgs fields are

$$\begin{aligned} &\left(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right)_{\mathbf{H},3,\mathrm{VI}} \oplus \left(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3}\right)_{\mathbf{H},3,\mathrm{VI}} \subset \ldots \subset \overline{\mathbf{8_H}}_{,3,\mathrm{VI}}, \\ &\left(\mathbf{1}, \mathbf{3}, -\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left[\left(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3}\right)'_{\mathbf{H},\dot{\omega}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}}, 0)_{\mathbf{H},\dot{\omega}}\right] \\ \oplus &\left[\left(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right)'_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{3}, \mathbf{1}, +\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}}\right] \subset \ldots \subset \overline{\mathbf{28_H}}_{,\dot{\omega}}, \dot{\omega} = (\dot{2}, \dot{\mathrm{V}}\mathrm{III}, \dot{\mathrm{I}}\mathrm{X}), \\ &\left(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3}\right)_{\mathbf{H}} \subset \ldots \subset \mathbf{70_H}. \end{aligned} \tag{2.10}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{331}$ IRs can be found in Ref. [24]. Consequently, we have the $\mathfrak{g}_{331}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(3)_c}, b^{(1)}_{\mathfrak{su}(3)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_2}}) = \left(-\frac{5}{3}, -\frac{3}{2}, +\frac{116}{9}\right), \qquad b^{(2)}_{\mathfrak{g}_{331}} = \begin{pmatrix} 256/3 & 36 & 58/9 \\ 36 & 89 & 22/3 \\ 464/9 & 176/3 & 728/27 \end{pmatrix}. \tag{2.11}$$

The $\mathfrak{su}(4)_s \oplus \mathfrak{u}(1)_{X_1}$ gauge couplings match with the $\mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_{X_2}$ gauge couplings according to the following conditions:

$$\alpha^{-1}_{4s}(v_{431}) = \alpha^{-1}_{3c}(v_{431}), \qquad \alpha^{-1}_{X_2}(v_{431}) = \frac{1}{6}\alpha^{-1}_{4s}(v_{431}) + \alpha^{-1}_{X_1}(v_{431}). \tag{2.12}$$

Between the $v_{\mathrm{EW}} \leq \mu \leq v_{331}$, the massless Higgs field is

$$\left(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2}\right)_{\mathbf{H}} \subset \left(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3}\right)_{\mathbf{H}} \subset \left(\mathbf{4}, \bar{\mathbf{3}}, +\frac{5}{12}\right)_{\mathbf{H}} \subset \left(\mathbf{10}, \bar{\mathbf{3}}, +\frac{4}{15}\right)_{\mathbf{H}} \subset \mathbf{70_H}. \tag{2.13}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{\mathrm{SM}}$ IRs can be found in Ref. [24]. Consequently, we have the $\mathfrak{g}_{\mathrm{SM}}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(3)_c}, b^{(1)}_{\mathfrak{su}(2)_W}, b^{(1)}_{\mathfrak{u}(1)_Y}) = \left(-7, -\frac{19}{6}, +\frac{41}{6}\right), \qquad b^{(2)}_{\mathfrak{g}_{\mathrm{SM}}} = \begin{pmatrix} -26 & 9/2 & 11/6 \\ 12 & 35/6 & 3/2 \\ 44/3 & 9/2 & 199/18 \end{pmatrix}. \tag{2.14}$$

The $\mathfrak{su}(3)_W \oplus \mathfrak{u}(1)_{X_2}$ gauge couplings of $(\alpha_{3W}, \alpha_{X_2})$ match with the $\mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_{\mathrm{Y}}$ gauge couplings according to the following conditions:

$$\alpha^{-1}_{3W}(v_{331}) = \alpha^{-1}_{2W}(v_{331}), \qquad \alpha^{-1}_{Y}(v_{331}) = \frac{1}{3}\alpha^{-1}_{3W}(v_{331}) + \alpha^{-1}_{X_2}(v_{331}). \tag{2.15}$$

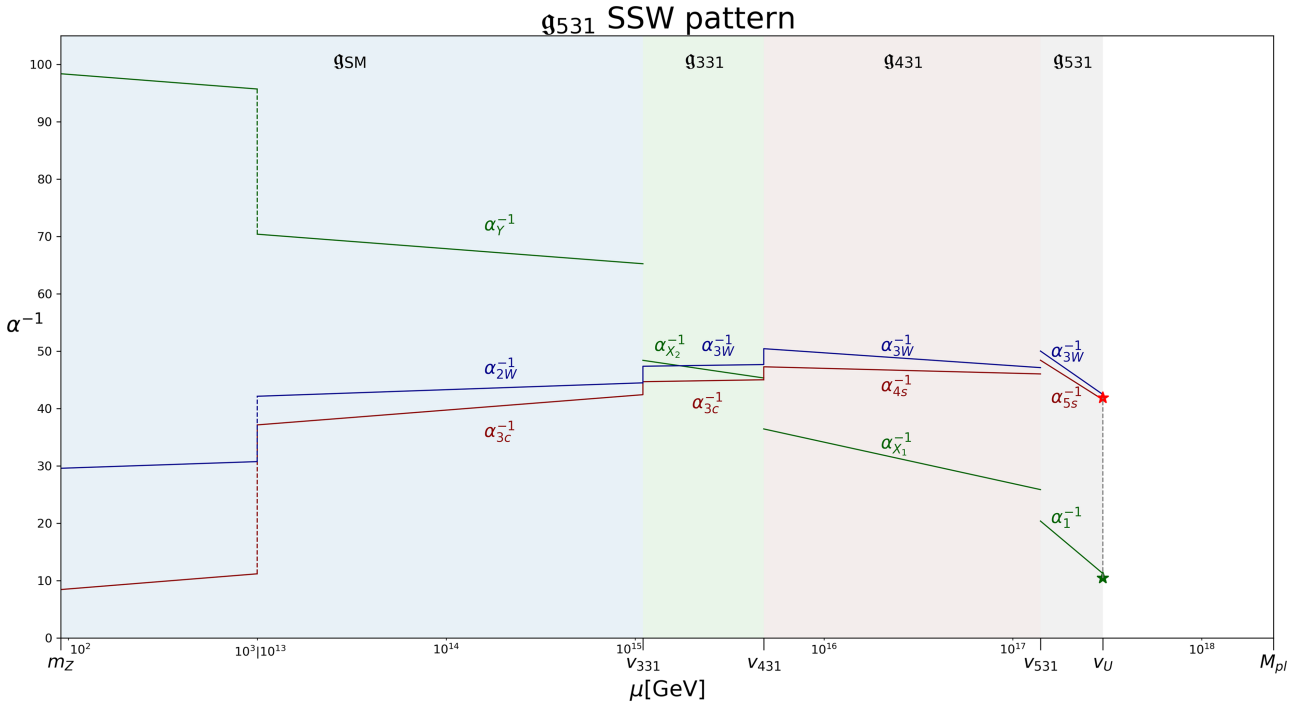


FIG. 1. The two-loop RGEs of the $\mathfrak{su}(8)$ setup according to the SSW symmetry breaking pattern. The RG evolutions within 10^3 GeV $\lesssim \mu \lesssim 10^{13}$ GeV are hidden in order to highlight the behaviors in three intermediate symmetry breaking scales given by the benchmark point in Eq. (2.16).

Based on constructing the SM quark/lepton mass matrices to reproduce the observed hierarchical masses and the CKM mixing pattern, we find the following benchmark point:

$$v_{531} \simeq 1.4 \times 10^{17}\ \text{GeV}, \qquad v_{431} \simeq 4.8 \times 10^{15}\ \text{GeV},$$
$$v_{331} \simeq 4.8 \times 10^{13}\ \text{GeV}. \tag{2.16}$$

With the one- and two-loop β coefficients defined in Eqs. (2.6), (2.8), (2.11), and (2.14), we plot the RGEs of the $\mathfrak{su}(8)$ setup along the SSW symmetry breaking pattern in Fig. 1. The discontinuities of the $\mathfrak{u}(1)$ gauge couplings at three intermediate scales follow from their definitions in Eqs. (2.9), (2.12), (2.15), respectively. The benchmark point in Fig. 1 is marked by ⋆ and reads

$$c_{\rm HSW} \approx 0.24, \qquad v_U \approx 2.28 \times 10^{17}\ \text{GeV},$$
$$\alpha_{5c}^{-1}(v_U) \approx 44.02, \qquad \alpha_{3W}^{-1}(v_U) \approx 45.18,$$
$$\alpha_1^{-1}(v_U) \approx 10.50. \tag{2.17}$$

III. THE SWS SYMMETRY BREAKING PATTERN

A. Decompositions of the $\mathfrak{su}(8)$ fermions

By following the SWS symmetry breaking pattern in Eq. (1.13b), we tabulate the fermion representations at various stages in Tables V–VII. Along this pattern, the global $\tilde{\rm U}(1)_T$ symmetry evolves to the global $\tilde{\rm U}(1)_{B-L}$ at the EW scale according to the following sequence:

$$\mathfrak{g}_{531}\colon\ \mathcal{T}' \equiv \mathcal{T} - 5t\mathcal{X}_0, \qquad \mathfrak{g}_{431}\colon\ \mathcal{T}'' \equiv \mathcal{T}' + 8t\mathcal{X}_1,$$
$$\mathfrak{g}_{421}\colon\ \mathcal{T}''' \equiv \mathcal{T}'' - 8t\mathcal{X}_2, \qquad \mathfrak{g}_{\rm SM}\colon\ \mathcal{B}-\mathcal{L} \equiv \mathcal{T}''' + 8t\mathcal{Y}. \tag{3.1}$$

B. Decompositions of the $\mathfrak{su}(8)$ Higgs fields

We decompose the Higgs fields in Eq. (1.6) into components that can be responsible for the sequential symmetry breaking stages in Eq. (1.13b). For Higgs fields of $\overline{\mathbf{8}_{\mathbf{H},\omega}}$, they read

$$\begin{aligned}\overline{\mathbf{8}_{\mathbf{H},\omega}} &\supset \underline{\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega}} \oplus \left\langle\left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\omega}\right\rangle \\ &\supset \left\langle\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega}\right\rangle \oplus \underline{\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\omega}} \\ &\supset \left\langle\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\omega}\right\rangle.\end{aligned} \tag{3.2}$$

For Higgs fields of $\overline{\mathbf{28}_{\mathbf{H},\dot{\omega}}}$, they read

TABLE V. The $\mathfrak{su}(8)$ fermion representation of $\overline{\mathbf{8_F}^{\Omega}}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{431}, \mathfrak{g}_{421}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{431}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\rm SM}$
$\overline{\mathbf{8_F}^{\Omega}}$	$(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},+\frac{1}{3})^{\Omega}_{\mathbf{F}}$: $\mathcal{D}_R^{\Omega c}$
				$(\mathbf{1},\mathbf{1},0)^{\Omega}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega}$
		$(\mathbf{1},\mathbf{1},0)^{\Omega'}_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)^{\Omega'}_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)^{\Omega'}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega'}$
	$(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2})^{\Omega}_{\mathbf{F}}$: $\mathcal{L}_L^{\Omega} = (\mathcal{E}_L^{\Omega}, -\mathcal{N}_L^{\Omega})^T$
			$(\mathbf{1},\mathbf{1},0)^{\Omega''}_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)^{\Omega''}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega''}$

TABLE VI. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{28_F}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{431}, \mathfrak{g}_{421}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{431}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{28_F}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})_{\mathbf{F}}$: $(\mathfrak{n}_R{}^c, -\mathfrak{e}_R{}^c)^T$
			$(\mathbf{1},\mathbf{1},+1)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)_{\mathbf{F}}$: $\tau_R{}^c$
	$(\mathbf{5},\mathbf{3},+\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{3},+\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})'_{\mathbf{F}}$: $(\mathfrak{e}_R'^c, \mathfrak{n}_R'^c)^T$
			$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},0)_{\mathbf{F}}$: $\check{\mathfrak{n}}_R^c$
		$(\mathbf{4},\mathbf{3},+\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{4},\mathbf{1},-\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})_{\mathbf{F}}$: $\mathfrak{D}_L$
				$(\mathbf{1},\mathbf{1},0)'_{\mathbf{F}}$: $\check{\mathfrak{n}}_R'^c$
			$(\mathbf{4},\mathbf{2},+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})''_{\mathbf{F}}$: $(\mathfrak{e}_R''^c, \mathfrak{n}_R''^c)^T$
				$(\mathbf{3},\mathbf{2},+\frac{1}{6})_{\mathbf{F}}$: $(t_L, b_L)^T$
	$(\mathbf{10},\mathbf{1},-\frac{2}{5})_{\mathbf{F}}$	$(\mathbf{4},\mathbf{1},-\frac{1}{4})'_{\mathbf{F}}$	$(\mathbf{4},\mathbf{1},-\frac{1}{4})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'_{\mathbf{F}}$: $\mathfrak{D}'_L$
				$(\mathbf{1},\mathbf{1},0)''_{\mathbf{F}}$: $\check{\mathfrak{n}}_R''^c$
		$(\mathbf{6},\mathbf{1},-\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{6},\mathbf{1},-\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''_{\mathbf{F}}$: $\mathfrak{D}''_L$
				$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})_{\mathbf{F}}$: $t_R{}^c$

TABLE VII. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{56_F}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{431}, \mathfrak{g}_{421}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{431}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{56_F}$	$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$: $\mu_R{}^c$
	$(\mathbf{5},\bar{\mathbf{3}},+\frac{7}{15})_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})'''_{\mathbf{F}}$	$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})'''_{\mathbf{F}}$: $(\mathfrak{n}_R'''^c, -\mathfrak{e}_R'''^c)^T$
			$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$: $e_R{}^c$
		$(\mathbf{4},\bar{\mathbf{3}},+\frac{5}{12})_{\mathbf{F}}$	$(\mathbf{4},\mathbf{1},+\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},+\frac{2}{3})_{\mathbf{F}}$: $\mathfrak{U}_L$
				$(\mathbf{1},\mathbf{1},+1)'''_{\mathbf{F}}$: $\mathfrak{E}_R{}^c$
			$(\mathbf{4},\bar{\mathbf{2}},+\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3},\bar{\mathbf{2}},+\frac{1}{6})_{\mathbf{F}}$: $(\mathfrak{d}_L, -\mathfrak{u}_L)^T$
				$(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2})''''_{\mathbf{F}}$: $(\mathfrak{n}_R''''^c, -\mathfrak{e}_R''''^c)^T$
	$(\overline{\mathbf{10}},\mathbf{1},-\frac{3}{5})_{\mathbf{F}}$	$(\bar{\mathbf{4}},\mathbf{1},-\frac{3}{4})_{\mathbf{F}}$	$(\bar{\mathbf{4}},\mathbf{1},-\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$: $\mathfrak{E}_L$
				$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$: $\mathfrak{U}_R{}^c$
		$(\mathbf{6},\mathbf{1},-\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{6},\mathbf{1},-\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''_{\mathbf{F}}$: $\mathfrak{D}'''_L$
				$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})''_{\mathbf{F}}$: $u_R{}^c$
	$(\mathbf{10},\mathbf{3},-\frac{1}{15})_{\mathbf{F}}$	$(\mathbf{4},\mathbf{3},+\frac{1}{12})'_{\mathbf{F}}$	$(\mathbf{4},\mathbf{1},-\frac{1}{4})''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
				$(\mathbf{1},\mathbf{1},0)'''_{\mathbf{F}}$: $\check{\mathfrak{n}}_R'''^c$
			$(\mathbf{4},\mathbf{2},+\frac{1}{4})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})'_{\mathbf{F}}$: $(u_L, d_L)^T$
				$(\mathbf{1},\mathbf{2},+\frac{1}{2})''''_{\mathbf{F}}$: $(\mathfrak{e}_R''''^c, \mathfrak{n}_R''''^c)^T$
		$(\mathbf{6},\mathbf{3},-\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{6},\mathbf{1},-\frac{1}{2})''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
				$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'''_{\mathbf{F}}$: $c_R{}^c$
			$(\mathbf{6},\mathbf{2},0)_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})''_{\mathbf{F}}$: $(c_L, s_L)^T$
				$(\bar{\mathbf{3}},\mathbf{2},-\frac{1}{6})_{\mathbf{F}}$: $(\mathfrak{d}_R{}^c, \mathfrak{u}_R{}^c)^T$

$$\begin{aligned}\overline{\mathbf{28}}_{\mathbf{H},\dot{\omega}} \supset & \underline{\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}}} \oplus \underline{\left(\bar{\mathbf{5}},\bar{\mathbf{3}},-\frac{2}{15}\right)_{\mathbf{H},\dot{\omega}}} \oplus \underline{\left(\overline{\mathbf{10}},\mathbf{1},+\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}}} \\ \supset & \underline{(\mathbf{1},\mathbf{3},-\frac{2}{3})_{\mathbf{H},\dot{\omega}}} \oplus \left[\left\langle\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}}\right\rangle \oplus \underline{\left(\bar{\mathbf{4}},\bar{\mathbf{3}},-\frac{1}{12}\right)_{\mathbf{H},\dot{\omega}}}\right] \oplus \underline{\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}}} \\ \supset & \underline{\left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left[\underline{\left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left\langle\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}}\right\rangle \oplus \underline{\left(\bar{\mathbf{4}},\bar{\mathbf{2}},-\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}}}\right] \\ & \oplus \left\langle\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)'_{\mathbf{H},\dot{\omega}}\right\rangle. \end{aligned} \tag{3.3}$$

For Higgs field of $\mathbf{70_H}$, it reads

$$\begin{aligned}\mathbf{70_H} \supset & \left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \underline{\left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}} \\ \supset & \underline{\left(\mathbf{4},\bar{\mathbf{3}},+\frac{5}{12}\right)_{\mathbf{H}}} \supset \underline{\left(\mathbf{4},\bar{\mathbf{2}},+\frac{1}{4}\right)_{\mathbf{H}}} \supset \left\langle\left(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2}\right)_{\mathbf{H}}\right\rangle. \end{aligned} \tag{3.4}$$

C. RGEs of the SWS symmetry breaking pattern

Between the $v_{531} \le \mu \le v_U$, almost all massless Higgs fields are in the first lines of Eqs. (3.2)–(3.4), which are

$$\begin{aligned}&\left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\omega} \oplus \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega} \subset \overline{\mathbf{8}}_{\mathbf{H},\omega}, \\ &\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{5}},\bar{\mathbf{3}},-\frac{2}{15}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\overline{\mathbf{10}},\mathbf{1},+\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}} \subset \overline{\mathbf{28}}_{\mathbf{H},\dot{\omega}}, \\ &\left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}} \subset \mathbf{70_H}, \\ &\left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}} \subset \overline{\mathbf{70_H}}, \\ &\left(\mathbf{5},\mathbf{1},-\frac{1}{5}\right)^{\omega}_{\mathbf{H}} \oplus \left(\mathbf{1},\mathbf{3},+\frac{1}{3}\right)^{\omega}_{\mathbf{H}} \subset \mathbf{8_H}^{\omega}, \\ &\left(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3}\right)^{\dot{\omega}}_{\mathbf{H}} \oplus \left(\mathbf{5},\mathbf{3},+\frac{2}{15}\right)^{\dot{\omega}}_{\mathbf{H}} \oplus \left(\mathbf{10},\mathbf{1},-\frac{2}{5}\right)^{\dot{\omega}}_{\mathbf{H}} \subset \mathbf{28_H}^{\dot{\omega}}. \end{aligned} \tag{3.5}$$

All fermionic components of $\overline{\mathbf{8_F}}^{\Omega} \oplus \mathbf{28_F} \oplus \mathbf{56_F}$ remain massless after the decomposition into the $\mathfrak{g}_{531}$ IRs. Consequently, we have the $\mathfrak{g}_{531}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(5)_s}, b^{(1)}_{\mathfrak{su}(3)_W}, b^{(1)}_{\mathfrak{u}(1)_{\chi_0}}) = \left(+54, +60, +\frac{368}{5}\right), \qquad b^{(2)}_{\mathfrak{g}_{531}} = \begin{pmatrix} 6984/5 & 216 & 672/25 \\ 648 & 728 & 352/15 \\ 16128/25 & 2816/15 & 98816/1125 \end{pmatrix}. \tag{3.6}$$

Between the $v_{431} \leq \mu \leq v_{531}$, the massless Higgs fields are the following:

$$\begin{aligned}
&\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},3,\mathrm{V},\mathrm{VI}} \oplus \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},3,\mathrm{V},\mathrm{VI}} \subset \ldots \subset \overline{\mathbf{8_H}}_{,3,\mathrm{V},\mathrm{VI}},\\
&\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left[\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{6},\mathbf{1},+\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}\right]\\
\oplus&\left[\left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{4}},\bar{\mathbf{3}},-\frac{1}{12}\right)_{\mathbf{H},\dot{\omega}}\right] \subset \ldots \subset \overline{\mathbf{28_H}}_{,\dot{\omega}}, \dot{\omega} = (\dot{\mathrm{I}},\dot{2},\dot{\mathrm{V}}\mathrm{II},\dot{\mathrm{V}}\mathrm{III},\dot{\mathrm{I}}\mathrm{X}),\\
&\left(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3}\right)^{\dot{\kappa}}_{\mathbf{H}} \oplus \left[\left(\mathbf{1},\mathbf{3},+\frac{1}{3}\right)^{\dot{\kappa}}_{\mathbf{H}} \oplus \left(\mathbf{4},\mathbf{3},+\frac{1}{12}\right)^{\dot{\kappa}}_{\mathbf{H}}\right]\\
\oplus&\left[\left(\mathbf{4},\mathbf{1},-\frac{1}{4}\right)^{\dot{\kappa}}_{\mathbf{H}} \oplus \left(\mathbf{6},\mathbf{1},-\frac{1}{2}\right)^{\dot{\kappa}}_{\mathbf{H}}\right] \subset \ldots \subset \mathbf{28_H}^{\dot{\kappa}}, \dot{\kappa} = (\dot{\mathrm{V}}\mathrm{II},\dot{\mathrm{I}}\mathrm{X}),\\
&\left(\mathbf{4},\bar{\mathbf{3}},+\frac{5}{12}\right)_{\mathbf{H}} \subset \ldots \subset \mathbf{70_H}.
\end{aligned} \tag{3.7}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{431}$ IRs can be found in Ref. [24]. Consequently, we have the $\mathfrak{g}_{431}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(4)_s}, b^{(1)}_{\mathfrak{su}(3)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_1}}) = \left(+\frac{8}{3},+\frac{13}{2},+\frac{61}{3}\right), \qquad b^{(2)}_{\mathfrak{g}_{431}} = \begin{pmatrix} 3907/12 & 84 & 99/8 \\ 315/2 & 255 & 155/12 \\ 1485/8 & 310/3 & 5855/144 \end{pmatrix}. \tag{3.8}$$

The $\mathfrak{su}(5)_s \oplus \mathfrak{u}(1)_{X_0}$ gauge couplings match with the $\mathfrak{su}(4)_s \oplus \mathfrak{u}(1)_{X_1}$ gauge couplings as follows:

$$\alpha^{-1}_{5s}(v_{531}) = \alpha^{-1}_{4s}(v_{531}), \qquad \alpha^{-1}_{X_1}(v_{531}) = \frac{1}{10}\alpha^{-1}_{5s}(v_{531}) + \alpha^{-1}_{X_0}(v_{531}). \tag{3.9}$$

Between the $v_{421} \leq \mu \leq v_{431}$, the massless Higgs fields are

$$\begin{aligned}
&\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},3,\mathrm{VI}} \oplus \left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},3,\mathrm{VI}} \subset \ldots \subset \overline{\mathbf{8_H}}_{,3,\mathrm{VI}},\\
&\left[\left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}} \oplus (\mathbf{1},\mathbf{1},-1)_{\mathbf{H},\dot{\omega}}\right] \oplus \left[\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)'_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{6},\mathbf{1},+\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}\right]\\
\oplus&\left[\left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{4}},\bar{\mathbf{2}},-\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}}\right] \subset \ldots \subset \overline{\mathbf{28_H}}_{,\dot{\omega}}, \dot{\omega} = (\dot{2},\dot{\mathrm{V}}\mathrm{III},\dot{\mathrm{I}}\mathrm{X}),\\
&\left[\left(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}} \oplus (\mathbf{1},\mathbf{1},+1)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}}\right] \oplus \left[\left(\mathbf{4},\mathbf{1},-\frac{1}{4}\right)^{\dot{\mathrm{I}}\mathrm{X}'}_{\mathbf{H}} \oplus \left(\mathbf{6},\mathbf{1},-\frac{1}{2}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}}\right]\\
\oplus&\left[\left(\mathbf{1},\mathbf{2},+\frac{1}{2}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}} \oplus \left(\mathbf{4},\mathbf{1},-\frac{1}{4}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}} \oplus \left(\mathbf{4},\mathbf{2},+\frac{1}{4}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}}\right] \subset \ldots \subset \mathbf{28_H}^{\dot{\mathrm{I}}\mathrm{X}},\\
&\left(\mathbf{4},\bar{\mathbf{2}},+\frac{1}{4}\right)_{\mathbf{H}} \subset \ldots \subset \mathbf{70_H}.
\end{aligned} \tag{3.10}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{421}$ IRs can be found in Ref. [24]. Consequently, we have the $\mathfrak{g}_{421}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(4)_s}, b^{(1)}_{\mathfrak{su}(2)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_2}}) = \left(-\frac{8}{3},+\frac{17}{3},+\frac{47}{3}\right), \qquad b^{(2)}_{\mathfrak{g}_{421}} = \begin{pmatrix} 991/6 & 45/2 & 35/4 \\ 225/2 & 353/3 & 39/4 \\ 525/4 & 117/4 & 307/8 \end{pmatrix}. \tag{3.11}$$

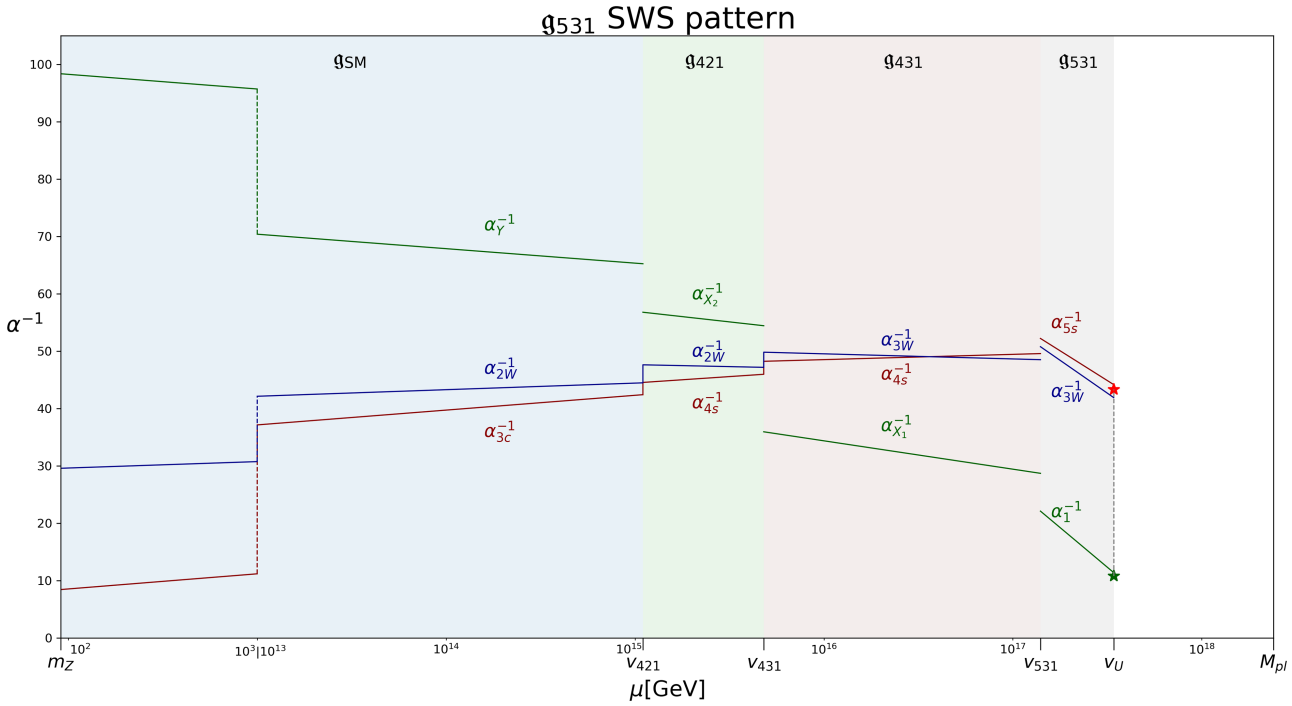


FIG. 2. The two-loop RGEs of the $\mathfrak{su}(8)$ setup according to the SWS symmetry breaking pattern. The RG evolutions within 10^3 GeV $\lesssim \mu \lesssim 10^{13}$ GeV are hidden in order to highlight the behaviors in three intermediate symmetry breaking scales given by the benchmark point in Eq. (3.14).

The $\mathfrak{su}(3)_W \oplus \mathfrak{u}(1)_{X_1}$ gauge couplings match with the $\mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_{X_2}$ gauge couplings as follows:

$$\alpha_{3W}^{-1}(v_{431}) = \alpha_{2W}^{-1}(v_{431}),$$
$$\alpha_{X_2}^{-1}(v_{431}) = \frac{1}{3}\alpha_{3W}^{-1}(v_{431}) + \alpha_{X_1}^{-1}(v_{431}). \quad (3.12)$$

Between the $v_{\rm EW} \leq \mu \leq v_{421}$, we have the same $\mathfrak{g}_{\rm SM}$ β coefficients as Eq. (2.14). The $\mathfrak{su}(4)_s \oplus \mathfrak{u}(1)_{X_2}$ gauge couplings match with the $\mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_Y$ gauge couplings as follows:

$$\alpha_{4s}^{-1}(v_{421}) = \alpha_{3c}^{-1}(v_{421}),$$
$$\alpha_{\rm Y}^{-1}(v_{421}) = \frac{1}{6}\alpha_{4s}^{-1}(v_{421}) + \alpha_{X_2}^{-1}(v_{421}). \quad (3.13)$$

Based on constructing the SM quark/lepton mass matrices to reproduce the observed hierarchical masses and the CKM mixing pattern, we find the following benchmark point:

$$v_{531} \simeq 1.4 \times 10^{17}\ \text{GeV}, \qquad v_{431} \simeq 4.8 \times 10^{15}\ \text{GeV},$$
$$v_{421} \simeq 1.1 \times 10^{15}\ \text{GeV}. \quad (3.14)$$

With the one- and two-loop β coefficients defined in Eqs. (3.6), (3.8), (3.11), and (2.14), we plot the RGEs of the $\mathfrak{su}(8)$ setup along the SWS symmetry breaking pattern in Fig. 2. The discontinuities of the $\mathfrak{u}(1)$ gauge couplings at three intermediate scales follow from their definitions in Eqs. (3.9), (3.12), (3.13), respectively. The benchmark point in Fig. 2 is marked by $\star$ and reads

$$c_{\rm HSW} \approx -0.31, \qquad v_U \approx 2.59 \times 10^{17}\ \text{GeV},$$
$$\alpha_{5c}^{-1}(v_U) \approx 46.70, \qquad \alpha_{3W}^{-1}(v_U) \approx 44.70,$$
$$\alpha_1^{-1}(v_U) \approx 10.82. \quad (3.15)$$

IV. THE WSS SYMMETRY BREAKING PATTERN

A. Decompositions of the $\mathfrak{su}(8)$ fermions

By following the WSS symmetry breaking pattern in Eq. (1.13c), we tabulate the fermion representations at various stages in Tables VIII–X. Along this pattern, the global $\tilde{\rm U}(1)_T$ symmetry evolves to the global $\tilde{\rm U}(1)_{B-L}$ at the EW scale according to the following sequence:

$$\mathfrak{g}_{531}: \mathcal{T}' \equiv \mathcal{T} + 3t\mathcal{X}_0, \qquad \mathfrak{g}_{521}: \mathcal{T}'' \equiv \mathcal{T}' - 8t\mathcal{X}_1,$$
$$\mathfrak{g}_{421}: \mathcal{T}''' \equiv \mathcal{T}'', \qquad \mathfrak{g}_{\rm SM}: \mathcal{B} - \mathcal{L} \equiv \mathcal{T}''' + 8t\mathcal{Y}. \quad (4.1)$$

B. Decompositions of the $\mathfrak{su}(8)$ Higgs fields

We decompose the Higgs fields in Eq. (1.6) into components that can be responsible for the sequential symmetry breaking stages in Eq. (1.13c). For Higgs fields of $\overline{\mathbf{8}}_{\mathbf{H},\omega}$, they read

TABLE VIII. The $\mathfrak{su}(8)$ fermion representation of $\overline{\mathbf{8}_\mathbf{F}}^{\Omega}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{521}, \mathfrak{g}_{421}, \mathfrak{g}_{\mathrm{SM}}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{521}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\mathrm{SM}}$
$\overline{\mathbf{8}_\mathbf{F}}^{\Omega}$	$(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{5}}, \mathbf{1}, +\frac{1}{5})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})^{\Omega}_{\mathbf{F}}$: $\mathcal{D}_R^{\Omega c}$
				$(\mathbf{1}, \mathbf{1}, 0)^{\Omega}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega}$
			$(\mathbf{1}, \mathbf{1}, 0)^{\Omega'}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)^{\Omega'}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega'}$
	$(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})^{\Omega}_{\mathbf{F}}$: $\mathcal{L}_L^{\Omega} = (\mathcal{E}_L^{\Omega}, -\mathcal{N}_L^{\Omega})^T$
		$(\mathbf{1}, \mathbf{1}, 0)^{\Omega''}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)^{\Omega''}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)^{\Omega''}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega''}$

TABLE IX. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{28}_\mathbf{F}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{521}, \mathfrak{g}_{421}, \mathfrak{g}_{\mathrm{SM}}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{521}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\mathrm{SM}}$
$\mathbf{28}_\mathbf{F}$	$(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}$: $(\mathfrak{n}_R{}^c, -\mathfrak{e}_R{}^c)^T$
		$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$: $\tau_R{}^c$
	$(\mathbf{5}, \mathbf{3}, +\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{5}, \mathbf{1}, -\frac{1}{5})_{\mathbf{F}}$	$(\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}$: $\mathfrak{D}_L$
				$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}$: $\check{\mathfrak{n}}_R^c$
			$(\mathbf{1}, \mathbf{1}, 0)'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)'_{\mathbf{F}}$: $\check{\mathfrak{n}}_R'^c$
		$(\mathbf{5}, \mathbf{2}, +\frac{3}{10})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})'_{\mathbf{F}}$: $(\mathfrak{e}_R'^c, \mathfrak{n}_R'^c)^T$
			$(\mathbf{4}, \mathbf{2}, +\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})''_{\mathbf{F}}$: $(\mathfrak{e}_R''{}^c, \mathfrak{n}_R''{}^c)^T$
				$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}$: $(t_L, b_L)^T$
	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})_{\mathbf{F}}$	$(\mathbf{4}, \mathbf{1}, -\frac{1}{4})'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'_{\mathbf{F}}$: $\mathfrak{D}'_L$
				$(\mathbf{1}, \mathbf{1}, 0)''_{\mathbf{F}}$: $\check{\mathfrak{n}}_R''{}^c$
			$(\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''_{\mathbf{F}}$: $\mathfrak{D}''_L$
				$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$: $t_R{}^c$

TABLE X. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{56}_\mathbf{F}$ under the $\mathfrak{g}_{531}, \mathfrak{g}_{521}, \mathfrak{g}_{421}, \mathfrak{g}_{\mathrm{SM}}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{531}$	$\mathfrak{g}_{521}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\mathrm{SM}}$
$\mathbf{56}_\mathbf{F}$	$(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}}$: $\mu_R{}^c$
	$(\mathbf{5}, \bar{\mathbf{3}}, +\frac{7}{15})_{\mathbf{F}}$	$(\mathbf{5}, \mathbf{1}, +\frac{4}{5})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}}$: $e_R{}^c$
			$(\mathbf{4}, \mathbf{1}, +\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)'''_{\mathbf{F}}$: $\mathfrak{E}_R{}^c$
				$(\mathbf{3}, \mathbf{1}, +\frac{2}{3})_{\mathbf{F}}$: $\mathfrak{U}_L$
		$(\mathbf{5}, \bar{\mathbf{2}}, +\frac{3}{10})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})'''_{\mathbf{F}}$: $(\mathfrak{n}_R'''{}^c, -\mathfrak{e}_R'''{}^c)^T$
			$(\mathbf{4}, \bar{\mathbf{2}}, +\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})''''_{\mathbf{F}}$: $(\mathfrak{n}_R''''{}^c, -\mathfrak{e}_R''''{}^c)^T$
				$(\mathbf{3}, \bar{\mathbf{2}}, +\frac{1}{6})_{\mathbf{F}}$: $(\mathfrak{d}_L, -\mathfrak{u}_L)^T$
	$(\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}}$	$(\overline{\mathbf{10}}, \mathbf{1}, -\frac{3}{5})_{\mathbf{F}}$	$(\bar{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}$: $\mathfrak{E}_L$
				$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}}$: $\mathfrak{U}_R{}^c$
			$(\mathbf{6}, \mathbf{1}, -\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}}$: $\mathfrak{D}'''_L$
				$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}}$: $u_R{}^c$
	$(\mathbf{10}, \mathbf{3}, -\frac{1}{15})_{\mathbf{F}}$	$(\mathbf{10}, \mathbf{1}, -\frac{2}{5})'_{\mathbf{F}}$	$(\mathbf{4}, \mathbf{1}, -\frac{1}{4})''_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
				$(\mathbf{1}, \mathbf{1}, 0)'''_{\mathbf{F}}$: $\check{\mathfrak{n}}_R'''{}^c$
			$(\mathbf{6}, \mathbf{1}, -\frac{1}{2})''_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''''_{\mathbf{F}}$: $\mathfrak{D}'''''_L$
				$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}}$: $c_R{}^c$
		$(\mathbf{10}, \mathbf{2}, +\frac{1}{10})_{\mathbf{F}}$	$(\mathbf{4}, \mathbf{2}, +\frac{1}{4})'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'_{\mathbf{F}}$: $(u_L, d_L)^T$
				$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})'''''_{\mathbf{F}}$: $(\mathfrak{e}_R'''''{}^c, \mathfrak{n}_R'''''{}^c)^T$
			$(\mathbf{6}, \mathbf{2}, 0)_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})''_{\mathbf{F}}$: $(c_L, s_L)^T$
				$(\bar{\mathbf{3}}, \mathbf{2}, -\frac{1}{6})_{\mathbf{F}}$: $(\mathfrak{d}_R{}^c, \mathfrak{u}_R{}^c)^T$

$$\overline{\mathbf{8}_{\mathbf{H},\omega}} \supset \left\langle \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega} \right\rangle \oplus \underline{\left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\omega}} \supset \underline{\left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},\omega}} \oplus \left\langle \left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\omega} \right\rangle$$
$$\supset \underline{\left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},\omega}} \oplus \left\langle \left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\omega} \right\rangle. \tag{4.2}$$

For Higgs fields of $\overline{\mathbf{28}_{\mathbf{H},\dot{\omega}}}$, they read

$$\overline{\mathbf{28}_{\mathbf{H},\dot{\omega}}} \supset \underline{\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}}} \oplus \underline{\left(\bar{\mathbf{5}},\bar{\mathbf{3}},-\frac{2}{15}\right)_{\mathbf{H},\dot{\omega}}} \oplus \underline{\left(\overline{\mathbf{10}},\mathbf{1},+\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}}}$$
$$\supset \underline{\left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left[\underline{\left(\bar{\mathbf{5}},\bar{\mathbf{2}},-\frac{3}{10}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left\langle \left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\dot{\omega}} \right\rangle\right] \oplus \underline{\left(\overline{\mathbf{10}},\mathbf{1},+\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}}}$$
$$\supset \underline{\left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left[\underline{\left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}} \oplus \underline{\left(\bar{\mathbf{4}},\bar{\mathbf{2}},-\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left\langle \left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}} \right\rangle\right] \oplus \left\langle \left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)'_{\mathbf{H},\dot{\omega}} \right\rangle. \tag{4.3}$$

For Higgs field of $\mathbf{70}_{\mathbf{H}}$, it reads

$$\mathbf{70}_{\mathbf{H}} \supset \left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \underline{\left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}}$$
$$\supset \underline{\left(\mathbf{10},\bar{\mathbf{2}},+\frac{1}{10}\right)_{\mathbf{H}}} \supset \underline{\left(\mathbf{4},\bar{\mathbf{2}},+\frac{1}{4}\right)_{\mathbf{H}}} \supset \left\langle \left(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2}\right)_{\mathbf{H}} \right\rangle. \tag{4.4}$$

C. RGEs of the WSS symmetry breaking pattern

Between the $v_{531} \le \mu \le v_U$, almost all massless Higgs fields are summarized as follows:

$$\left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\omega} \oplus \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega} \subset \overline{\mathbf{8}_{\mathbf{H},\omega}},$$
$$\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{5}},\bar{\mathbf{3}},-\frac{2}{15}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\overline{\mathbf{10}},\mathbf{1},+\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}} \subset \overline{\mathbf{28}_{\mathbf{H},\dot{\omega}}},$$
$$\left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}} \subset \mathbf{70}_{\mathbf{H}},$$
$$\left(\bar{\mathbf{5}},\mathbf{1},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{5},\mathbf{1},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\overline{\mathbf{10}},\mathbf{3},-\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\mathbf{10},\bar{\mathbf{3}},+\frac{4}{15}\right)_{\mathbf{H}} \subset \overline{\mathbf{70}_{\mathbf{H}}},$$
$$\left(\mathbf{5},\mathbf{1},-\frac{1}{5}\right)^{\omega}_{\mathbf{H}} \oplus \left(\mathbf{1},\mathbf{3},+\frac{1}{3}\right)^{\omega}_{\mathbf{H}} \subset \mathbf{8}_{\mathbf{H}}{}^{\omega},$$
$$\left(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3}\right)^{\dot{\omega}}_{\mathbf{H}} \oplus \left(\mathbf{5},\mathbf{3},+\frac{2}{15}\right)^{\dot{\omega}}_{\mathbf{H}} \oplus \left(\mathbf{10},\mathbf{1},-\frac{2}{5}\right)^{\dot{\omega}}_{\mathbf{H}} \subset \mathbf{28}_{\mathbf{H}}{}^{\dot{\omega}}. \tag{4.5}$$

All fermionic components of $\overline{\mathbf{8}_{\mathbf{F}}}^{\Omega} \oplus \mathbf{28}_{\mathbf{F}} \oplus \mathbf{56}_{\mathbf{F}}$ remain massless after the decomposition into the $\mathfrak{g}_{531}$ IRs. Consequently, we have the $\mathfrak{g}_{531}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(5)_s}, b^{(1)}_{\mathfrak{su}(3)_W}, b^{(1)}_{\mathfrak{u}(1)_{\chi_0}}) = \left(+54,+60,+\frac{368}{5}\right), \qquad b^{(2)}_{\mathfrak{g}_{531}} = \begin{pmatrix} 6984/5 & 216 & 672/25 \\ 648 & 728 & 352/15 \\ 16128/25 & 2816/15 & 98816/1125 \end{pmatrix}. \tag{4.6}$$

Between the $v_{521} \leq \mu \leq v_{531}$, the massless Higgs fields are

$$\begin{aligned}
&\left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},3,\mathrm{V},\mathrm{VI}} \oplus \left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},3,\mathrm{V},\mathrm{VI}} \subset \ldots \subset \overline{\mathbf{8_H}}_{,3,\mathrm{V},\mathrm{VI}},\\
&\left(\overline{\mathbf{10}},\mathbf{1},+\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}} \oplus \left[\left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}} \oplus (\mathbf{1},\mathbf{1},-1)_{\mathbf{H},\dot{\omega}}\right]\\
\oplus&\left[\left(\bar{\mathbf{5}},\mathbf{1},+\frac{1}{5}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{5}},\bar{\mathbf{2}},-\frac{3}{10}\right)_{\mathbf{H},\dot{\omega}}\right] \subset \ldots \subset \overline{\mathbf{28_H}}_{,\dot{\omega}},\ \dot{\omega} = (\dot{\mathrm{I}},\dot{2},\dot{\mathrm{V}}\mathrm{II},\dot{\mathrm{V}}\mathrm{III},\dot{\mathrm{I}}\mathrm{X}),\\
&\left(\mathbf{10},\mathbf{1},-\frac{2}{5}\right)^{\dot{\kappa}}_{\mathbf{H}} \oplus \left[\left(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2}\right)^{\dot{\kappa}}_{\mathbf{H}} \oplus (\mathbf{1},\mathbf{1},+1)^{\dot{\kappa}}_{\mathbf{H}}\right]\\
\oplus&\left[\left(\mathbf{5},\mathbf{1},-\frac{1}{5}\right)^{\dot{\kappa}}_{\mathbf{H}} \oplus \left(\mathbf{5},\mathbf{2},+\frac{3}{10}\right)^{\dot{\kappa}}_{\mathbf{H}}\right] \subset \ldots \subset \mathbf{28_H}^{\dot{\kappa}},\ \dot{\kappa} = (\dot{\mathrm{V}}\mathrm{II},\dot{\mathrm{I}}\mathrm{X}),\\
&\left(\mathbf{10},\bar{\mathbf{2}},+\frac{1}{10}\right)_{\mathbf{H}} \subset \ldots \subset \mathbf{70_H}.
\end{aligned} \tag{4.7}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{521}$ IRs can be found in Ref. [24]. Consequently, we have the $\mathfrak{g}_{521}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(5)_s}, b^{(1)}_{\mathfrak{su}(2)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_1}}) = \left(-\frac{1}{2},+\frac{67}{6},+\frac{709}{30}\right), \qquad b^{(2)}_{\mathfrak{g}_{521}} = \begin{pmatrix} 2149/5 & 75/2 & 679/50 \\ 300 & 1129/6 & 29/2 \\ 8148/25 & 87/2 & 14233/250 \end{pmatrix}. \tag{4.8}$$

The $\mathfrak{su}(3)_W \oplus \mathfrak{u}(1)_{X_0}$ gauge couplings match with the $\mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_{X_1}$ gauge couplings as follows:

$$\alpha^{-1}_{3W}(v_{531}) = \alpha^{-1}_{2W}(v_{531}), \qquad \alpha^{-1}_{X_1}(v_{531}) = \frac{1}{3}\alpha^{-1}_{3W}(v_{531}) + \alpha^{-1}_{X_0}(v_{531}). \tag{4.9}$$

Between the $v_{421} \leq \mu \leq v_{521}$, the massless Higgs fields are

$$\begin{aligned}
&\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},3,\mathrm{VI}} \oplus \left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},3,\mathrm{VI}} \subset \ldots \subset \overline{\mathbf{8_H}}_{,3,\mathrm{VI}},\\
&\left[\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)'_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{6},\mathbf{1},+\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}\right] \oplus \left[\left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}} \oplus (\mathbf{1},\mathbf{1},-1)_{\mathbf{H},\dot{\omega}}\right]\\
\oplus&\left[\left(\bar{\mathbf{4}},\mathbf{1},+\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{1},\bar{\mathbf{2}},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{4}},\bar{\mathbf{2}},-\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}}\right] \subset \ldots \subset \overline{\mathbf{28_H}}_{,\dot{\omega}},\ \dot{\omega} = (\dot{2},\dot{\mathrm{V}}\mathrm{III},\dot{\mathrm{I}}\mathrm{X}),\\
&\left[\left(\mathbf{4},\mathbf{1},-\frac{1}{4}\right)^{\dot{\mathrm{I}}\mathrm{X}'}_{\mathbf{H}} \oplus \left(\mathbf{6},\mathbf{1},-\frac{1}{2}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}}\right] \oplus \left[\left(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}} \oplus (\mathbf{1},\mathbf{1},+1)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}}\right]\\
\oplus&\left[\left(\mathbf{4},\mathbf{1},-\frac{1}{4}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}} \oplus \left(\mathbf{1},\mathbf{2},+\frac{1}{2}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}} \oplus \left(\mathbf{4},\mathbf{2},+\frac{1}{4}\right)^{\dot{\mathrm{I}}\mathrm{X}}_{\mathbf{H}}\right] \subset \ldots \subset \mathbf{28_H}^{\dot{\mathrm{I}}\mathrm{X}},\\
&\left(\mathbf{4},\bar{\mathbf{2}},+\frac{1}{4}\right)_{\mathbf{H}} \subset \ldots \subset \mathbf{70_H}.
\end{aligned} \tag{4.10}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{421}$ IRs can be found in Ref. [24]. Consequently, we have the $\mathfrak{g}_{421}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(4)_s}, b^{(1)}_{\mathfrak{su}(2)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_2}}) = \left(-\frac{8}{3},+\frac{17}{3},+\frac{47}{3}\right), \qquad b^{(2)}_{\mathfrak{g}_{421}} = \begin{pmatrix} 991/6 & 45/2 & 35/4 \\ 225/2 & 353/3 & 39/4 \\ 525/4 & 117/4 & 307/8 \end{pmatrix}. \tag{4.11}$$

The $\mathfrak{su}(5)_s \oplus \mathfrak{u}(1)_{X_1}$ gauge couplings match with the $\mathfrak{su}(4)_s \oplus \mathfrak{u}(1)_{X_2}$ gauge couplings as follows:

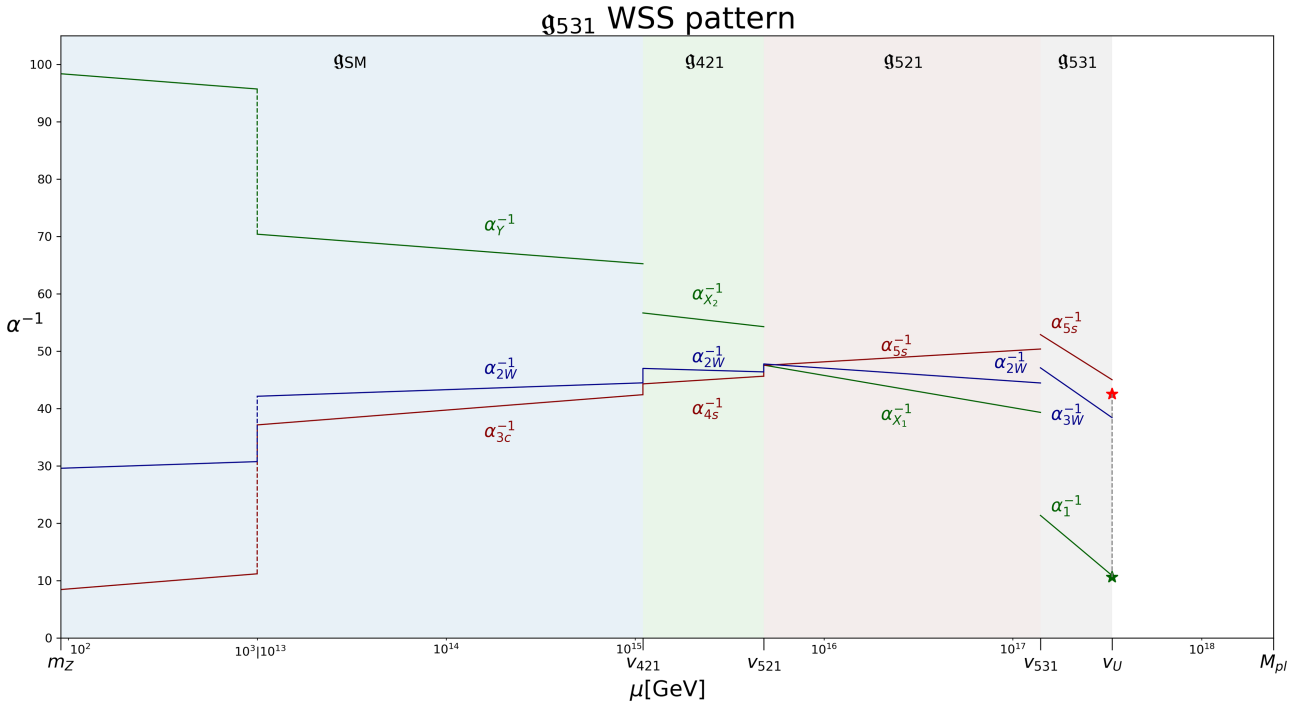


FIG. 3. The two-loop RGEs of the $\mathfrak{su}(8)$ setup according to the WSS symmetry breaking pattern. The RG evolutions within 10^3 GeV $\lesssim \mu \lesssim 10^{13}$ GeV are hidden in order to highlight the behaviors in three intermediate symmetry breaking scales given by the benchmark point in Eq. (4.14).

$$\alpha_{5s}^{-1}(v_{521}) = \alpha_{4s}^{-1}(v_{521}),$$
$$\alpha_{X_2}^{-1}(v_{521}) = \frac{1}{10}\alpha_{5s}^{-1}(v_{521}) + \alpha_{X_1}^{-1}(v_{521}). \quad (4.12)$$

Between the $v_{\rm EW} \leq \mu \leq v_{421}$, we have the same $\mathfrak{g}_{\rm SM}$ β coefficients as Eq. (2.14). The $\mathfrak{su}(4)_s \oplus \mathfrak{u}(1)_{X_2}$ gauge couplings match with the $\mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_{\rm Y}$ gauge couplings as follows:

$$\alpha_{4s}^{-1}(v_{421}) = \alpha_{3c}^{-1}(v_{421}),$$
$$\alpha_{\rm Y}^{-1}(v_{421}) = \frac{1}{6}\alpha_{4s}^{-1}(v_{421}) + \alpha_{X_2}^{-1}(v_{421}). \quad (4.13)$$

Based on constructing the SM quark/lepton mass matrices to reproduce the observed hierarchical masses and the CKM mixing pattern, we find the following benchmark point:

$$v_{531} \simeq 1.4 \times 10^{17}\ \text{GeV}, \qquad v_{521} \simeq 4.8 \times 10^{15}\ \text{GeV},$$
$$v_{421} \simeq 1.1 \times 10^{15}\ \text{GeV}. \quad (4.14)$$

With the one- and two-loop β coefficients defined in Eqs. (4.6), (4.8), (4.11), and (2.14), we plot the RGEs of the $\mathfrak{su}(8)$ setup along the WSS symmetry breaking pattern in Fig. 3. The discontinuities of the $\mathfrak{u}(1)$ gauge couplings at three intermediate scales follow from their definitions in Eqs. (4.9), (4.12), (4.13), respectively. The benchmark point in Fig. 3 is marked by $\star$ and reads

$$c_{\rm HSW} \approx -0.98, \qquad v_U \approx 2.56 \times 10^{17}\ \text{GeV},$$
$$\alpha_{5c}^{-1}(v_U) \approx 47.45, \qquad \alpha_{3W}^{-1}(v_U) \approx 41.09,$$
$$\alpha_1^{-1}(v_U) \approx 10.63. \quad (4.15)$$

V. THE WWW SYMMETRY BREAKING PATTERN

A. Decompositions of the $\mathfrak{su}(8)$ fermions

By following the WWW symmetry breaking pattern in Eq. (1.13d), we tabulate the fermion representations at various stages in Tables XI–XIII. Along this pattern, the global $\tilde{\rm U}(1)_T$ symmetry evolves to the global $\tilde{\rm U}(1)_{B-L}$ at the EW scale according to the following sequence:

$$\mathfrak{g}_{351}: \mathcal{T}' \equiv \mathcal{T} + 5\mathcal{X}_0, \qquad \mathfrak{g}_{341}: \mathcal{T}'' \equiv \mathcal{T}',$$
$$\mathfrak{g}_{331}: \mathcal{T}''' \equiv \mathcal{T}'', \qquad \mathfrak{g}_{\rm SM}: \mathcal{B} - \mathcal{L} \equiv \mathcal{T}'''. \quad (5.1)$$

B. Decompositions of the $\mathfrak{su}(8)$ Higgs fields

We decompose the Higgs fields in Eq. (1.6) into components that can be responsible for the sequential symmetry breaking stages in Eq. (1.13d). For Higgs fields of $\overline{\mathbf{8}_{\mathbf{H},\omega}}$, they read

TABLE XI. The $\mathfrak{su}(8)$ fermion representation of $\overline{\mathbf{8_F}}^{\Omega}$ under the $\mathfrak{g}_{351}, \mathfrak{g}_{341}, \mathfrak{g}_{331}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{351}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\overline{\mathbf{8_F}}^{\Omega}$	$(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})^{\Omega}_{\mathbf{F}}$: $\mathcal{D}_R^{\Omega c}$
	$(\mathbf{1}, \bar{\mathbf{5}}, -\frac{1}{5})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, -\frac{1}{2})^{\Omega}_{\mathbf{F}}$: $\mathcal{L}_L^{\Omega} = (\mathcal{E}_L^{\Omega}, -\mathcal{N}_L^{\Omega})^T$
				$(\mathbf{1}, \mathbf{1}, 0)^{\Omega'}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega'}$
			$(\mathbf{1}, \mathbf{1}, 0)^{\Omega''}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)^{\Omega''}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega''}$
		$(\mathbf{1}, \mathbf{1}, 0)^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)^{\Omega}_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)^{\Omega}_{\mathbf{F}}$: $\check{\mathcal{N}}_L^{\Omega}$

TABLE XII. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{28_F}$ under the $\mathfrak{g}_{351}, \mathfrak{g}_{341}, \mathfrak{g}_{331}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{351}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{28_F}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$: ${t_R}^c$
	$(\mathbf{3}, \mathbf{5}, -\frac{2}{15})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}$: $(t_L, b_L)^T$
				$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'_{\mathbf{F}}$: $\mathfrak{D}'_L$
			$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''_{\mathbf{F}}$: $\mathfrak{D}''_L$
		$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}$: $\mathfrak{D}_L$
	$(\mathbf{1}, \mathbf{10}, +\frac{2}{5})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{4}, +\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})''_{\mathbf{F}}$: $({\mathfrak{e}''_R}^c, {\mathfrak{n}''_R}^c)^T$
				$(\mathbf{1}, \mathbf{1}, 0)'_{\mathbf{F}}$: $\check{\mathfrak{n}}'^c_R$
			$(\mathbf{1}, \mathbf{1}, 0)''_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)''_{\mathbf{F}}$: $\check{\mathfrak{n}}''^c_R$
		$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)''_{\mathbf{F}}$: $\check{\mathfrak{n}}^c_R$
				$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}$: $({\mathfrak{e}_R}^c, {\mathfrak{n}_R}^c)^T$
			$(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$: ${\tau_R}^c$
				$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})'_{\mathbf{F}}$: $(\mathfrak{n}'^c_R, -\mathfrak{e}'^c_R)^T$

TABLE XIII. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{56_F}$ under the $\mathfrak{g}_{351}, \mathfrak{g}_{341}, \mathfrak{g}_{331}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{351}$	$\mathfrak{g}_{341}$	$\mathfrak{g}_{331}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{56_F}$	$(\bar{\mathbf{3}}, \mathbf{5}, -\frac{7}{15})_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{4}, -\frac{5}{12})_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{3}, -\frac{1}{3})_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{2}, -\frac{1}{6})_{\mathbf{F}}$: $({\mathfrak{d}_R}^c, {\mathfrak{u}_R}^c)^T$
				$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}}$: ${\mathfrak{U}_R}^c$
			$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}}$: ${c_R}^c$
		$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}}$: ${u_R}^c$
	$(\mathbf{1}, \overline{\mathbf{10}}, +\frac{3}{5})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}$: $(\mathfrak{n}'''^c_R, -\mathfrak{e}'''^c_R)^T$
				$(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}}$: ${\mu_R}^c$
			$(\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}}$: ${\mathfrak{E}_R}^c$
		$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})''_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})''''_{\mathbf{F}}$: $({\mathfrak{e}''''_R}^c, {\mathfrak{n}''''_R}^c)^T$
				$(\mathbf{1}, \mathbf{1}, 0)'''_{\mathbf{F}}$: $\check{\mathfrak{n}}'''^c_R$
			$(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})''_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})''''_{\mathbf{F}}$: $(\mathfrak{n}''''^c_R, -\mathfrak{e}''''^c_R)^T$
				$(\mathbf{1}, \mathbf{1}, +1)'''_{\mathbf{F}}$: ${e_R}^c$
	$(\mathbf{3}, \mathbf{10}, +\frac{1}{15})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{4}, -\frac{1}{12})'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
				$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'_{\mathbf{F}}$: $(u_L, d_L)^T$
			$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
		$(\mathbf{3}, \mathbf{6}, +\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}}$: $\mathfrak{D}'''_L$
				$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})''_{\mathbf{F}}$: $(c_L, s_L)^T$
			$(\mathbf{3}, \bar{\mathbf{3}}, +\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, +\frac{2}{3})_{\mathbf{F}}$: $\mathfrak{U}_L$
				$(\mathbf{3}, \bar{\mathbf{2}}, +\frac{1}{6})'''_{\mathbf{F}}$: $(\mathfrak{d}_L, -\mathfrak{u}_L)^T$
	$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}$: $\mathfrak{E}_L$

$$\overline{\mathbf{8_H}}_{,\omega} \supset \left\langle \left(\mathbf{1},\bar{\mathbf{5}},-\frac{1}{5}\right)_{\mathbf{H},\omega} \right\rangle \supset \left\langle \left(\mathbf{1},\bar{\mathbf{4}},-\frac{1}{4}\right)_{\mathbf{H},\omega} \right\rangle \supset \left\langle \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\omega} \right\rangle. \tag{5.2}$$

For Higgs fields of $\overline{\mathbf{28_H}}_{,\dot{\omega}}$, they read

$$\begin{aligned}\overline{\mathbf{28_H}}_{,\dot{\omega}} &\supset \underline{\left(\mathbf{1},\overline{\mathbf{10}},-\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}}} \supset \underline{\left(\mathbf{1},\mathbf{6},-\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left\langle \left(\mathbf{1},\bar{\mathbf{4}},-\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}} \right\rangle \\ &\supset \left[\underline{\left(\mathbf{1},\mathbf{3},-\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}}} \oplus \left\langle \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}} \right\rangle\right] \oplus \left\langle \left(\mathbf{1},\bar{\mathbf{3}},-\frac{1}{3}\right)'_{\mathbf{H},\dot{\omega}} \right\rangle.\end{aligned} \tag{5.3}$$

For Higgs field of $\mathbf{70_H}$, it reads

$$\begin{aligned}\mathbf{70_H} &\supset \left(\mathbf{1},\mathbf{5},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \underline{\left(\mathbf{1},\bar{\mathbf{5}},+\frac{4}{5}\right)_{\mathbf{H}}} \oplus \left(\mathbf{3},\overline{\mathbf{10}},+\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{3}},\mathbf{10},-\frac{4}{15}\right)_{\mathbf{H}} \\ &\supset \underline{\left(\mathbf{1},\bar{\mathbf{4}},+\frac{3}{4}\right)_{\mathbf{H}}} \supset \underline{\left(\mathbf{1},\bar{\mathbf{3}},+\frac{2}{3}\right)_{\mathbf{H}}} \supset \left\langle \left(\mathbf{1},\bar{\mathbf{2}},+\frac{1}{2}\right)_{\mathbf{H}} \right\rangle.\end{aligned} \tag{5.4}$$

C. RGEs of the WWW symmetry breaking pattern

Between the $v_{351} \le \mu \le v_U$, almost all massless Higgs fields are summarized as follows:

$$\begin{aligned}&\left(\bar{\mathbf{3}},\mathbf{1},+\frac{1}{3}\right)_{\mathbf{H},\omega} \oplus \left(\mathbf{1},\bar{\mathbf{5}},-\frac{1}{5}\right)_{\mathbf{H},\omega} \subset \overline{\mathbf{8_H}}_{,\omega}, \\ &\left(\mathbf{3},\mathbf{1},+\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{3}},\bar{\mathbf{5}},+\frac{2}{15}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{1},\overline{\mathbf{10}},-\frac{2}{5}\right)_{\mathbf{H},\dot{\omega}} \subset \overline{\mathbf{28_H}}_{,\dot{\omega}}, \\ &\left(\mathbf{1},\mathbf{5},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{1},\bar{\mathbf{5}},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{3},\overline{\mathbf{10}},+\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{3}},\mathbf{10},-\frac{4}{15}\right)_{\mathbf{H}} \subset \mathbf{70_H}, \\ &\left(\mathbf{1},\bar{\mathbf{5}},+\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\mathbf{1},\mathbf{5},-\frac{4}{5}\right)_{\mathbf{H}} \oplus \left(\bar{\mathbf{3}},\mathbf{10},-\frac{4}{15}\right)_{\mathbf{H}} \oplus \left(\mathbf{3},\overline{\mathbf{10}},+\frac{4}{15}\right)_{\mathbf{H}} \subset \overline{\mathbf{70_H}}, \\ &\left(\mathbf{3},\mathbf{1},-\frac{1}{3}\right)^{\omega}_{\mathbf{H}} \oplus \left(\mathbf{1},\mathbf{5},+\frac{1}{5}\right)^{\omega}_{\mathbf{H}} \subset \mathbf{8_H}^{\omega}, \\ &\left(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3}\right)^{\dot{\omega}}_{\mathbf{H}} \oplus \left(\mathbf{3},\mathbf{5},-\frac{2}{15}\right)^{\dot{\omega}}_{\mathbf{H}} \oplus \left(\mathbf{1},\mathbf{10},+\frac{2}{5}\right)^{\dot{\omega}}_{\mathbf{H}} \subset \mathbf{28_H}^{\dot{\omega}}.\end{aligned} \tag{5.5}$$

All fermionic components of $\overline{\mathbf{8_F}}^{\Omega} \oplus \mathbf{28_F} \oplus \mathbf{56_F}$ remain massless after the decomposition into the $\mathfrak{g}_{351}$ IRs. Consequently, we have the $\mathfrak{g}_{351}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(3)_c}, b^{(1)}_{\mathfrak{su}(5)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_0}}) = \left(+60,+54,+\frac{368}{5}\right), \qquad b^{(2)}_{\mathcal{G}_{351}} = \begin{pmatrix} 728 & 648 & 352/15 \\ 216 & 6984/5 & 672/25 \\ 2816/15 & 16128/25 & 98816/1125 \end{pmatrix}. \tag{5.6}$$

Between the $v_{341} \leq \mu \leq v_{351}$, the massless Higgs fields are

$$\begin{aligned}
&\left(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right)_{\mathbf{H},3,\mathrm{V},\mathrm{VI}} \oplus \left(\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4}\right)_{\mathbf{H},3,\mathrm{V},\mathrm{VI}} \subset \ldots \subset \overline{\mathbf{8_H}}_{,3,\mathrm{V},\mathrm{VI}}, \\
&\left(\mathbf{3}, \mathbf{1}, +\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left[\left(\bar{\mathbf{3}}, \bar{\mathbf{4}}, +\frac{1}{12}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}}\right] \\
\oplus &\left[\left(\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{1}, \mathbf{6}, -\frac{1}{2}\right)_{\mathbf{H},\dot{\omega}}\right] \subset \ldots \subset \overline{\mathbf{28_H}}_{,\dot{\omega}}, \qquad \dot{\omega} = (\dot{1}, \dot{2}, \dot{\mathrm{V}}\mathrm{II}, \dot{\mathrm{V}}\mathrm{III}, \dot{\mathrm{I}}\mathrm{X}), \\
&\left(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3}\right)_{\mathbf{H}}^{\dot{\mathrm{V}}\mathrm{II}} \oplus \left[\left(\mathbf{3}, \mathbf{4}, -\frac{1}{12}\right)_{\mathbf{H}}^{\dot{\mathrm{V}}\mathrm{II}} \oplus \left(\mathbf{3}, \mathbf{1}, -\frac{1}{3}\right)_{\mathbf{H}}^{\dot{\mathrm{V}}\mathrm{II}}\right] \\
\oplus &\left[\left(\mathbf{1}, \mathbf{4}, +\frac{1}{4}\right)_{\mathbf{H}}^{\dot{\mathrm{V}}\mathrm{II}} \oplus \left(\mathbf{1}, \mathbf{6}, +\frac{1}{2}\right)_{\mathbf{H}}^{\dot{\mathrm{V}}\mathrm{II}}\right] \subset \ldots \subset \mathbf{28_H}^{\dot{\mathrm{V}}\mathrm{II}}, \\
&\left(\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4}\right)_{\mathbf{H}} \subset \ldots \subset \mathbf{70_H}.
\end{aligned} \tag{5.7}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{341}$ IRs can be found in Ref. [24]. Consequently, we have the $\mathfrak{g}_{341}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(3)_c}, b^{(1)}_{\mathfrak{su}(4)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_1}}) = \left(+\frac{29}{6}, +\frac{4}{3}, +\frac{173}{9}\right), \qquad b^{(2)}_{\mathfrak{g}_{341}} = \begin{pmatrix} 655/3 & 255/2 & 373/36 \\ 68 & 3389/12 & 271/24 \\ 746/9 & 1355/8 & 17363/432 \end{pmatrix}. \tag{5.8}$$

The $\mathfrak{su}(5)_W \oplus \mathfrak{u}(1)_{X_0}$ gauge couplings match with the $\mathfrak{su}(4)_W \oplus \mathfrak{u}(1)_{X_1}$ gauge couplings as follows:

$$\alpha^{-1}_{5W}(v_{351}) = \alpha^{-1}_{4W}(v_{351}), \qquad \alpha^{-1}_{X_1}(v_{351}) = \frac{1}{10}\alpha^{-1}_{5W}(v_{351}) + \alpha^{-1}_{X_0}(v_{351}). \tag{5.9}$$

Between the $v_{331} \leq \mu \leq v_{341}$, the massless Higgs fields are

$$\begin{aligned}
&\left(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right)_{\mathbf{H},3,\mathrm{VI}} \oplus \left(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3}\right)_{\mathbf{H},3,\mathrm{VI}} \subset \ldots \subset \overline{\mathbf{8_H}}_{,3,\mathrm{VI}}, \\
&\left(\mathbf{3}, \mathbf{1}, +\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left[\left(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right)'_{\mathbf{H},\dot{\omega}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}}, 0)_{\mathbf{H},\dot{\omega}}\right] \\
\oplus &\left[\left(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3}\right)_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3}\right)'_{\mathbf{H},\dot{\omega}} \oplus \left(\mathbf{1}, \mathbf{3}, -\frac{2}{3}\right)_{\mathbf{H},\dot{\omega}}\right] \subset \ldots \subset \overline{\mathbf{28_H}}_{,\dot{\omega}}, \qquad \dot{\omega} = (\dot{2}, \dot{\mathrm{V}}\mathrm{III}, \dot{\mathrm{I}}\mathrm{X}), \\
&\left(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3}\right)_{\mathbf{H}} \subset \ldots \subset \mathbf{70_H}.
\end{aligned} \tag{5.10}$$

All $\mathfrak{su}(8)$ fermions that remain massless after the decomposition into the $\mathfrak{g}_{331}$ IRs can be found in Ref. [24]. Consequently, we have the $\mathfrak{g}_{331}$ β coefficients of

$$(b^{(1)}_{\mathfrak{su}(3)_c}, b^{(1)}_{\mathfrak{su}(3)_W}, b^{(1)}_{\mathfrak{u}(1)_{X_2}}) = \left(-\frac{5}{3}, -\frac{3}{2}, +\frac{122}{9}\right), \qquad b^{(2)}_{\mathfrak{g}_{331}} = \begin{pmatrix} 256/3 & 36 & 58/9 \\ 36 & 89 & 22/3 \\ 464/9 & 176/3 & 782/27 \end{pmatrix}. \tag{5.11}$$

The $\mathfrak{su}(4)_W \oplus \mathfrak{u}(1)_{X_1}$ gauge couplings match with the $\mathfrak{su}(3)_W \oplus \mathfrak{u}(1)_{X_2}$ gauge couplings as follows:

$$\alpha^{-1}_{4W}(v_{341}) = \alpha^{-1}_{3W}(v_{341}), \qquad \alpha^{-1}_{X_2}(v_{341}) = \frac{1}{6}\alpha^{-1}_{4W}(v_{341}) + \alpha^{-1}_{X_1}(v_{341}). \tag{5.12}$$

Between the $v_{\mathrm{EW}} \leq \mu \leq v_{331}$, we have the same $\mathfrak{g}_{\mathrm{SM}}$ β coefficients as Eq. (2.14). The $\mathfrak{su}(3)_W \oplus \mathfrak{u}(1)_{X_2}$ gauge couplings match with the $\mathfrak{su}(2)_W \oplus \mathfrak{u}(1)_Y$ gauge couplings as follows:

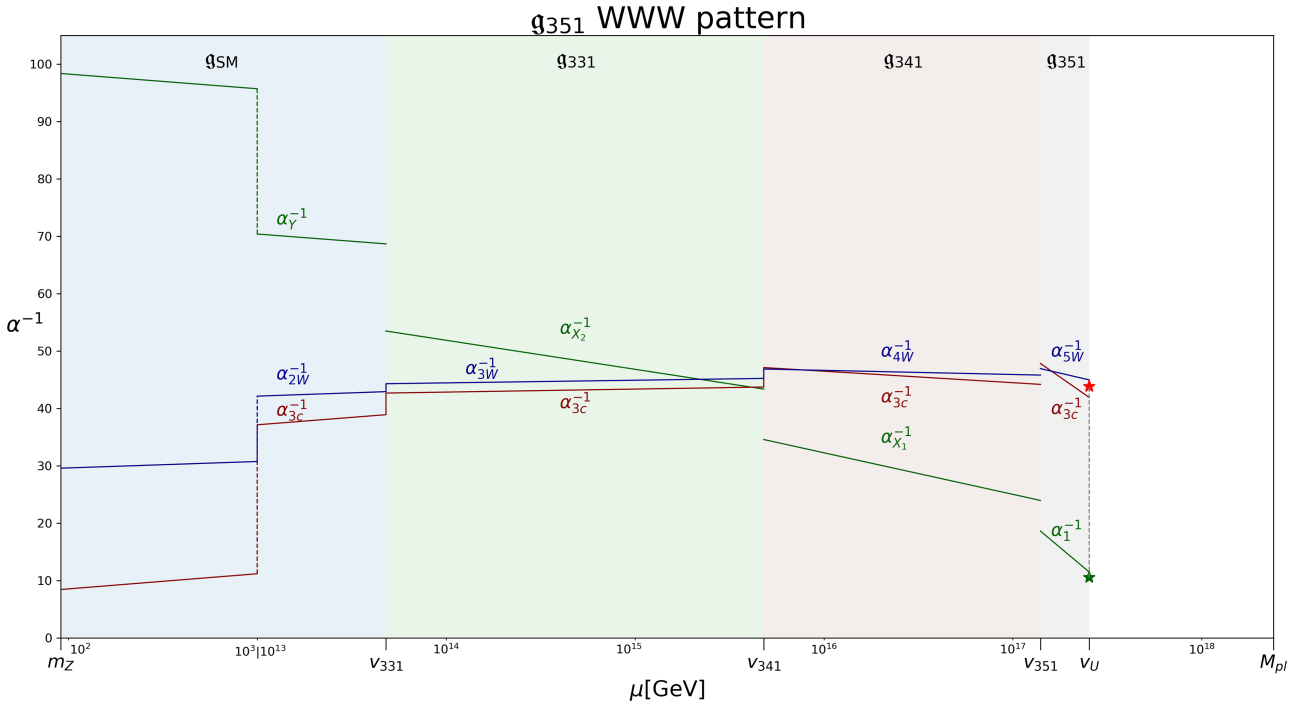


FIG. 4. The two-loop RGEs of the $\mathfrak{su}(8)$ setup according to the WWW symmetry breaking pattern. The RG evolutions within 10^3 GeV $\lesssim \mu \lesssim 10^{13}$ GeV are hidden in order to highlight the behaviors in three intermediate symmetry breaking scales given by the benchmark point in Eq. (5.14).

$$\alpha_{3W}^{-1}(v_{331}) = \alpha_{2W}^{-1}(v_{331}),$$
$$\alpha_Y^{-1}(v_{331}) = \frac{1}{3}\alpha_{3W}^{-1}(v_{331}) + \alpha_{X_2}^{-1}(v_{331}). \quad (5.13)$$

Based on constructing the SM quark/lepton mass matrices to reproduce the observed hierarchical masses and the CKM mixing pattern, we find the following benchmark point:

$$v_{351} \simeq 1.4 \times 10^{17}\ \text{GeV}, \qquad v_{341} \simeq 4.8 \times 10^{15}\ \text{GeV},$$
$$v_{331} \simeq 4.8 \times 10^{13}\ \text{GeV}. \quad (5.14)$$

With the one- and two-loop β coefficients defined in Eqs. (5.6), (5.8), (5.11), and (2.14), we plot the RGEs of the $\mathfrak{su}(8)$ setup along the WWW symmetry breaking pattern in Fig. 4. The discontinuities of the $\mathfrak{u}(1)$ gauge couplings at three intermediate scales follow from their definitions in Eqs. (5.9), (5.12), and (5.13). The benchmark point in Fig. 4 is marked by $\star$ and reads

$$c_{\text{HSW}} \approx 0.51, \qquad v_U \approx 2.72 \times 10^{17}\ \text{GeV},$$
$$\alpha_{3c}^{-1}(v_U) \approx 41.21, \qquad \alpha_{5W}^{-1}(v_U) \approx 44.73,$$
$$\alpha_1^{-1}(v_U) \approx 10.57. \quad (5.15)$$

VI. THE NO-GO PATTERN OF $\mathfrak{su}(8) \to \mathfrak{g}_{621}$

In this section, we prove that the nonmaximal symmetry breaking pattern of $\mathfrak{su}(8) \to \mathfrak{g}_{621}$ cannot be realistic, since the vectorlike quarks of $(\mathfrak{u}, \mathfrak{d})$ from the $\mathbf{56_F}$ are massless at the EWSB scale. For this purpose, we only tabulate the fermion representation of $\mathbf{56_F}$ at various stages in Table XIV by following the SSS symmetry breaking pattern in Eq. (1.11c).

Analogous the other symmetry breaking patterns analyzed before, the potential vectorlike fermion mass terms are due to the $d = 5$ operator from the superpotential in Eq. (1.6), where a Higgs field of $\mathbf{28_H}^{\dot\omega}$ is involved. The corresponding decompositions are the following:

$$\begin{aligned}\mathbf{28_H}^{\dot\omega} &\supset \underline{\left(\mathbf{15},\mathbf{1},-\frac{1}{3}\right)_{\mathbf{H}}^{\dot\omega}} \oplus \left(\mathbf{6},\mathbf{2},+\frac{1}{3}\right)_{\mathbf{H}}^{\dot\omega} \oplus (\mathbf{1},\mathbf{1},+1)_{\mathbf{H}}^{\dot\omega} \\ &\supset \underline{\left(\mathbf{10},\mathbf{1},-\frac{2}{5}\right)_{\mathbf{H}}^{\dot\omega}} \oplus \left\langle \left(\mathbf{5},\mathbf{1},-\frac{1}{5}\right)_{\mathbf{H}}^{\dot\omega} \right\rangle \\ &\supset \left(\mathbf{6},\mathbf{1},-\frac{1}{2}\right)_{\mathbf{H}}^{\dot\omega} \oplus \left\langle \left(\mathbf{4},\mathbf{1},-\frac{1}{4}\right)_{\mathbf{H}}^{\dot\omega} \right\rangle \\ &\oplus \left\langle \left(\mathbf{4},\mathbf{1},-\frac{1}{4}\right)_{\mathbf{H}}^{\prime\dot\omega} \right\rangle. \end{aligned} \quad (6.1)$$

Accordingly, we decompose the Yukawa coupling along the SSS symmetry breaking pattern between two $\mathbf{56_F}$'s due to the following $d = 5$ operator from the superpotential in Eq. (1.6):

TABLE XIV. The $\mathfrak{su}(8)$ fermion representation of $\mathbf{56_F}$ under the $\mathfrak{g}_{621}, \mathfrak{g}_{521}, \mathfrak{g}_{421}, \mathfrak{g}_{\rm SM}$ subalgebras.

$\mathfrak{su}(8)$	$\mathfrak{g}_{621}$	$\mathfrak{g}_{521}$	$\mathfrak{g}_{421}$	$\mathfrak{g}_{\rm SM}$
$\mathbf{56_F}$	$(\mathbf{20},\mathbf{1},-\frac{1}{2})_{\mathbf{F}}$	$(\overline{\mathbf{10}},\mathbf{1},-\frac{3}{5})_{\mathbf{F}}$	$(\bar{\mathbf{4}},\mathbf{1},-\frac{3}{4})_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'_{\mathbf{F}}$: ${c_R}^c$
				$(\mathbf{1},\mathbf{1},-1)_{\mathbf{F}}$: $\mathfrak{E}_L$
			$(\mathbf{6},\mathbf{1},-\frac{1}{2})'_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})''_{\mathbf{F}}$: ${u_R}^c$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''_{\mathbf{F}}$: $\mathfrak{D}'''_L$
		$(\mathbf{10},\mathbf{1},-\frac{2}{5})'_{\mathbf{F}}$	$(\mathbf{6},\mathbf{1},-\frac{1}{2})''_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{1},-\frac{2}{3})'''_{\mathbf{F}}$: ${\mathfrak{U}_R}^c$
				$(\mathbf{3},\mathbf{1},-\frac{1}{3})''''_{\mathbf{F}}$: $\mathfrak{D}''''_L$
			$(\mathbf{4},\mathbf{1},-\frac{1}{4})''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},-\frac{1}{3})'''''_{\mathbf{F}}$: $\mathfrak{D}'''''_L$
				$(\mathbf{1},\mathbf{1},0)''_{\mathbf{F}}$: $\check{\mathfrak{n}}'''^c_R$
	$(\mathbf{15},\mathbf{2},+\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{10},\mathbf{2},+\frac{1}{10})_{\mathbf{F}}$	$(\mathbf{6},\mathbf{2},0)_{\mathbf{F}}$	$(\bar{\mathbf{3}},\mathbf{2},-\frac{1}{6})_{\mathbf{F}}$: $({\mathfrak{d}_R}^c,{\mathfrak{u}_R}^c)^T$
				$(\mathbf{3},\mathbf{2},+\frac{1}{6})'_{\mathbf{F}}$: $(\mathfrak{u}_L,\mathfrak{d}_L)^T$
			$(\mathbf{4},\mathbf{2},+\frac{1}{4})'_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})''_{\mathbf{F}}$: $(c_L,s_L)^T$
				$(\mathbf{1},\mathbf{2},+\frac{1}{2})'''_{\mathbf{F}}$: $({\mathfrak{e}_R}'''^c,{\mathfrak{n}_R}'''^c)^T$
		$(\mathbf{5},\mathbf{2},+\frac{3}{10})'_{\mathbf{F}}$	$(\mathbf{4},\mathbf{2},+\frac{1}{4})''_{\mathbf{F}}$	$(\mathbf{3},\mathbf{2},+\frac{1}{6})'''_{\mathbf{F}}$: $(u_L,d_L)^T$
				$(\mathbf{1},\mathbf{2},+\frac{1}{2})''''_{\mathbf{F}}$: $({\mathfrak{e}_R}''''^c,{\mathfrak{n}_R}''''^c)^T$
			$(\mathbf{1},\mathbf{2},+\frac{1}{2})''''''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{2},+\frac{1}{2})''''''_{\mathbf{F}}$: $({\mathfrak{e}_R}''''''^c,{\mathfrak{n}_R}''''''^c)^T$
	$(\mathbf{6},\mathbf{1},+\frac{5}{6})_{\mathbf{F}}$	$(\mathbf{5},\mathbf{1},+\frac{4}{5})_{\mathbf{F}}$	$(\mathbf{4},\mathbf{1},+\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{3},\mathbf{1},+\frac{2}{3})_{\mathbf{F}}$: $\mathfrak{U}_L$
				$(\mathbf{1},\mathbf{1},+1)'_{\mathbf{F}}$: ${e_R}^c$
			$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)''_{\mathbf{F}}$: ${\mu_R}^c$
		$(\mathbf{1},\mathbf{1},+1)'''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)'''_{\mathbf{F}}$	$(\mathbf{1},\mathbf{1},+1)'''_{\mathbf{F}}$: ${\mathfrak{E}_R}^c$

$$
\begin{aligned}
&\frac{1}{M_{\rm pl}}\mathbf{56_F}\mathbf{56_F}\langle\mathbf{63_H}\rangle\mathbf{28_H}^{\dot\omega}\\
&\supset \zeta_0\left(\mathbf{15},\mathbf{2},+\frac{1}{6}\right)_{\mathbf{F}}\otimes\left(\mathbf{15},\mathbf{2},+\frac{1}{6}\right)_{\mathbf{F}}\otimes\left(\mathbf{15},\mathbf{1},-\frac{1}{3}\right)^{\dot\omega}_{\mathbf{H}}\\
&\supset \zeta_0\left(\mathbf{10},\mathbf{2},+\frac{1}{10}\right)_{\mathbf{F}}\otimes\left(\mathbf{10},\mathbf{2},+\frac{1}{10}\right)_{\mathbf{F}}\\
&\otimes\left\langle\left(\mathbf{5},\mathbf{1},-\frac{1}{5}\right)^{\dot\omega}_{\mathbf{H}}\right\rangle, \qquad (6.2)
\end{aligned}
$$

where we use the decompositions of the Higgs fields of $\mathbf{28_H}^{\dot\omega}$ in Eq. (6.1). To see why this term vanishes, let us denote relevant fields in terms of their components as $\Psi_\alpha^{[\bar a\,\bar b],i}\equiv(\mathbf{10},\mathbf{2},+\frac{1}{10})_{\mathbf{F}}$ and $\Phi^{\bar e,\dot\omega}\equiv(\mathbf{5},\mathbf{1},-\frac{1}{5})^{\dot\omega}_{\mathbf{H}}$, with $\bar a=1,\ldots,5$ being the $\mathfrak{su}(5)_s$ indices, $i=1,2$ being the $\mathfrak{su}(2)_W$ indices, and $\alpha=1,\ 2$ being the two-component Weyl spinor indices. The above gauge-invariant Yukawa coupling term can be explicitly expressed as

$$
\begin{aligned}
&\epsilon^{\alpha\beta}\epsilon_{\bar a\,\bar b\,\bar c\,\bar d\,\bar e}\epsilon_{ij}\Psi_\alpha^{[\bar a\,\bar b],i}\Psi_\beta^{[\bar c\,\bar d],j}\Phi^{\bar e,\dot\omega}\\
&=\epsilon^{\beta\alpha}\epsilon_{\bar c\,\bar d\,\bar a\,\bar b\,\bar e}\epsilon_{ji}\Psi_\beta^{[\bar c\,\bar d],j}\Psi_\alpha^{[\bar a\,\bar b],i}\Phi^{\bar e,\dot\omega}\\
&=+\epsilon^{\alpha\beta}\epsilon_{\bar a\,\bar b\,\bar c\,\bar d\,\bar e}\epsilon_{ij}\Psi_\beta^{[\bar c\,\bar d],j}\Psi_\alpha^{[\bar a\,\bar b],i}\Phi^{\bar e,\dot\omega}\\
&=-\epsilon^{\alpha\beta}\epsilon_{\bar a\,\bar b\,\bar c\,\bar d\,\bar e}\epsilon_{ij}\Psi_\alpha^{[\bar a\,\bar b],i}\Psi_\beta^{[\bar c\,\bar d],j}\Phi^{\bar e,\dot\omega}\\
&\Rightarrow\left(\mathbf{10},\mathbf{2},+\frac{1}{10}\right)_{\mathbf{F}}\otimes\left(\mathbf{10},\mathbf{2},+\frac{1}{10}\right)_{\mathbf{F}}\otimes\left(\mathbf{5},\mathbf{1},-\frac{1}{5}\right)^{\dot\omega}_{\mathbf{H}}\\
&=0, \qquad (6.3)
\end{aligned}
$$

where we have swapped all gauge and spinor indices of two Weyl fermions in the second line. By further looking for the potential mass mixing terms between the $(\mathfrak{u},\mathfrak{d})$ quarks and other flavors of quarks [24], we find that the $\mathfrak{d}$ quark remains massless since its left-handed component cannot form a gauge-invariant mixing term with any other down-type quarks.

VII. SUMMARY

In this paper, we focus on the possible nonmaximal symmetry breaking patterns of the affine Lie algebra $\widehat{\mathfrak{su}}(8)_{k_U=1}\to\mathfrak{g}_{531}/\mathfrak{g}_{351}$ allowed by the SUSY extension. Based on the analyses to the two-loop RGEs of each pattern, we found that the gauge coupling unification can be achieved below the reduced Planck scale according to the relations given in Eqs. (1.16) in the context of the affine Lie algebra $\widehat{\mathfrak{su}}(8)_{k_U=1}$. The corresponding unification scales and the unified gauge couplings have been obtained and summarized in Table XV. With the joint effects of the threshold effects and the $d=5$ gravity-induced HSW operator in Eq. (1.14), we find that the gauge coupling unification can be realized around $\mathcal{O}(10^{17})$ GeV for four distinctive patterns. By using the naive estimation [25,26] of

$$
\tau[p\to\pi^0+e^+]\approx\frac{v_U^4}{\alpha_U^2 m_p^5}, \qquad m_p=938\ \text{MeV}, \qquad (7.1)
$$

purely from the gauge sector contribution, we have found significantly enhanced proton lifetime of

TABLE XV. The benchmark points of the affine Lie algebra $\widehat{\mathfrak{su}}(8)_{k_U=1}$ through four possible nonmaximal symmetry breaking patterns with the $\mathcal{N}=1$ SUSY extension. We use the $(\Delta^s_{\rm IR}, \Delta^W_{\rm IR}, \Delta^1_{\rm IR})$ to represent the reasonable threshold effects estimated according to Eq. (A6) at each symmetry breaking stage from the UV scale to the IR scale successively.

Patterns	$\Delta^s_{\rm IR}$	$\Delta^W_{\rm IR}$	$\Delta^1_{\rm IR}$	$c_{\rm HSW}$	v_U [GeV]	$\alpha_U(v_U)$	τ_p [yrs]
SSW	90.01	180.96	−33.16	0.24	2.28×10^{17}	0.021	$\gtrsim1.76\times10^{41}$
	85.27	104.22	−52.11				
	85.30	108.96	−75.80				
SWS	99.49	85.29	−61.58	−0.31	2.59×10^{17}	0.022	$\gtrsim2.67\times10^{41}$
	85.27	99.50	−104.22				
	80.53	118.43	−52.10				
WSS	23.69	23.68	−118.42	−0.98	2.56×10^{17}	0.022	$\gtrsim2.55\times10^{41}$
	71.06	52.11	−80.54				
	71.10	94.75	−56.85				
WWW	137.39	42.63	−28.42	0.51	2.72×10^{17}	0.023	$\gtrsim2.97\times10^{41}$
	127.91	61.58	−47.37				
	142.12	52.11	−33.16				

$\tau[p\to\pi^0+e^+]\gtrsim\mathcal{O}(10^{41})$ yrs, even without considering the mixing effects. This means several ongoing experimental probes of the proton decay modes are unlikely to discover such a signal, with the typical exclusion limit of $\sim\mathcal{O}(10^{34})$ yrs from the Super-Kamiokande [27] or DUNE Collaboration [28]. In addition, we have also proved that the nonmaximal symmetry breaking pattern of $\widehat{\mathfrak{su}}(8)_{k_U=1}\to\mathfrak{g}_{621}$ is not realistic, since the vectorlike quarks of $\mathfrak{d}$ within the $\mathbf{56_F}$ remain massless.

Besides the gauge coupling unifications, the flavor identifications from Tables II through XIII according to the SM fermion mass hierarchies for each symmetry breaking pattern have been analyzed in Ref. [24]. There are several differences in the corresponding mass spectra among these patterns, as well as compared to the maximal breaking patterns analyzed previously [6,20].

(i) The vectorlike fermions of $(\mathfrak{E},\mathfrak{U},\mathfrak{u},\mathfrak{d})$ within the $\mathbf{56_F}$ may obtain their masses at different stages along different patterns, which are summarized in Ref. [24].
(ii) Within the $\mathbf{56_F}$, we found different SM flavor identifications of the first- and second-generational SM fermions along the SSW, SWS, and the WSS symmetry breaking patterns, which can be found in Tables IV, VII, and X. The third-generational SM fermions of $(t_L, b_L, {t_R}^c, {\tau_R}^c)$ always live in the $\mathbf{28_F}$, regardless of the specific symmetry breaking patterns.
(iii) Given the SM flavor identifications along the SWS and WSS symmetry breaking patterns, we have already observed the tree-level mass splitting between the strange quark and the muon. The origin of the Cabbibo angle is interpreted as the ratio of ζ_{23} between two different symmetry breaking scales along the SWS and WSS patterns.

Above all, to determine the realistic symmetry breaking patterns, one should rely on the RGEs of the corresponding SM fermion mass terms based on the current results together with the previous ones along the maximal symmetry breaking patterns in Refs. [6,10]. We defer these to future work of performing the detailed analyses of the RGEs of the SM fermion masses.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: THE GAUGE COUPLING UNIFICATION IN THE $\widehat{\mathfrak{su}}(8)$ THEORY

Generically, the two-loop RGE of a gauge coupling of α_Υ in the $\overline{\rm MS}$ scheme for the Lie algebra $\mathfrak{g}_\Upsilon$ is given by [29]

$$\frac{{\rm d}\alpha_\Upsilon^{-1}(\mu)}{{\rm d}\log\mu}=-\frac{b^{(1)}_\Upsilon}{2\pi}-\sum_{\Upsilon'}\frac{b^{(2)}_{\Upsilon\Upsilon'}}{8\pi^2\alpha^{-1}_{\Upsilon'}(\mu)}. \tag{A1}$$

The nonSUSY one- and two-loop β coefficients for the non-Abelian Lie algebras are given by

$$b_\Upsilon^{(1)} = -\frac{11}{3}C_2(\mathfrak{g}_\Upsilon) + \frac{2}{3}\sum_F T(\mathcal{R}_\Upsilon^F) + \frac{1}{3}\sum_S T(\mathcal{R}_\Upsilon^S), \tag{A2a}$$

$$\begin{aligned} b_{\Upsilon\Upsilon'}^{(2)} = &-\frac{34}{3}\delta_{\Upsilon\Upsilon'}C_2(\mathfrak{g}_\Upsilon)^2 \\ &+ \sum_F \left[2\sum_{\Upsilon'} C_2(\mathcal{R}_{\Upsilon'}^F) + \frac{10}{3}\delta_{\Upsilon\Upsilon'}C_2(\mathfrak{g}_\Upsilon)\right] T(\mathcal{R}_\Upsilon^F) \\ &+ \sum_S \left[4\sum_{\Upsilon'} C_2(\mathcal{R}_{\Upsilon'}^S) + \frac{2}{3}\delta_{\Upsilon\Upsilon'}C_2(\mathfrak{g}_\Upsilon)\right] T(\mathcal{R}_\Upsilon^S), \end{aligned} \tag{A2b}$$

with $\mathcal{R}_\Upsilon^{F/S} \in \mathfrak{g}_\Upsilon$. Here, $C_2(\mathfrak{g}_\Upsilon)$ is the quadratic Casimir of the Lie algebra $\mathfrak{g}_\Upsilon$, and $T(\mathcal{R}_\Upsilon^{F/S})$ represent the trace invariants of the chiral fermions in the IRs of $\mathcal{R}_\Upsilon^F$ and complex scalars in the IRs of $\mathcal{R}_\Upsilon^S$. For the Abelian Lie algebras with the corresponding charges denoted as $\mathcal{X}_\Upsilon^{F/S}$, the one- and two-loop β coefficients are

$$b_\Upsilon^{(1)} = \frac{2}{3}\sum_F (\mathcal{X}_\Upsilon^F)^2 + \frac{1}{3}\sum_S (\mathcal{X}_\Upsilon^S)^2, \tag{A3a}$$

$$b_{\Upsilon\Upsilon'}^{(2)} = 2\sum_{F,\Upsilon'} (\mathcal{X}_{\Upsilon'}^F)^2 \cdot (\mathcal{X}_\Upsilon^F)^2 + 4\sum_{S,\Upsilon'} (\mathcal{X}_{\Upsilon'}^S)^2 \cdot (\mathcal{X}_\Upsilon^S)^2. \tag{A3b}$$

We will also assume a set of SUSY RGEs between the GUT scale and the first symmetry breaking stage, and the corresponding one- and two-loop β coefficients [30] are given by

$$b_\Upsilon^{(1)} = -3C_2(\mathfrak{g}_\Upsilon) + \sum_\Phi T(\mathcal{R}_\Upsilon^\Phi), \tag{A4a}$$

$$\begin{aligned} b_{\Upsilon\Upsilon'}^{(2)} = &-6\delta_{\Upsilon\Upsilon'}C_2(\mathfrak{g}_\Upsilon)^2 \\ &+ \sum_\Phi \left[4\sum_{\Upsilon'} C_2(\mathcal{R}_{\Upsilon'}^\Phi) + 2\delta_{\Upsilon\Upsilon'}C_2(\mathfrak{g}_\Upsilon)\right] T(\mathcal{R}_\Upsilon^\Phi), \end{aligned} \tag{A4b}$$

for the non-Abelian $\mathfrak{g}_\Upsilon$, and

$$b_\Upsilon^{(1)} = \sum_\Phi (\mathcal{X}_\Upsilon^\Phi)^2, \tag{A5a}$$

$$b_{\Upsilon\Upsilon'}^{(2)} = 4\sum_{\Phi,\Upsilon'} (\mathcal{X}_{\Upsilon'}^\Phi)^2 \cdot (\mathcal{X}_\Upsilon^\Phi)^2, \tag{A5b}$$

for the Abelian $\mathfrak{g}_\Upsilon$. In practice, we derive the two-loop RGEs numerically by using the PyR@TE code [31].

The one-loop threshold effect was usually considered as a major source to modify the RG behavior within the field theory [12,13,32], and it can be generically expressed as

$$\begin{aligned} \alpha_{\rm UV}^{-1}(\mu) &= \alpha_{\rm IR}^{-1}(\mu) + \frac{\Delta_{\rm IR}(\mu)}{12\pi}, \\ \Delta_{\rm IR}(\mu) &= C_2(\mathcal{G}_{\rm UV}) - C_2(\mathcal{G}_{\rm IR}) - 21\sum_{V_k \in \mathcal{G}_{\rm IR}} \dim(V_k)\log\frac{M_{V_k}}{\mu} \\ &\quad + 2\sum_{\mathcal{S}_k \in \mathcal{G}_{\rm IR}} \dim(\mathcal{S}_k)\log\frac{M_{\mathcal{S}_k}}{\mu} \\ &\quad + 4\sum_{\mathcal{F}_k \in \mathcal{G}_{\rm IR}} \dim(\mathcal{F}_k)\log\frac{M_{\mathcal{F}_k}}{\mu}, \end{aligned} \tag{A6}$$

where V_k, S_k, and $\mathcal{F}_k$ represent the vector bosons, complex scalars, and Weyl fermions that are integrated out at the threshold scale μ. Their dimensions are always given in terms of the broken subgroup of $\mathcal{G}_{\rm IR}$. One may evaluate contributions due to the threshold effects in Eq. (A6) to see if they can alter the gauge coupling evolutions significantly. By taking the renormalization scale of $\mu = \bar{M}_{V_k}$ (with $\bar{M}_{V_k}$ being the average of the gauge boson masses) in Eq. (A6), we list the contributions from the massive scalars and fermions at each symmetry breaking stage. Accordingly, the reasonable one-loop threshold effects in terms of $\Delta_{\rm IR}$ at each stage can be estimated. For simplicity, we take a common mass for both massive fermions and massive scalar fields at each stage, and we estimate the largest possible threshold effects by assuming these masses within the range of $[\frac{1}{10}, 10]$ to the renormalization scale of $\mu = \bar{M}_{V_k}$.

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