

Meson scattering and tetraquarks in two-dimensional QCD

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(Received 30 April 2025; accepted 20 August 2025; published 17 September 2025)

Two-quark–two-antiquark systems with four different quark flavors are considered in the framework of two-dimensional QCD, in the light-cone gauge, at leading orders of the $1/N_c$ expansion. Introducing basis functions with color-singlet mesonic clusters, integral equations are established for the Green's functions and the related scattering amplitudes involved in the sectors of the direct and quark-exchange channels. The problem of infrared divergences is dealt with via a systematic use of an infrared regulator cutoff introduced in the gluon propagator. It is shown that the on-mass shell scattering amplitudes are free of infrared divergences up to order $1/N_c^2$. In the limit of vanishing of the infrared cutoff, they can be represented by effective four-meson contact-type interaction terms with unitarity loops, calculable in terms of the meson wave functions and propagators. The results, obtained at order $1/N_c^2$, are summed with the constraint of unitarization. The unitarized scattering amplitudes are continued in the total mass-squared variable below the two-meson thresholds, leading to a tetraquark bound state equation, which generally has one solution. Spectroscopic applications of the above results are sketched and discussed.

DOI: [10.1103/fmk2-jj72](https://doi.org/10.1103/fmk2-jj72)

I. INTRODUCTION

The experimental discovery, during the last two decades, of many candidates of “exotic hadrons” [1–13], which are assumed to contain more valence quarks than the ordinary mesons and baryons [14,15], has led to intense investigations about the analysis of their internal structures [16–41]. Contrary to ordinary hadrons, exotic hadrons have the possibility of being internally decomposed into ordinary hadronic clusters, which could then screen the action of the confining forces between quarks and could deform the expected compact structure into a loosely bound molecular-type structure [42–47].

The latter phenomenon complicates the task of analyzing the properties of the exotic hadrons. The main reason of this comes from our incomplete control of the confining forces. QCD is a nonperturbative theory in the infrared region, or, equivalently, at large distances, where it becomes confining. The tools that we dispose of are either empirical, or approximate, or at best numerical, with severe limitations. Conventional methods that hinge on the use of additive confining potentials, or additive empirical gluon propagators for the study of the

many-body bound state problem, have the drawback of producing residual long-range van der Waals forces that are not present on experimental grounds [48–53] and that, therefore, might alter some of the qualitative features of the problem under study.

From this point of view, two-dimensional QCD in the large- N_c limit, where N_c is the parameter of the color-gauge group $SU(N_c)$, which was first introduced and studied by 't Hooft [54,55], has been revealed as an efficient tool for probing many of the problems related with the confinement of quarks. In two dimensions, confinement is a basic property of the theory with controllable infrared behavior. The large- N_c limit ensures the damping of all inelasticity and pair creation effects and reduces the class of dominant Feynman diagrams to that of “planar” ones for color-singlet irreducible systems [54,56–59]. In noncovariant gauges [60] in two dimensions, trilinear and quadrilinear gluon couplings disappear and the single gluon propagator becomes the only dynamical object of the theory; in particular, in the many-body case, the interactions become free of the residual long-range van der Waals forces. Two-dimensional QCD at large N_c has later been considered by many authors in its various aspects related with confinement [61–87].

The aim of this paper is to investigate, in the large- N_c limit of two-dimensional QCD, the properties of the theory concerning the four-body sector, generated by two quark and two antiquark fields and the possible emergence of tetraquark bound states or resonances. As is known from

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the works of Witten and Coleman [56–58], because of cluster factorization, multi-quark states could not exist in the large- N_c limit. This severe theoretical prediction was, however, amended at a later stage by Weinberg [88], who argued that even if multi-quark states would not appear in leading-order terms of N_c , they might still produce poles in Green's functions and scattering amplitudes at non-leading orders of N_c . This would have also consequences for the corresponding couplings and decay widths of possibly existing such states. Investigations along this line of approach have been undertaken by several authors [37,89–95].

Our analysis is done in the light-cone gauge, following 't Hooft's approach [54,55,61,62]. Concentrating on the case of four different quark flavors, we derive the four-particle Green's function equations in the color-singlet sector and the corresponding integral equations satisfied by the scattering amplitudes. These are classified as belonging to the “direct” or to the “recombination” channels, according to whether the final and initial states have the same color combinations with respect to the quark flavors or whether they have undergone quark exchanges.

One of the main challenges in dealing with confining theories is the demonstration that the theory at hand is infrared finite. Two-dimensional QCD offers the possibility of explicitly checking that issue. To this aim, we introduce in the gluon propagator an infrared regulator cutoff parameter, playing the role of a mass term, which, at the end of the calculations is taken to zero.

The infrared finiteness of the theory for quark-anti-quark color-singlet systems has been shown in [54]. A similar property has been shown at order $1/N_c$ in [61], concerning meson-meson scattering amplitudes. In the present work, with four different quark flavors, the recombination-type meson scattering amplitudes are of order $1/N_c + O(1/N_c^3)$ and the derivation of their finiteness is established in a similar way as in [61]. We show that, in the infrared limit, the scattering amplitudes reduce to finite effective four-meson contact terms, whose expressions are calculable in terms of overlapping integrals involving the meson wave functions. The direct-type meson scattering amplitudes are of order $1/N_c^2 + O(1/N_c^4)$. They are plagued by several dozens of infrared diverging terms, involving up to four gluon propagators exchanged between different meson clusters. These can be grouped into several categories in which the cancelation properties of the divergences are more easily shown. We demonstrate that each category is globally free of divergences and the scattering amplitudes reduce to the unitarity corrective terms arising from the recombination contact term and a new four-meson contact term.

While the demonstration of the finiteness of the scattering amplitudes beyond the order $1/N_c^2$ seems out of reach, we conjecture that the natural emergence of the unitarity correction in the direct channel is the sign of a global unitarity property satisfied by the on-mass shell scattering amplitudes. This leads us to complete, by a unitarization operation, the results obtained so far and to provide finite unitarized scattering amplitudes.

Finally, the continuation of the scattering amplitudes in the total mass-squared variable below the two-meson thresholds allows us to obtain a bound state equation for possibly existing tetraquark states. It is shown that the bound state equation has generally one solution. The properties of the latter are sensitive, however, to the values of the effective contact terms and a detailed study of their properties is left for a separate work.

The plan of the paper is the following. In Sec. II, we introduce definitions and review the properties of the quark-antiquark systems. In Sec. III, we consider diquark systems and display the differences that emerge, in the infrared region, with respect to the color-singlet quark-antiquark case. In Sec. IV, we consider two-quark–two-antiquark systems and derive the integral equations of the corresponding Green's functions and the related meson scattering amplitudes. Section V is devoted to the demonstration of the finiteness of the scattering amplitudes up to order $1/N_c^2$. The problem of the unitarization of the finite scattering amplitudes is considered in Sec. VI. Section VII provides the bound state equation resulting from the continuation of the scattering amplitudes below the two-meson thresholds. In Sec. VIII, we summarize our results and mention open questions to be resolved. Two Appendixes provide details about the regularization scheme and the technical operations related to the cancelation of infrared divergences.

II. QUARK-ANTIQUARK SYSTEMS

We briefly sketch in this section the main results obtained so far for quark-antiquark systems in the large- N_c limit [55,61,62], the quark fields belonging to the fundamental representation of the color-gauge group $SU(N_c)$. They will be useful for the treatment of the more general case of two-quark–two-antiquark systems.

A. Definitions and conventions

The definitions and color dependences of propagators and vertices are presented in Fig. 1. Small Latin letters are used for color indices in the fundamental representation, while capital Latin letters refer to the adjoint representation. Greek letters are related to the Dirac spinorial indices.

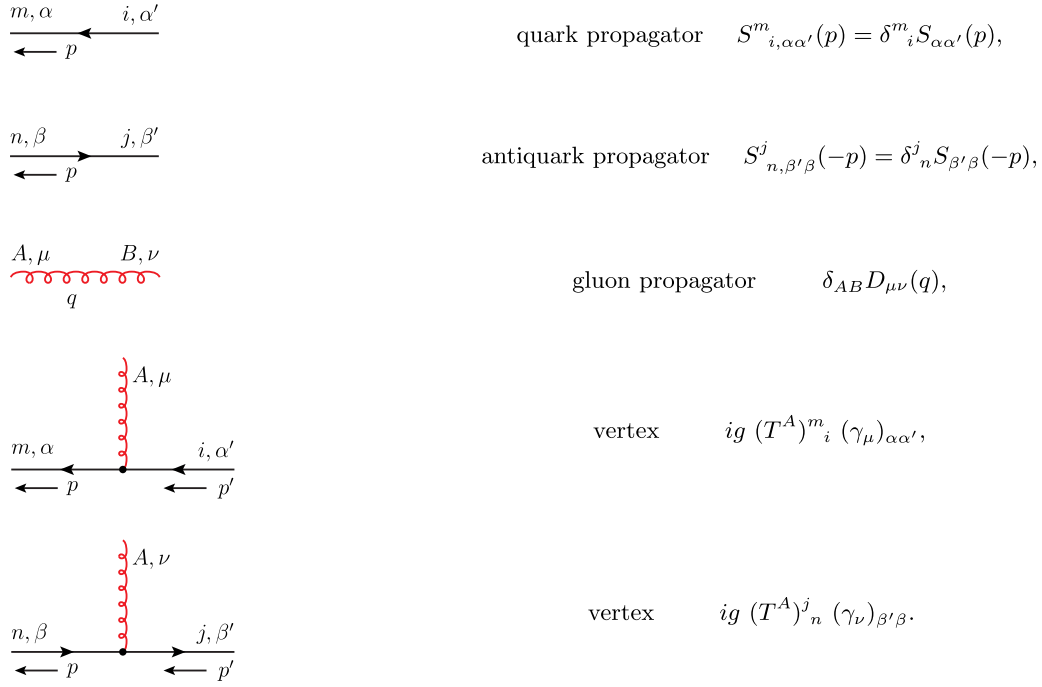


FIG. 1. Color dependences of the quark, antiquark, and gluon propagators, and of the corresponding vertices. Small Latin letters refer to the color indices of the fundamental representation and capital Latin letters to the adjoint representation. Greek letters are related to the Dirac spinorial indices. T^A are the color-gauge-group generators.

T^A are the color-gauge-group generators in the fundamental representation.

The color-gauge-group generators T^A satisfy, among others, the following properties:

$$(T^A)^m_i (T^A)^j_n = \frac{1}{2} \left(\delta^m_n \delta^j_i - \frac{1}{N_c} \delta^m_i \delta^j_n \right),$$

$$\text{tr}(T^A T^B) = \frac{1}{2} \delta_{AB}. \quad (2.1)$$

The string tension is defined through the relation¹

$$\sigma \equiv \frac{g^2 N_c}{4} \left(1 - \frac{1}{N_c^2} \right). \quad (2.2)$$

The light-cone components and matrices are defined as follows:

¹Notice that a difference of a factor $\sqrt{2}$ exists in our definition of the coupling constant g and that of Refs. [55,61,62]; this is due to the difference in the definitions of the gluon fields; in Refs. [55,61,62] the gluon fields are defined with the double-index notation, while here, we use the more conventional single-index notation. The relationship between the two definitions is given by the formula $(T^B A^B_{\mu})^a_b = A^a_{b,\mu}/\sqrt{2}$.

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1), \quad x_\pm = x^\mp,$$

$$p_\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_1), \quad A_\pm = \frac{1}{\sqrt{2}}(A_0 \pm A_1),$$

$$\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^1), \quad \gamma_\pm = \gamma^\mp,$$

$$(\gamma_\pm)^2 = 0, \quad (\gamma_+ \gamma_- + \gamma_- \gamma_+) = 2. \quad (2.3)$$

One may also introduce two unit lightlike vectors n and $\bar{n}$, such that

$$n^2 = \bar{n}^2 = 0, \quad n \cdot \bar{n} = 1, \quad (2.4)$$

and for any vectors V and W one has the decompositions

$$V_\mu = V_- \bar{n}_\mu + V_+ n_\mu, \quad W_\mu = W_- \bar{n}_\mu + W_+ n_\mu,$$

$$n \cdot V = V_-, \quad \bar{n} \cdot V = V_+, \quad V \cdot W = V_- W_+ + V_+ W_-. \quad (2.5)$$

The light-cone gauge is specified with the gauge fixing condition

$$n \cdot A^B = A^B_- = 0, \quad B = 1, 2, \dots, N_c^2 - 1. \quad (2.6)$$

The vertex part $A^\mu \gamma_\mu$ (Fig. 1) then reduces to $A_+ \gamma_-$. In this gauge, as well as in noncovariant gauges, ghost fields are absent [60]. Furthermore, in two dimensions, in the

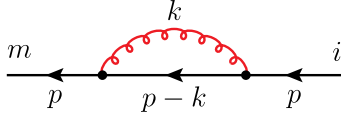


FIG. 2. Quark self-energy.

light-cone gauge, cubic and quartic gluon couplings are also absent. The gluon interacts only with quarks.

B. Gluon and quark propagators

The gluon propagator, in the light-cone gauge (in any dimension), is [60]

$$D_{\mu\nu}^{AB}(q) = \frac{-i\delta_{AB}}{(q^2 + i\epsilon)} \left(g_{\mu\nu} - \frac{(n_\mu q_\nu + n_\nu q_\mu)}{n \cdot q} \right) = \delta_{AB} D_{\mu\nu}(q). \quad (2.7)$$

In two dimensions, it becomes

$$D_{\mu\nu}(q) = n_\mu n_\nu D_{++}(q), \quad D_{++}(q) = \frac{i}{q_-^2}. \quad (2.8)$$

The full quark propagator S is calculated with the inclusion of the self-energy Σ , which, being a planar diagram, contributes at leading order in N_c . Defining

$$S_i^m = \delta_i^m S, \quad \Sigma_i^m = \delta_i^m \Sigma, \quad (2.9)$$

one has

$$S(p) = \frac{i}{\gamma \cdot p - m - \Sigma + i\epsilon}. \quad (2.10)$$

Σ is calculated from the Feynman diagram of Fig. 2, with the full quark propagator in it. It is the only leading diagram at large N_c .

Decomposing S and Σ along the γ matrices,

$$S = S_- \gamma_+ + S_+ \gamma_- + S_0 1, \quad \Sigma = \Sigma_- \gamma_+ + \Sigma_+ \gamma_- + \Sigma_0 1, \quad (2.11)$$

one finds that Σ is proportional to the matrix γ_- and hence $\Sigma_- = \Sigma_0 = 0$,

$$\Sigma(p) = \gamma_- \Sigma_+(p). \quad (2.12)$$

Furthermore, the loop integration with respect to k_+ removes the possible dependence of Σ upon p_+ ; one finds

$$\Sigma_+(p_-) = \sigma \int \frac{dk_-}{2\pi} \frac{\varepsilon(p_- + k_-)}{k_-^2}, \quad (2.13)$$

where $\varepsilon(p_-) = \text{sgn}(p_-)$.

The last integral is infrared divergent. 't Hooft [55] has introduced an infrared cutoff $\lambda > 0$ to regulate the k_- integral. We shall rather introduce a small mass term λ in the gluon propagator to effect the regulation. It turns out that this procedure brings a better control of the infrared limit when we are in the presence of multiple integrals involving several gluon propagators in convolution with quark propagators (cf. Appendix A). Generically, we make the substitution

$$\frac{1}{k_-^2} \longrightarrow \frac{1}{k_-^2 + \lambda^2}. \quad (2.14)$$

We shall, however, replace λ in the following by another quantity, Λ , which is more suitably incorporated in the various formulas,

$$\Lambda \equiv \frac{\sigma}{\lambda}. \quad (2.15)$$

(The limit $\lambda \rightarrow 0$ is now transcribed into the limit $\Lambda \rightarrow \infty$.) One finally finds

$$\Sigma_+(p_-) = \frac{1}{2} \varepsilon(p_-) \left(\Lambda - \frac{2\sigma}{\pi|p_-|} \right). \quad (2.16)$$

The quark propagator is then

$$S(p) = \frac{i \left\{ \gamma_+ p_- + \gamma_- \left[p_+ - \varepsilon(p_-) \left(\frac{\Lambda}{2} - \frac{\sigma}{\pi|p_-|} \right) \right] + m \right\}}{2p_+ p_- - (|p_-| \Lambda - 2\sigma/\pi) - m^2 + i\epsilon}. \quad (2.17)$$

In the dynamical equations that follow, because of the particular structure of the vertices in the light-cone gauge, where only the γ_- matrix appears, it is the component S_- of S that plays a fundamental role. We display more explicitly its expression,

$$S_-(p) = \frac{i p_-}{[2p_+ p_- - |p_-| \Lambda - m'^2 + i\epsilon]}, \quad (2.18)$$

where we have defined

$$m'^2 \equiv m^2 - \frac{2\sigma}{\pi}. \quad (2.19)$$

The quantity $|p_-| \Lambda$ is a scalar and contributes to the mass renormalization. When $\Lambda \rightarrow \infty$, the renormalized mass tends to ∞ , signifying that the quark, which corresponds to a colored field, becomes unobservable in that limit. This has occurred without making any primary hypothesis about the physical outcome of the theory. It is then crucial to check whether the physically observable quantities, which are expected to be generated by color-singlet operators, do remain finite in the above limit. Here appears one of the

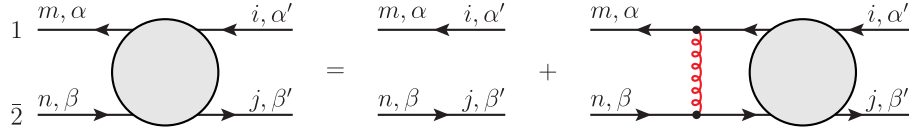


FIG. 3. Integral equation of the quark-antiquark Green's function.

main advantages of using the infrared cutoff regularization procedure, which provides us with an explicit criterion for the distinction of observable quantities. In other regularization schemes, such as those using the principal value definition, one must supplement them with the requirement of gauge invariance of the related quantities [62].

C. Bound-state equation

We consider two quark fields with different flavors and free masses m_1 and m_2 , and specialize to the transition

amplitude where the ingoing and outgoing particles are made of the quark-antiquark system. The quark and the antiquark are referred to by the indices 1 and $\bar{2}$, respectively.

We designate by $G_{1\bar{2}}$ the corresponding two-body Green's function, by $G_{1\bar{2},0}$ its free part and by $K_{1\bar{2}}$ the kernel of the corresponding integral equation; at leading order of large N_c , the latter reduces to the one-gluon exchange term.

The integral equation has the form²

$$G_{1\bar{2};\alpha\beta,\beta'\alpha'}^{mn,ji}(r; p, p') = G_{1\bar{2},0;\alpha\beta,\beta'\alpha'}^{mn,ji}(r; p)(2\pi)^2\delta^2(p - p') + G_{1\bar{2},0;\alpha\beta,\xi\sigma}^{mn,sr}(r; p) \int \frac{d^2 p''}{(2\pi)^2} K_{1\bar{2};\sigma\xi,\zeta\eta}^{rs,ut}(p'' - p) G_{1\bar{2};\eta\zeta,\beta'\alpha'}^{tu,ji}(r; p'', p'), \quad (2.20)$$

where

$$r = \text{total momentum}, \quad p = \text{outgoing quark momentum}, \quad p' = \text{ingoing quark momentum}, \quad (2.21)$$

$$\begin{aligned} G_{1\bar{2},0;\alpha\beta,\beta'\alpha'}^{mn,ji}(r; p) &= S_{1;\alpha\alpha'}^{mi}(p) S_{2;\beta'\beta}^{jn}(p - r) \\ &= \delta_{mi} S_{1;\alpha\alpha'}(p) \delta_{jn} S_{2;\beta'\beta}(p - r) \\ &= \delta_{mi} \delta_{jn} G_{1\bar{2},0;\alpha\beta,\beta'\alpha'}(r; p), \end{aligned} \quad (2.22)$$

S_1 and S_2 being the full quark propagators; it is represented in Fig. 3.

The expression of $K_{1\bar{2}}$ is

$$\begin{aligned} K_{1\bar{2};\sigma\xi,\zeta\eta}^{rs,ut}(p'' - p) &= -g^2 (T^A)_{rt} (T^A)_{us} (\gamma^\mu)_{\sigma\eta} (\gamma^\nu)_{\zeta\xi} D_{\mu\nu}(p'' - p) \\ &= -\frac{g^2}{2} \left(\delta_{rs} \delta_{ut} - \frac{1}{N_c} \delta_{rt} \delta_{us} \right) (\gamma_-)_{\sigma\eta} (\gamma_-)_{\zeta\xi} D_{++}(p'' - p). \end{aligned} \quad (2.23)$$

The Green's functions G and G_0 can be decomposed along the basis of the tensor products of the two sets of independent γ matrices (γ_+ , γ_- , $\gamma_- \gamma_+$, $\gamma_+ \gamma_-$) related to the quark 1 and the antiquark $\bar{2}$, respectively. Because of the particular structure of the kernel K , which is proportional to two γ_- matrices, each acting on the quark or the antiquark lines, and the fact that $\gamma_\pm^2 = 0$, it is only the component $G_{-,-}$ of G accompanying the matrices $\gamma_+ \times \gamma_+$ that contributes to the dynamical part of the equation, the other components being

determined from it by kinematic relations. Similarly, it is the components $S_{1,-}$ and $S_{2,-}$ of S_1 and S_2 , respectively, that participate in the above equation. It is therefore sufficient to project the integral equation on the above sector, simplifying it from redundant indices and concentrating on the color indices. On the other hand, the quark propagators that surround a gluon propagator at its vertices produce by contraction of the γ_+ matrices a factor 2 at each vertex [55], and hence a factor 4 for the gluon propagator contribution in the sector of the $\gamma_+ \times \gamma_+$ matrices. To simplify notations, one can remove from the various quantities the corresponding γ -matrix indices and adopt the following redefinitions:

$$\begin{aligned} S_{a,-} &\rightarrow S_a, \quad a = 1, 2, \quad D_{++} \rightarrow D = \frac{i}{q_-^2 + \lambda^2}, \\ G_{-,-} &\rightarrow G, \quad S_{1,-} S_{2,-} = G_{0,-,-} \rightarrow G_0. \end{aligned} \quad (2.24)$$

Removing the system indices $1\bar{2}$ and the integration symbols, the integral equation takes the following compact form:

$$G_{n,i}^{m,j} = G_{0,n,i}^{m,j} + G_{0,n,r}^{m,s} 4K_{s,t}^{r,u} G_{u,i}^{t,j}, \quad (2.25)$$

²In some formulas, when too many indices are present, we write the color indices of the quarks and of the antiquarks on the same line.

with

$$\begin{aligned}
G_0^{m, j}_{n, i} &= \delta^m_i \delta^j_n G_0, \\
K^{r, u}_{s, t} &= \left(-\frac{g^2}{2} D \right) \left(\delta^r_s \delta^u_t - \frac{1}{N_c} \delta^r_t \delta^u_s \right), \\
(G_0 K)^{m, u}_{n, t} &= G_0 K \frac{1}{N_c} \left[\delta^m_n \delta^u_t - \frac{N_c}{(N_c^2 - 1)} \right. \\
&\quad \times \left. \left(\delta^m_t \delta^u_n - \frac{1}{N_c} \delta^m_n \delta^u_t \right) \right], \\
K &= (-2\sigma D). \tag{2.26}
\end{aligned}$$

Since the quark and antiquark fields belong to the fundamental and antifundamental representations of the color-gauge group $SU(N_c)$, the quark-antiquark system should belong either to the singlet or to the adjoint representation. Global color being preserved by the interaction, this feature is also exhibited on the decomposition of the quantity $(G_0 K)$ of Eq. (2.26) and transmitted to the intermediate states that saturate the Green's function. The latter can also be decomposed along the above representations. We therefore introduce a similar decomposition of G ,

$$G^{m, j}_{n, i} = \delta^m_n \delta^j_i G_s + \left(\delta^m_i \delta^j_n - \frac{1}{N_c} \delta^m_n \delta^j_i \right) G_{\text{adj}}, \tag{2.27}$$

where G_s and G_{adj} represent the parts of the Green's function that contain intermediate states belonging to the singlet and adjoint representations, respectively. The integral equation (2.25) then splits into two decoupled independent equations,

$$G_s = \frac{1}{N_c} G_0 + G_0 (4K) G_s, \tag{2.28}$$

$$G_{\text{adj}} = G_0 - \frac{1}{(N_c^2 - 1)} G_0 (4K) G_{\text{adj}}. \tag{2.29}$$

It turns out, as will be shown in the following, that the kernel part, $G_0(4K)$, with a positive sign corresponds to an attractive interaction and may therefore lead to the existence of bound states. This is the case of G_s . On the other hand, the quark-antiquark system in its adjoint representation, with a negative sign in front of $G_0(4K)$, is submitted to an internal repulsive interaction and could not lead to the appearance of bound states. To be observable, unbound states should display properties close to those of scattering states. However, we have seen in Sec. II B, under the effect of the infrared singularity, individual quark states escape observational criteria. Also note that the kernel in the adjoint representation is damped by a factor of $1/(N_c^2 - 1)$. In the large- N_c limit, it tends to zero. We shall concentrate in the following on the color-singlet sector of the quark-antiquark system.

In Eq. (2.28), the factor $1/N_c$ in front of G_0 fixes the normalization constant of the possibly existing bound state wave functions. Otherwise, it does not have any influence on the dynamics of the system. This is easily seen by redefining G_s with a similar coefficient (dropping the subscript s from the redefined function),

$$G_s = G/N_c \Rightarrow G = G_0 + G_0(4K)G. \tag{2.30}$$

Assuming the existence of bound states with wave functions ϕ , their equation is determined by the homogeneous part of Eq. (2.30), which we display in explicit form,

$$\begin{aligned}
\phi(r, p) &= -\frac{p_-}{[p^2 - |p_-| \Lambda - m_1^2 + i\epsilon]} \\
&\quad \times \frac{(r_- - p_-)}{[(r-p)^2 - |r_- - p_-| \Lambda - m_2^2 + i\epsilon]} \\
&\quad \times (8\sigma) \int \frac{d^2 p''}{(2\pi)^2} \frac{i}{(p'' - p_-)^2 + \lambda^2} \phi(r, p''), \tag{2.31}
\end{aligned}$$

Λ and m'^2 being defined in Eqs. (2.15) and (2.19). The Dirac matrix structure of the total wave function can be reconstructed by relating it, together with its adjoint, to the tensor basis of the γ matrices of the outgoing and ingoing sectors, respectively. To this end, it is sufficient to use the relationship $(\gamma_+)_{\alpha\alpha'} (\gamma_+)_{\beta\beta'} = (\gamma_+)_{\alpha\beta} (\gamma_+)_{\beta'\alpha'}$. In this basis, ϕ corresponds to the component ϕ_- .

The interaction being lightlike instantaneous, the integration with respect to p''_+ can be done, removing p_+ from the integral. Defining the integrated wave function as

$$\varphi(r, p_-) = \int \frac{dp_+}{2\pi} \phi(r, p), \tag{2.32}$$

which represents the lightlike instantaneous limit of ϕ , one can integrate Eq. (2.31) with respect to p_+ on both sides. It concerns on the right-hand side the external propagators and gives a nonzero result only if the conditions $p_- > 0$ and $p_- < r_-$ are satisfied (r_- and r_+ being both positive), which are then reflected in the wave function of the left-hand side [cf. Eq. (B2) of Appendix B]. We hence assume that the wave function $\varphi(r, p_-)$ satisfies those conditions,

$$\varphi(r, p_-) = \varphi(r, p_-) \theta(p_-(r_- - p_-)). \tag{2.33}$$

One obtains

$$\begin{aligned}
&\left[r_+ - \Lambda - \frac{m_1^2}{2p_-} - \frac{m_2^2}{2(r_- - p_-)} \right] \varphi(r, p_-) \\
&= -2\sigma \int_0^{r_-} \frac{dp''_-}{(2\pi)} \frac{1}{(p''_- - p_-)^2 + \lambda^2} \varphi(r, p''_-). \tag{2.34}
\end{aligned}$$

The integral in Eq. (2.34) is divergent in the limit $\lambda \rightarrow 0$. The details of the separation of the divergent part are presented in Eqs. (A1)–(A4) of Appendix A. One obtains for the right-hand side of Eq. (2.34) the expression

$$-2\sigma \int_0^{r_-} \frac{dp_-''}{(2\pi)(p_-'' - p_-)^2} (\varphi(r, p_-'') - \varphi(r, p_-)) - \left(\Lambda - \frac{2\sigma}{\pi} \left(\frac{1}{2p_-} + \frac{1}{2(r_- - p_-)} \right) \right) \varphi(r, p_-). \quad (2.35)$$

The divergent part cancels the factor Λ of the left-hand side, while the last two finite contributions cancel the finite renormalization parts contained in m_1^2 and m_2^2 , bringing them to their initial values m_1^2 and m_2^2 . The equation becomes

$$\left[r_+ - \frac{m_1^2}{2p_-} - \frac{m_2^2}{2(r_- - p_-)} \right] \varphi(r, p_-) = -2\sigma \int_0^{r_-} \frac{dp_-''}{(2\pi)(p_-'' - p_-)^2} (\varphi(r, p_-'') - \varphi(r, p_-)). \quad (2.36)$$

Introducing new variables x and y ,

$$x = \frac{p_-}{r_-}, \quad 0 \leq x \leq 1, \quad y = \frac{p_-''}{r_-}, \quad 0 \leq y \leq 1, \quad (2.37)$$

the equation takes the form

$$\left[r^2 - \frac{m_1^2}{x} - \frac{m_2^2}{(1-x)} \right] \varphi(x) = -\left(\frac{2\sigma}{\pi} \right) \int_0^1 dy \frac{(\varphi(y) - \varphi(x))}{(y-x)^2}. \quad (2.38)$$

This is the 't Hooft equation. It may take slightly different forms, according to the way of grouping the regularized terms.

't Hooft has shown that the mass-squared operator is positive definite and self-adjoint in the space of functions $\varphi(x)$ that vanish at the boundaries $x = 0$ and $x = 1$. The spectrum of eigenvalues is discrete, characterized with an integer quantum number $n (= 0, 1, 2, \dots)$, and corresponds to a confining potential. The eigenfunctions form a complete set and are orthogonal among themselves for different quantum numbers.

For large values of n , the eigenfunctions can be approximated by

$$\varphi_n(x) \sim \sin(n\pi x), \quad (2.39)$$

and the spectrum is approximately

$$r_n^2 \simeq 2\sigma\pi n + (m_1^2 + m_2^2) \ln(n) + \text{const.} \quad (2.40)$$

The leading term in n justifies the identification of σ with the string tension [96]. When the quark masses tend to zero, the ground state mass also tends to zero, with φ tending to a constant.

One important property of Eq. (2.38) is the disappearance of the cutoff factor Λ ; this is a consistency check of the present formalism for the evaluation of all observable quantities.

The complete wave function ϕ can be determined from Eq. (2.31), using in its right-hand side Eqs. (2.32) and (2.34). One finds

$$\phi(r, p) = 4iG_0 \left[r_+ - \Lambda - \frac{m_1^2}{2p_-} - \frac{m_2^2}{2(r_- - p_-)} \right] \varphi(r, p_-). \quad (2.41)$$

The normalization condition of the wave function ϕ according to the Bethe–Salpeter equation, after restoration of the γ matrices and of the factor $1/N_c$ of Eq. (2.28), imposes that of the reduced wave functions φ in the limit $\Lambda \rightarrow \infty$,

$$\int_0^1 dx \varphi_m^*(x) \varphi_n(x) = \delta_{mn} \frac{\pi}{N_c}. \quad (2.42)$$

The same result can also be found directly from Eq. (2.30), after restoration of the factor $1/N_c$ in the inhomogeneous part, with the use of the wave function (2.41).

The quark-antiquark scattering amplitude $\mathcal{T}$, which is generated by the one-gluon-exchange diagram, is proportional, in its γ -matrix form, to the tensor product $\gamma_- \times \gamma_-$. Using, in matrix form, its defining equation, $G = G_0/N_c + G_0 \mathcal{T} G_0$, one again finds that it is the component $G_{-,-}$ of G , accompanying the matrices $\gamma_+ \times \gamma_+$, that contributes to the dynamical part of the equation. The integral equation of the component $\mathcal{T}_{+,+}$, which accompanies the $\gamma_- \times \gamma_-$ matrices of $\mathcal{T}$, takes the following form for the color-singlet sector (omitting the indices),

$$\mathcal{T} = \frac{1}{N_c} K + 4KG_0 \mathcal{T}, \quad (2.43)$$

or more explicitly

$$\mathcal{T}(r; p, p') = -\frac{2\sigma}{N_c} D(p - p') - 8\sigma \int \frac{d^2k}{(2\pi)^2} D(k - p) S_1(k) S_2(k - r) \mathcal{T}(r; k, p'). \quad (2.44)$$

This equation has been solved in [61].³ The solution is

$$\mathcal{T}(r; x, x') = \frac{2\sigma}{N_c} \frac{(-i)}{r_-^2 (x - x')^2} + \frac{i}{N_c} \sum_n \frac{\tilde{\phi}_n(r, x) \tilde{\phi}_n^*(r, x')}{(r^2 - r_n^2)}. \quad (2.45)$$

$\tilde{\phi}_n(r, x)$ is related to the eigenfunction $\varphi_n(x)$ through the relation

$$\tilde{\phi}_n(r, x) = -\frac{i}{r_-} \left(\frac{2\sigma}{\sqrt{\pi}} \right) \int_0^1 dy \frac{\varphi_n(y)}{(y - x)^2}, \quad (2.46)$$

where now the canonical orthonormality condition $\int_0^1 dx \varphi_m^*(x) \varphi_n(x) = \delta_{mn}$ has been used for the φ_n s to make the $1/N_c$ dependence of $\mathcal{T}$ explicit.

For $x < 0$ or $x > 1$, the right-hand side of Eq. (2.46) is a regular function of x and allows the analytic continuation of $\tilde{\phi}_n(r, x)$ to that domain and hence of $\mathcal{T}(r; x, x')$ when it is considered in unphysical domains of the momenta. Using Eq. (2.38), which also allows the analytic continuation of $\varphi_n(x)$ to the domains $x < 0$ and $x > 1$, the relation (2.46) takes, for any x , the form

$$\begin{aligned} \tilde{\phi}_n(r, x) = \frac{i\sqrt{\pi}}{r_-} \left[r_n^2 - 2\Lambda|r_-|\theta(x(1-x)) \right. \\ \left. - \frac{m_1^2}{x} - \frac{m_2^2}{(1-x)} \right] \varphi_n(x). \end{aligned} \quad (2.47)$$

Notice that the divergent part of $\tilde{\phi}_n(r, x)$ in the limit $\Lambda \rightarrow \infty$ comes from the contribution of the domain $0 \leq x \leq 1$.

The relationship between the bound state wave functions defined from the Green's function and those related to $\mathcal{T}$ is

$$\phi_n(r, x) = \frac{2}{\sqrt{N_c}} G_0 \tilde{\phi}_n(r, x). \quad (2.48)$$

For completeness, we also display explicitly the relation between the Green's function and the scattering amplitude after restoration of the γ -matrix component indices,

$$G_{-,-} = \frac{G_{0,-,-}}{N_c} + G_{0,-,-} 4\mathcal{T}_{+,+} G_{0,-,-}. \quad (2.49)$$

³In that reference, the multiplicative factor 4 of the kernel has been absorbed in a redefinition of the quark propagators, with a multiplicative factor 2 for each.

One notices from Eqs. (2.45), (2.47), and (2.48) that the complete wave function ϕ and the scattering amplitude $\mathcal{T}$ are cutoff dependent and therefore not directly observable. The quark-antiquark scattering amplitude contributes, however, to the calculation of the meson-meson scattering amplitudes. One then has to check that the latter, on the mass-shell, are cutoff independent. This has been verified, in simple cases of quark flavors, at leading order $1/N_c$ [61]. This results from genuine compensations between the divergent terms of the quark-antiquark scattering amplitude and the damping factors of the quark propagators, after the loop integrations with respect to the $+$ component of the loop momenta are done.

Finally, we consider the effects of $1/N_c$ -order subleading contributions into the results obtained thus far. The gluon propagator receives radiative corrections from quark loop insertions. Considering N_f quark flavors, the gluon propagator obtains the form [61]

$$D_{++}(q) = \frac{i}{q_-^2 + \lambda^2 + \lambda|q_-|N_f/(\pi N_c)}. \quad (2.50)$$

It can be verified that the net effect of the modification of the gluon propagator on the evaluation of singular contributions amounts to the change of the infrared cutoff parameter in the form

$$\Lambda \rightarrow \Lambda \left(1 - \frac{N_f}{\pi^2 N_c} \right). \quad (2.51)$$

This change is transmitted to the expression of the quark self-energy (2.16), which now becomes

$$\Sigma_+(p_-) = \frac{1}{2} \varepsilon(p_-) \left(\Lambda \left(1 - \frac{N_f}{\pi^2 N_c} \right) - \frac{2\sigma}{\pi|p_-|} \right). \quad (2.52)$$

One notices that the finite part of Σ_+ has not been modified, which means that the mass redefinition (2.19) has not been affected by the presence of the quark loops. One therefore may redefine the infrared cutoff Λ as represented by the right-hand side of the formula (2.51) and ignore the presence of quark loops at the next subleading order in $1/N_c$.

Crossed diagrams of gluon propagators and vertex corrections to the gluon-quark coupling, representing non-planar diagrams, contribute only to subleading orders in $1/N_c$. However, in two dimensions and in the light-cone gauge, because of the changes of sign in front of one of the $i\varepsilon$ factors of the quark propagators, their contribution

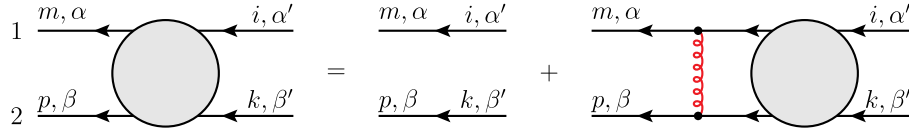


FIG. 4. Integral equation of the diquark Green's function.

generally vanishes during loop integrations. In more complicated diagrams, remainders may survive with coefficients depending on the infrared cutoff Λ , such that in the infrared limit $\lambda \rightarrow 0$, because of the spectral conditions satisfied by the meson wave functions and due to the quark momentum routings, they also vanish.

In conclusion, when evaluating below multi-quark Green's functions or meson-meson scattering amplitudes, the presence of quark loops, crossed diagrams, and vertex corrections can be ignored at the next subleading order in $1/N_c$.

III. DIQUARK SYSTEMS

We consider in this section diquark systems, made of two quark fields. Each quark belonging to the fundamental representation of the color-gauge group $SU(N_c)$, it is evident that such systems cannot be in the color-singlet

representation. Rather, they belong either to the symmetric representation [1, 1] (in the terminology of Young tableaux) or to the antisymmetric one [2], corresponding, in the case of $SU(3)$, to the representations **6** and **$\bar{3}$** , respectively. One would expect, on the basis of a possible infrared cutoff presence, that they could not be observed in isolation. However, diquark systems, together with antidiquark ones, might be embedded in larger color-singlet systems, in which case a global cutoff cancelation might occur. This is why a separate study of diquark systems would be of interest in its own.

The integral equation satisfied by the diquark Green's function is very similar to that of the quark-antiquark system, except for the routings of the color and Dirac indices (cf. Fig. 4), the second quark being represented by the index 2.

One has,

$$G_{12,0;\alpha\beta,\beta'\alpha'}^{mp,ki}(r; p) = \delta_{mi} S_{1;\alpha\alpha'}(p) \delta_{pk} S_{2;\beta\beta'}(r - p), \quad (3.1)$$

$$\begin{aligned} K_{12;\sigma\xi,\zeta\eta}^{rs,ut}(p'' - p) &= -g^2 (T^A)_{rt} (T^A)_{su} (\gamma^\mu)_{\sigma\eta} (\gamma^\nu)_{\xi\zeta} D_{\mu\nu}(p'' - p) \\ &= -\frac{g^2}{2} \left[\frac{1}{2} \left(1 - \frac{1}{N_c} \right) (\delta_{rt} \delta_{su} + \delta_{ru} \delta_{st}) - \frac{1}{2} \left(1 + \frac{1}{N_c} \right) (\delta_{rt} \delta_{su} - \delta_{ru} \delta_{st}) \right] (\gamma_-)_{\sigma\eta} (\gamma_-)_{\xi\zeta} D_{++}(p'' - p). \end{aligned} \quad (3.2)$$

Here again, the Green's functions G and G_0 can be decomposed along the basis of the tensor products of the two sets of independent Dirac matrices related to the quarks 1 and 2, respectively. As in the quark-antiquark case, one finds that it is only the component $G_{-,-}$ of G accompanying the matrices $\gamma_+ \times \gamma_+$ that contributes to the dynamical part of the equation, together with the components $S_{1,-}$ and $S_{2,-}$ of S_1 and S_2 , respectively. Projections similar to those of Eq. (2.24) can then be adopted. The integral equation takes the following compact form:

$$G^{mp,ki}_{-,-} = G_0^{mp,ki} + G_0^{mp,ki} 4K^{rs,ut}_{-,-} G^{tu,ki}_{-,-}, \quad (3.3)$$

with

$$\begin{aligned} G_0^{mp,ki} &= \delta^m_i \delta^p_k G_0, \\ K^{rs,ut}_{-,-} &= \left(-\frac{g^2}{2} D \right) \left[\frac{1}{2} \left(1 - \frac{1}{N_c} \right) (\delta^r_t \delta^s_u + \delta^r_u \delta^s_t) - \frac{1}{2} \left(1 + \frac{1}{N_c} \right) (\delta^r_t \delta^s_u - \delta^r_u \delta^s_t) \right], \\ (G_0 K)^{mp,ut}_{-,-} &= G_0 K \left[\frac{1}{2(N_c + 1)} (\delta^m_t \delta^p_u + \delta^m_u \delta^p_t) - \frac{1}{2(N_c - 1)} (\delta^m_t \delta^p_u - \delta^m_u \delta^p_t) \right], \\ K &= (-2\sigma D). \end{aligned} \quad (3.4)$$

As stated earlier, the diquark sytem should belong either to the symmetric [1, 1] or to the antisymmetric [2] representations of the color group $SU(N_c)$. This feature is also exhibited in the decomposition of the quantity $(G_0 K)$ of Eqs. (3.4). The Green's function itself can then be decomposed along the above representations,

$$G^{mp, ki} = \frac{1}{2}(\delta^m_i \delta^p_k + \delta^m_k \delta^p_i) G_S - \frac{1}{2}(\delta^m_i \delta^p_k - \delta^m_k \delta^p_i) G_A \quad (3.5)$$

where G_S and G_A refer to the symmetric and antisymmetric parts of the Green's function, respectively. The integral equation (3.5) then splits into two decoupled independent equations,

$$G_S = G_0 + \frac{4}{(N_c + 1)} G_0 K G_S, \quad (3.6)$$

$$G_A = G_0 - \frac{4}{(N_c - 1)} G_0 K G_A. \quad (3.7)$$

We observe, as expected, that the infrared cutoff Λ is not canceled between both sides of the equation. This means that the diquark sytem, considered in isolation, cannot lead to observable bound states, since in the limit $\Lambda \rightarrow \infty$ the only solution of the equation is $\varphi_A = 0$. Nevertheless, one might consider situations where the diquark system is embedded within a larger color-singlet system [16,17,97–100], in which case its residual cutoff Λ might be canceled by contributions coming from the other subsystems. What would then be the contribution of the finite part of the diquark system, considered as if it was in isolation? For this, it is sufficient to remove from Eq. (3.10) its divergent part, as well as, for simplicity, the finite renormalization parts proportional to $(2\sigma/\pi)$, which could only affect the values of the low-lying bound-state masses, to obtain

$$\begin{aligned} \left[r_+ - \frac{m_1^2}{2p_-} - \frac{m_2^2}{2(r_- - p_-)} \right] \varphi_A(r, p_-) \\ = -\frac{2\sigma}{(N_c - 1)} \left[\int_0^{r_-} \frac{dp_-''}{(2\pi)} \frac{1}{(p_-'' - p_-)^2} (\varphi_A(r, p_-'') \right. \\ \left. - \varphi_A(r, p_-)) \right]. \end{aligned} \quad (3.11)$$

This equation has the same structure as that of the quark-antiquark system, except that the coupling constant σ is now

Among the kernels of the two equations, it is that of G_A that is attractive and could lead to the formation of a bound state.⁴ We shall therefore concentrate on the antisymmetric representation. Proceeding as in the quark-antiquark case and assuming the existence of bound states with wave functions ϕ_A , one obtains the equation

$$\begin{aligned} \phi_A(r, p) = & -\frac{4}{(N_c - 1)} \frac{p_-}{[p_-^2 - |p_-| \Lambda - m_1^2 + i\epsilon]} \\ & \times \frac{(r_- - p_-)}{[(r - p)^2 - |r_- - p_-| \Lambda - m_2^2 + i\epsilon]} \\ & \times (2\sigma) \int \frac{d^2 p''}{(2\pi)^2} \frac{i}{(p_-'' - p_-)^2 + \lambda^2} \phi_A(r, p''). \end{aligned} \quad (3.8)$$

Defining the lightlike instantaneous limit of ϕ_A as in Eq. (2.32),

$$\varphi_A(r, p_-) = \int \frac{dp_+}{2\pi} \phi_A(r, p), \quad (3.9)$$

one arrives at the equation

$$\begin{aligned} \left[r_+ - \Lambda - \frac{m_1^2}{2p_-} - \frac{m_2^2}{2(r_- - p_-)} \right] \varphi_A(r, p_-) = & -\frac{2\sigma}{(N_c - 1)} \left[\int_0^{r_-} \frac{dp_-''}{(2\pi)} \frac{1}{(p_-'' - p_-)^2} (\varphi_A(r, p_-'') \right. \\ & \left. - \varphi_A(r, p_-)) \right] \\ & - \frac{1}{(N_c - 1)} \left(\Lambda - \frac{2\sigma}{\pi} \left(\frac{1}{2p_-} + \frac{1}{2(r_- - p_-)} \right) \right) \varphi_A(r, p_-). \end{aligned} \quad (3.10)$$

replaced by $\sigma/(N_c - 1)$. This means that the spectrum of the bound states of the diquark system is very similar to that of the quark-antiquark system with a damping of the energy gaps due to the scaling $\sigma \rightarrow \sigma/(N_c - 1)$. In particular, in the limit $N_c \rightarrow \infty$, the energy gaps between the bound states tend to zero and the whole spectrum shrinks to the two-quark mass threshold. In the real world, where $N_c = 3$, the damping factor $1/(N_c - 1)$ is $1/2$.

IV. TWO-QUARK–TWO-ANTIQUARK SYSTEMS

We consider in this section the case of two-quark–two-antiquark systems, out of which tetraquarks are expected to be formed. The quark flavors are assumed to be different from each other; this case avoids mixing problems with ordinary meson states. The quarks are denoted with the labels 1 and 3, with masses m_1 and m_3 , and the antiquarks with labels $\bar{2}$ and $\bar{4}$, with masses m_2 and m_4 , respectively.

In relation with the present investigation and the large- N_c considerations, let us, at this stage, outline the following property of multi-quark states. Contrary to the ordinary

⁴In comparison to Eqs. (2.28) and (2.29), here the quark 2 propagator in G_0 introduces an additional minus sign with respect to that of the antiquark $\bar{2}$.

baryonic case, where large N_c requires increasing of the number of constituent quarks with N_c , the description of multi-quark states, other than ordinary baryons, can be realized by several different types of representation (cf. Ref. [37], Sec. III.2). Tetraquarks, in a diquark anti-symmetric representation, can be described in $(N_c - 2)$ different ways. One way corresponds to the case where one has $(N_c - 1)$ external quarks linked with a single Wilson line to $(N_c - 1)$ external antiquarks. Other representations are obtained by decreasing the number of external quarks and antiquarks, but increasing in parallel the number of links between them. The extreme case to this corresponds to two external quarks and two external antiquarks linked to each other with $(N_c - 2)$ Wilson lines (cf. Ref. [37], Eq. (43) and Fig. 19). This last case is the most practical one, since it preserves the same external description of the state as in the case $N_c = 3$; the variation of N_c affects only the parameters related to the internal structures, in much analogy with the ordinary meson case. We shall stick, in the present work, to that representation of tetraquark states, also including the symmetric diquark representation and other equivalent representations obtained by Fierz transformations.

A. Green's functions

The Green's function corresponding to the four ingoing and outgoing particles made of the quarks 1 and 3 and the antiquarks $\bar{2}$ and $\bar{4}$ is represented in Fig. 5.

Most of the properties, encountered with the two-body systems in Secs. II and III, concerning N_c -leading powers of diagrams and Dirac matrices, are also met in the present case. The N_c -leading diagrams are produced by one-gluon exchange kernels. Because the latter are proportional to the tensor product of two Dirac γ_- matrices, the final dynamical equation is governed by the component $G_{-,-,-,-}$, which accompanies the tensor product of four γ_+ matrices, each acting on a single quark line; at the same time, the quark propagators of the free part of the Green's function act in this equation with their $S_{a,-}$ component ($a = 1, \dots, 4$) [Eq. (2.18)]. Notations, similar to those of Eq. (2.24), can then be adopted,

$$\begin{aligned} S_{a,-} &\rightarrow S_a \quad (a = 1, 2, 3, 4), & D_{++} &\rightarrow D, \\ G_{-,-,-,-} &\rightarrow G, \\ S_{1,-}S_{2,-}S_{3,-}S_{4,-} &= G_{0,-,-,-,-} \rightarrow G_0, \\ S_{a,-}S_{b,-} &\rightarrow G_{ab,0} \quad (a, b = 1, 2, 3, 4, a \neq b). \end{aligned} \quad (4.1)$$

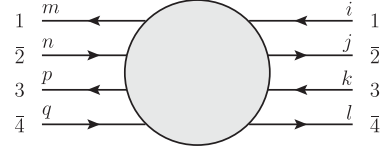


FIG. 5. Four-body Green's function with two quarks, labeled 1 and 3, and two antiquarks, labeled $\bar{2}$ and $\bar{4}$. Latin letters refer to color indices.

The contributions of the one-gluon exchange kernels are similar to those met in the quark-antiquark and the diquark cases, Eqs. (2.26) and (3.4), in which we have to include now the corresponding particle indices,

$$\begin{aligned} (G_{ab,0}K_{ab})^{m,u}_{n,t} &= G_{ab,0}K_{ab} \frac{1}{N_c} \left[\delta^m_n \delta^u_t - \frac{N_c}{(N_c^2 - 1)} \right. \\ &\quad \left. \times \left(\delta^m_t \delta^u_n - \frac{1}{N_c} \delta^m_n \delta^u_t \right) \right] \\ (a = 1, 3; b = \bar{2}, \bar{4}), \end{aligned} \quad (4.2)$$

$$\begin{aligned} (G_{ab,0}K_{ab})^{mp}_{vt} &= G_{ab,0}K_{ab} \left[\frac{1}{2(N_c + 1)} (\delta^m_t \delta^p_v + \delta^m_v \delta^p_t) \right. \\ &\quad \left. - \frac{1}{2(N_c - 1)} (\delta^m_t \delta^p_v - \delta^m_v \delta^p_t) \right] \\ (a = 1, b = 3, \text{ or } a = \bar{2}, b = \bar{4}), \end{aligned} \quad (4.3)$$

$$K_{ab} = (-2\sigma D_{ab}) \quad (a \neq b), \quad (4.4)$$

where the indices of D correspond to the lines to which the gluon propagator is connected.

It can be checked that the iteration of the sum of the kernels of two disjoint diagrams, for instance $(G_{1\bar{2},0}K_{1\bar{2}} + G_{3\bar{4},0}K_{3\bar{4}})$, produces a double counting with the interference term. This is cured at all orders of perturbation theory by subtracting from the above kernel the product of the two kernels $(G_{1\bar{2},0}K_{1\bar{2}}G_{3\bar{4},0}K_{3\bar{4}})$ for the example above) [38–40,101–105].

The total Green's function then satisfies the following integral equation,

$$\begin{aligned} G_{nq,ki}^{mp,lj} &= G_{nq,ki,0}^{mp,lj} + \left\{ ((G_{1\bar{2},0}4K_{1\bar{2}})\mathbf{1}_{3\bar{4}} + \mathbf{1}_{1\bar{2}}(G_{3\bar{4},0}4K_{3\bar{4}}) - (G_{1\bar{2},0}4K_{1\bar{2}})(G_{3\bar{4},0}4K_{3\bar{4}}))_{nq,vt}^{mp,wu} \right. \\ &\quad + ((G_{13,0}4K_{13})\mathbf{1}_{\bar{2}\bar{4}} + \mathbf{1}_{13}(G_{\bar{2}\bar{4},0}4K_{\bar{2}\bar{4}}) - (G_{13,0}4K_{13})(G_{\bar{2}\bar{4},0}4K_{\bar{2}\bar{4}}))_{nq,vt}^{mp,wu} \\ &\quad \left. + ((G_{1\bar{4},0}4K_{1\bar{4}})\mathbf{1}_{3\bar{2}} + \mathbf{1}_{1\bar{4}}(G_{3\bar{2},0}4K_{3\bar{2}}) - (G_{1\bar{4},0}4K_{1\bar{4}})(G_{3\bar{2},0}4K_{3\bar{2}}))_{nq,vt}^{mp,wu} \right\} G_{uw,ki}^{tv,lj}. \end{aligned} \quad (4.5)$$

We shall be interested in the color-singlet sectors of the incoming and outgoing particles in the Green's function of Fig. 5, which are expected to be the only observable sectors on experimental grounds. However, contrary to the case of ordinary mesons and baryons, color singlets of multi-quark states are not color irreducible, in the sense that their color structure can be decomposed in general along combinations of products of clusters with simpler color structures, representing color singlets of quark-antiquark or three-quark states [42,45,46,58]. The decomposition of the Green's function (4.5) along components involving color-singlet tensors can be done in two ways. First, using the mesonic clusters $(1\bar{2})(3\bar{4})$ and $(1\bar{4})(3\bar{2})$, or, second, using the diquark-antidiquark combinations in their symmetric and antisymmetric representations. One advantage of the first type of decomposition is its direct relationship with the meson-meson scattering amplitudes, which are other quantities of interest, while an advantage of the second type of decomposition is its relationship with a possible existence of compact tetraquark bound states. The two types of decomposition are related to each other by linear transformations. In the following, we shall consider the first type

of decomposition to establish the connection of the Green's function with the meson-meson scattering amplitudes.

Associating the incoming and outgoing states with the mesonic clusters, one can distinguish four different channels in the transition process, two of which will be called “direct,” the ingoing and outgoing clusters being the same, and the two others “recombination,” the outgoing clusters having undergone a quark exchange. The four different processes are,

$$(1\bar{2}) + (3\bar{4}) \rightarrow (1\bar{2}) + (3\bar{4}), \quad \text{direct 1 (D1)}, \quad (4.6)$$

$$(1\bar{2}) + (3\bar{4}) \rightarrow (1\bar{4}) + (3\bar{2}), \quad \text{recomb. 1 (R1)}, \quad (4.7)$$

$$(1\bar{4}) + (3\bar{2}) \rightarrow (1\bar{4}) + (3\bar{2}), \quad \text{direct 2 (D2)}, \quad (4.8)$$

$$(1\bar{4}) + (3\bar{2}) \rightarrow (1\bar{2}) + (3\bar{4}), \quad \text{recomb. 2 (R2)}. \quad (4.9)$$

Choosing a basis that projects, by contraction of color indices, the Green's function on the above channels, one has the following decomposition:

$$\begin{aligned} G_{nq,ki}^{mp,lj} = & \frac{N_c^2}{(N_c^2 - 1)^2} \left\{ \left(\delta_n^m \delta_q^p - \frac{1}{N_c} \delta_q^m \delta_n^p \right) \left(\delta_i^j \delta_k^l - \frac{1}{N_c} \delta_k^j \delta_i^l \right) G_{D1} + \left(\delta_q^m \delta_n^p - \frac{1}{N_c} \delta_n^m \delta_q^p \right) \left(\delta_i^j \delta_k^l - \frac{1}{N_c} \delta_k^j \delta_i^l \right) G_{R1} \right. \\ & \left. + \left(\delta_q^m \delta_n^p - \frac{1}{N_c} \delta_n^m \delta_q^p \right) \left(\delta_k^j \delta_i^l - \frac{1}{N_c} \delta_i^j \delta_k^l \right) G_{D2} + \left(\delta_n^m \delta_q^p - \frac{1}{N_c} \delta_q^m \delta_n^p \right) \left(\delta_k^j \delta_i^l - \frac{1}{N_c} \delta_i^j \delta_k^l \right) G_{R2} \right\} + \dots, \end{aligned} \quad (4.10)$$

where the dots stand for the existing higher representations than the singlets. The normalization is such that

$$\begin{aligned} \delta_n^m \delta_q^p \delta_i^j \delta_k^l G_{nq,ki}^{mp,lj} &= N_c^2 G_{D1}, \\ \delta_q^m \delta_n^p \delta_i^j \delta_k^l G_{nq,ki}^{mp,lj} &= N_c^2 G_{R1}, \end{aligned} \quad (4.11)$$

and similarly with the other two components, after appropriate permutation of indices. The decomposition of the free Green's function $G_{nq,ki,0}^{mp,lj}$ [Eq. (4.5)] is

$$\begin{aligned} G_{nq,ki,0}^{mp,lj} &= \delta_i^m \delta_j^n \delta_k^p \delta_l^q G_0 \\ &= \frac{N_c^2}{(N_c^2 - 1)^2} \left\{ \left(\delta_n^m \delta_q^p - \frac{1}{N_c} \delta_q^m \delta_n^p \right) \left(\delta_i^j \delta_k^l - \frac{1}{N_c} \delta_k^j \delta_i^l \right) G_0 \right. \\ &\quad + \left(\delta_q^m \delta_n^p - \frac{1}{N_c} \delta_n^m \delta_q^p \right) \left(\delta_i^j \delta_k^l - \frac{1}{N_c} \delta_k^j \delta_i^l \right) \frac{1}{N_c} G_0 + \left(\delta_q^m \delta_n^p - \frac{1}{N_c} \delta_n^m \delta_q^p \right) \left(\delta_k^j \delta_i^l - \frac{1}{N_c} \delta_i^j \delta_k^l \right) G_0 \\ &\quad \left. + \left(\delta_n^m \delta_q^p - \frac{1}{N_c} \delta_q^m \delta_n^p \right) \left(\delta_k^j \delta_i^l - \frac{1}{N_c} \delta_i^j \delta_k^l \right) \frac{1}{N_c} G_0 \right\} + \dots. \end{aligned} \quad (4.12)$$

Taking into account the decompositions (4.10) and (4.12) and the kernel structures (4.2) and (4.3), the integral equation (4.5) is decomposed into two independent sets of equations, each involving two coupled equations. The first set concerns the components G_{D1} and G_{R1} , the second set G_{D2} and G_{R2} . The latter set is obtained from the former one by the interchange of the indices $\bar{2}$ and $\bar{4}$. This is why we shall concentrate in the following on the first set only.

Introducing the notation

$$k_{ab} \equiv 4G_{ab,0}K_{ab} \quad (a \neq b), \quad (4.13)$$

[Eqs. (4.1) and (4.4)], the coupled integral equations of G_{D1} and G_{R1} take the following form:

$$\begin{aligned} G_{D1} = & G_0 + (k_{1\bar{2}} + k_{3\bar{4}} - k_{1\bar{2}}k_{3\bar{4}})G_{D1} + (k_{1\bar{4}} + k_{3\bar{2}})\frac{N_c}{(N_c^2 - 1)}\left(-\frac{1}{N_c}G_{D1} + G_{R1}\right) \\ & - k_{1\bar{4}}k_{3\bar{2}}\left[\frac{1}{N_c}G_{R1} + \frac{1}{(N_c^2 - 1)^2}\left(G_{D1} - \frac{1}{N_c}G_{R1}\right)\right] + (k_{13} + k_{2\bar{4}})\frac{N_c}{(N_c^2 - 1)}\left(-\frac{1}{N_c}G_{D1} + G_{R1}\right) \\ & - k_{13}k_{2\bar{4}}\frac{N_c^2}{(N_c^2 - 1)^2}\left[\left(1 + \frac{1}{N_c^2}\right)G_{D1} - \frac{2}{N_c}G_{R1}\right]. \end{aligned} \quad (4.14)$$

$$\begin{aligned} G_{R1} = & \frac{1}{N_c}G_0 + (k_{1\bar{4}} + k_{3\bar{2}} - k_{1\bar{4}}k_{3\bar{2}})G_{R1} + (k_{1\bar{2}} + k_{3\bar{4}})\frac{N_c}{(N_c^2 - 1)}\left(G_{D1} - \frac{1}{N_c}G_{R1}\right) \\ & - k_{1\bar{2}}k_{3\bar{4}}\left[\frac{1}{N_c}G_{D1} - \frac{1}{(N_c^2 - 1)^2}\left(G_{D1} - G_{R1}\right)\right] + (k_{13} + k_{2\bar{4}})\frac{N_c}{(N_c^2 - 1)}\left(G_{D1} - \frac{1}{N_c}G_{R1}\right) \\ & - k_{13}k_{2\bar{4}}\frac{N_c^2}{(N_c^2 - 1)^2}\left[-\frac{2}{N_c}G_{D1} + \left(1 + \frac{1}{N_c^2}\right)G_{R1}\right]. \end{aligned} \quad (4.15)$$

Let us notice that one-gluon exchanges between two color-singlet states are generally forbidden by color conservation. Here, this process concerns the Green's function G_{D1} , which describes the interaction between the color-singlet states $(1\bar{2})$ and $(3\bar{4})$ [Eq. (4.6)]. It can be checked that the one-gluon-exchange terms between the two states that appear through the quantities $(k_{1\bar{4}} + k_{3\bar{2}})$ and $(k_{13} + k_{2\bar{4}})$, do disappear in lowest order of perturbation theory, when one replaces in the accompanying terms G_{D1} and G_{R1} by their lowest-order expressions G_0 and G_0/N_c , respectively.

B. Scattering amplitudes

To obtain the integral equations of the scattering amplitudes of the meson clusters, one first has to restore the γ matrix structure of the Green's function equations, as in the two-body case [before Eq. (2.43)]. Because of the particular structure of the kernels K_{ab} in γ matrices (tensor products $\gamma_- \times \gamma_-$) and the dynamical role of the components $G_{-,-,-}$ of the Green's functions, the scattering amplitudes are tensor products of four γ_- matrices. After contraction of the γ matrices among themselves, one comes

back to the basis that was used for the Green's functions [cf. Eqs. (4.1)].

The off-mass shell scattering amplitudes of the processes (4.6)–(4.9) are obtained by subtracting the disconnected parts due to the corresponding clusters in the direct channels and then factorizing out the related two-body Green's functions. We adopt for the channels $D1$ and $R1$, in the projected space of the γ matrices, the definitions,

$$G_{D1} = G_{1\bar{2}}G_{3\bar{4}} + G_{1\bar{2}}G_{3\bar{4}}\mathcal{T}_{D1}G_{1\bar{2}}G_{3\bar{4}}, \quad (4.16)$$

$$G_{R1} = G_{1\bar{4}}G_{3\bar{2}}\mathcal{T}_{R1}G_{1\bar{2}}G_{3\bar{4}}. \quad (4.17)$$

In replacing these expressions in Eqs. (4.14) and (4.15), one uses in a few N_c -leading terms the equations satisfied by $G_{1\bar{2}}G_{3\bar{4}}$ and $G_{1\bar{4}}G_{3\bar{2}}$, namely [Eq. (2.30)]

$$\begin{aligned} (1 - k_{1\bar{2}})(1 - k_{3\bar{4}})G_{1\bar{2}}G_{3\bar{4}} &= G_0, \\ (1 - k_{1\bar{4}})(1 - k_{3\bar{2}})G_{1\bar{4}}G_{3\bar{2}} &= G_0. \end{aligned} \quad (4.18)$$

One obtains,

$$\begin{aligned} G_0\mathcal{T}_{D1}G_0 = & -(k_{1\bar{4}} + k_{3\bar{2}} + k_{13} + k_{2\bar{4}})\frac{1}{(N_c^2 - 1)}[1 + G_{1\bar{2}}G_{3\bar{4}}\mathcal{T}_{D1} - N_cG_{1\bar{4}}G_{3\bar{2}}\mathcal{T}_{R1}]G_0 \\ & - k_{1\bar{4}}k_{3\bar{2}}\left[\frac{1}{N_c}G_{1\bar{4}}G_{3\bar{2}}\mathcal{T}_{R1} + \frac{1}{(N_c^2 - 1)^2}\left(1 + G_{1\bar{2}}G_{3\bar{4}}\mathcal{T}_{D1} - \frac{1}{N_c}G_{1\bar{4}}G_{3\bar{2}}\mathcal{T}_{R1}\right)\right]G_0 \\ & - k_{13}k_{2\bar{4}}\frac{1}{(N_c^2 - 1)^2}[(N_c^2 + 1) + (N_c^2 + 1)G_{1\bar{2}}G_{3\bar{4}}\mathcal{T}_{D1} - 2N_cG_{1\bar{4}}G_{3\bar{2}}\mathcal{T}_{R1}]G_0. \end{aligned} \quad (4.19)$$

$$\begin{aligned}
G_0 \mathcal{T}_{R1} G_0 &= \frac{G_0}{N_c} + \frac{1}{N_c} (k_{1\bar{2}} + k_{3\bar{4}} - k_{1\bar{2}} k_{3\bar{4}}) G_{1\bar{2}} G_{3\bar{4}} \mathcal{T}_{D1} G_0 \\
&+ \frac{1}{(N_c^2 - 1)} \left(k_{1\bar{2}} + k_{3\bar{4}} + \frac{1}{(N_c^2 - 1)} k_{1\bar{2}} k_{3\bar{4}} \right) \left(\frac{1}{N_c} + \frac{1}{N_c} G_{1\bar{2}} G_{3\bar{4}} \mathcal{T}_{D1} - G_{1\bar{4}} G_{3\bar{2}} \mathcal{T}_{R1} \right) G_0 \\
&+ (k_{13} + k_{2\bar{4}}) \left[\frac{N_c}{(N_c^2 - 1)} (1 + G_{1\bar{2}} G_{3\bar{4}} \mathcal{T}_{D1}) - \frac{1}{(N_c^2 - 1)} G_{1\bar{4}} G_{3\bar{2}} \mathcal{T}_{R1} \right] G_0 \\
&- k_{13} k_{2\bar{4}} \frac{1}{(N_c^2 - 1)^2} [-2N_c - 2N_c G_{1\bar{2}} G_{3\bar{4}} \mathcal{T}_{D1} + (N_c^2 + 1) G_{1\bar{4}} G_{3\bar{2}} \mathcal{T}_{R1}] G_0.
\end{aligned} \tag{4.20}$$

The two-body Green's functions G_{ab} satisfy in turn Eqs. (2.49),

$$G_{ab} = G_{ab,0} + G_{ab,0} 4\mathcal{T}_{ab} G_{ab,0} \quad (a \neq b). \tag{4.21}$$

[The factor $1/N_c$ has been removed from Eq. (2.49), being now included in the normalization conditions (4.11) and (4.12) of the four-body Green's functions; $\mathcal{T}_{ab}$ has expression (2.45) without the factors $1/N_c$.]

The on-mass shell scattering amplitudes are obtained by projecting Eqs. (4.19) and (4.20) on the two-body wave functions of the type of (2.46), with factors 2 [Eq. (2.48) without the factor $1/\sqrt{N_c}$], and imposing on the corresponding total momenta the meson mass-shell constraints. Thus,

$$(\mathcal{T}_{D1})_{nm,n'm'} = 2^4 \tilde{\phi}_{1\bar{2},n}^* \tilde{\phi}_{3\bar{4},m}^* (G_0 \mathcal{T}_{D1} G_0) \tilde{\phi}_{1\bar{2},n'} \tilde{\phi}_{3\bar{4},m'}, \tag{4.22}$$

$$(\mathcal{T}_{R1})_{pq,n'm'} = 2^4 \tilde{\phi}_{1\bar{4},p}^* \tilde{\phi}_{3\bar{2},q}^* (G_0 \mathcal{T}_{R1} G_0) \tilde{\phi}_{1\bar{2},n'} \tilde{\phi}_{3\bar{4},m'}, \tag{4.23}$$

where n, m, p, q, n', m' are the quantum numbers of the meson states.

Equations (4.19) and (4.20) display the explicit dependence on N_c of the scattering amplitudes. The recombination scattering amplitude $\mathcal{T}_{R1}$ is of order N_c^{-1} , while $\mathcal{T}_{D1}$ is of order N_c^{-2} , with subleading orders decreasing by orders of N_c^{-2} in both cases. The leading-order term of $\mathcal{T}_{D1}$ is generated by the iteration of the leading-order term of $\mathcal{T}_{R1}$, a feature that is also valid in four dimensions, noticed from the analysis of the structure of the corresponding Feynman diagrams [37,106].

At leading orders one has

$$G_0 \mathcal{T}_{R1} G_0 = \frac{1}{N_c} (1 + k_{13} + k_{2\bar{4}}) G_0 + O(N_c^{-3}), \tag{4.24}$$

$$\begin{aligned}
G_0 \mathcal{T}_{D1} G_0 &= -\frac{1}{N_c^2} (k_{1\bar{4}} + k_{3\bar{2}} + k_{13} + k_{2\bar{4}}) \\
&\times [1 - N_c G_{1\bar{4}} G_{3\bar{2}} \mathcal{T}_{R1}] G_0 \\
&- k_{1\bar{4}} k_{3\bar{2}} \frac{1}{N_c} G_{1\bar{4}} G_{3\bar{2}} \mathcal{T}_{R1} G_0 \\
&- \frac{1}{N_c^2} k_{13} k_{2\bar{4}} G_0 + O(N_c^{-4}).
\end{aligned} \tag{4.25}$$

We notice that, as in the case of G_{D1} [Eq. (4.14)], the terms in the brackets in Eq. (4.25) are themselves proportional to interaction terms, after $\mathcal{T}_{R1}$ has been replaced by its lowest-order expression (4.25) and decompositions (4.21) are used, and therefore $\mathcal{T}_{D1}$ does not contain one-gluon exchange terms.

V. FINITENESS OF THE SCATTERING AMPLITUDES TO ORDER $1/N_c^2$

One of the important tests of the present theory is the check that physical observables are finite quantities, independent of the infrared cutoff introduced in intermediate calculations. This was verified in the case of the quark-antiquark system for color-singlet states (Sec. II). The meson-meson scattering amplitudes on the mass-shell being also observable quantities, one must also check their independence of the cutoff parameter. The N_c -leading-order expressions of the scattering amplitudes, given in Eqs. (4.24) and (4.25), allow us to proceed to this check and to extract from them the physical content they convey.

Another check, which goes in parallel to the previous one, is the verification that the meson-meson scattering amplitudes, even if they are finite, are free of long-range van der Waals type forces. This is not evident from Eqs. (4.24) and (4.25), since they exhibit many gluon propagators combined with various other quantities.

We shall mainly present in this section the qualitative aspects of the calculations with their final results, leaving the technical details to Appendix B. It is understood that expressions (4.24) and (4.25) are projected, as in Eqs. (4.22) and (4.23), on the meson wave functions; however, for the simplicity of notation, we shall often omit them in the formulas.

We start with the recombination scattering amplitude (4.25). It is composed of three contributions, which are graphically represented in Fig. 6.

In this figure, the circles represent the external meson wave functions, such as $\tilde{\phi}_{1\bar{2}}$ and $\tilde{\phi}_{3\bar{4}}$ for the incoming ones, and $\tilde{\phi}_{1\bar{4}}^*$ and $\tilde{\phi}_{3\bar{2}}^*$ for the outgoing ones; in the present discussion, the specific bound state quantum numbers of the wave functions are irrelevant and are omitted. The arrows indicate the momentum routings of the quarks and

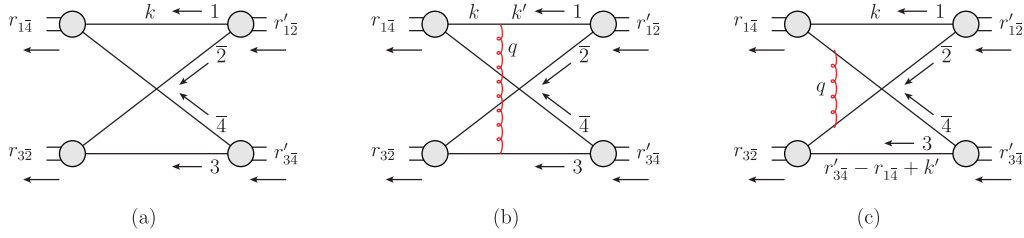


FIG. 6. Three diagrams contributing to the recombination scattering amplitude $R1$ [Eq. (4.7)] at order $1/N_c$.

antiquarks; the flavors of the quarks and antiquarks are also indicated in the internal lines. Solid lines represent quark and antiquark propagators and the curly lines the gluon propagator. The loop momenta are indicated on some quark lines and the whole momentum distributions can then easily be reconstituted from the knowledge of the external momenta, which are the only physical momenta in the scattering processes. Meson wave functions will be specified by the total momentum they carry and by their internal quark momentum. The wave functions $\tilde{\phi}$ have a pair of quark and antiquark outgoing toward the interior of the diagrams, while the $\tilde{\phi}^*$'s have an incoming pair from the interior of the diagrams. Thus the wave function $\tilde{\phi}_{1\bar{2}}$ in diagram (a) has total momentum $r'_{1\bar{2}}$ and quark momentum k ; the wave function $\tilde{\phi}_{1\bar{4}}^*$ has total momentum $r'_{1\bar{4}}$ and quark momentum k . In diagrams (b) and (c), the momentum carried by the gluon is $q = k' - k$.

One important remark concerns the spectral conditions imposed on the wave functions. It is assumed that the external mesons have physical momenta, which means that the components $r_{a\bar{b},-}$ of their total momentum $r_{a\bar{b}}$ are positive. On the other hand, it is seen from Eq. (2.47) that, for the divergent part of the wave function, each quark and antiquark should also separately satisfy such a condition. When the latter condition is not satisfied at the quark and antiquark levels, it is the subleading part in Λ of the wave function that contributes. However, for the class of quark flavors that we are considering in the present work (four different flavors), the same phenomenon also occurs at a second wave function connected to the previous one with the concerned quark or antiquark lines. For instance, if, in diagram (a) of Fig. 6, $k_- < 0$, then this feature will concern the two wave functions $\tilde{\phi}_{1\bar{2}}$ and $\tilde{\phi}_{1\bar{4}}^*$, and, therefore, the reduction of the power in Λ will be of two degrees. As the detailed calculations show, such a reduction eliminates the corresponding contributions from the physical quantity when the limit $\Lambda \rightarrow \infty$ is taken. The same phenomenon also occurs in the presence of gluon propagators. For instance, in diagram (b) of Fig. 6, one may have $k'_- < 0$; the leading divergence in the diagram will come from the singularity of the gluon propagator, which implies the limit $k'_- = k_-$, and thus the contribution of the quark propagator reaching the meson $(1\bar{4})$ will also reduce the power of Λ by one degree; one again falls in the case

considered above. Therefore, one may calculate the leading and subleading powers of Λ by retaining only that part of each wave function [Eq. (2.47)] that satisfies the above spectral conditions.

In diagrams having gluon propagators, the leading divergences will receive contributions from the singularities of the latter. As in Eqs. (2.34) and (2.35), the finite parts of the gluon propagators will contribute to the subleading terms in Λ of the diagrams.

We can now evaluate the contributions of the diagrams of Fig. 6. Diagram (a) contains four quark propagators, each carrying a damping factor Λ^{-1} [Eq. (2.18)]. The global damping factor is therefore Λ^{-4} . However, these propagators are submitted to the integration with respect to the component k_+ of the loop momentum, which transforms the damping factor into Λ^{-3} . Taking into account the Λ^4 diverging factor coming from the external wave functions, one finds a final diverging factor Λ^1 , plus additional finite contributions. The diagram is therefore divergent and could not account alone for the description of a physical process. One therefore has to associate with it the two other diagrams, (b) and (c), of Fig. 6.

Diagram (b) has six quark propagators and an associated damping factor Λ^{-6} . There are two loop integrations with respect to k_+ and k'_+ , which transform Λ^{-6} into Λ^{-4} . The singularity of the gluon propagator increases Λ^{-4} up to Λ^{-3} . With the contribution of the external wave functions, this is transformed into Λ^1 . A similar result is also obtained with diagram (c).

It turns out that the sum of the divergences coming from the three diagrams vanishes and the global result is a finite quantity, coming from the finite parts of the quark and gluon propagators and from the instantaneous part, φ , of the wave functions [Eqs. (2.32) and (2.47)]. The details of the above calculations can be found in Appendix B. We simply describe here the main qualitative features of the resulting finite expression of the scattering amplitude [cf. the discussion after Eq. (B10), leading to Eq. (B11)].

The latter can be expressed as a double integral, in the $-$ components of the loop momenta, within finite bounds, involving the four external wave functions and the finite part of the gluon propagator. The expression does not display any singularity in the momentum transfer of the scattering process, which would be the signal of the

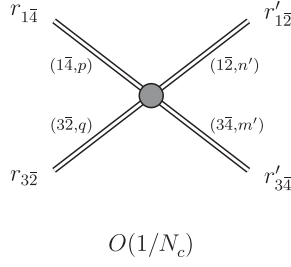


FIG. 7. Effective contact interaction representing the recombination scattering amplitude $R1$ at order $1/N_c$. The flavor and excitation quantum numbers of the mesons are also indicated.

existence of long-range van der Waals type forces. Rather, it displays a smooth dependence on the momenta, dominated by a constant term, of the scale of the string tension σ . The domains of the spectral conditions of the four meson wave functions do not generally coincide in the integrals and lead to complicated momentum-dependent expressions, which could be determined only by detailed numerical calculations. It is the nonsingular property of these expressions that allows us to approximate them by a local expression [Eq. (B11)], represented by its scalar, momentum-independent, part. In this respect, it can be viewed as a generalized effective contact term. This is represented in Fig. 7.

We next consider the direct process represented in Eq. (4.25). Here, the right-hand side contains the two-particle propagators $G_{1\bar{4}}$ and $G_{3\bar{2}}$, whose structure in terms of the two-body scattering amplitudes is displayed in Eq. (4.21). The latter, in turn, have the structure represented in Eq. (2.45), composed of the contributions of one-gluon exchange and meson-bound-state poles. We separate in a more explicit form the latter contributions by adopting the following notation:

$$G_{ab,0}4\tilde{T}_{ab} \equiv k_{ab} + G_{ab,0}4\tilde{T}_{ab}, \quad (5.1)$$

where k_{ab} is defined in Eqs. (4.4) and (4.13) and $\tilde{T}_{ab}$ contains the contributions of the meson poles. Equation (4.25) can then be expressed in the following form:

$$N_c^2 G_0 \mathcal{T}_{D1} G_0 = \{(a) + (b) + (c) + (d) + (e)\}, \quad (5.2)$$

where the symbols in the right-hand side have the expressions

$$(a) = (1 + k_{13} + k_{\bar{2}\bar{4}})k_{1\bar{4}}k_{3\bar{2}}(1 + k_{13} + k_{\bar{2}\bar{4}})G_0, \quad (5.3)$$

$$(b) = (1 + k_{13} + k_{\bar{2}\bar{4}})(k_{1\bar{4}}G_{3\bar{2},0}4\tilde{T}_{3\bar{2}} + k_{3\bar{2}}G_{1\bar{4},0}4\tilde{T}_{1\bar{4}}) \times (1 + k_{13} + k_{\bar{2}\bar{4}})G_0, \quad (5.4)$$

$$(c) = (1 + k_{13} + k_{\bar{2}\bar{4}})G_{3\bar{2},0}4\tilde{T}_{3\bar{2}}G_{1\bar{4},0}4\tilde{T}_{1\bar{4}} \times (1 + k_{13} + k_{\bar{2}\bar{4}})G_0, \quad (5.5)$$

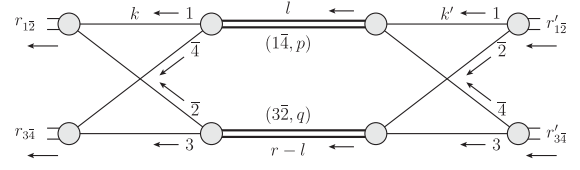


FIG. 8. Diagrammatic representation of the part of expression (c), Eq. (5.5), not containing gluon propagators. Double lines represent meson propagators. r is the total momentum.

$$(d) = (1 + k_{13} + k_{\bar{2}\bar{4}})(G_{3\bar{2},0}4\tilde{T}_{3\bar{2}} + G_{1\bar{4},0}4\tilde{T}_{1\bar{4}}) \times (1 + k_{13} + k_{\bar{2}\bar{4}})G_0, \quad (5.6)$$

$$(e) = (1 + k_{13} + k_{\bar{2}\bar{4}})(k_{1\bar{4}} + k_{3\bar{2}})(1 + k_{13} + k_{\bar{2}\bar{4}})G_0 + [(k_{13}^2 + k_{\bar{2}\bar{4}}^2 + k_{13}k_{\bar{2}\bar{4}}) - (k_{1\bar{4}} + k_{3\bar{2}})]G_0. \quad (5.7)$$

We first consider expression (c), Eq. (5.5). It contains nine different contributions. The part not containing gluon propagators is graphically represented in Fig. 8.

This diagram is made of three parts: a central part, containing two meson propagators with quark-antiquark content $(1\bar{4})$ and $(3\bar{2})$, respectively, and summed over the infinite number of mesons appearing in the quark-antiquark spectrum [Eq. (2.45)], and two side parts, corresponding to the recombination diagram (a) of Fig. 6. The other parts of Eq. (5.5) correspond to the inclusion of the gluon propagator in the side parts, as in diagrams (b) and (c) of Fig. 6. Therefore, as is also evident from Eq. (4.24), expression (5.5) represents the inclusion of the two meson propagators between the two recombination amplitudes $R2$ and $R1$ [Eqs. (4.9) and (4.7)], each at order $1/N_c$. It is not necessary in this case to calculate separately the contribution of each diagram. For a fixed value of the momentum l , which appears in the meson propagators, the integration of the momenta of the left and right sets of recombination diagrams can be done independently, provided the momentum l is considered as part of the external momenta. As we have seen previously, each set of the three recombination diagrams leads to a finite result, represented by the generalized contact-type interaction of Fig. 7. At this stage, one should not yet take the limit $\Lambda \rightarrow \infty$, awaiting the integration with respect to the component $+$ of the momentum l . The latter operation yields the once-integrated (with respect to l_+) finite expression of the two-meson loop amplitude, taken between the two effective contact terms of the recombination amplitudes. The result is therefore equivalent to the generalized form of the unitarity diagram, including the infinite number of intermediate mesons, and generated by the recombination process at order $1/N_c$, the whole contribution being of order $1/N_c^2$. This is represented in Fig. 9.

We next consider expression (a), Eq. (5.3). It is composed of nine diagrams, with quark and gluon propagators, not involving mesons. A typical diagram of this category is

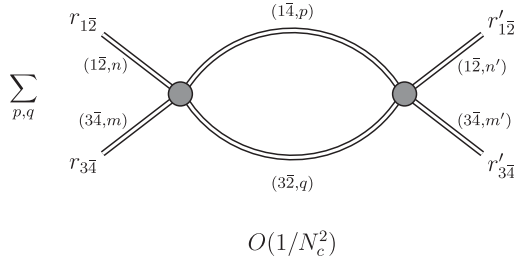


FIG. 9. Effective unitarity diagrams arising from expression (5.5) in the direct scattering amplitude $D1$ at order $1/N_c^2$. The flavor and excitation quantum numbers of the mesons are also indicated.

represented in Fig. 10. Here we encounter diagrams that may contain several gluon lines.

One of the main properties in the diagrams of the direct processes $D1$ and $D2$, except for the category (c) that we have met earlier, is that the sum of the momenta of the exchanged gluons is equal to the momentum transfer $(r'_{1\bar{2},-} - r_{1\bar{2},-})$; as a consequence, when the leading singularities of the gluon propagators are considered for the evaluation of the leading effects in Λ , one of the gluon propagators factorizes out of the integrals with the expression $-2i\sigma/(r'_{1\bar{2},-} - r_{1\bar{2},-})^2$. Therefore, the gluon propagators provide singularities with one degree less in Λ than superficially expected. The whole contribution of the diagram becomes in that case finite. However, the counterpart of this result is that one faces now an expression where an explicit gluon propagator is present, as if the interaction between the mesons was a confining one. This, of course, would be unacceptable, since the mutual meson interactions not only are not confining, but also are expected to be free of long-range van der Waals type forces. It is therefore mandatory, as for the infrared diverging quantities, to verify that the final sum of such quantities does indeed vanish. The details of the calculation of the contributions in the case of several gluon propagators is presented in Appendixes A and B.

Coming back to the example of Fig. 10 and to the counting rules of the damping and diverging factors, we have there ten quark propagators, yielding a damping factor

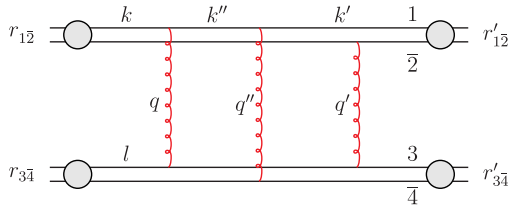


FIG. 10. Typical diagram contributing to Eq. (5.3), with three gluon exchanges, corresponding to k_{13} , $k_{1\bar{4}}$, and $k_{3\bar{2}}$. The momentum routings of the quarks and the antiquarks are from right to left.

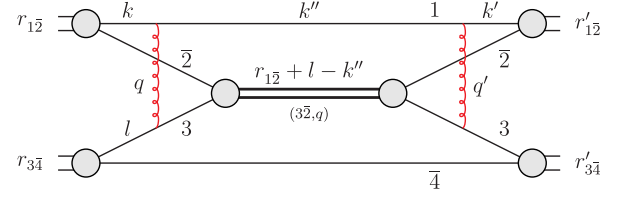


FIG. 11. Typical diagram contributing to Eq. (5.4), with two gluon exchanges, each corresponding to k_{13} . Double lines represent meson propagators, whose flavor and excitation quantum numbers are also indicated. The momentum routings of the quarks and the antiquarks are from right to left.

Λ^{-10} . The four loop momenta increase, by integration with respect to the $+$ components, this factor up to Λ^{-6} . Among the three gluon propagators, only two of them provide diverging singularities, transforming the latter factor into Λ^{-4} , which itself is balanced by the diverging factor Λ^4 coming from the external wave functions. The final quantity is therefore finite with the explicit presence of a gluon propagator carrying the momentum transfer of the process. It is verified that the sum of leading singularities of the nine contributions of expression (a), Eq. (5.3), vanishes in the limit $\Lambda \rightarrow \infty$. One is left with finite nonsingular contributions, which induce a generalized effective contact-type interaction at the mesonic level.

The next category in Eq. (5.2) is expression (b), Eq. (5.4), containing 18 contributions. A typical diagram of it is represented in Fig. 11. Here, we have the presence of only one type of meson propagators.

In Fig. 11, we have ten quark propagators, yielding the damping factor Λ^{-10} . We have four loop momenta, which, however, produce only a diverging factor Λ^3 , because of the presence of the meson propagator, which recuperates a damping factor Λ^{-1} (or prevents one damping factor from disappearing from a quark propagator). Among the two gluon propagators, only one of them produces a diverging singularity, transforming the damping factor into Λ^{-6} , which is balanced by the diverging factor Λ^6 coming from the six wave functions. One remains with the explicit presence of a gluon propagator, carrying the momentum transfer of the process. It is verified that the sum of leading singularities of the contributions of the nine diagrams, grouped around the present meson propagator $[(3\bar{2}, q)]$, and that of the other nine diagrams, grouped around the meson propagator $(1\bar{4}, p)$, vanish separately, leaving finite regular quantities, which induce generalized contact-type interactions at the mesonic level.

The expressions (d) and (e) of Eq. (5.2), given in Eqs. (5.6) and (5.7), have structures that are very similar to those of expressions (b) and (a), respectively, from which they are distinguished only by the difference in the connections of the gluon propagators. The same counting rules are applied and again a factorized gluon propagator is found. One again finds for the sum of contributions in

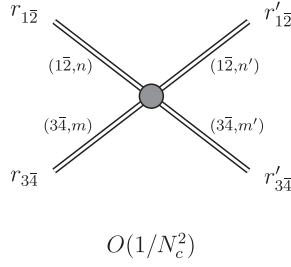


FIG. 12. Effective contact interaction resulting in the direct scattering amplitude $D1$ at order $1/N_c^2$. The flavor and excitation quantum numbers of the mesons are also indicated.

expressions (d) and (e) zero for the singular parts, with finite regular remainders leading to contact terms.

The resulting generalized contact term of the direct channel is represented in Fig. 12.

In summary, to order $1/N_c^2$, the meson-meson scattering amplitudes reduce, after the infrared cutoff is taken to zero, to three effective finite quantities: (1) and (2), generalized contact terms in the recombination and direct channels (Figs. 7 and 12), of order $1/N_c$ and $1/N_c^2$, respectively; (3), a sum of unitarity diagrams in the direct channels (Fig. 9), of order $1/N_c^2$, generated by the recombination-channel contact terms and the meson propagators.

VI. THE SCATTERING AMPLITUDES IN UNITARIZED FORM

The check of the finiteness of the scattering amplitudes at higher orders in $1/N_c$ is not a straightforward task, because of the continuous increasing of the number of the diagrams and the appearance of new categories of topologies. The latter, however, are expected to have weaker impact than those met above at large N_c . It is reasonable at this stage to stick to the approximation that consists in selecting from higher-order terms those which contribute to the unitarization of the scattering amplitudes. We already found that the theory has the tendency to reach that structure, by providing in the direct channels the unitarity diagrams generated by the recombination channel contact terms.

We consider that procedure by adopting two approximations. First, we consider for the external mesons the ground state mesons and we neglect in the unitarity diagrams the contributions of higher excited mesons. This is justified by the facts that the latter contribute with larger mass damping factors and have smaller contributions to the contact terms, due to the overlapping integrals between wave functions of the ground states and their radial excitations. Second, the contact terms are considered in their scalar approximation, which is expected to provide the dominant contribution, and are treated as constants. With these approximations, the unitarization procedure amounts to dealing with four coupled equations, which can be treated in matrix form.

Before proceeding further, we shall slightly change the convention of the scattering amplitude with respect to the one adopted earlier, by explicitly factorizing the coefficient i from it, as well as in the kernels of the related integral equations,

$$\mathcal{T} = iT' \equiv iT. \quad (6.1)$$

We shall continue using the same notation $\mathcal{T}$ for the newly defined one.

Designating by K_{R1} and K_{R2} the real contact terms of the recombination channels (unitarity requiring the equality $K_{R1} = K_{R2}$), by K_{D1} and K_{D2} those of the direct channels, and by $J_{1\bar{2},3\bar{4}} \equiv J_{D1}$ and $J_{1\bar{4},3\bar{2}} \equiv J_{D2}$ the two-meson loop functions, the scattering amplitudes and the kernels of the integral equations they satisfy can be represented in the following matrix forms:

$$\begin{aligned} \mathcal{T} &= \begin{pmatrix} \mathcal{T}_{D1} & \mathcal{T}_{R2} \\ \mathcal{T}_{R1} & \mathcal{T}_{D2} \end{pmatrix}, & K &= \begin{pmatrix} K_{D1} & K_{R2} \\ K_{R1} & K_{D2} \end{pmatrix}, \\ J &= \begin{pmatrix} J_{D1} & 0 \\ 0 & J_{D2} \end{pmatrix}. \end{aligned} \quad (6.2)$$

The expressions of the two-meson loop functions, in terms of the Mandelstam variable s , are

$$\begin{aligned} J_{a\bar{b},c\bar{d}}(s, M_{a\bar{b}}, M_{c\bar{d}}) &= \begin{cases} -\frac{i}{2\sqrt{-\lambda_{a\bar{b},c\bar{d}}(s)}} \left[1 - \frac{1}{\pi} \arctan \left(\frac{\sqrt{-\lambda_{a\bar{b},c\bar{d}}(s)}}{s - (M_{a\bar{b}}^2 + M_{c\bar{d}}^2)} \right) \right], & (M_{a\bar{b}} - M_{c\bar{d}})^2 < s < (M_{a\bar{b}} + M_{c\bar{d}})^2, \\ +\frac{1}{2\sqrt{\lambda_{a\bar{b},c\bar{d}}(s)}} \left[\pm 1 + \frac{i}{\pi} \ln \left(\frac{\sqrt{s - (M_{a\bar{b}} - M_{c\bar{d}})^2} + \sqrt{s - (M_{a\bar{b}} + M_{c\bar{d}})^2}}{\sqrt{s - (M_{a\bar{b}} - M_{c\bar{d}})^2} - \sqrt{s - (M_{a\bar{b}} + M_{c\bar{d}})^2}} \right) \right], & \text{Re}(s) > (M_{a\bar{b}} + M_{c\bar{d}})^2, \text{Im}(s) = \pm i\epsilon, \epsilon > 0, \end{cases} \end{aligned} \quad (6.3)$$

with

$$\lambda_{a\bar{b},c\bar{d}}(s) = (s - (M_{a\bar{b}} + M_{c\bar{d}})^2)(s - (M_{a\bar{b}} - M_{c\bar{d}})^2). \quad (6.4)$$

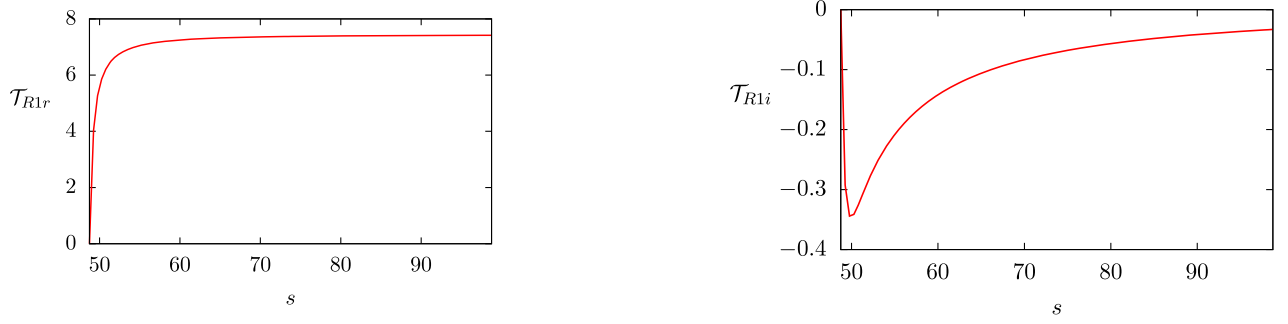


FIG. 13. Real and imaginary parts of the recombination scattering amplitude $\mathcal{T}_{R1}$ in the low energy region (left and right panels, respectively). The units of s and $\mathcal{T}$ are in $(\frac{2\sigma}{\pi}) \simeq 0.114 \text{ GeV}^2$.

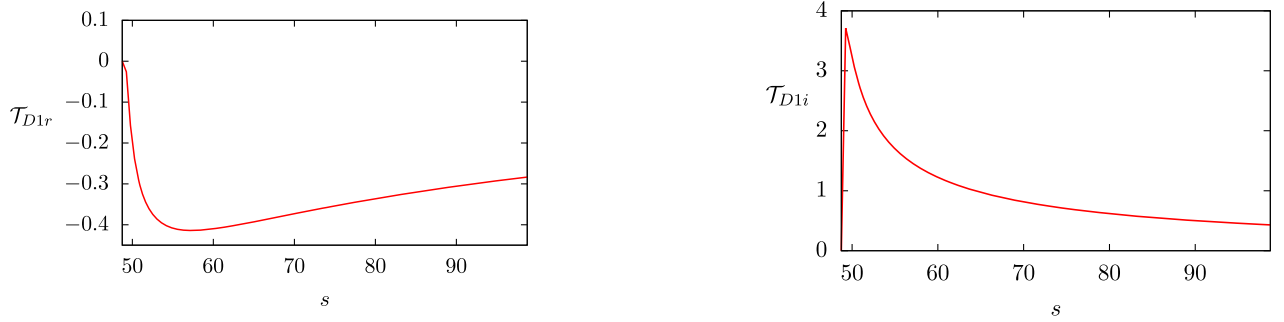


FIG. 14. Real and imaginary parts of the direct scattering amplitude $\mathcal{T}_{D1}$ in the low energy region (left and right panels, respectively). The units of s and $\mathcal{T}$ are in $(\frac{2\sigma}{\pi}) \simeq 0.114 \text{ GeV}^2$.

The unitarization operation is realized by the iteration of the kernel K with the aid of the two-meson loop functions,

$$\begin{aligned} \mathcal{T} &= (1 - iKJ)^{-1}K \\ &= \frac{1}{\det(1 - iKJ)} \begin{pmatrix} (1 - iK_{D2}J_{D2})K_{D1} + iK_{R2}J_{D2}K_{R1} & (1 - iK_{D2}J_{D2})K_{R2} + iK_{R2}J_{D2}K_{D2} \\ iK_{R1}J_{D1}K_{D1} + (1 - iK_{D1}J_{D1})K_{R1} & iK_{R1}J_{D1}K_{R2} + (1 - iK_{D1}J_{D1})K_{D2} \end{pmatrix}, \end{aligned} \quad (6.5)$$

with

$$\det(1 - iKJ) = 1 - iK_{D1}J_{D1} - iK_{D2}J_{D2} - K_{D1}J_{D1}K_{D2}J_{D2} + K_{R1}J_{D1}K_{R2}J_{D2}, \quad (6.6)$$

from which one may identify, with (6.2), the different components of $\mathcal{T}$.

We recall that the contact terms K_R and K_D are of order $1/N_c$ and $1/N_c^2$, respectively. A first approximation would consist in neglecting K_D in front of K_R . In this case, one has the simplified expressions

$$\begin{aligned} \mathcal{T} &= \begin{pmatrix} \mathcal{T}_{D1} & \mathcal{T}_{R2} \\ \mathcal{T}_{R1} & \mathcal{T}_{D2} \end{pmatrix} \\ &= \frac{1}{(1 + K_{R1}J_{D1}K_{R2}J_{D2})} \begin{pmatrix} iK_{R2}J_{D2}K_{R1} & K_{R2} \\ K_{R1} & iK_{R1}J_{D1}K_{R2} \end{pmatrix}. \end{aligned} \quad (6.7)$$

To have a qualitative idea of the behaviors of the scattering amplitudes, we consider the case of a system

made of $c\bar{u}s\bar{d}$ constituents ($1\bar{2}3\bar{4}$) in the isospin limit. In the $D1$ channel, the two mesons are $D^0\bar{K}^0$ and in the $D2$ channel, they are D^+K^- . Because of isospin symmetry, we have the mass equalities $M_{1\bar{2}} = M_{1\bar{4}}$, $M_{3\bar{4}} = M_{3\bar{2}}$, and the two-meson thresholds in both channels are the same: $(M_{1\bar{2}} + M_{3\bar{4}})^2$. The value of the contact-term coupling constant K_{R1} has been estimated from Eq. (B11), using the expressions of the meson wave functions, as obtained from the bound state equation (2.38), normalized according to the condition $\int_0^\infty dx \varphi_m^*(x) \varphi_n(x) = \delta_{mn}$. One finds $K_{R1} \simeq 7.5(\frac{2\sigma}{\pi})$. The behaviors of the real and imaginary parts of the scattering amplitudes $\mathcal{T}_{R1}$ and $\mathcal{T}_{D1}$, as functions of s , are presented in Figs. 13 and 14.

It is to be noted that the loop functions J [Eq. (6.3)] are singular at threshold. However, the unitarization operation

softens the behavior of the scattering amplitudes, which then vanish there. On the other hand, the loop functions vanish at high energies; therefore, the asymptotic behavior of the scattering amplitudes is governed by their leading terms in $1/N_c$: The real part of $\mathcal{T}_{R1}$ tends to K_{R1} , while its imaginary part vanishes; $\mathcal{T}_{D1}$ vanishes with its real and imaginary parts; had we kept in it the contact term K_{D1} , its real part would asymptotically tend to K_{D1} .

The real part of $\mathcal{T}_{D1}$ has a tiny positive value, with a peak near the threshold, before vanishing again (not visible with the scale of the figure); this also causes the appearance of the narrow peak in the imaginary part at the zero of the real part. However, these peaks have a purely kinematic origin and do not correspond to a pole of the scattering amplitude in the complex plane of s for values of $\text{Re}(s)$ above the threshold. More generally, we have not found resonances in the neighborhood of the two-meson thresholds in other cases of quark flavors. This might be due to the scalar nature of the interaction that is used in the evaluation of the contact term K_R .

VII. TETRAQUARK BOUND STATE EQUATION

Existence of tetraquark bound states can be searched for through the possible occurrence of pole-type singularities in the scattering amplitude expressions. These correspond to zeros of the denominator of $\mathcal{T}$ [Eq. (6.5)], which occur at the zeros of the determinant (6.6). One obtains the equation

$$1 - iK_{D1}J_{D1} - iK_{D2}J_{D2} - K_{D1}J_{D1}K_{D2}J_{D2} + K_{R1}J_{D1}K_{R2}J_{D2} = 0. \quad (7.1)$$

To have a qualitative idea of the content of this equation, we consider the case where the meson masses are the same in both channels $D1$ and $D2$: $M_{1\bar{2}} = M_{1\bar{4}}$, $M_{3\bar{4}} = M_{3\bar{2}}$. In that case, the two meson loop functions are equal, $J_{D1} = J_{D2} \equiv J_D$ and similarly for the direct-channel contact terms, $K_{D1} = K_{D2} \equiv K_D$; generally, by unitarity, one has $K_{R1} = K_{R2} \equiv K_R$. Equation (7.1) becomes

$$(1 - i(K_R + K_D)J_D)(1 + i(K_R - K_D)J_D) = 0. \quad (7.2)$$

K_D is of order $1/N_c^2$ and is expected to be smaller in modulus than $|K_R|$, which is of order $1/N_c$. Within this situation, and noticing that J_D below threshold is imaginary [Eq. (6.3)], one finds that whatever the sign of K_R is, Eq. (7.2) has generally a bound state solution in s . This conclusion is not modified in the more general case of unequal meson masses in the two channels $D1$ and $D2$.

Neglecting K_{D1} and K_{D2} in Eq. (7.1) and after introducing the expressions of J_{D1} and J_{D2} [Eq. (6.3)], the bound state equation takes the form

$$\begin{aligned} & \sqrt{16(-\lambda_{D1})(-\lambda_{D2})} \\ & - K_R^2 \left[1 - \frac{1}{\pi} \arctan \left(\frac{\sqrt{-\lambda_{D1}}}{(s - (M_{1\bar{2}}^2 + M_{3\bar{4}}^2))} \right) \right] \\ & \times \left[1 - \frac{1}{\pi} \arctan \left(\frac{\sqrt{-\lambda_{D2}}}{(s - (M_{1\bar{4}}^2 + M_{3\bar{2}}^2))} \right) \right] = 0, \quad (7.3) \end{aligned}$$

where $\lambda_{D1} = \lambda_{1\bar{2},3\bar{4}}$ and $\lambda_{D2} = \lambda_{1\bar{4},3\bar{2}}$ [Eq. (6.4)].

The value of K_R , which is determined from Eq. (B11), turns out to be sensitive to the flavor content of the system under study. We defer to a separate work the detailed analysis of this equation.

One main conclusion can be drawn, however, at the present stage. The coupling constant K_R in Eq. (7.3), being of order $1/N_c$, vanishes in the extreme limit $N_c \rightarrow \infty$ and hence the existing tetraquark approaches the nearest two-meson threshold and disappears there. This confirms Weinberg's conjecture [88] that if multiquark states exist, then they can only appear in nonleading terms in $1/N_c$. This is in contrast with the ordinary meson spectra, which remain almost unchanged in the large- N_c limit.

VIII. CONCLUDING REMARKS

The main result of the present work is the demonstration, in two-dimensional QCD, on the basis of an infrared regularization scheme and in the case of four different quark flavors, of the finiteness of the meson scattering amplitudes up to order $1/N_c^2$. The possibility of their unitarization allows us to investigate tetraquark spectroscopic problems and their internal dynamics in a more explicit framework. The results are free of external empirical parameters. The parameters of the theory are the coupling constant, or equivalently the string tension, which fixes the mass scale of the theory, and the quark masses. The equations that are obtained are fully relativistic and allow probing various energy regions with appropriate expansions.

One of the striking features of the infrared limiting procedure is the possibility of transforming a four-quark scattering process into an effective two-meson scattering process with contact-type interactions, representing here short-range interactions, generating unitarity diagrams and where the physical outcomes are expressible in terms of the meson wave functions and propagators. This is what one expects from QCD, however its two-dimensional version displays here explicitly the corresponding dynamical mechanism. The latter is dragged by the lowest-order quark-exchange diagrams in $1/N_c$, which, through unitarization, provide the dominant contributions.

The bound state equation (7.3) obtained from the on-mass shell scattering amplitudes does not, however, provide us with complete information about the bound state wave functions. Apart from the mass of the bound state, it gives information, by means of the residues of the bound state

pole in the four channels, about the couplings of the bound state to the two-meson clusters, which are represented by particular moments of the wave functions. To obtain full information about the wave functions, one has to go back to the integral equations (4.14) and (4.15) satisfied by the Green's functions and convert them into bound state equations through their homogeneous parts. Here, again, one faces the infrared divergence problem, which cannot, however, be solved in a parallel way with the procedure adopted for the scattering amplitudes, because of the differences in their structures. The way to reach infrared finite expressions is to pass to the instantaneous reduction of the equations and of the wave functions. This was the case for the quark-antiquark systems: it is the instantaneous wave function that is finite and leads to observable quantities. The problem, however, is more complicated in the present many-body case; simple global factorization of the wave function is no longer sufficient to reach that aim, because that initial ansatz is not self-reproducible. The correct method necessitates more complicated transformations, which gradually bring the above equations into instantaneously reduced finite forms. This aspect of the problem will be presented in a separate work.

The present study focused on the case of four different quark flavors, which offers the simplest theoretical framework, avoiding mixing problems with the ordinary meson sectors. The case containing hidden flavors, like $c\bar{c}$ or $b\bar{b}$, involves additional contributions, represented by quark-exchange diagrams in the direct scattering channels (cf. Refs. [61,63,64]), leading to mixings with ordinary meson sectors. To establish integral equations, one needs the inclusion of additional kernels or couplings with two-quark sectors, which leads to the enlargement of the space of the initially introduced Green's functions. This aspect of the problem requires a more specific treatment. Nevertheless, one can consider direct calculations of the scattering amplitudes with such diagrams; their main contribution starts at order $1/N_c$ and their finite part, in the limit of vanishing cutoff, reduces to contact terms plus meson-exchange contributions. Approximating also the latter by contact terms, the whole contribution can be incorporated in the kernels K_D , previously introduced [Eq. (6.2)], and through them into the unitarized scattering amplitudes. The only novelty is that the kernels K_D are now of order $1/N_c$ and enter into competition with K_R . In particular, contrary to the case of K_R , the sign of K_D , in case it is dominant, plays a crucial role for the existence of bound states.

The particular case when two quarks or two antiquarks have identical flavors, like cc or bb , also merits attention. Here, recombination and direct channels are identical and, therefore, one has a single-channel problem.

The ability of the two-dimensional theory to produce confinement and a linear meson spectrum at high energies, with a linear potential in coordinate space in the

nonrelativistic limit (the latter demonstrated more easily in the axial gauge), is an indication that some of the predictions obtained in multi-quark sectors might also be relevant in four dimensions. Two-dimensional models cannot, however, describe rotational motion, or transverse motion in light-cone bases. One is limited with sectors of pseudoscalar or scalar mesons with their radial excitations. Nevertheless, the mechanism of confinement and the explicit disappearance of infrared divergences in physical quantities in two dimensions might provide us with hints about similar mechanisms in four dimensions. Another point of relevance is the possibility of extracting from the spectrum of tetraquarks in two dimensions a more detailed description of their structure and their properties.

ACKNOWLEDGMENTS

The author thanks Wolfgang Lucha, Dmitri Melikhov, and Bachir Moussallam for stimulating discussions on the subject. This research has received financial support from the EU research and innovation programme Horizon 2020, under Grant Agreement No. 824093.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A

In this Appendix, we evaluate the contributions of the gluon propagators to the various types of Feynman diagrams.

We first consider the generic case of a gluon propagator acting on a wave function f by means of a convolution operation. We adopt for the gluon propagator the regularization method of introducing a small mass term λ in the denominator [Eq. (2.14)]. The structure of the resulting integral is of the type

$$I = \int_c^r dk \frac{f(k+p)}{2\pi k^2 + \lambda^2}, \quad (\text{A1})$$

where c and r are bounds coming from the spectral conditions satisfied by the wave function [cf. example of Eq. (2.33)]. The leading effect of the singularity of the gluon propagator is obtained by isolating the function $f(k+p)$ with its value at the position of the singularity, i.e., $f(p)$, and factorizing it outside the integral, obtaining

$$I = f(p) \int_c^r dk \frac{1}{2\pi k^2 + \lambda^2} + \int_c^r dk \frac{(f(k+p) - f(p))}{2\pi k^2}. \quad (\text{A2})$$

In the second integral, the mass term has been suppressed, since the integral is finite (with the principal value prescription). The first integral can be evaluated,

$$\int_c^r \frac{dk}{2\pi k^2 + \lambda^2} = \frac{1}{2\pi\lambda} \left(\arctan\left(\frac{r}{\lambda}\right) - \arctan\left(\frac{c}{\lambda}\right) \right). \quad (\text{A3})$$

In the limit $\lambda \rightarrow 0$, one has

$$\int_c^r \frac{dk}{2\pi k^2 + \lambda^2} \xrightarrow{\lambda \rightarrow 0} \frac{1}{4\lambda} (\varepsilon(r) - \varepsilon(c)) - \frac{1}{2\pi} \left(\frac{1}{r} - \frac{1}{c} \right), \quad (\text{A4})$$

where $\varepsilon(x) = \text{sgn}(x)$.

It is seen that the singularity shows up only when r and c are of opposite signs. In case one is interested in the leading term, it is then sufficient to replace the bounds by $\pm\infty$, according to the signs of r and c .

In many Feynman diagrams, some of the internal quark lines may not be subjected to the spectral conditions and may involve wider domains of integration, with negative p_- components. The latter, following the structure of the quark propagators [Eq. (2.18)], change the sign of the ie factor of the denominator and modify the evaluation of the corresponding Cauchy integral related to the p_+ component. It turns out that the resulting bounds of integration in integrals of the type of (A1) become now of the same sign and therefore do not contribute to the leading singular terms. For the evaluation of the latter, one may stick from the start to bounds satisfying everywhere the spectral conditions.

We next consider the case of several gluon propagators that appear in multiple integrals, in the direct channels, in convolution with each other. We concentrate on the treatment of the singular part of such integrals, since it is only that part that contributes in the finite infrared limit of multigluon diagrams that we meet in the case of direct processes of Sec. V.

A generic form of the integrals with n ($n \geq 2$) gluon propagators, after changes of variables by translation, is the following:

$$I_n(q) = \int \left(\prod_{i=1}^{n-1} \frac{dk_i}{2\pi (k_i^2 + \lambda^2)} \right) \frac{1}{((q - k_1 - k_2 - \dots - k_{n-1})^2 + \lambda^2)}, \quad (\text{A5})$$

where q is an external momentum, typically a momentum transfer between an ingoing meson and an outgoing one, of the form $q = r'_{12} - r_{12}$, for instance. The variables k_i ($i = 1, 2, \dots, n-1$) are themselves differences of momenta of neighboring quark or antiquark lines and, therefore, in reference to the integral (A4), their bounds, according to the spectral conditions of the wave functions and their related propagators, contain the point 0, with $r_i > 0$ and $c_i < 0$. It is only in the last propagator of Eq. (A5) that the bounds may play a crucial role.

Without taking into account the question of the bounds and making the calculation of the integral (A5) by

recursion, one finds $I_n(q) = \frac{1}{(2\lambda)^{n-1}} \frac{n}{(q^2 + (n\lambda)^2)}$. One notices the modification of the infrared parameter with the accompanying number of gluon propagators. In general, the diverging factor $1/(2\lambda)^{n-1}$ is canceled by a similar damping factor coming from the quark propagators and their integration.

Considering now the bounds of the variables k_i and taking for q^2 large values, one may reach domains of momenta where the sum $\sum_i k_i$ cannot cancel the momentum transfer q and, therefore, in these circumstances, the last propagator does not contribute to the leading singularity of the integral. Since the other singularities of k_i are located at the value 0, the final contribution of the last propagator would be $1/(q^2 + \lambda^2)$, while the other integrals would give the factor $1/(2\lambda)^{n-1}$. This result contradicts the one obtained above without taking into account the integration bounds and for which the behavior of the integral for large values of q^2 , without the diverging factor, would be n/q^2 and not $1/q^2$. It is obvious that the latter result is the correct one, since it takes into account the existence of the integration bounds. How does one reconcile, however, the two results for small values of q^2 , since in these domains the former result seems also correct, the bounds no longer playing a decisive role? The answer is related to the property that for small values of q^2 and in the limit of vanishing λ , one has the equivalence relation

$$\left(\frac{n\lambda}{q^2 + (n\lambda)^2} \right) \xrightarrow{\lambda \rightarrow 0} \frac{1}{\pi} \delta(q) \xrightarrow{\lambda \rightarrow 0} \left(\frac{\lambda}{q^2 + \lambda^2} \right). \quad (\text{A6})$$

We therefore conclude that the expression of the integral (A5) is

$$I_n(q) = \frac{1}{(2\lambda)^{n-1}} \frac{1}{(q^2 + \lambda^2)}. \quad (\text{A7})$$

We finally add the following remark. Apart from the diverging or singular expressions that we have found in the above calculations, one also finds finite nonsingular expressions, which do not disappear in the infrared limit $\lambda \rightarrow 0$ and which arise from the presence of various numerators in the integrals (A5), depending on the total momentum transfer, which we have factorized for simplicity. The existence of such terms should be taken into account, at least qualitatively, for the physical interpretation of the results. Generally, they produce, in the mesonic sectors, contact-type interactions.

APPENDIX B

This Appendix is devoted to the presentation of the details of the calculations of the various contributions to the meson-meson scattering amplitudes to order $1/N_c^2$ in the limit $\Lambda \rightarrow \infty$ (vanishing of the infrared cutoff).

We first consider the recombination channel scattering amplitude, represented by the diagrams of Fig. 6. To simplify the formulas, we shall adopt the following notations,

$$a'_i(k_{i-}) = \frac{m_i'^2}{2k_{i-}}, \quad a_i(k_{i-}) = \frac{m_i^2}{2k_{i-}}, \quad i = 1, \bar{2}, 3, \bar{4}, \quad (\text{B1})$$

a'_i being quantities that appear in the quark propagators [Eq. (2.18)] and the relation between m'^2 and m^2 is given in Eq. (2.19). We shall often omit, when no ambiguities arise, the mentioning of their momentum, the latter being easily

recognized from the corresponding graphs. The convention of Eq. (4.1) ($S_{a,-} \rightarrow S_a$) will be adopted. Furthermore, according to the analysis of the spectral conditions (cf. Sec. V), coming from the wave functions, and their incidence on the internal quark lines in the limit $\Lambda \rightarrow \infty$, the quarks and antiquarks of the loops are assumed to satisfy individually those conditions. This has a simplifying consequence on the expressions of the quark propagators, concerning the signs of the - components of the momenta.

Of particular help is the simplest formula for two-propagator integrations,

$$\begin{aligned} \int \frac{dk_+}{2\pi} S_1(k) S_2(k - r_{1\bar{2}}) &= \int \frac{dk_+}{2\pi} \frac{ik_-}{(k^2 - |k_-|\Lambda - m_1'^2 + i\epsilon)} \frac{i(k_- - r_{1\bar{2}-})}{((k - r_{1\bar{2}})^2 - |k_- - r_{1\bar{2}-}|\Lambda - m_2'^2 + i\epsilon)} \\ &= -\frac{i}{4} \frac{\theta(k_-(r_{1\bar{2}-} - k_-))}{(r_{1\bar{2}+} - \Lambda - a'_1(k_-) - a'_2(r_{1\bar{2}-} - k_-))}. \end{aligned} \quad (\text{B2})$$

Diagram (a) of Fig. 6 contains the following integral:

$$\begin{aligned} \tilde{I}_{(a)} &= \int \frac{dk_+}{2\pi} S_1(k) S_2(k - r'_{1\bar{2}}) S_3(r_{3\bar{2}} - r'_{1\bar{2}} + k) S_4(k - r_{1\bar{4}}) \\ &= \frac{i^4}{16} \int \frac{dk_+}{2\pi} \frac{1}{(k_+ - a'_1 - \frac{\Lambda}{2} + i\epsilon)} \frac{1}{(k_+ - r'_{1\bar{2}+} + a'_2 + \frac{\Lambda}{2} - i\epsilon)} \\ &\quad \times \frac{1}{(r_{3\bar{2}+} - r'_{1\bar{2}+} + k_+ - a'_3 - \frac{\Lambda}{2} + i\epsilon)} \frac{1}{(k_+ - r_{1\bar{4}+} + a'_4 + \frac{\Lambda}{2} - i\epsilon)}. \end{aligned} \quad (\text{B3})$$

(The full diagram contribution, including the external wave functions and the integration over k_- , will be designated by $I_{(a)}$.) After integrating with respect to k_+ with the Cauchy theorem and expanding terms with respect to $1/\Lambda$, one finds

$$\tilde{I}_{(a)} = \frac{2i}{16\Lambda^3} \frac{[1 - (r_+ - a'_1 - a'_2 - a'_3 - a'_4)/(2\Lambda)]}{(1 - (r'_{1\bar{2}+} - a'_1 - a'_2)/\Lambda)(1 - (r'_{3\bar{4}+} - a'_3 - a'_4)/\Lambda)(1 - (r_{1\bar{4}+} - a'_1 - a'_4)/\Lambda)(1 - (r_{3\bar{2}+} - a'_3 - a'_2)/\Lambda)}. \quad (\text{B4})$$

Notice that we have not expanded the denominator in terms of $1/\Lambda$. The reason is that each factor of the denominator is equal to the contribution of the finite corrective part of the corresponding wave function, according to Eq. (2.47), the latter contributing at the end by a multiplicative factor. This shows that the finite corrective part of the wave functions will be canceled and will not contribute to the result.

Equation (B4) shows that $\tilde{I}_{(a)}$, after multiplication by the factor Λ^4 coming from the four multiplicative external wave functions, leads to a linearly divergent quantity in Λ , plus a finite part represented by the contribution of the second term of the numerator of the fraction.

The integrals coming from diagrams (b) and (c) of Fig. 6 can be calculated in a similar way. Here, one has two loop momenta, k and k' , and six quark propagators. The integration with respect to the + component of each of them will concern three quark propagators. One then considers the contribution of the gluon propagator. We shall concentrate for the moment on the singular part of it, whose contribution is taken into account with the prescriptions of Eqs. (A1)–(A4), or (2.34) and (2.35). The singular part simply equates the components k_- and k'_- . We shall designate the corresponding integrals by $\tilde{I}_{(b/c)}$. The contributions coming from the regular part of the gluon propagator, having a different structure, will be treated separately. One finds

$$\tilde{I}_{(b)} = \tilde{I}_{(c)} = -\frac{i}{16\Lambda^3} \frac{1}{(1 - (r'_{1\bar{2}+} - a'_1 - a'_2)/\Lambda)(1 - (r'_{3\bar{4}+} - a'_3 - a'_4)/\Lambda)(1 - (r_{1\bar{4}+} - a'_1 - a'_4)/\Lambda)(1 - (r_{3\bar{2}+} - a'_3 - a'_2)/\Lambda)}. \quad (\text{B5})$$

One observes that the diverging part of $\tilde{I}_{(a)}$ is canceled by the sum of $\tilde{I}_{(b)}$ and $\tilde{I}_{(c)}$,

$$\tilde{I}_{(a)} + \tilde{I}_{(b)} + \tilde{I}_{(c)} = -\frac{i}{16\Lambda^4} (r_+ - a'_1 - a'_2 - a'_3 - a'_4). \quad (\text{B6})$$

[There is no need now to keep the denominators of Eqs. (B4) and (B5), since it is only the diverging parts of the wave functions that can contribute to maintain the quantity different from zero.]

One also has to include, according to Eq. (A4), with the diverging contribution of the gluon propagator, a finite part coming from the integration bounds. The latter contribution transforms, as in the case of the wave equation (2.38), m'^2 into m^2 [Eq. (2.19)]. One obtains

$$\tilde{I}_{(a)} + \tilde{I}_{(b)} + \tilde{I}_{(c)} = -\frac{i}{16\Lambda^4} (r_+ - a_1 - a_2 - a_3 - a_4). \quad (\text{B7})$$

The term in the parentheses in Eq. (B7) can be divided into four parts,

$$\begin{aligned} & r_+ - a_1 - a_2 - a_3 - a_4 \\ &= \frac{1}{2} [(r'_{1\bar{2}+} - a_1 - a_2) + (r'_{3\bar{4}+} - a_3 - a_4) \\ &+ (r_{1\bar{4}+} - a_1 - a_4) + (r_{3\bar{2}+} - a_3 - a_2)], \end{aligned} \quad (\text{B8})$$

each representing a part of the wave equation satisfied by the corresponding wave function having the same quark content. Equation (B7) becomes

$$\begin{aligned} \tilde{I}_{(a)} + \tilde{I}_{(b)} + \tilde{I}_{(c)} &= -\frac{i}{32\Lambda^4} [(r'_{1\bar{2}+} - a_1 - a_2) \\ &+ (r'_{3\bar{4}+} - a_3 - a_4) + (r_{1\bar{4}+} - a_1 - a_4) \\ &+ (r_{3\bar{2}+} - a_3 - a_2)]. \end{aligned} \quad (\text{B9})$$

We next consider the contribution of the regular part of the gluon propagator. Considering, for definiteness, diagram (b) of Fig. 6, we notice that we have now integrals with respect to k_- and k'_- , with the gluon propagator momentum q_- equal to $(k'_- - k_-)$. Considering k_- as fixed, the integration with respect to k'_- will concern the wave functions $\varphi_{1\bar{2}}(r'_{1\bar{2}-}, k'_-)$ and $\varphi_{3\bar{2}}(r_{3\bar{2}-}, r'_{3\bar{4}-} - r_{1\bar{4}-} + k'_-)$, which we shall shorten by the notations $\varphi_{1\bar{2}}(k'_-)$ and $\varphi_{3\bar{2}}(k'_-)$. A similar shorthand notation is also used for the two other wave functions, $\varphi_{1\bar{4}}(k_-)$ and $\varphi_{3\bar{4}}(k_-)$.

After having integrated with respect to k_+ and k'_+ , one remains with four quark propagators, each having the damping factor $(-\Lambda^{-1})$. The global damping factor Λ^{-4} is then canceled by the diverging factor Λ^4 , coming from the external wave functions. The contribution of the regular part of the gluon propagator is then

$$\begin{aligned} I_{(b),gl} &= 16\pi^2 (-2\sigma) \int \frac{dk_-}{2\pi} \varphi_{1\bar{4}}(k_-) \varphi_{3\bar{4}}(k_-) \\ &\times \int \frac{dk'_-}{2\pi} \frac{i}{(k'_- - k_-)^2} \\ &\times (\varphi_{1\bar{2}}(k'_-) \varphi_{3\bar{2}}(k'_-) - \varphi_{1\bar{2}}(k_-) \varphi_{3\bar{2}}(k_-)). \end{aligned} \quad (\text{B10})$$

The contribution of diagram (c) has a similar expression, obtained by appropriate interchange of indices.

The integration bounds of the variable k_- in (B9) and of k_- and k'_- in (B10) have to respect the spectral conditions of the wave functions. However, each wave function depends, in general, on a different total momentum $r_{a\bar{b}-}$ and on a different quark momentum, k_{a-} , with a combined dependence upon the variable $x_{a\bar{b}} = k_{a-}/r_{a\bar{b}-}$. Converting for instance the integration momentum variable k_- of Fig. 6(a) into the variable $x_{1\bar{2}}$, one is left in the other wave functions with the presence of the ratios $r'_{1\bar{2}-}/r_{1\bar{4}-}$, $r'_{1\bar{2}-}/r_{3\bar{2}-}$ and $r'_{1\bar{2}-}/r'_{3\bar{4}-}$, which, generally are not equal to 1. The domain of integration of $x_{1\bar{2}}$ is $[0, 1]$, but taking into account the values of the latter ratios, which themselves may change according to the external conditions, it is restricted to a smaller domain.

The amplitudes resulting from expressions (B9) and (B10) are equivalent to generalized momentum-dependent contact interactions, for they do not induce long-range type interactions, remaining finite in various momentum limits. However, the exact analytic calculation of the restricted domain of integration and of its momentum dependence is a rather complicated task, which, in its general case demands detailed numerical evaluations. Furthermore, we will be interested in the present work by the unitary summation of the low-order amplitudes to take into account the most important higher-order effects. This involves integrations of meson-loop momenta; the presence of momentum transfer variables inside the wave functions, which generally are known numerically, does not allow the realization of such a summation in analytic form. It is therefore preferable to stick, at the present stage, to the scalar approximation of the contact interactions. This can be done by adopting for the external momentum ratios the following simplifications: $r'_{1\bar{2}-}/r_{1\bar{4}-} = 1$, $r'_{1\bar{2}-}/r_{3\bar{2}-} = 1$ and $r'_{1\bar{2}-}/r'_{3\bar{4}-} = 1$. In terms

of the Mandelstam variables, they would correspond to $t = 0$ and $u = 0$ and in the equal-mass case of mesons to $s = 4M^2$, where M is the mass of the external mesons. For the case of unequal masses, the above choices of momentum ratios may not correspond to physical values. However, that simplification concerns only the internal domain of integration of the wave functions and not the external momenta, which might still appear in other multiplicative expressions.

Another remark concerns the case of bound states. For values of the variable s lying below two-meson thresholds,

some of the momentum ratios mentioned above may become complex. This is only an apparent difficulty, since the analyticity property in s , which should allow continuation to the bound state domain, concerns the scattering amplitude itself and not intermediate or auxiliary variables. One therefore has first to calculate the scattering amplitude in its physical domain and then proceed to its continuation to the bound state region.

Taking into account the abovementioned simplifications, the final expression for the contribution of the diagrams of Fig. 6 is,

$$\begin{aligned}
 I &= I_{(a)} + I_{(b)} + I_{(c)} \equiv iN_c K_{R1} \\
 &= i2\pi \left\{ \int_0^1 dx \varphi_{1\bar{2}}(x) \varphi_{3\bar{4}}(x) \varphi_{1\bar{4}}(x) \varphi_{3\bar{2}}(x) \left[(M_{1\bar{2}}^2 + M_{3\bar{4}}^2 + M_{1\bar{4}}^2 + M_{3\bar{2}}^2) - 2 \frac{(m_1^2 + m_3^2)}{x} - 2 \frac{(m_2^2 + m_4^2)}{(1-x)} \right] \right\} \\
 &\quad - i4\pi \left(\frac{2\sigma}{\pi} \right) \left\{ \int_0^1 dx \varphi_{1\bar{4}}(x) \varphi_{3\bar{4}}(x) \int_0^1 dx' \frac{1}{(x' - x)^2} (\varphi_{1\bar{2}}(x') \varphi_{3\bar{2}}(x') - \varphi_{1\bar{2}}(x) \varphi_{3\bar{2}}(x)) \right. \\
 &\quad \left. + \int_0^1 dx \varphi_{1\bar{2}}(x) \varphi_{3\bar{2}}(x) \int_0^1 dx' \frac{1}{(x' - x)^2} (\varphi_{1\bar{4}}(x') \varphi_{3\bar{4}}(x') - \varphi_{1\bar{4}}(x) \varphi_{3\bar{4}}(x)) \right\}, \tag{B11}
 \end{aligned}$$

where M_{ab}^2 are the external meson masses squared. A similar expression is also valid for the recombination amplitude R_2 , which is obtained from R_1 by the interchanges $(1\bar{2})(3\bar{4}) \leftrightarrow (1\bar{4})(3\bar{2})$.

We next consider the direct channel scattering amplitude and the corresponding parts, represented in Eqs. (5.2)–(5.7).

The way of calculating expression (c), represented for one of its parts in Fig. 8, has been explained in the main text. The last technical point that remains is the evaluation of the integral with respect to the loop momentum l_+ . The momentum l appears in the meson propagators and in two quark propagators in each recombination diagram, after the integrations with respect to the other momenta have been done. We recall that the limit $\Lambda \rightarrow \infty$ should not be taken before the last integration is done (at least for the quantities depending on l). The integration with respect to l_+ receives two types of contribution: a first one comes from the meson propagators, a second one from the quark propagators.

The meson propagators do not contain Λ . In analogy with (B1), we introduce, for the meson having the flavor and excitation quantum numbers $(a\bar{b})$ and n , the following notation:

$$A_{a\bar{b},n}(l_{a\bar{b}-}) = \frac{r_{a\bar{b},n}^2}{2l_{a\bar{b}-}}, \tag{B12}$$

where $l_{a\bar{b}}$ is a generic momentum, carried by the meson, and $r_{a\bar{b},n}^2$ is the mass squared of the meson. The contribution coming from one of the meson propagators produces with the other meson propagator the term

$$\begin{aligned}
 J_{1\bar{4},3\bar{2}}(r, l_-) &= \frac{i}{4l_-(r_- - l_-)} \\
 &\quad \times \frac{1}{[r_+ - A_{1\bar{4},p}(l_-) - A_{3\bar{2},q}(r_- - l_-) + i\epsilon]}, \tag{B13}
 \end{aligned}$$

which is the two-meson once-integrated loop contribution. The Λ dependence of the quark propagators of the recombination-diagram parts is not modified by this contribution. Taking then the limit $\Lambda \rightarrow \infty$, one finds a finite contribution for each recombination part on both sides of the meson loop contribution, given by expressions of the type of Eq. (B11), where the external momenta contained in the meson wave functions have to be fixed according to the figure. In particular, the momenta of the wave functions coming from the internal meson propagators depend on l_- . Because of the spectral conditions that are satisfied at the quark-propagator level, the integration bounds of l_- are 0 and r_- . r_+ is fixed by the external ingoing or outgoing meson momenta; for instance, $r_+ = r'_{1\bar{2}+} + r'_{3\bar{4}+} = r_{1\bar{2},n'}^2 / (2r'_{1\bar{2}-}) + r_{3\bar{4},m'}^2 / (2r'_{3\bar{4}-})$. l_- itself enters in the integration bounds of the internal momenta of the recombination-channel parts of the effective diagram. In general, one has an off-energy type integration.

The contribution coming from the quark propagators during the integration of l_+ has two effects: (i) the degree of the damping factor in Λ decreases simultaneously by one power in each recombination part on both sides of the meson propagators; (ii) each meson propagator catches one damping power in Λ . The global damping power in Λ

remains therefore the same (Λ^{-8}). When the limit $\Lambda \rightarrow \infty$ is taken, the quark and meson propagators disappear in favor of the factor Λ^{-8} . One remains, at leading order, with the difference of two equal quantities, which reduces to zero. Therefore, expression (c) produces in the limit $\Lambda \rightarrow \infty$ only the effective two-meson loop contribution between the two effective recombination contact terms (Fig. 9).

Expressions (a), (b), (d), and (e) of Eqs. (5.2)–(5.4), (5.6), and (5.7) can be evaluated with the same methods and power countings as for the cases met before. The calculation of integrals involving several gluon propagators is done in Appendix A [Eqs. (A5) and (A7)]. The only novelty with respect to the previous cases is that in the corresponding diagrams one finds at the end a finite quantity, factorized by a gluon propagator carrying the momentum transfer of the process, which, therefore, is no longer involved in the internal integrations [cf. Eq. (A7)]. Such a term, considered separately, would produce long-range forces in coordinate space. It is therefore necessary to check, in each category of diagrams mentioned above, the cancelation of such terms or their reduction to short-range type expressions.

The detailed calculation of the contributions of the corresponding diagrams shows that the sum of contributions of terms factorized with an external-type gluon propagator in each category, (a), (b), (d), (e), is equal to zero. As emphasized at the end of Appendix A, one still has contributions coming from finite nonsingular terms, which play the role of contact-type interactions in the effective meson theory. Because of the big number of such contributions, the precise evaluation of the expression of the contact term in terms of meson wave functions requires lengthy calculations that go beyond the limitations of the present work. We shall be content with admitting, in addition to the unitarity diagrams resulting from the category (c) above, the existence of such an interaction in the direct channels.

Finally, let us mention that, according to the way of calculation, one may find in some particular diagrams, such as in category (d), instead of a gluon propagator, a delta function of the momentum transfer. In such cases, one may use the equivalence relation (A6), to convert the delta function into a propagator in the infrared limit $\lambda \rightarrow 0$.

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