

Pathway to decay and fission of orthosymplectic quiver theories

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We present an algorithm to extract the Coulomb branch Hasse diagram of orthosymplectic three-dimensional (3D) $\mathcal{N} = 4$ quiver gauge theories. The algorithm systematically predicts all descendant theories arising from Coulomb branch Higgsing, thereby detailing the stratification of the symplectic singularity defined by the initial Coulomb branch. Leveraging the Lie algebra isomorphism $\mathfrak{su}(4) \cong \mathfrak{so}(6)$, we validate our algorithm via the 3D mirror of the 4D theories of class $\mathcal{S}$ of such type. This comparison involves moduli spaces that admit both orthosymplectic and unitary quiver realizations, the latter being well understood via standard techniques such as decay and fission. Higgsing on the Coulomb branch of the 3D mirror or the magnetic quiver translates to Higgs branch renormalization group flows of the corresponding higher-dimensional superconformal field theories. Thus, we benchmark our method via Higgsing 6D $\mathcal{N} = (1, 0)$ D-type orbi-instanton theories, predicting novel Higgsing patterns involving products of interacting fixed points, and class $\mathcal{S}$ theories of type $\mathfrak{so}(2N)$, demonstrating Higgsing to products of theories of types specified by Levi subalgebras of $\mathfrak{so}(2N)$.

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I. INTRODUCTION

The connection between the Higgsing patterns on the Higgs branch of supersymmetric quantum field theories (SQFTs) with eight supercharges and the stratification of symplectic singularities has been elucidated in greater depth following [1]. Physically, the Englert-Brout-Higgs-Guralnik-Hagen-Kibble mechanism [2–5], commonly referred to as the Higgs mechanism, describes the process of a scalar field acquiring a vacuum expectation value (VEV), breaking the gauge group G to a subgroup $H \subset G$.

In SQFTs with eight supercharges, the hypermultiplet contains four scalars that are rotated under an $SU(2)$ factor of the R-symmetry. The region of the moduli space where these scalars acquire a VEV, called the Higgs branch, defines a variety which is both a hyper-Kähler [6] and a symplectic singularity [7]. Mathematically, a symplectic singularity X admits a finite stratification [8] in *symplectic leaves*, X_i ,

partially ordered by inclusion $X_0 \subset X_1 \subset \cdots \subset X_n$, with $X_0 = \{\emptyset\}$ and $X_n = X$. Each stratum is such that the singular part of X_i is in X_{i-1} , and its normalization is a symplectic singularity. Moreover, near each stratum, X_i , the variety admits a local decomposition in the product of the closed strata itself and a transverse slice T_i , which is itself a symplectic singularity. The physical and mathematical description interpolates the Higgsing pattern with the stratification in symplectic leaves. In a physical theory, each leaf X_i is spanned by the distinct set of VEVs that Higgs the initial theory to a specific residual theory. The transverse slice T_i (to the entire space X) is then the moduli space of the residual theory. This information can be effectively encoded in a Hasse diagram, which captures the structure of a partially ordered set after transitive reduction.

While in three and four dimensions, the Higgs branch $\mathcal{H}$ is “classical,” or at most well behaved, and can be computed directly once given the physical content of the theory. In five and six dimensions, this space receives infinite corrections from instantons; therefore, it is necessary to resort to a method that can resum all these contributions. A solution is to utilize what is known as the magnetic quiver for the Higgs branch [9,10], where the demanding problem of summing instantons is traded for the much better under-control task of resumming monopoles in three-dimensional (3D) $\mathcal{N} = 4$ Lagrangian theories. This technique consists of engineering a set of 3D $\mathcal{N} = 4$ magnetic theories M_i whose Coulomb branch $\mathcal{C}$ is

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isomorphic to the Higgs branch $\mathcal{H}$ of the original *electric* theory E ,

$$\mathcal{H}_{3/4/5/6D}(E) \cong \bigcup_i \mathcal{C}_{3D}(M_i). \quad (1.1)$$

In fact, due to Poincaré duality in the 3D, the Abelian gauge bosons can be dualized to (periodic) scalars, thus giving the space spanned by the vacuum expectation values of the vector multiplet scalar fields, known as the Coulomb branch, and also giving the structure of a symplectic singularity. This variety is parametrized by monopole operators that get “dressed” by gauge invariant operators of the residual massless theory in a nontrivial monopole background; all of which is repackaged in the monopole formula [11].

Such monopole operators [12–14] have emerged as a crucial class of half Bogomol’nyi-Prasad-Sommerfield (half-BPS) operators that are utilized to provide a quantum-corrected description of the 3D $\mathcal{N} = 4$ Coulomb branch. Given a UV G gauge theory, (bare) monopole operators are in one-to-one correspondence with the highest weight representations of the Goddard-Nuyts-Olive (GNO) dual group $G^\vee$ [15] due to a quantization condition [16]. A given monopole operator is, moreover, labeled by its conformal dimension (equal to the R-symmetry charge) and its charges under Coulomb branch symmetries, provided such are at least partially manifest in the UV description. Most notably, monopole operators play a key role in the phenomenon of the enhancement of the Coulomb branch global symmetry in the infrared (see [14,17,18] and subsequent works). The rapid development of Coulomb branch techniques has laid the foundation for the study of Higgs branches via magnetic quivers.

Although the magnetic quiver technique has been fruitfully applied to a plethora of cases, studying the stratification of the variety, and thus the Higgsing pattern of the theory, requires an additional ingredient. The key point is defining a *quiver algorithm*, which is a generic term for any procedure that allows for the extraction of both the leaves and the transverse slices that characterize the singularity, resorting only to its formulation in terms of quiver gauge theory. While various algorithms have been proposed [19–23], a full solution has been found only for unitary quiver gauge theories. In [21,22] it was shown that studying the stratification of the Coulomb branch moduli space, a 3D $\mathcal{N} = 4$ quiver theory Q , with only unitary gauge nodes, can only undergo two possible processes:

- (i) *Decay* to a quiver Q' having the same shape as Q but gauge nodes of lower ranks.
- (ii) *Fission* to a product of quivers Q_1 and Q_2 .

The resulting quivers correspond to magnetic quivers for the partially Higgsed theory (i.e., they keep track of the transverse slices) and are all *good* theories in the sense of [17], that is, they all have monopole operators whose conformal dimension sits above the unitarity bound, guaranteeing that the UV and IR R-symmetries are the same.

In this paper, we propose an extension of this algorithm to the realm of less-understood orthosymplectic quiver gauge theories. Framed orthosymplectic quivers (i.e., quivers with flavor nodes) played a pivotal role in realizing 3D mirror symmetry via S-duality on type IIB brane configurations with orientifold 3-planes [17,24–29]. In contrast, unframed orthosymplectic quivers (the class relevant for this work) appeared, for example, as 3D mirrors of class $\mathcal{S}$ theories [30,31] and, more generally, as magnetic quivers of higher-dimensional theories [20,32–37]. Similar to their unitary counterparts, foliating the Coulomb branch forces the orthosymplectic quiver to undergo either decay or fission. Special to this case is that an orthosymplectic quiver does not need to remain of this type. It can, in fact, undergo *unitarization* and become a unitary theory, or more generically, fission to a product of an orthosymplectic quiver and a unitary one. Moreover, as expected from [35], an orthosymplectic quiver can decay to an orthosymplectic quiver whose Coulomb branch is a product.

The results of this work provide concrete predictions for the Higgs branch renormalization group (RG) flows of higher-dimensional superconformal field theories (SCFTs) that are both nonperturbative and non-Lagrangian. These flows, which are often difficult to access using conventional methods, are elucidated via the decay and fission patterns of their associated 3D magnetic quivers. We demonstrate the utility of this approach in two key contexts: First, for 4D $\mathcal{N} = 2$ theories of class $\mathcal{S}$, where our algorithm recovers and extends known Higgsing patterns. Second, for 6D $\mathcal{N} = (1, 0)$ D-type orbi-instanton theories, where it leads to new predictions involving interacting fixed points and novel patterns of symmetry breaking.

The paper is structured as follows. In Sec. II, we explain the orthosymplectic decay and fission algorithm that we propose and provide a practical example of the algorithm and its range of applicability. In Sec. III, we introduce class $\mathcal{S}$ theories of types $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ and show that the algorithm predicts the same result for the orthosymplectic realization as the unitary one. Next, in Sec. IV, we provide some deeper explanation for the physics underpinning the proposed algorithm. Then, in Sec. V, we benchmark the algorithm against quivers and Hasse diagrams obtained independently, focusing, in particular, on a new result in 6D $\mathcal{N} = (1, 0)$ D-type orbi-instanton theories. Finally, we summarize our results and discuss some future avenues for research in Sec. VI.

II. THE ORTHOSYMPLECTIC DECAY AND FISSION ALGORITHM

The goal of this section is to present a set of heuristic rules governing the decay and fission of orthosymplectic quivers, analogous to the rules presented for unitary quivers in [21,22]. Before stating these rules, we outline the physical logic that motivates them, and we leave a detailed discussion of the derivation for Sec. IV. From the perspective of the 3D magnetic quiver theory, Q , decay and fission correspond to

RG flows triggered by the VEVs of the Coulomb branch operators. Since such (dressed or bare) monopole operators are IR objects with no known UV Lagrangian counterpart for non-Abelian G , tracking these flows is highly nontrivial. To study them, we draw on insights from 3D mirror symmetry and from well-understood Higgs branch flows involving nilpotent orbit closures.

From the complementary viewpoint of the higher-dimensional theory, E , Higgs branch RG flows are a key tool for mapping out the landscape of theories with eight supercharges. These flows are often encoded in geometric features of a space Y on which E is realized via string/M/F theory. In this way, the decay and fission of the magnetic quiver Q can both reflect and predict Higgs flows in the higher-dimensional parent theory E . For instance, it has long been known that class $\mathcal{S}$ theories admit Higgsings that align with nilpotent orbit closures, but decay and fission of unitary quivers have uncovered more general flows. The same philosophy applies here to orthosymplectic quivers. By analyzing the auxiliary 3D theory, Q , we gain powerful insight into the structure and dynamics of E .

In the opposite direction, one of the powerful features of the higher-dimensional theories arising from string theory is geometrization: subtle nonperturbative physics can be captured by the geometry of the compactification space Y . Specifically, certain aspects of the Higgs branch can be

understood through the geometric properties of Y . This alternative, geometric approach to the Higgs branches of higher-dimensional theories, E , also underpins our motivation for developing and testing the proposed rules in such contexts. For instance, in six dimensions, the Higgs branch corresponds to the complex structure deformations of Y . This geometric perspective allows us to cross-check the validity of the proposed decay and fission rules, going beyond the derivation from brane constructions in Sec. IV, by comparing them with geometric data.

The rules we formulate are thus extracted and motivated from known examples of Higgs branch flows across dimensions three to six. For clarity and concreteness, we first present the rules in their entirety. Their physical origin and justification are then developed in later sections, after which we apply the algorithm to extract new predictions.

In the introduction, we have stressed the importance of having an algorithm that is able to predict the stratification of the Coulomb branch of a 3D $\mathcal{N} = 4$ quiver theory. In the remainder of this work, we refer to a quiver theory by the following standard dictionary:

- (i) Circular nodes are gauge nodes. They are depicted in a different color corresponding to the associated gauge algebra. In this paper, we fix all the global forms, and we implicitly use the color coding to symbolize the gauge group¹:

$$\text{red circle}_k \leftrightarrow SO(k) \quad , \quad \text{blue circle}_{2k} \leftrightarrow USp(2k) \quad , \quad \text{white circle}_k \leftrightarrow U(k). \quad (2.1)$$

- (ii) Square nodes denote the presence of a flavor symmetry algebra. The same color coding used for the gauge nodes is employed for the flavor algebras.
- (iii) An edge between a square and a circular node signals the presence of hypermultiplets in the bi-fundamental representation of the gauge and flavor symmetry algebras.

From [17], we have a prescription, based only on the quiver's shape, to assess whether the conformal dimension of the theory's monopole operators falls within the unitarity bound. The quantity that controls this property is called the *balance*, usually denoted by b . It depends on the flavors N_f , i.e., the number of (full) hypermultiplets, a certain gauge node G sees

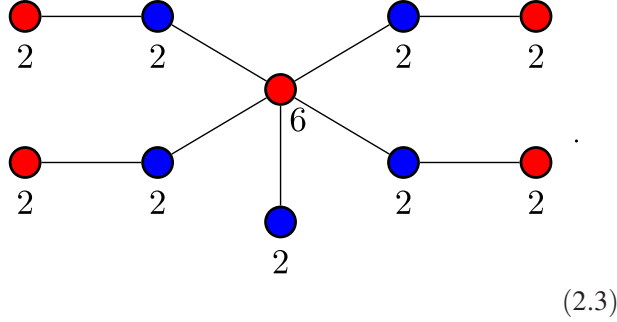
$$b = \begin{cases} N_f - 2N & \text{iff } G = U(N), \\ N_f - N + 1 & \text{iff } G = SO(N), \\ N_f - 2N - 1 & \text{iff } G = USp(2N). \end{cases} \quad (2.2)$$

When $b \geq 0$, the node is *good*. In this case, all the monopole operators are above the unitarity bound. When $b = 0$, the node is referred to as *balanced*. Such a node is

still good, but it admits an operator contributing to the flavor symmetry, leading to an enhancement. The case $b = -1$ is called *ugly*; here, in the unitary case, a monopole operator exists at the unitarity bound, implying that this is a decoupled twisted hypermultiplet in disguise. For orthosymplectic nodes, $b = -1$ already falls within the *bad* category. Lastly, when $b < -1$, the node is *bad* and admits monopole operators below the unitarity bound. This signals that the R-symmetry in the UV is not the one in the IR. Therefore, we are counting operators under the wrong symmetry. The good, ugly, and bad classification, albeit being a local statement for each gauge node, is then inherited by the full theory. If no bad or ugly nodes are encountered, the theory is good. However, when making global statements on the goodness of a quiver theory, it may happen that extended monopoles, i.e., monopoles valued in the magnetic lattice of multiple gauge group factors, may

¹Different choices of the global form of a gauge group affect the weight lattice of the GNO-dual group, which labels the monopole operators parametrizing the Coulomb branch (see, for example, [38,39]).

render an apparently good theory bad. This is a known case of affine Dynkin-shaped unitary quiver theories whose Coulomb branch is related to the moduli space of instantons of the respective Dynkin algebra in flat space [40]. To elucidate this discussion with an example, we depict the following good orthosymplectic quiver:



Here, we can see that the $USp(2)$ gauge node at the bottom of the quiver, as well as the four $SO(2)$ nodes, are balanced, and the remaining gauge nodes are all good. We return to this quiver as an example anon.

The orthosymplectic quivers we discuss in this paper have a special feature compared to their unitary counterparts. The monopole operators can take charges both in an integer lattice *and* a shifted lattice where all the charges are half-integer. In the unitary case, the goodness condition on each gauge node is equivalent to the statement that no monopole charged in the integer lattice has an R-charge below the unitarity bound. In particular, in the orthosymplectic case, monopoles charged in the half-integer lattice can fall below the unitarity bound in a way that is invisible to the balance of the individual gauge nodes. A more accurate notion of the scope of applicability of our algorithm is when the Coulomb branch is such that it contains exactly one singular point.² Although this usually corresponds to the goodness of the theory in the Gaiotto-Witten sense, i.e., the balance requirement, this definition is more general. Extended monopole operators with nontrivial fluxes under multiple gauge group factors, half-integer lattice effects not visible from the balance, and other subtleties are all allowed under this alternative definition of applicability. However, since a notion of balance is useful in stating the algorithm, we postulate a re-adjustment of the balance notion for $SO(6)$ gauge nodes to partially be sensitive to the half-integer monopole charges. This is captured in the following rules, for which a justification is given in Sec. IV:

- (1) *$USp(2)$ contribution:* A stand-alone $USp(2)$ gauge node, i.e., connected to only one gauge node, next to an $SO(6)$ gauge node, effectively contributes to N_f as $\frac{5}{8}$, instead of 1. A $USp(2)$ in the between $SO(6)$

and $SO(2)$ gauge nodes effectively contributes to N_f as $\frac{5}{4}$ towards the balance of the $SO(6)$ node.

Once the goodness of the theory is correctly assessed, it is possible to move forward and decay the quiver in accordance with the following prescription:

- (2) *Decay existence:* A decay is admissible only if it leads to an orthosymplectic quiver with nodes of balance greater than or equal to -1 . In the $SO(6)$, where fractional balances between -1 and 0 are allowed, these cases are not admissible as decay products.
- (3a) *Allowed decays:* A quiver with subsets of balanced nodes decays only by subtracting the maximal set of labels allowed in the list of Table I from the balanced nodes. In this case, the transverse slice can be uniquely determined from the quiver's shape and the subtracted labels. Whereas nodes with $b > 0$ can decay to gauge nodes of the same group but with lowered rank.
- (3b) *Allowed decays:* Gauge nodes with $b = -1$ can only decay to nodes of the same kind with a rank lowered by 1.

Another option that must be considered at any stage is the possibility of fissioning the quiver into a product of two theories, with one possibly being degenerate.

- (4a) *Fission:* An $\mathfrak{so}(2N)$ algebra can fission to any $\mathfrak{so}(2N - 2K) \oplus \mathfrak{u}(K)$ algebras for $K \leq N$. Similarly, a $\mathfrak{usp}(2N)$ algebra can fission to any $\mathfrak{usp}(2N - 2K) \oplus \mathfrak{u}(K)$ algebras; these fission processes uplift from the direct sum of algebras to the product groups. Said fission process is possible only if the resulting unitary and orthosymplectic quivers are good.
- (4b) *Unitarization:* The extreme case of the fission product when $N = K$ for all nodes in the quiver is called *unitarization*. In this process, different from a standard fission, the obtained unitary gauge groups may need to be lowered such that the resulting unitary quiver is globally good.

After a fission operation, the unitary quiver that is generated can be Higgsed further via standard decay and fission.

To get a feeling of a practical application of the orthosymplectic decay and fission algorithm, we consider the orthosymplectic quiver theory $\mathcal{Q}$ of Eq. (2.3) and derive the Hasse diagram of the Coulomb branch. This is depicted in Fig. 1(a).

We comment on each derivation step:

- (i) Step 0. Checking the overall balance of the quiver according to the new prescription, we can start from the central $SO(6)$ node of $\mathcal{Q}$ and compute the balance,

$$b = 4 \cdot \frac{5}{4} + 5/8 - 5 = 5/8. \quad (2.4)$$

Therefore, this node is good; here, it was important that we used the new balance-contribution prescription given in Rule 1. The stand-alone $USp(2)$ is balanced ($b = 3 - 2 - 1 = 0$), as are all the other $SO(2)$ gauge nodes ($b = 1 - 2 + 1 = 0$). The remaining

²Higgs branch algorithms such as [23] are intrinsically free from this shortcoming due to the “classical” nature of the space.

TABLE I. Table of admitted decay structures from orthosymplectic quiver theories. Each entry in the table is associated with the closure of the minimal nilpotent orbit of the manifest said symmetry in the quiver that is supposed to be Higgsed via the decay process.

Slice	Quiver
$d_{n \geq 3}$	$n-1 SO(2)$ nodes and $n-2 USp(2)$ nodes
$b_{n \geq 3}$	$n-1 SO(2)$ nodes and $n-1 USp(2)$ nodes
a_7	
a_5	
a_4	
a_2	
a_1 or A_k or D_k	2 OR 2
e_7	
e_8	

$USp(2)$ nodes are all good with balance $b = 1 + 3 - 2 - 1 = 1$.

- (ii) Step 1. We now analyze all the possible decay options. The stand-alone $USp(2)$ node, according to Table I, can be decayed, giving rise to the leftmost quiver in the first row from the bottom of Fig. 1(a). Any of the $SO(2)$ nodes can similarly be decayed, leading to the central quiver of the same row. Both these options lead to good quivers. A third option consists of decaying the central node to $SO(4)$. This leads to the rightmost quiver in the first row; in this case, the stand-alone $USp(2)$ has balance $b = -1$. Therefore, the quiver is bad, but we allow it due to Rule 2. Any other subtraction of the allowed labels in Table I leads to

theories with at least one node of badness less than -1 . Hence, they are not admissible.

It is now time to turn our attention to fission processes. The central $\mathfrak{so}(6)$ algebra can fission to $\mathfrak{so}(4) \oplus \mathfrak{u}(1)$.³ Due to the Lie algebra isomorphism

³The idea behind this fission can be observed from brane setups with orientifold planes. A Higgsing related to fission takes a subset of dynamical branes and moves them away from the orientifold planes. Brane dynamics tells us this exactly leads to the decomposition of the gauge algebras $\mathfrak{so}(2n) \rightarrow \mathfrak{so}(2l) \oplus \mathfrak{u}(k)$ and $\mathfrak{usp}(2n) \rightarrow \mathfrak{usp}(2l) \oplus \mathfrak{u}(k)$, with $n = k + l$. In the case where $l = 0$, this leads to unitarization, which is akin to moving *all* the branes away from the orientifold plane. This explanation is discussed in more detail in Sec. IV C.

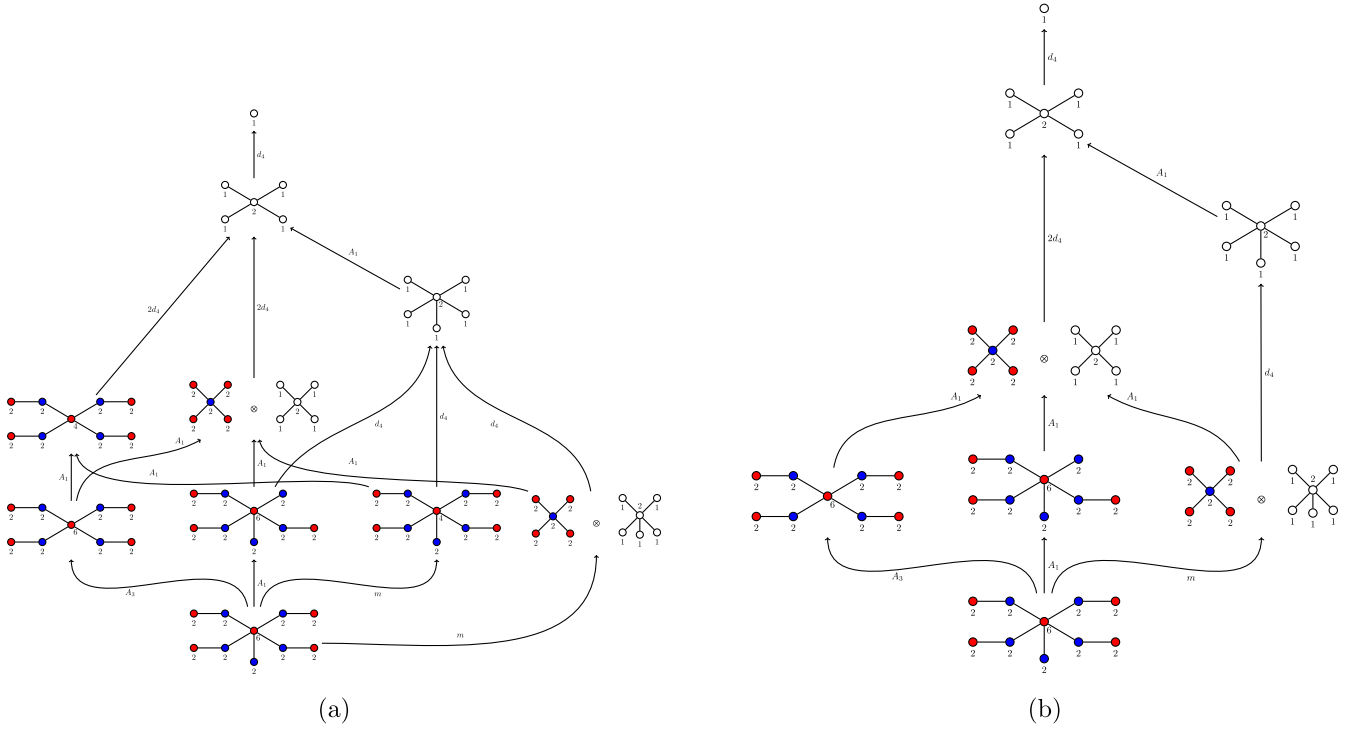


FIG. 1. (a) Tentative Hasse diagram for the Coulomb branch of the quiver 3D $\mathcal{N} = 4$ orthosymplectic quiver $\mathcal{Q}$ of Eq. (2.3) extracted via the Orthosymplectic decay and fission algorithm. The transverse slices are inferred and double-checked via the standard decay and fission on an alternative unitary realization of the same moduli space. (b) Hasse diagram for the Coulomb branch of the quiver 3D $\mathcal{N} = 4$ orthosymplectic quiver $\mathcal{Q}$ of Eq. (2.3) obtained by removing the redundant quivers obtained via the orthosymplectic decay and fission.

between $\mathfrak{so}_4$ and $\mathfrak{su}_2 \oplus \mathfrak{su}_2$, one of the product factors can recombine with the $\mathfrak{u}_1$, leading to the following fission of the central node:

$$\mathfrak{so}(6) \rightarrow \mathfrak{usp}(2) \oplus \mathfrak{u}(2), \quad (2.5)$$

where we implicitly used the algebra isomorphism $\mathfrak{su}(2) \cong \mathfrak{usp}(2)$. The $\mathfrak{usp}(2)$ algebras attached to the central node can fission to $\mathfrak{u}(1)$, which can be thought of as either an $\mathfrak{so}(2)$ or as product algebra of the “trivial” orthosymplectic algebra $\emptyset$ and $\mathfrak{u}(1)$. Explicitly, in the case of the leftmost quiver,

$$\begin{array}{c} \text{Quiver 1} \end{array} \rightarrow \begin{array}{c} \text{Quiver 2} \end{array} \rightarrow \begin{array}{c} \text{Quiver 3} \end{array} \otimes \begin{array}{c} \text{Quiver 4} \end{array}, \quad (2.6)$$

The diagram (2.6) illustrates the fission process of the central node in the leftmost quiver. It shows a sequence of three quivers connected by arrows. The first quiver has a central node labeled 6, connected to four nodes labeled 2. The second quiver shows the central node fissioning into two nodes labeled $\mathfrak{usp}(2) \oplus \mathfrak{u}(2)$ and $\mathfrak{u}(1)$. The third quiver shows the final result, which is a tensor product of two quivers: one with a central node labeled 2 and four nodes labeled 2, and another with a central node labeled 2 and four nodes labeled 1.

where we have conveniently written the product algebras such that the leftmost would be associated with the orthosymplectic quiver theory and the rightmost to the unitary one. No minimal unitarization processes are available at this stage.

(iii) Step 2. Starting again from studying the possible decays, the only option consistent with the rules for the leftmost quiver in the first row is decaying the

central $SO(6)$ gauge node. All tentative decays on the central theory would result in a bad quiver theory. Likewise, the rightmost nonproduct quiver in the first row can decay the $b = -1$ node, leading to the leftmost quiver in the second row. It is now time to turn our attention to fission processes. Both the leftmost and the central theories in the first row can fission. For instance, in the case of the leftmost quiver, we have

$$(2.7)$$

where we have adopted the previous employed convention of denoting the product algebras such that the leftmost would be associated with the orthosymplectic quiver theory and the rightmost to the unitary one.

Alongside fission, it is possible to consider quiver unitarizations. These occur for both the central and the rightmost quiver of the first row. For example,

$$(2.8)$$

- (iv) Step 3. Apart from the leftmost quiver and the orthosymplectic quiver in the central position in the second row of Fig. 1(a), which undergo unitarization (albeit the latter unitarizes to the trivial theory), from here on it is possible to resort to standard decay and fission as all the quivers are of the unitary type.

The diagram of Fig. 1(a) does not qualify as a Hasse diagram in the standard sense. Consider the two rightmost

quivers belonging to the first row from the bottom and the two leftmost quivers belonging to the second row from the bottom of Fig. 1(a). Both pairs have the same Coulomb branch dimension, the same descendants obtained via Higgsing, and in the latter case, where the Coulomb branch Hilbert series is computable, the same Coulomb branch (two copies of d_4 intersecting at the origin). In particular, we have shown that

$$(2.9)$$

where HS_C denotes the Coulomb branch Hilbert series.

We conjecture that, in general, when a pair of theories, either good or bad, engineers the same Hasse diagram, those two theories are actually different presentations of the same theory. However, it is impossible for the algorithm alone to correctly identify them. The algorithm thus predicts the decay and fission products of an orthosymplectic quiver, but with the caveat of possibly engineering a plurality of theories in between that must be identified in order to extract the correct Hasse diagram [see Fig. 1(b)]. Moreover, the algorithm is only applicable to simply laced theories, for which the Coulomb branch symmetry is composed of algebras other than $\mathfrak{g}_2$ and $\mathfrak{f}_4$. In the latter case, this limitation intrinsically reflects the absence of a simply laced linear quiver gauge theory encoding in its Coulomb branch, Slodowy slices from the regular orbit to a generic orbit O , or from O to the trivial orbit.

Having highlighted the range of applicability of orthosymplectic decay and fission, we can now turn our attention to justifying the rules for the algorithm. In order to do as proposed, we need to introduce a further ingredient: the 4D $\mathcal{N} = 2$ class $\mathcal{S}$ theories.

III. LESSONS FROM $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ CLASS $\mathcal{S}$

In Sec. II, we proposed an algorithm by which the structure of the Coulomb branch of a 3D $\mathcal{N} = 4$ orthosymplectic quiver gauge theory can be elucidated. Of course, this algorithm is not something that Moses carried down the mountain in his other hand but is instead motivated initially by reference to myriad examples. In particular, we can exploit the Lie algebra isomorphism

$$\mathfrak{su}(4) \cong \mathfrak{so}(6), \quad (3.1)$$

and we can compare the results of orthosymplectic decay and fission to the structure for the Coulomb branch for the corresponding unitary theory obtained via unitary decay and fission. In this section, we consider the Higgs branch of class $\mathcal{S}$ theories of types $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$ and demonstrate that the algorithm in Sec. II produces the expected result.

Class $\mathcal{S}$ [41,42] is a construction of 4D $\mathcal{N} = 2$ SCFTs, which starts from the 6D (2,0) SCFT of type $\mathfrak{g}$, where $\mathfrak{g}$ is a simple and simply laced Lie algebra, i.e., it is of the ADE type, that is further compactified on a n -punctured genus g Riemann surface with a topological twist to preserve eight supercharges. We denote such a 4D theory as

$$\mathcal{S}_{\mathfrak{g}}\langle C_{g,n} \rangle \{O_1, \dots, O_n\}, \quad (3.2)$$

where O_i capture the data defining the n punctures. In this paper, we are concerned only with regular untwisted punctures, in which case each O_i corresponds to a choice of nilpotent orbit of $\mathfrak{g}$ [43] (see [44] for a recent summary of the pertinent details and terminology of the class $\mathcal{S}$ construction). An important point about the class $\mathcal{S}$ construction is that the flavor symmetry algebra of the resulting 4D $\mathcal{N} = 2$ SCFT decomposes over the set of punctures, and we have

$$\mathfrak{f} \supseteq \bigoplus_{i=1}^n \mathfrak{f}(O_i), \quad (3.3)$$

where $\mathfrak{f}(O_i)$ is the flavor locally associated to the puncture O_i by the choice of nilpotent orbit. The generic case is for the inequality in Eq. (3.3) to be saturated; when it is not saturated, the theory is said to have “enhanced” flavor symmetry.

Typically, class $\mathcal{S}$ theories are non-Lagrangian, and understanding their Higgs branches is *a priori* challenging. However, when $\mathfrak{g}$ is a classical Lie algebra, one can compactify on an S^1 and perform the mirror duality to obtain a 3D $\mathcal{N} = 4$ Lagrangian quiver, which is known as the 3D mirror. The Coulomb branch of this 3D mirror is isomorphic to the Higgs branch of the class $\mathcal{S}$ theory, and since the 3D mirror is Lagrangian, we can utilize the well-developed toolkits to understand the Higgs branch of the non-Lagrangian 4D SCFT.⁴ For a class $\mathcal{S}$ theory of the form in Eq. (3.2) with only regular untwisted punctures constructed on genus zero Riemann surfaces,⁵ the 3D mirror was given in [31] and is schematically shown in Fig. 2.⁶

⁴However, one can also study the 4D Higgs branches of class $\mathcal{S}$ directly, without exploiting the 3D mirror (see, for example, [45] for details).

⁵The 3D mirror is given in [31] for generic genus. However, for simplicity, we focus only on $g = 0$.

⁶The 3D mirrors for class $\mathcal{S}$ with twisted regular punctures are also known, see [46,47] for the case where $\mathfrak{g} = \mathfrak{su}(2n+1)$. Such quivers are often (special-)unitary-orthosymplectic; such theories would be a productive test for a further extension of the decay and fission algorithm to these more general quivers.

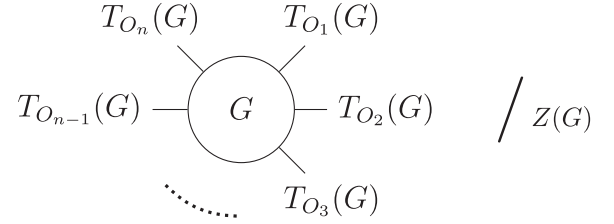


FIG. 2. Star-shaped mirror dual quiver for the class $\mathcal{S}$ theory in Eq. (3.2) compactified on a circle. The $T_{\rho}(G)$, which are the 3D $\mathcal{N} = 4$ theories, have their G (Higgs branch) flavor symmetries diagonally gauged. We explicitly write the quotient action by the center of the group G , denoted as $Z(G)$, which acts diagonally on all the gauge groups in the theory; the subtlety of this extra quotient was observed in [48].

As we can see from Fig. 2, the 3D mirrors involve the diagonal gauging of the flavour symmetry of certain 3D $\mathcal{N} = 4$ SCFTs labeled as $T_{\rho}(G)$ [17], where O_{ρ} is a nilpotent orbit of G . These theories have a Higgs branch which is the closure of nilpotent orbit $O_{\rho^{\vee}}$ of the Langlands dual of G^7 and a Coulomb branch which is the Slodowy slice, S_{ρ} , transverse to O_{ρ} inside of the nilpotent cone of G .

For $\mathfrak{g}$, a classical Lie algebra, the 3D mirror of the class $\mathcal{S}$ theories are Lagrangian as the $T_{\rho}(G)$ theories are themselves linear Lagrangian quivers. To determine these quiver gauge theories, we first note that nilpotent orbits of classical Lie algebras are associated with integer partitions.⁸ In terms of this partition, ρ , an algorithm is given to construct the $T_{\rho}(G)$ in [17].⁹ We write the partitions associated with the nilpotent orbits and the corresponding linear quivers in Table II. In particular, we see that $\mathfrak{su}(4)$ nilpotent orbits are in one-to-one correspondence with $\mathfrak{so}(6)$ nilpotent orbits, consistent with the Lie algebra isomorphism. We conclude that the 3D mirrors of the class $\mathcal{S}$ theories we are considering are star-shaped quivers, where the tails are T_{ρ} theories associated with the nilpotent orbits describing the punctures.

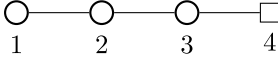
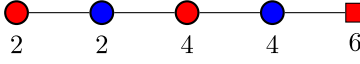
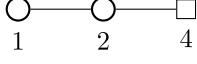
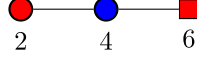
We notice that the class $\mathcal{S}$ description, and thus the Coulomb branch of the 3D mirror, is completely specified by the Lie algebra $\mathfrak{g}$ as well as the collection of nilpotent orbits capturing the punctures. As such, by the isomorphism in Eq. (3.1), we expect that the $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$ theories are identical, where we also map the nilpotent orbits as in Table II. *A priori*, this is not obvious when looking at the Lagrangian quivers describing the 3D mirror, as the $T_{\rho}(SO(6))$ can involve gauge nodes which do not themselves have a dual unitary description.

⁷The orbit $\rho^{\vee}$ of $G^{\vee}$ is determined by the action of the Spaltenstein map $\vee: G \rightarrow G^{\vee}$ [49], exchanging the nilpotent orbit, specified by ρ , of G with a nilpotent orbit of $G^{\vee}$.

⁸For a textbook treatment of nilpotent orbits of (semi-)simple Lie algebras, see [50].

⁹We often use ρ to refer to both the nilpotent orbit and the associated integer partition.

TABLE II. The nilpotent orbits of $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$ written as integer partitions of 4 and 6, respectively. We also write the linear quiver descriptions of the T_ρ theories associated to each nilpotent orbit. The nilpotent orbits written in the same row are isomorphic under the $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ isomorphism. We write the partitions using multiplicative notation throughout this paper.

O_{A_3}	$T_{O_{A_3}}(SU(4))$	O_{D_3}	$T_{O_{D_3}}(SO(6))$
$[1^4]$		$[1^6]$	
$[2, 1^2]$		$[2^2, 1^2]$	
$[2^2]$		$[3, 1^3]$	
$[3, 1]$		$[3^2]$	
$[4]$		$[5, 1]$	

Thus, the central point of this construction is having equivalent unitary and orthosymplectic interpretations of the class $\mathcal{S}$ theories $\mathcal{S}_{\mathfrak{so}_6}\langle C_{g,n} \rangle \{O_1, \dots, O_n\}$. From the unitary 3D mirror via the unitary decay and fission algorithm, we can derive the Higgsing pattern for the unitary 4D theory and map each partially Higgsed theory to its orthosymplectic counterpart. In this way, we can extensively test the orthosymplectic decay and fission algorithm presented in Sec. II.

The generic Higgsing pattern in class $\mathcal{S}$ is known as a partial puncture closure. This occurs when there is a Higgs branch renormalization group flow between theories

$$\mathcal{S}_{\mathfrak{g}}\langle C_{g,n} \rangle \{O_1, \dots\} \rightarrow \mathcal{S}_{\mathfrak{g}}\langle C_{g,n} \rangle \{O'_1, \dots\}, \quad (3.4)$$

where the punctures in the $\dots$ remain the same, i.e., only one puncture is modified. The set of nilpotent orbits of $\mathfrak{g}$ is known to admit a partial ordering, and it is believed that the physical interpretation of this partial ordering is that there exists a Higgs branch renormalization flow between theories as in Eq. (3.4) whenever $O'_1 < O_1$ under this partial ordering on nilpotent orbits of $\mathfrak{g}$ [43].¹⁰ Indeed, this can be justified by recalling that the puncture/nilpotent orbit corresponds to an expansion of the so-called Higgs field in the class $\mathcal{S}$ construction, and if $O' < O$ in the partial ordering on the set of nilpotent orbits, then the expansion of the Higgs field associated with O' can be obtained by tuning the coefficients in the expansion associated with O [43].

Eventually, after sufficient partial puncture closure, the resulting class $\mathcal{S}$ theory develops an enhanced flavor

symmetry; at this point, the partial puncture closure can be obstructed as half-BPS operators that enter the spectrum from one puncture can mix with those of another puncture to form the representations of the enhanced flavor symmetry. Thus, giving vacuum expectation values to operators arising from one puncture can affect the other punctures. Since the enhanced symmetry is not manifest in the class $\mathcal{S}$ geometry and thus does not have a geometric interpretation like the partial puncture closure, the Higgs branch of such theories requires individual analysis. However, enhanced symmetry only occurs in a small number of cases, and always occurs near the end of the Higgs branch Hasse diagram for a generic class $\mathcal{S}$ theory.

These Higgsings obtained from partially closing punctures do not cover all possible Higgsings of a class $\mathcal{S}$ theory. The missing Higgsings are precisely the fission processes, and therefore unitarization, in the orthosymplectic decay and fission algorithm.

However, they are sufficient to describe all possible Higgsing when $\mathfrak{g} = \mathfrak{su}(4) \cong \mathfrak{so}(6)$. More general possibilities, in fact these are the fissions and the unitarizations, which can be associated with the generators of the Higgs branch chiral ring [52] that are not moment-map operators, are discussed from the perspective of the 4D Higgs branch in [45] and can partially be related to the complex deformation theory of the $\mathbb{C}^2/\Gamma$ orbifolds that engineer the 6D (2,0) SCFTs in type IIB string theory (cf. [53,54]).

Let us focus first on just one tail of the star-shaped quiver and assume that the balance of the central $\mathfrak{so}(6)$ node is sufficiently positive and that there does not exist any enhanced flavor symmetry. We assume that the puncture that we are interested in is O_1 , and let $O'_1 < O_1$ such that the balance of the $\mathfrak{so}(6)$ after replacing O_1 with O'_1 is such that the $\mathfrak{so}(6)$ node has non-negative balance. Then, it is

¹⁰It is, however, not entirely clear that all such Higgsings exist from the four-dimensional perspective (see [51] for some of the subtleties).

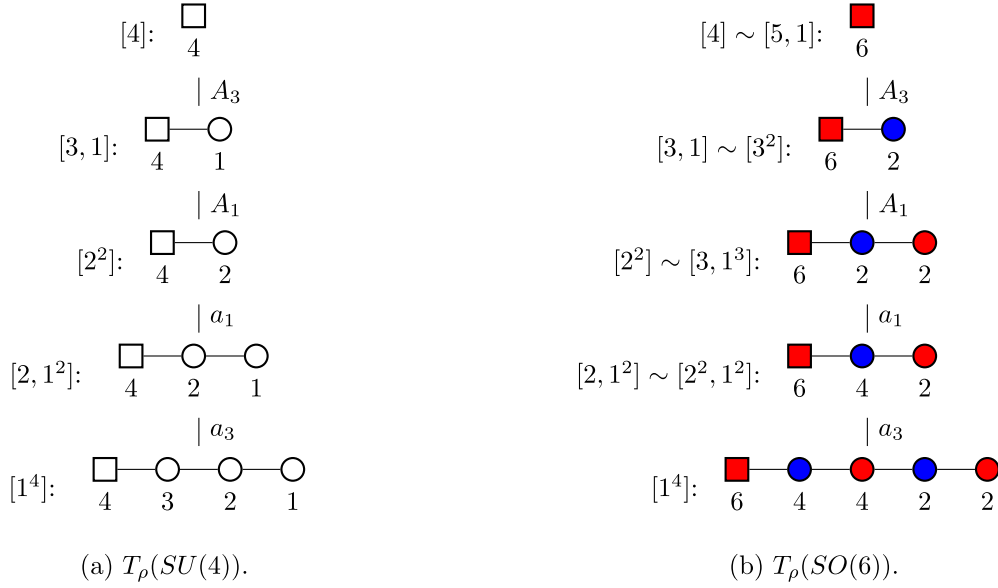
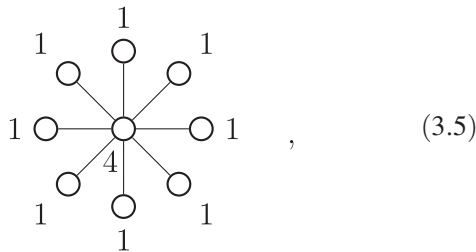


FIG. 3. Hasse diagram under the dominance ordering on partitions of the nilpotent orbits of (a) $\mathfrak{su}(4)$ and (b) $\mathfrak{so}(6)$. We also depict the associated T_ρ theories, and we can see that there exists both unitary and orthosymplectic decays, respectively, which reproduce the Hasse diagram from the dominance ordering.

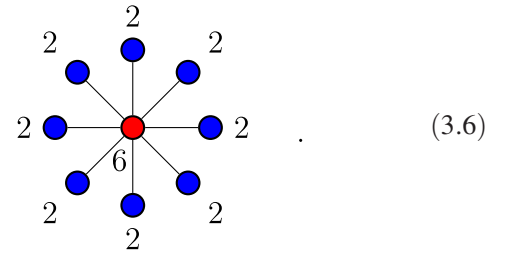
clear that the decay and fission algorithm of Sec. II allows one to decay $T_{O_1}(SO(6))$ to $T_{O'_1}(SO(6))$ following the Hasse diagram presented in Fig. 3(b). This directly matches with the unitary quivers, as the Hasse diagrams for $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$ nilpotent orbits match via the isomorphism.

Let us now consider the cases of $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ class $\mathcal{S}$ theories where there is an enhanced flavor symmetry, that is, where the inequality in Eq. (3.3) is strict. In the unitary 3D mirror of such theories, the central $U(4)$ gauge node of the quiver becomes balanced, which leads to the additional Coulomb symmetries. There are only a small number of examples of gaugings of $T_{O_i}(SU(4))$ such that the $U(4)$ node is balanced, and thus we are able to consider the cases exhaustively here. We depict each of these quivers in Fig. 4.

First, we consider the class $\mathcal{S}$ of type $\mathfrak{su}(4)$ on a sphere with eight punctures associated to the nilpotent orbits $[3, 1]$. The unitary 3D mirror of this theory is



and the structure of the Higgs branch, as obtained from performing decay and fission on the unitary 3D mirror, is given in Fig. 5. Using the isomorphism between $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$, we can see that there also exists an orthosymplectic 3D mirror of this class $\mathcal{S}$ theory, which is



The first transition in the Coulomb branch of the quiver in eq. (3.6) involves unitarization, where

$$\mathfrak{so}(6) \rightarrow \mathfrak{u}(3), \quad \mathfrak{usp}(2) \rightarrow \mathfrak{u}(1). \quad (3.7)$$

This leads to

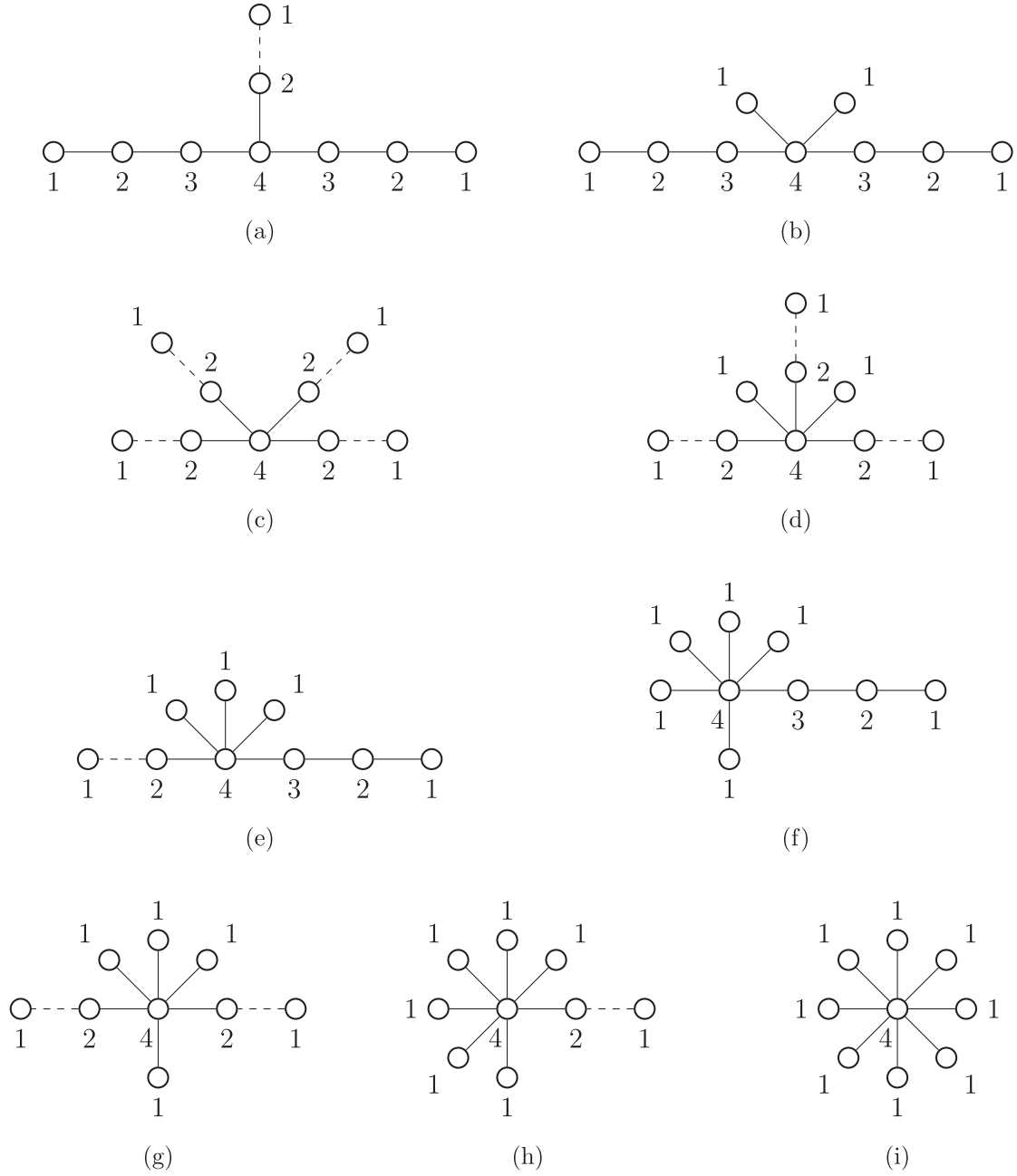


FIG. 4. All unitary 3D mirrors for the class $\mathcal{S}$ theories of type $\mathfrak{su}(4)$ on a sphere with regular untwisted punctures and enhanced flavor symmetry. The partitions associated with the punctures can be read off from Table II. To prevent an unnecessary proliferation, we have used a dashed line to indicate that the attached tail may be either that of $T_{[2,1^2]}(SU(4))$ or $T_{[2^2]}(SU(4))$.

$$\begin{array}{ccc}
 \begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 6 \end{array} & \longrightarrow & \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 3 \end{array} , \quad (3.8)
 \end{array}$$

which manifestly reproduces one of the two first steps in Fig. 5, and since the result of the unitarization is a unitary quiver, the further decay and fission evidently leads to an identical structure. The second allowed decay and fission of Eq. (3.6) again involves a fission step, in particular

$$\mathfrak{so}(6) \rightarrow \mathfrak{usp}(2) \oplus \mathfrak{u}(2), \quad \mathfrak{usp}(2) \rightarrow \mathfrak{so}(2) \cong \mathfrak{u}(1). \quad (3.9)$$

It is easy to see that there is only one way to attach the eight $SO(2) \cong U(1)$ nodes to the $USp(2)$ and $U(2)$ central nodes such that the resulting quivers are sufficiently good, following the notions of goodness that were explained in Sec. II.¹¹ Therefore, we obtain the following transition:

Diagram (3.10) shows a transition from a star-shaped quiver to a tensor product of two quivers. The left quiver has a central red node labeled 6, with eight blue nodes labeled 2 connected to it. An arrow points to the right, where two quivers are shown separated by a tensor product symbol $\otimes$. The first quiver on the right has a central white node labeled 2, with four white nodes labeled 1 connected to it. The second quiver on the right has a central blue node labeled 2, with four red nodes labeled 2 connected to it.

It is already well-known that both quivers on the right-hand side have identical Coulomb branches, which, as symplectic singularities, are just the closure of the minimal nilpotent orbit of $\mathfrak{so}(8)$. It remains to argue that the rightmost quiver in Eq. (3.10), under the orthosymplectic decay and fission algorithm, has a Coulomb branch consisting of only a single slice which leads to a trivial quiver. We can see this, for example, by noting that a unitarization step would lead to a quiver with no remaining

gauge nodes and that none of the other rules presented in Sec. II lead to a nontrivial quiver. Therefore, we have reproduced the full structure (given in Fig. 5) of the Coulomb branch of the unitary quiver in Eq. (3.5) starting from the orthosymplectic quiver in Eq. (3.6) and applying the algorithm for orthosymplectic decay and fission.

Applying the orthosymplectic decay and fission algorithm of Sec. II, it would appear that there exists a third consistent transition,

Diagram (3.11) shows a transition from a star-shaped quiver to another star-shaped quiver. The left quiver has a central red node labeled 6, with eight blue nodes labeled 2 connected to it. An arrow points to the right, where a quiver is shown with a central red node labeled 4, with eight blue nodes labeled 2 connected to it.

How can we explain the presence of this transition when we know from the unitary quiver description that there exists only two inequivalent elementary transitions from the theory in Eq. (3.6)? In fact, this is an example of one of the properties of the orthosymplectic decay and fission algorithm highlighted in Sec. II. Specifically, the algorithm produces sometimes two different quivers describing the same sublocus in the Coulomb branch. In this case, the Coulomb branches of

Diagram (3.12) shows two quivers separated by the word "and". The left quiver has a central white node labeled 3, with six white nodes labeled 1 connected to it. The right quiver has a central red node labeled 4, with eight blue nodes labeled 2 connected to it.

¹¹Note that the star-shaped quiver with $\mathfrak{usp}(2)$ in the center and three $\mathfrak{so}(2)$ legs has a nonlocal monopole operator that decouples, despite each of the four gauge nodes being locally good. As such, it is ruled out.

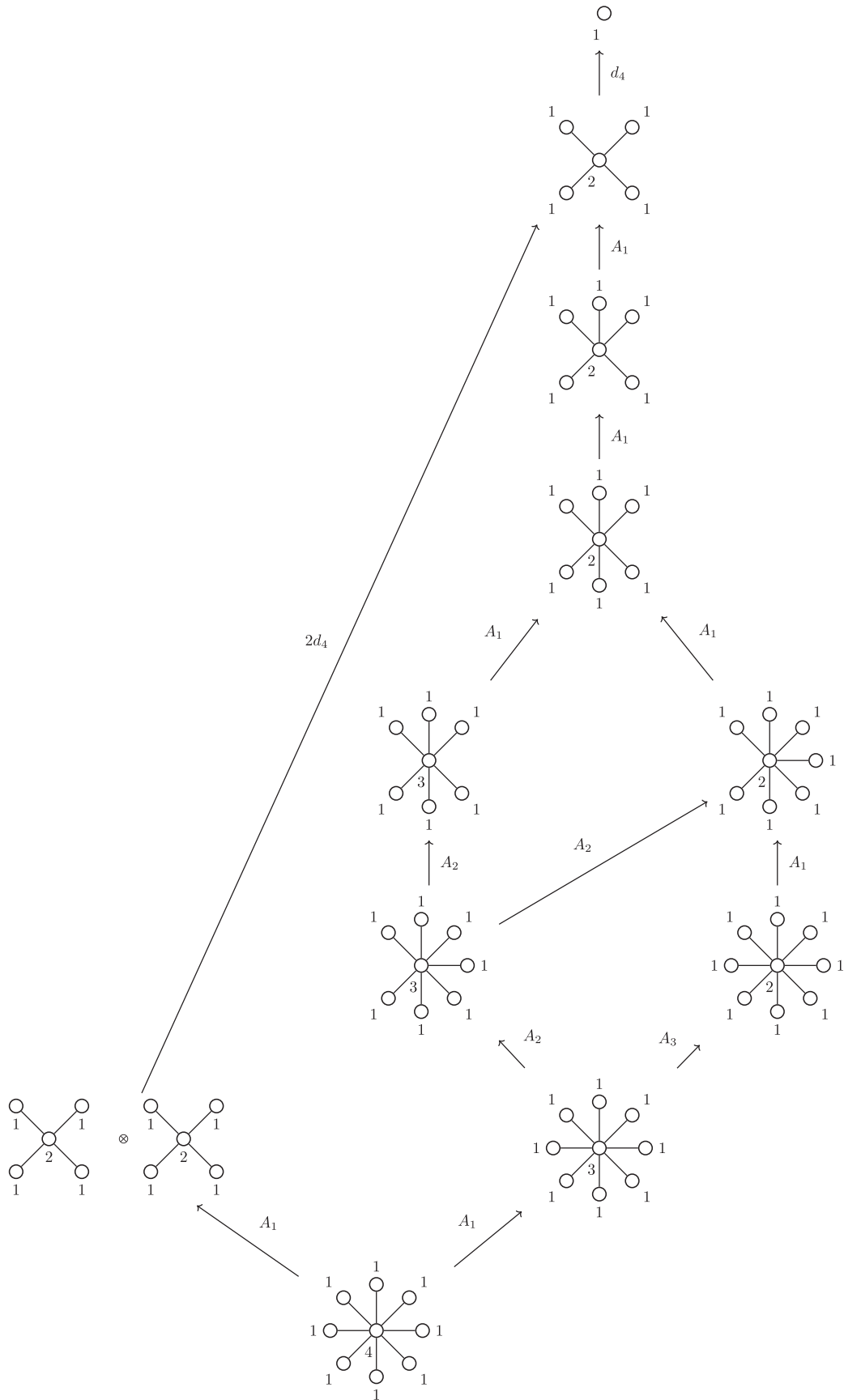


FIG. 5. Applying the unitary decay and fission algorithm to the quiver in Eq. (3.5) generates this Coulomb branch Hasse diagram. This structure is reproduced by applying the orthosymplectic decay and fission algorithm to the quiver in Eq. (3.6).

are identical.¹² It is straightforward to observe this by comparing the subgraph of Fig. 5 starting from the unitary quiver on the left (generated by the unitary decay and fission algorithm) with the Coulomb branch Hasse diagram of the orthosymplectic quiver on the right (generated by the orthosymplectic decay and fission algorithm) which we depict in Fig. 6.

There is one further subtlety in the derivation of the Hasse diagram in Fig. 6 that we briefly highlight here. Since we allow $USp(2)$ gauge nodes with balance -1 in the algorithm in Sec. II, we can no longer use standard notions of badness to know when a quiver should not appear in the decay and fission algorithm. Instead, we ask when the corresponding 4D class $\mathcal{S}$ theory is good and use that as a diagnostic to rule out certain 3D quivers. Consider class $\mathcal{S}$ of type $\mathfrak{so}(4)$. Since the partition $[2^2]$ of 4 is very even,¹³ it is associated with two distinct nilpotent orbits, which we label as

$$[2^2]_I \quad \text{and} \quad [2^2]_{II}. \quad (3.13)$$

While these two different nilpotent orbits generally lead to different class $\mathcal{S}$ theories with different Higgs branch chiral rings (see [55] for the explicit difference in the half-BPS operator spectrum), the naive 3D mirror appears to be agnostic to this since

$$T_{[2^2]_I}(SO(4)) = T_{[2^2]_{II}}(SO(4)) = \begin{array}{c} \text{red square} \text{---} \text{blue circle} \\ 4 \qquad 2 \end{array}. \quad (3.14)$$

From the class $\mathcal{S}$ perspective, one can compute the Hall-Littlewood index [56–60] which, for genus zero with regular untwisted punctures [47], counts Higgs branch operators. Schematically, it is determined as

$$\text{HL}(\tau) = \text{Tr}_{\mathcal{H}_{\text{HL}}} \tau^{2(\Delta-R)} (-1)^F = \sum_{\Lambda} \dots, \quad (3.15)$$

where the sum runs over all irreducible representations of $\mathfrak{g}$, and $\dots$ represents a complicated product of Hall-Littlewood polynomials over the punctures. For the details, we refer to [55,61] where a straightforward prescription was given to determine the lowest order n_{Λ} that the representation Λ contributes to the Hall-Littlewood index; i.e., the Hall-Littlewood index contains a

¹²In this case, the orthosymplectic quiver has bad $USp(2)$ nodes, and thus we cannot use the monopole formula to compute the Coulomb branch Hilbert series and compare it to that of the unitary quiver. Here, we use the weaker condition that the Coulomb branch Hasse diagrams are identical, though this is not in general sufficient to demonstrate that the Coulomb branches themselves are identical.

¹³A very even partition has only even entries, each with even multiplicity.

term $\tau^{n_{\Lambda}}$. If $n_{\Lambda} < 1$ for any Λ , then the theory is bad. Consider now $\mathfrak{so}(4)$ class $\mathcal{S}$ on a sphere with n_R punctures $[2^2]_I$ and n_B punctures $[2^2]_{II}$. The two spinor representations of $\mathfrak{so}(4)$ contribute to the index at the following orders:

$$n_2 = -2 + n_B \quad \text{and} \quad n_{2'} = -2 + n_R. \quad (3.16)$$

As we can see, it is necessary to have at least six punctures, three $[2^2]_I$ and three $[2^2]_{II}$ for the class $\mathcal{S}$ theory to not be bad. Therefore, we propose that the 3D mirror of such a bad 4D theory is also bad and should be removed from the results of the decay and fission algorithm. With this computation in hand, it is tempting to suggest an interpretation directly in the 3D Lagrangian quiver. If we consider

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2), \quad (3.17)$$

and let n_R and n_B denote the number of fundamental flavor attached to the two $\mathfrak{su}(2)$ factors, then Eq. (3.16) is precisely the balances of the $\mathfrak{su}(2)$ factors, respectively. It is not clear why this *should* be the correct interpretation, and since it is orthogonal to the analysis of this paper, we leave a further exploration for future work.

In the example, we might have expected, from the orthosymplectic quiver, that there exists another transition,

$$\begin{array}{c} \begin{array}{c} 2 \qquad 2 \qquad 2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{red circle} \\ \diagup \quad \diagdown \quad \diagdown \\ 2 \qquad 2 \qquad 2 \end{array} \quad \longrightarrow \quad \begin{array}{c} 2 \qquad 2 \qquad 2 \\ \diagdown \quad \diagup \quad \diagup \\ \text{red circle} \\ \diagup \quad \diagdown \quad \diagdown \\ 2 \qquad 2 \qquad 2 \end{array} \end{array}. \quad (3.18)$$

On the right-hand-side of Eq. (3.18), each of the $\mathfrak{usp}(2)$ and $\mathfrak{so}(6)$ gauge nodes have a non-negative balance, in the conventional sense of [17]. Instead, if we use the rules from Sec. II, each isolated $USp(2)$ should contribute to the balance of the $SO(6)$ gauge node as $5/8$ full hypermultiplets in the $\mathbf{6}$ of $\mathfrak{so}(6)$ (i.e., Rule 1). Thus, the effective balance of the $SO(6)$ node should be considered as

$$b = 7 \times \frac{5}{8} - 6 + 1 = -\frac{5}{8} < 0. \quad (3.19)$$

Since the modified balance of the special orthogonal node is negative, it is treated as bad and this configuration is ruled out. Indeed, this is a good opportunity to explain the motivation for the modified balance contribution in Rule 1. While each of the nodes is individually good, there still exists a monopole operator which crosses the unitarity bound; this operator arises from the half-integer

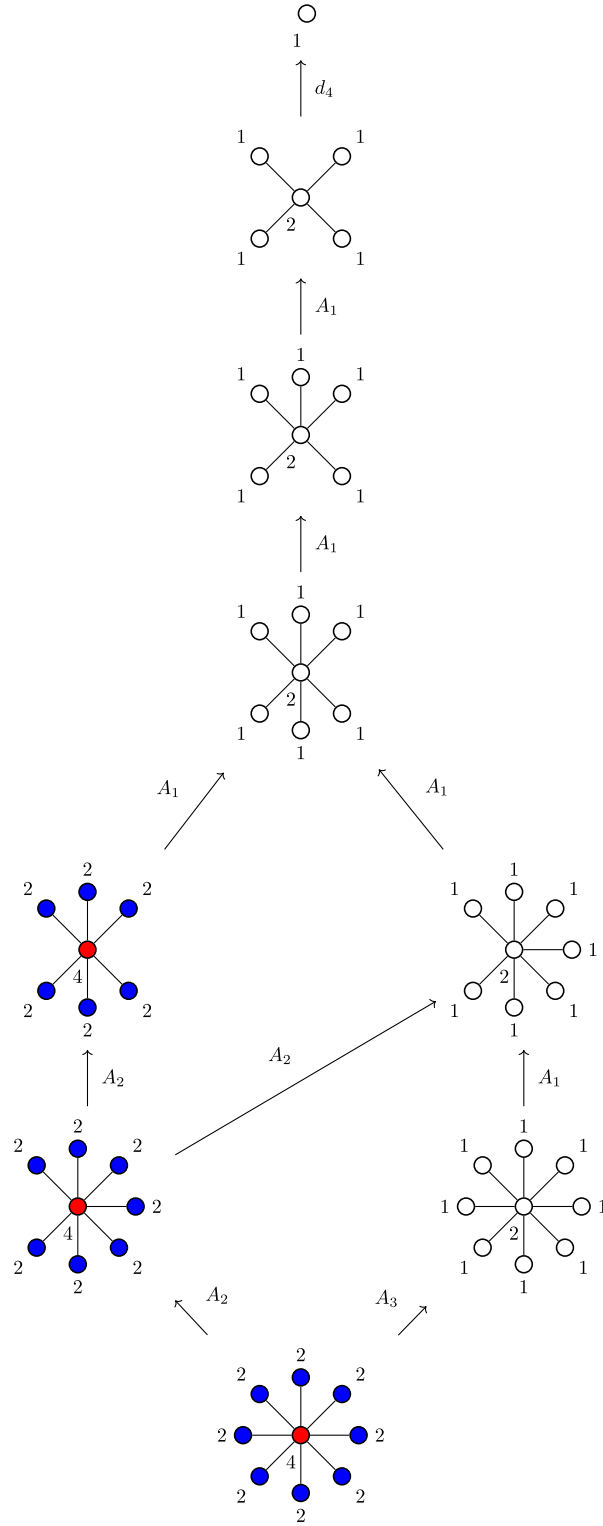


FIG. 6. Applying the orthosymplectic decay and fission algorithm to the quiver on the right in Eq. (3.11), we find this Coulomb branch Hasse diagram. This is identical to the subdiagram of Fig. 5 starting from the quiver on the left in Eq. (3.11).

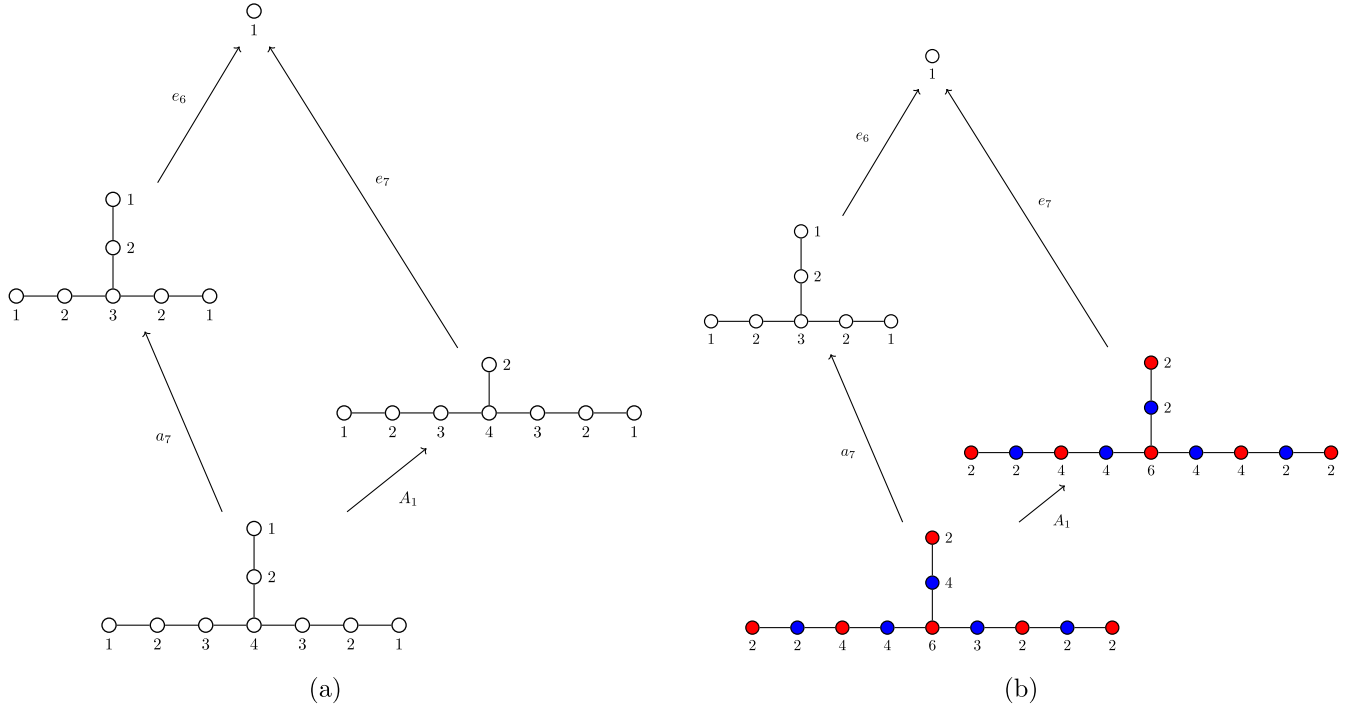


FIG. 7. Higgs branch Hasse diagram of $\mathcal{S}_{\mathfrak{su}_4}\langle C_{0,3}\rangle\{[1^4], [1^4], [2, 1^2]\} \cong \mathcal{S}_{\mathfrak{so}_6}\langle C_{0,3}\rangle\{[1^6], [1^6], [2^2, 1^2]\}$ derived via decay and fission in (a) through the unitary 3D mirror and in (b) through the orthosymplectic 3D mirror.

lattice.¹⁴ We consider first the general theory of a $SO(2K)$ gauge node attached to an n $USp(2)$ gauge node,

$$(3.20)$$

The minimally charged monopole operator inside of the half-integer lattice crosses the unitarity bound if $n < 8$ for any $K \geq 3$, except $K = 4$; in the latter case, the bound is $n < 7$. We can see that the condition for the monopole not to decouple when $\mathfrak{so}(6)$ is imposed by Rule 1.

We now consider another $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ class $\mathcal{S}$ theory with enhanced flavor symmetry from Fig. 4. We turn to the theory

$$\mathcal{S}_{\mathfrak{su}_4}\langle C_{0,3}\rangle\{[1^4], [1^4], [2, 1^2]\}, \quad (3.21)$$

which has the following unitary 3D mirror:

¹⁴Notice that although it stays true for $n \leq 7$ that a monopole operator arising from the half-integer lattice crosses the unitarity bound, only for $n = 7$, the theory we get is, in fact, free.

$$(3.22)$$

We can see that this quiver manifests an $\mathfrak{su}(8) \oplus \mathfrak{su}(2)$ Coulomb symmetry. Applying the unitary decay and fission algorithm, we see that there are two inequivalent minimal Higgsings of this quiver, and we depict the full Hasse diagram of the Coulomb branch in Fig. 7(a). Since this is an $\mathfrak{su}(4)$ class $\mathcal{S}$ theory, there is an isomorphic $\mathfrak{so}(6)$ class $\mathcal{S}$ theory, which is

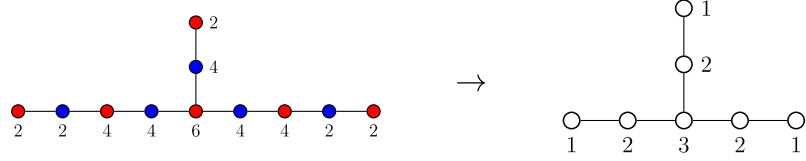
$$\mathcal{S}_{\mathfrak{so}_6}\langle C_{0,3}\rangle\{[1^6], [1^6], [2^2, 1^2]\}. \quad (3.23)$$

The 3D mirror from this perspective is the following orthosymplectic quiver:

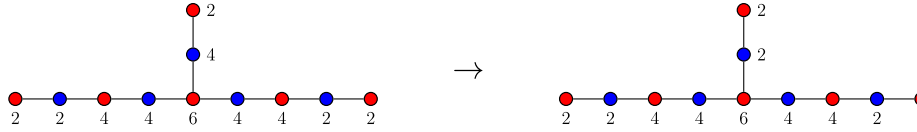
$$(3.24)$$

We can similarly identify the enhanced $\mathfrak{su}(8) \oplus \mathfrak{su}(2)$ flavor symmetry from the balanced nodes in this quiver.

We now wish to discuss the orthosymplectic decay and fission algorithm from Sec. II as applied to the quiver in Eq. (3.24). The first transition to consider is a unitarization step,


(3.25)

As we can see, we land directly on the quiver in the shape of an affine E_6 Dynkin diagram from Fig. 7a, which also appears identically from the unitary decay and fission algorithm. We note that this is the only fission operation which produces (a product of) quivers which are all sufficiently good according to Sec. II. The only other nontrivial orthosymplectic decay and fission of the quiver in Eq. (3.24) is


(3.26)

where the $USp(4)$ node on the vertical tail decayed to $USp(2)$. It is well known that the quiver on the right has a Coulomb branch which is the closure of the minimal nilpotent orbit of $\mathfrak{e}_7$, exactly the same as the affine- E_7 -shaped quiver that appears in Fig. 7(a). We have drawn the corresponding orthosymplectic Coulomb branch Hasse diagram in Fig. 7(b), which we see has an identical structure to the unitary case, as required by the $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ isomorphism.

We can perform a similar analysis of all of the class $\mathcal{S}$ theories of type $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ where the flavor symmetry is enhanced. The unitary 3D mirrors for each such case are listed in Fig. 4, and the corresponding orthosymplectic 3D mirrors can be directly worked out from the equivalence of the $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$ nilpotent orbits given in Table II. In all cases, the orthosymplectic decay and fission algorithm of Sec. II produces the same Coulomb branch Hasse diagram as that of the unitary decay and fission algorithm applied to the unitary 3D mirrors. The Higgs branch of class $\mathcal{S}$ theories has a generic structure. First, the transitions are given by partial puncture closure, until there exists an enhanced flavor symmetry. After the enhancement, the transitions are as one of the quivers in Fig. 4. As such, we have demonstrated that the structure of the Higgs branch of all $\mathfrak{so}(6)$ class $\mathcal{S}$ theories (with only regular untwisted punctures) can be recovered from the orthosymplectic decay and fission algorithm presented in Sec. II.

IV. WALKING THE PATH TO THE RULES

As explained in Sec. III, although we heavily leveraged on the predictions following from class $\mathcal{S}$ and the algebra isomorphism $\mathfrak{su}_4 \cong \mathfrak{so}_6$ to lay down the rule of an

orthosymplectic decay and fission algorithm, some more general insights follow from venturing deeper in the physics behind each of the rules stated in Sec. II.

A. Symmetry through monopoles

Already at the beginning of Sec. II, we defined the *balance* as a quantity locally controlling the conformal dimension of the theories' monopole operators in relation to the unitary bound. The definition given in (2.2) stems from a general formula that takes into account the type of gauge theory under consideration in the presence of fundamental matter. As already noticed by the original authors [17], for a significant number of orthosymplectic gauge theories, the goodness condition $b \geq 0$ does not suffice to ensure the goodness of the entire theory. When the quiver theory is comprised of only gauge nodes, there exists a redundant $\mathbb{Z}_2$ symmetry acting trivially on the product of the gauge groups, which we always choose to ungauged [39]. This choice is reflected in the flux lattice of the monopole operators that, together with their usual contribution in a chamber of a $\mathbb{Z}$ -valued lattice of dimension dictated by the product of the rank of the various gauge groups, also receive a contribution from a chamber in a lattice valued on the shifted half integers $\mathbb{Z} + \frac{1}{2}$. Explicitly, the balance condition of (2.2) is locally sensitive to the contribution of $\mathbb{Z}$ -valued fluxes. Thus, whereas for quiver theories with special orthogonal nodes of rank $k \geq 4$, this condition suffices to assess the global goodness of a theory, for $k = 3$, i.e., $SO(6)$ gauge nodes, this condition fails as the minimal monopole falling below the unitarity bound corresponds to fluxes valued in the half-integer lattice. These kinds of “bad” contributions are not localized on the single gauge node but are spread along

the whole quiver theory. It is possible to circumvent this problem by having an empirical definition of balance that mimics the inclusion of these half-integers valued mono-poles by readjusting the contribution of a neighboring symplectic node towards the number of flavours. Exploiting the aforementioned algebra isomorphism and class $\mathcal{S}$, such an empirical statement about the goodness of the theory regardless of the quiver's shape can be derived.

The theories $\mathcal{S}_{\mathfrak{su}_4}\langle C_{0,n}\rangle\{[3,1], \dots, [3,1]\}$ and $\mathcal{S}_{\mathfrak{so}_6}\langle C_{0,n}\rangle\{[3^2], \dots, [3^2]\}$ in (4.1) are isomorphic by construction for every n ,

$$\mathcal{S}_{\mathfrak{su}_4}\langle C_{0,n}\rangle\{[3,1], \dots, [3,1]\} \quad \mathcal{S}_{\mathfrak{so}_6}\langle C_{0,n}\rangle\{[3^2], \dots, [3^2]\} \quad (4.1)$$

The unitary theory dictates that for $n > 8$, both theories must be good, for $n = 8$ balanced, and for $n \leq 7$ bad.¹⁵ Reproducing the same behavior from the balance notion for special-orthogonal theories of (2.2) requires that all the stand-alone $USp(2)$ in the orthosymplectic construction must be effectively counted as contributing $\frac{5}{8}$ -s towards N_f . The same reasoning can be applied to the theories $\mathcal{S}_{\mathfrak{su}_4}\langle C_{0,n}\rangle\{[2^2], \dots, [2^2]\}$ and $\mathcal{S}_{\mathfrak{so}_6}\langle C_{0,n}\rangle\{[3,1^3], \dots, [3,1^3]\}$ to assess that a $USp(2)$ between an $SO(6)$ and an $SO(2)$ gauge node contributes to N_f as $\frac{5}{4}$. The just-performed analysis sheds light behind Rule 1 of our algorithm, justifying it as an empirical readjustment of (2.2) to be sensitive to all the monopoles that may fall below the unitarity bound.

The role of half-integer lattice flux-valued monopoles extends beyond the scope of $SO(6)$ gauge theories. It is a known phenomenon that flavor symmetries can be enhanced due to the presence of extra contribution stemming from monopoles valued in this shifted lattice [39]. Classically, the Coulomb branch flavor symmetry algebra¹⁶

¹⁵Actually, for $n = 7$ the unitary theory is ugly, but since this definition does not translate to general orthosymplectic theories, we incorporate it into the bad notion. Moreover, ugly theories do not appear as decay and fission products of good/balanced unitary theories. Hence, it is sensible to exclude the orthosymplectic counterpart from appearing in the orthosymplectic decay and fission algorithm.

¹⁶Coherently, with a choice of global structure for the gauge algebra that determines the group nature, we denote the flavor algebra by the associated group even though we are not determining the presence of discrete actions.

TABLE III. List of flavor symmetry groups G that can be realized as enhancement in orthosymplectic quiver gauge theories.

G	$\prod_i SO(r_i)$	adj Decomposition
$SU(8)$	$SO(6) \times SO(6) \times SO(2)$	$63 = 15 + 15 + 1 + 2^5$
$SU(6)$	$SO(6) \times SO(3) \times SO(2)$	$35 = 15 + 3 + 1 + 2^4$
$SU(5)$	$SO(6) \times SO(2)$	$24 = 15 + 1 + 2^3$
$SU(3)$	$SO(3) \times SO(2)$	$8 = 3 + 1 + 2^2$
F_4	$SO(9)$	$52 = 36 + 2^4$
E_6	$SO(10) \times SO(2)$	$78 = 45 + 1 + 2^5$
E_7	$SO(12) \times SO(3)$	$133 = 66 + 3 + 2^6$
E_8	$SO(16)$	$248 = 120 + 2^7$

in an orthosymplectic quiver is read from a chain of p -balanced nodes to be

- (i) $SO(p+3)$ if the chain starts and ends with an $SO(2)$ gauge node.
- (ii) $SO(p+2)$ if the chain starts or ends with an $SO(2)$ gauge node.
- (iii) $SO(p+1)$ in all the other cases.

This symmetry arises from monopoles valued in the integer lattice and corresponds, in general, to the full symmetry of the theory. In specific cases, the half-integer-valued monopoles recombine into a spinor representation of the Coulomb branch symmetry read from the aforementioned rule and can interplay with the integer-valued monopoles to enhance the symmetry. We can keep track of this possibility by noticing that, coherently, with a spinor representation's dimension, half-integer valued monopoles contribute to the adjoint action of the flavour symmetry (i.e., to the flavor current) at most 2^d , where d is the number of $SO(\text{even})$ gauge nodes in the quiver. The group theory of the problem requires finding an embedding of G of the form $\prod_i SO(r_i)$, where the spinor representation of the obtained algebras recombine with the adjoint to form the adjoint representation of the original group G . We tabulated all such occurrences¹⁷ in Table III.

All these examples, apart from the E_8 and the F_4 ones, can be realized in the controlled $\mathcal{S}_{\mathfrak{su}_4}\langle C_{0,n}\rangle$ framework. The E_8 case is known from the 6D (1,0) E-string theory [62] and its magnetic quiver [20], whereas an $F_4 \times F_4$ example was realized in [35] by studying balanced simply-laced forked-quivers of orthosymplectic type. Studying the unitary class $\mathcal{S}$ Hasse diagram, we found the list of admitted decays found in Table I and thus justify Rule 3a.

B. Vacuum of bad theories

We emphasized in multiple instances that we are reticent toward bad theories. The main reason supporting our

¹⁷The case of $G_2 \rightarrow SU(2) \times SU(2) \rightsquigarrow 14 = 3 + 3 + 2^3$ is a special embedding; therefore, it does not entail a naive recombination of spinors.

attitude lies in the nature of their vacuum¹⁸ structure. The Coulomb branch moduli space of vacua, different from good theories where it supports a one and only singular locus, admits a set of (possibly continuous) singular loci. The understanding of the global geometry of this space is hindered by the impossibility of an explicit counting of the theory's monopole operators due to the IR R-symmetry being different from the UV one. Nevertheless, it is still possible to probe the local geometry of the Coulomb branch moduli space near each of their singular loci. In [63,64], a first approach has been made in this direction, identifying, in the unitary case, a Seiberg-like duality that maps the local geometry near each singular point to the Coulomb branch moduli space of a good unitary theory plus some decoupled hypermultiplets responsible for the flat directions in the bad theory. This result was also recovered from index-like computations in [65,66], where the index of bad theories behaves as a sum of distributions, each identifying a particular singular locus of the bad theory.

Shifting our attention to bad orthosymplectic theories, already a simple case such as the $USp(2N)$ SQCD reveals itself to be more subtle with respect to its unitary counterpart. Symplectic gauge theories, as well as special-orthogonal ones, do not admit an ugly notion, and their moduli space of vacua drifts from the good single-cone-behavior already when $b = -1$, i.e., when the $USp(2N)$ theory is paired with $N_f = 2N$ fundamental hypermultiplets. In this case, the vacuum consists of two singular loci that are quantum mechanically separated but swapped by a $\mathbb{G}_2 \cong \mathbb{Z}_2$ global symmetry on the Coulomb branch [64] without any flat direction. The origin of this discrete symmetry is explained when interpreting the bad symplectic SQCD theory as a $T_\rho(SO(4N))$ theory. There exist two inequivalent nilpotent orbits (mapped under a $\mathbb{Z}_2$ symmetry, O_I, O_{II}) in the nilpotent cone of $SO(4N)$ cone which are associated with the same (very-)even partition $\rho = [(2N)^2]$ of $4N$. Studying the local IR SCFT living at each of the singular points in the moduli space results in it being the source of a Higgs branch that admits a unitary 3D mirror [64,67]. For more severe badnesses, i.e., $b < -1$ and $N_f < 2N$, the vacuum behavior shown by $USp(2N)$ SQCDs matches the classic expectations for bad theories. There exist multiple singular points where the local geometry looks like a good theory with extra flat directions.

This can be extended to general $T_\rho(SO(2N))$ theories. Leveraging the well definiteness of the nilpotent cone of $\mathfrak{so}_{2N}$ algebras as a singular space with the singularity localized in a single locus, any slice to it retains the singularity structure; thus, for any even partition ρ of

$2N$, the singularity which is the Coulomb branch moduli space of the associated $T_\rho(SO(2N))$ stays well behaved. The mathematical definition of these spaces signals that the appearance of bad $USp(2m)$ nodes with $N_f = 2m$ flavors in $T_\rho(SO(2N))$ theories does not and needs not to disrupt the goodness¹⁹ of the theory. A pathway to opening a physical understanding of this phenomenon can be opened thinking that the $\mathbb{Z}_2$ symmetry exchanging the local two singular points of the bad symplectic gauge node, in reality, identifies them, despite their quantum mechanical inequivalence. Another example is to consider the 3D mirror pair $T(SO(2n+1))$ and $T(USp(2n))$ established in [17], which have as maximal branches the maximal/regular nilpotent orbit closures of $\mathfrak{so}(2n+1)$ and $\mathfrak{usp}(2n)$, respectively. The Higgs branch of $T(USp(2n))$ is a well-defined hyper-Kähler cone. Therefore, the Coulomb branch of $T(SO(2n+1))$ is also a well-defined hyper-Kähler cone with one most singular point despite the fact that all its symplectic gauge nodes have balance $b = -1$. Consequently, we generically admit bad nodes with $b = -1$ in our Hasse Diagrams, as stated in Rule 2. Notice, however, that although an explicit analysis has been carried out for symplectic gauge nodes, an alternative one for even special-orthogonal nodes is lacking, and we leave it for future analysis.

Equipped with this new understanding, we can push the study of $T_\rho(SO(2N))$ theories to also justify the decay pattern predicted by Rule 3b. The natural inclusion ordering on the closure of nilpotent orbits induces an ordering in the $T_\rho(SO(2N))$ theories on the set of partitions ρ -s associated with the orbits O -s. In fact, starting from a partition ρ_B , such that the resulting $T_{\rho_B}(SO(2N))$ theory has m gauge nodes of balance $b = -1$,²⁰ it is possible to flow to the next special partition²¹ $\rho_>$ in the aforementioned partial ordering. The $T_{\rho_>}(SO(2N))$ theory has the same structure of $T_{\rho_B}(SO(2N))$ but has $n < m$ of the previous bad nodes with the rank lowered by one, i.e., the bad node in $T_{\rho_B}(SO(2N))$ has now become balanced. This transition is characterized by a c_n transverse slice, i.e., the closure of the minimal nilpotent orbit of $USp(2n)$, with n given by the number of -1 balance nodes that are now balanced.

¹⁹Here, this is meant as having a Coulomb branch moduli space constituting of a single singular locus and not in the balance sense of (2.2), which pertains to the conformal dimension of the theory's monopole operators.

²⁰From the construction of [17], it is clear that the minimum balance for a gauge node achievable in a T_ρ theory is $b = -1$.

²¹Special partitions are such that they are mapped into themselves after having applied twice the Barbasch-Vogan map, which takes partitions of G to special partitions of its GNO dual $G^\vee$. This requirement is physically equivalent to demanding $(G^\vee)^\vee = G$ [15]. Ee can go from the electric to the magnetic phase of a theory, i.e., trading monopoles for multiplets in the 3D language and back and trading again multiplets for monopoles, consistently.

¹⁸The word “vacuum” has been used in the sense of [63], where it stands for the global collection of all the local geometries, i.e., Coulomb branch moduli space of vacua, admitted near each singular point in the space.

In the Appendix, the algorithm is showcased in the context of $T_\rho^\sigma(SO(2N))$ theories. We require said c_m pattern to be transported to more general orthosymplectic quiver theories consistently with observations, leading to Rule 3b.

C. Unitary within orthosymplectic

In dealing with orthosymplectic quiver theories, it is crucial to encompass the possibility of partial Higgsing directions leading to unitary theories. As remarked in Sec. II, Rule 4a and Rule 4b, play the fundamental role of explaining the appearance of unitary quiver gauge theories as fission or unitarization products of their orthosymplectic ancestors. The formal justification of said procedure lies in the representation theory of orthosymplectic algebras, each of which admits a Levi decomposition as a sum of subalgebras of the form

$$\begin{aligned}\mathfrak{so}_{2N} &\rightarrow \mathfrak{so}_{2N-2K} \oplus \mathfrak{su}_K \oplus \mathfrak{u}_1, \\ \mathfrak{usp}_{2N} &\rightarrow \mathfrak{usp}_{2N-2K} \oplus \mathfrak{su}_K \oplus \mathfrak{u}_1.\end{aligned}\quad (4.2)$$

Under this decomposition, a nilpotent orbit O of the parent algebra fractionates and induces the nilpotent orbits $O_{1^{\text{st}}}$ and $O_{2^{\text{nd}}}$ in the first and second addendum in the decomposition. In terms of physics, it would be improper to talk about this decomposition in terms of $T_\rho(G)$ theories. The reason, we soon thoroughly illustrate, lies in the presence of the G flavor node.

In the standard decay and fission algorithm [21,22], quiver gauge theories of the unitary type are the main object of study; a subset of these theories admit a brane construction in a type IIB D3-D5-NS5 system [68] or, when magnetic quivers are also taken into account, as type IIB (p, q) -webs of 5-branes [69,70], or type IIA D6-D8-NS5 brane engineering [26,71]. When a brane system is available, a clear interpretation is given to each possible Higgsing process, and a *decay* constitutes a sequence of Kraft-Procesi transitions [72], i.e., in the standard type IIB D3-D5-NS5 system, they correspond to moving D3-branes segments, suspended in between the NS5-branes, along the transverse spatial directions spanned by the D5-branes. A more complete explanation of such brane configurations and transitions are in the original papers [72,73]. Conversely, a *fission* process accounts for partitioning the total number of light branes suspended between heavy branes such that two distinct noninteracting theories arise [74,75].

Inspired by this interpretation of fission processes in a brane picture, the analogous phenomenon for orthosymplectic quiver theories requires the aforementioned superstring theory constructions in the presence of an additional Op -orientifold plane. Different from the unitary case, the orientifold is spatially localized; thus, a separation in stacks of Dp -branes followed by a careful infinite distance limit,

i.e., the partitioning process of the unitary case, leads to having only one of the stacks remaining on top of the orientifold and giving rise to an orthosymplectic theory. Oppositely, the stack that has been pulled away realizes a unitary theory since it does not sense the presence of the orientifold any more.²² In the case of $T_\rho(G)$ realized in a type IIB D3-D5-NS5 system, a transverse pulling away action on D3-branes would not be able to break the flavor symmetry determined by D5-branes, as the D5s would be unmovable and do not share any spatial direction with the NS5-branes other than the one also shared with D3-branes.

There is yet one process that needs to be understood: unitarization, which corresponds to having $N = K$ in the subalgebra splitting. The phenomenology of this choice corresponds to the extreme action of pulling away all the allowed branes from the orientifold plane. Here, the term allowed includes the possibility of having fractional branes remaining stuck on the orientifold plane and not contributing to physical gauge degrees of freedom.

An analogous Higgsing pattern compatible with the phenomena just described has been encountered in [54], where studying $(2,0)$ 6D SCFTs in type D_N there exist complex structure deformations, the analogous of Higgsings in F-theory construction, leading to a product of theories of type D_{N-k} and type A_{k-1} for $1 \leq k \leq n$. Type D_N theories admit a magnetic quiver constituted by a single $SO(2N)$ gauge node with antisymmetric matter, whereas the magnetic quiver for the type A_{N-1} theory has a single $U(N)$ with an adjoint matter field. Therefore, the F-theory predictions of the Hasse diagram are mimicked in Rule 4a and Rule 4b of the orthosymplectic decay and fission algorithm. Arguments from the F-theory in support of these rules also stem from the possibility of turning special orthogonal algebras, corresponding to fibrations of type I_n^* in F-theory constructions, to (special-)unitary ones, corresponding to fibrations of type I_n in F-theory constructions. The analog for symplectic algebra a has been discussed in [74] and substitutes the I_n^* fibers with I_n ones but with a different monodromy with respect to the (special-)unitary ones. This geometrical phenomenon is the electric analog of the fission and unitarization process described via magnetic quivers.

V. BEYOND $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ CLASS $\mathcal{S}$

The development of the orthosymplectic decay and fission algorithm that we presented in Sec. II was influenced by an exhaustive analysis of the Higgs branch renormalization group flows of class $\mathcal{S}$ of type $\mathfrak{su}(4) \cong \mathfrak{so}(6)$. The key which unlocked this analysis was the Lie algebra isomorphism, which allowed us to

²²Although for a reader not familiar with brane engineering this description may appear somewhat quick, we will soon provide, in Sec. V, explicit examples of brane systems undergoing fission processes. We refer the reader to that section for an illustration of the brane dynamics.

use the unitary decay and fission algorithm to predict the required structure obtained from the orthosymplectic decay and fission algorithm.

In this section, we demonstrate that the algorithm holds beyond these class $\mathcal{S}$ examples. We consider both examples of higher rank class $\mathcal{S}$ theories and examples of 6D (1,0) SCFTs, whose Higgs branch is given by a known orthosymplectic 3D mirror/magnetic quiver and for which the Higgs branch renormalization group flows can be understood via the complex structure deformations of the singularity realizing the 6D SCFT in the F-theory. In these examples, we observe that the orthosymplectic decay and fission algorithm on the magnetic quiver/3D mirror produces the expected result for the structure of the Higgs branch of the higher dimensional theory. To test the orthosymplectic decay and fission algorithm, as presented here, in the highly constrained space of 6D SCFTs with (1,0) supersymmetry, we seek 6D SCFTs that have a known orthosymplectic magnetic quiver for their Higgs branches and such that the magnetic quiver does not have any two-index-tensor-valued matter. The latter requirement is because we have not proposed decay and fission rules in the presence of such matter, though this should be a straightforward extension that we leave for future work. Only one such class of 6D (1,0) SCFTs is known, they are called the D-type orbi-instanton theories, and we discuss them in detail in Sec. VA.

In fact, there are many more constructions that lead to Coulomb branches which have simultaneous unitary and orthosymplectic quiver descriptions. For example, in [76,77], subgroups of the Coulomb symmetry of unitary and orthosymplectic quivers have been gauged, and quiver descriptions for the resulting theories has been proposed. In this way, new orthosymplectic quiver theories whose moduli space coincides with a unitary one have been derived (see [77], Tables 2 and 3). It is straightforward to apply the orthosymplectic decay and fission algorithm to these examples and see that they agree with the unitary decay and fission. Hence, providing another nontrivial consistency check of the rules proposed here. Yet another source of 3D $\mathcal{N}=4$ quivers with both unitary and orthosymplectic descriptions is [33]. We leave this additional construction as a future testing ground for the orthosymplectic decay and fission algorithm.

A. Higgsing the D-type Orbi-instanton

We now turn to the study of 3D $\mathcal{N}=4$ orthosymplectic quivers, which are magnetic quivers for the Higgs branch of certain 6D (1,0) SCFTs. Related to the anomaly cancellation constraints, 6D (1,0) SCFTs are believed to be classified via a construction in terms of F-theory compactified on noncompact elliptically fibered Calabi-Yau threefolds satisfying certain restrictive conditions [78–80]. The effective theories that live on the subloci of the Higgs branch can be determined by studying the space of complex structure deformations of the Calabi-Yau threefold engineering the 6D (1,0) SCFT in question. Therefore, the Orthosymplectic decay and fission algorithm proposed in Sec. II can be geometrically tested. In fact, applying the Orthosymplectic decay and fission algorithm allows for both new predictions about Higgs branch renormalization group flows of 6D SCFTs, as well as recovering and generalizing fission processes predicted in [37].

We consider a class of 6D (1,0) SCFTs, which not only have a geometric construction in the F-theory but also possess a realization in the M-theory. Consider a stack of N M5-branes probing a $\mathbb{C}^2/\Gamma$, with Γ being a finite subgroup of $SU(2)$, with orbifold singularity and contained inside of an M9-plane, as depicted in Fig. 8. The SCFT living on the world volume of the M5-branes is known as the rank N ($\mathfrak{e}_8, \mathfrak{g}$) orbi-instanton [79], where $\mathfrak{g}$ is the simple and simply laced Lie algebra associated to Γ via the McKay correspondence. To describe such a theory, it is necessary to specify a flat connection on the E_8 -bundle associated with the M9-plane on the S^3/Γ boundary of the orbifold. The trivial flat connection leads to an $\mathfrak{e}_8$ global symmetry factor, whereas other choices generally have a reduced global symmetry.

When $\mathfrak{g} = \mathfrak{so}(2n+8)$, there exists an alternative construction of the rank N ($\mathfrak{e}_8, \mathfrak{g}$) orbi-instanton in type IIA. The orbifold becomes a stack of $n+4$ full D6-branes on top of an $O6^-$ -orientifold-plane, the M5-branes are replaced by NS5-branes that fractionate on the orientifold, and the M9-plane is reduced to an $O8^-$ -orientifold-plane paired with 8 full D8-branes plus an ON^0 plane living at the intersection of the two orientifold planes [28]. For $n=2k$, a particular choice of flat connection for the E_8 -bundle associated with the M9-plane exists such that the type IIA brane engineering takes the following form:

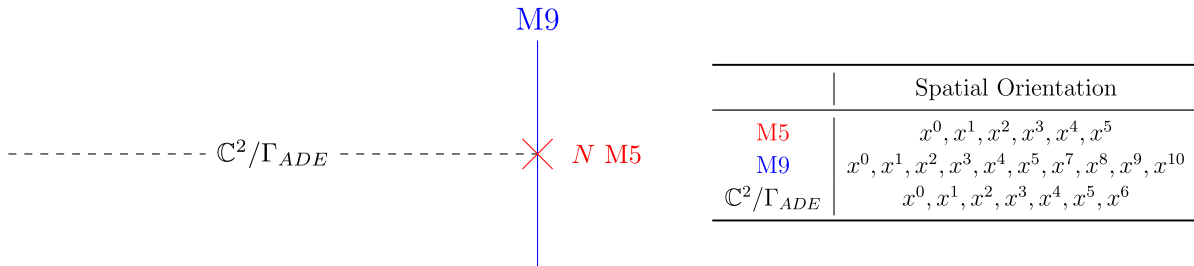


FIG. 8. M-theory configuration for a rank N ($\mathfrak{e}_8, \mathfrak{g}_{ADE}$) orbi-instanton theory.

$$(5.1)$$

The 6D (1,0) SCFT engineered by this brane configuration can be captured by writing the following generalized quiver:

$$(5.2)$$

In 6D SQFTs, there are vector multiplets, here associated with the written Lie algebra, and tensor multiplets, one for each simple gauge algebra and associated with self-dual strings, whose string charge is given by the corresponding bold integer. There are also hypermultiplets; however, the hypermultiplet spectrum is generally fixed by anomaly cancellation after the vector and tensor spectrum is specified. The SCFT is obtained by taking all the gauge couplings simultaneously to infinity. In Calabi-Yau geometry, the string charge associated with the tensor multiplets are the self-intersection numbers of smooth rational curves,

and the geometry giving rise to the SCFT is obtained via simultaneous contraction of each of these curves. For a detailed introduction to the language of tensor branch curve configurations in F-theory compactifications to six dimensions, we refer the reader to the review [81].

We can now apply the technique of passing to the magnetic phase of the type IIA brane system to extract a magnetic quiver for the Higgs branch of the 6D (1,0) SCFT (i.e., where all the gauge couplings in the tensor branch description are taken to infinity) [20,35]. The magnetic brane system is

$$(5.3)$$

where the Higgs branch moduli near the intersection of the orientifolds are arranged as follows:

$$(5.4)$$

On the right, we have color-coded the nodes in the resulting quiver to highlight which stacks of D6-branes they correspond to. The complete magnetic quiver for the Higgs branch of the 6D (1,0) SCFT, which is the 3D $\mathcal{N} = 4$ object we are interested in, is thus read off as

$$(5.5)$$

We note that for $N = 0$. This coincides with the magnetic quiver derived in [35]. As we can determine from the balance of the nodes in this quiver, the global symmetry is generically (for $N > 0$) given by

$$\mathfrak{f} = \mathfrak{so}(4k+8) \oplus \mathfrak{so}(8) \oplus \mathfrak{so}(8). \quad (5.6)$$

For the rank zero D-type orbi-instanton, the 6D (1,0) SCFT is in fact a product theory, and correspondingly, the global symmetry is enhanced. The product is two copies of the 6D (1,0) SCFTs known as rank one $[\mathfrak{so}(2k), \mathfrak{so}(2k)]$ conformal matter; it would be interesting to apply the orthosymplectic decay and fission algorithm to this magnetic quiver (which is not itself a product) and to recover the product nature of the Higgs branch.

From the perspective of the M-theory realization of the orbi-instanton theories, a Higgs branch deformation that separates ℓ of the N M5-branes out of the M9-brane is

expected. As a benchmark of the orthosymplectic decay and fission algorithm from Sec. II, we would like to be able to reproduce this Higgs branch RG flow from the magnetic quiver for the Higgs branch. For simplicity, we consider the case where $k = 0$; then, the SCFTs is specified by the following curve configuration at the generic point of the tensor branch:²³

$$[\mathfrak{so}_8] \begin{matrix} 1 \\ \mathfrak{so}_8 \end{matrix} \underbrace{1 \ 4 \ \cdots \ 1 \ 4 \ 1}_{N-1} [\mathfrak{so}_8]. \quad (5.7)$$

The magnetic quiver for the Higgs branch then takes the following form:

$$(5.8)$$

This is a star-shaped orthosymplectic quiver, and applying it to the orthosymplectic decay and fission algorithm is straightforward. While the resulting Coulomb branch Hasse diagram is rather large and intricate, it is easy to show that, for each $0 < \ell \leq N$, there exists a fission to the following product of quivers:

$$(5.9)$$

²³It would be excessively cumbersome to provide a careful introduction for this notation to capture 6D (1,0) SCFTs in this paper. We instead refer the reader to [81–83] for comprehensive reviews.

The unitary quiver appearing in the product after fission is nothing other than a magnetic quiver for the Higgs branch of the 6D (1,0) SCFT engineered as a stack of ℓ M5-branes, not probing any orbifold, inside of an M9-plane [10,75]. This is the rank ℓ E-string theory. In the F-theory language, this transition corresponds to the Higgsing

$$\begin{array}{c} [\mathfrak{so}_8] \\ 1 \\ \mathfrak{so}_8 \\ \underbrace{[\mathfrak{so}_8] \ 1 \ 4 \ \cdots \ 1 \ 4 \ 1 [\mathfrak{so}_8]}_{N-1} \end{array} \rightarrow \begin{array}{c} [\mathfrak{so}_8] \\ 1 \\ \mathfrak{so}_8 \\ \underbrace{[\mathfrak{so}_8] \ 1 \ 4 \ \cdots \ 1 \ 4 \ 1 [\mathfrak{so}_8]}_{N-1-\ell} \end{array} \sqcup \begin{array}{c} [\mathfrak{e}_8] \ 12 \ \cdots \ 2 [\mathfrak{su}_2] \\ \ell \end{array}. \quad (5.10)$$

Although this feature is something expected from all rank N ($\mathfrak{e}_8, \mathfrak{g}$) orbi-instanton theories [74,75,84], this was the first time that an explicit example of this phenomenon was carried out beyond the $\mathfrak{g} = \mathfrak{su}(K)$ case.²⁴

B. Higher-rank Class $\mathcal{S}$: Beyond Partial Puncture Closure

General class $\mathcal{S}_{\mathfrak{g}}\langle C_{g,n} \rangle \{O_1, \dots, O_n\}$ theories with $\mathfrak{g} = \mathfrak{so}_{2k}$ and their Higgsings have been studied in [89]. In the literature, there has been control over such theories leveraging the association between certain Higgsings, nilpotent orbits, and partial puncture closure in class $\mathcal{S}$. Such interplays allow us to predict Higgs branch RG flows of the 4D $\mathcal{N} = 2$ SCFTs that are localized on a puncture via the corresponding nilpotent data, leading in the extreme case to the full close of the puncture itself, which changes the Riemann surface to $C_{g,n-1}$.

However, there are more general Higgsings in class $\mathcal{S}$ going beyond just partial puncture closure. More general Higgsing directions take the class $\mathcal{S}$ of type $\mathfrak{g}$ to (possibly a product of) class $\mathcal{S}$ of type $\mathfrak{g}'$ for $\mathfrak{g}' \subset \mathfrak{g}$, a Levi subalgebra of $\mathfrak{g}$. Such possibilities were fully explored only for $\mathfrak{g} = \mathfrak{su}(n)$, where exploiting the unitary-nature of the 3D mirrors for the associated class $\mathcal{S}$ theories [31], the decay and fission algorithm [21,22] demonstrates the transition

$$\begin{aligned} \mathcal{S}_{\mathfrak{su}_n}\langle C_{g,n} \rangle \{O_1, \dots, O_n\} &\rightarrow \mathcal{S}_{\mathfrak{su}_{n-k}}\langle C_{g',n} \rangle \{O'_1, \dots, O'_n\} \\ &\times \mathcal{S}_{\mathfrak{su}_k}\langle C_{g'',n} \rangle \{O''_1, \dots, O''_n\}. \end{aligned} \quad (5.11)$$

Since our algorithm implements decay and fission for orthosymplectic quiver theories (without additional anti-symmetric matter), we can explore the full Higgs branch of

$$\mathcal{S}_{\mathfrak{so}_{2n}}\langle C_{0,n} \rangle \{O_1, \dots, O_n\} \quad (5.12)$$

and appreciate the Higgs branch renormalization group flow corresponding to the decomposition of $\mathfrak{so}(2n)$ into the following Levi subalgebras:

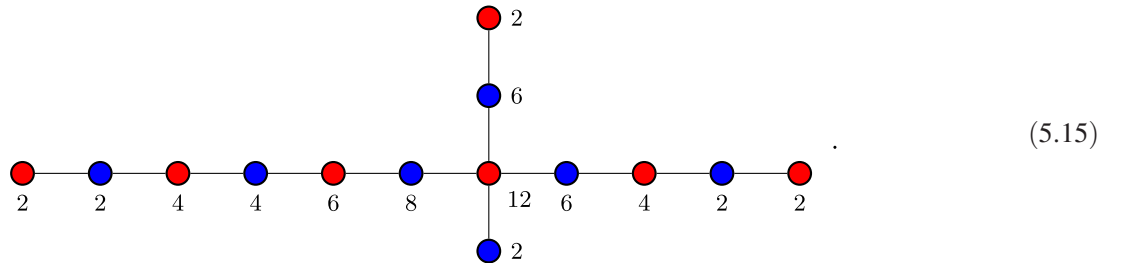
$$\mathfrak{so}(2n) \rightarrow \mathfrak{so}(2n-2k) \oplus \mathfrak{su}(k). \quad (5.13)$$

In this section, we highlight an interesting example of the type-changing Higgs branch RG flows expected to exist from the orthosymplectic decay and fission algorithm.

For simplicity, we consider only a subset of all the possible Higgs branch renormalization group flows of the theory,

$$\mathcal{S}_{\mathfrak{so}_{12}}\langle C_{0,4} \rangle \{[3^2, 1^6], [5, 3, 1^4], [9, 3], [5^2, 1^2]\}, \quad (5.14)$$

which are sufficient to demonstrate the most notable features of higher rank class $\mathcal{S}$. The 3D mirror for this theory is the following star-shaped orthosymplectic quiver:



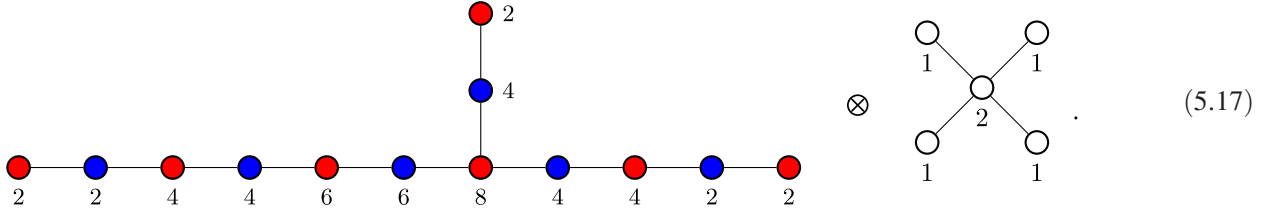
²⁴Intriguingly, there is a growing body of evidence that torus compactifications (with Stiefel-Whitney and outer-automorphism twists) followed by mass deformations of orbi-instanton SCFTs provide a higher-dimensional origin for all 4D $\mathcal{N} = 2$ SCFTs (see, for example, [85–88]). While we remain agnostic on such a conjecture, it highlights that understanding the Higgs branches of orbi-instanton SCFTs can have far-reaching implications.

We note that all the gauge nodes are either balanced or good, leading to an explicit non-Abelian $\mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(4)$ global symmetry.²⁵

The orthosymplectic decay and fission algorithm applied to the quiver in Eq. (5.15) indicates the existence of a fission process determined by the following algebras splitting

$$\begin{aligned}\mathfrak{so}(12) &\rightarrow \mathfrak{so}(8) \oplus \mathfrak{u}(2), \\ \mathfrak{usp}(8) &\rightarrow \mathfrak{usp}(6) \oplus \mathfrak{u}(1), \\ \mathfrak{usp}(6) &\rightarrow \mathfrak{usp}(4) \oplus \mathfrak{u}(1), \\ \mathfrak{usp}(2) &\rightarrow \emptyset \oplus \mathfrak{u}(1),\end{aligned}\tag{5.16}$$

where the fission happens on the central $SO(12)$ gauge node and the four surrounding gauge nodes only. The end result of this fission is the following product theory:



This replicates the following Higgs branch renormalization group flow in class $\mathcal{S}$:

$$\mathcal{S}_{\mathfrak{so}_{12}}\langle C_{0,4} \rangle \{[3^2, 1^6], [5, 3, 1^4], [9, 3], [5^2, 1^2]\} \longrightarrow \mathcal{S}_{\mathfrak{so}_8}\langle C_{0,3} \rangle \{[1^8], [3^2, 1^2], [3, 1^5]\} \times \mathcal{S}_{\mathfrak{u}_2}\langle C_{0,4} \rangle \{[1^2], [1^2], [1^2], [1^2]\},\tag{5.18}$$

which itself stems from the existence of an $\mathfrak{so}(8) \oplus \mathfrak{su}(2)$ Levi subalgebra in $\mathfrak{so}(12)$.

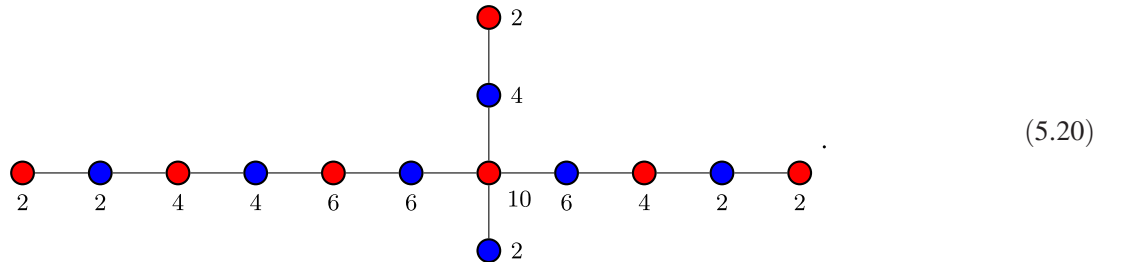
Let us now further apply the orthosymplectic decay and fission algorithm to the 3D mirror of the $\mathcal{S}_{\mathfrak{so}_8}\langle C_{0,3} \rangle \{[1^8], [3^2, 1^2], [3, 1^5]\}$ class $\mathcal{S}$ theory appearing in Eq. (5.18). We point out that this theory also appears in the class $\mathcal{S}$ analysis of [89]; hence, we have two independent ways to verify that the global symmetry is $\mathfrak{so}(10) \oplus \mathfrak{usp}(4) \oplus \mathfrak{u}(1)$. The $\mathfrak{so}(10)$ symmetry can be understood from the magnetic quiver by noticing that the central node is balanced; therefore, minimal nilpotent Higgsing of the moment map of this symmetry is an action that cannot be reproduced by only partial puncture closure in the class $\mathcal{S}$ picture. From the magnetic quiver, it corresponds to decaying the set of balanced nodes generating this symmetry via Table I. This operation recovers the quiver in Eq. (3.24), and this justifies the class $\mathcal{S}$ Higgs

branch transition,

$$\begin{aligned}\mathcal{S}_{\mathfrak{so}_8}\langle C_{0,3} \rangle \{[1^8], [3^2, 1^2], [3, 1^5]\} \\ \xrightarrow{d_5} \mathcal{S}_{\mathfrak{so}_6}\langle C_{0,3} \rangle \{[1^6], [1^6], [2^2, 1^2]\}.\end{aligned}\tag{5.19}$$

We note that the dimension of the conjectured transverse slice $[\dim_{\mathbb{H}} d_n = 2n - 3]$, where d_n again denotes the closure of the minimal nilpotent orbit of $\mathfrak{so}(2n)$ correctly reproduces the change in Coulomb branch dimension of the 3D quiver.

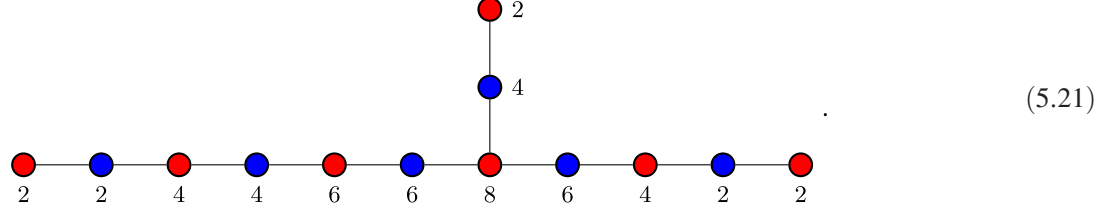
Alternatively, we can decay the $\mathfrak{so}(4)$ symmetry involving the balanced $SO(12)$ node of Eq. (5.15) and reduce by 1 the rank of the bad USp -type gauge node resulting from the process. The outcome of this procedure is the 3D mirror of the $\mathfrak{so}(10)$ class $\mathcal{S}$ theory $\mathcal{S}_{\mathfrak{so}_{10}}\langle C_{0,4} \rangle \{[3, 1^7], [3^2, 1^4], [5, 3, 1^2], [7, 3]\}$, which is



²⁵We have not analyzed extra Abelian factors not emerging from balanced subsets.

This 3D mirror exhibits an $\mathfrak{so}(7)$ factor in the global symmetry from the balanced nodes on the left tail, agreeing with expectations from the Hall-Littlewood index of the class $\mathcal{S}$ theory.

We can perform a further sequence of decays on the quiver in Eq. (5.20). Fully closing the puncture associated with the stand-alone $USp(2)$ node, i.e., decaying this node, renders the central $SO(10)$ balanced, which, together with the balanced $USp(6)$, give rise to an $\mathfrak{so}(3)$ symmetry that can be further decayed, leading to the 3D mirror of the $\mathfrak{so}(8)$ class $\mathcal{S}$ theory, $\mathcal{S}_{\mathfrak{so}_8}\langle C_{0,3} \rangle \{[1^8], [3^2, 1^2], [2^2, 1^4]\}$,



Finally, a further decay of the bad $USp(6)$ node on the right tail leads to the orthosymplectic quiver in Eq. (5.17). As we have discussed, this theory can be further decayed to the orthosymplectic quiver shown in Eq. (3.24).

We have depicted these sequences of decays and fissions in Fig. 9. This result is particularly interesting as this Higgs branch structure, in particular the existence of at least two

separate Higgs branch renormalization group flows between the class $\mathcal{S}$ theories

$$\mathcal{S}_{\mathfrak{so}_{12}}\langle C_{0,4} \rangle \{[3^2, 1^6], [5, 3, 1^4], [9, 3], [5^2, 1^2]\} \quad (5.22)$$

and

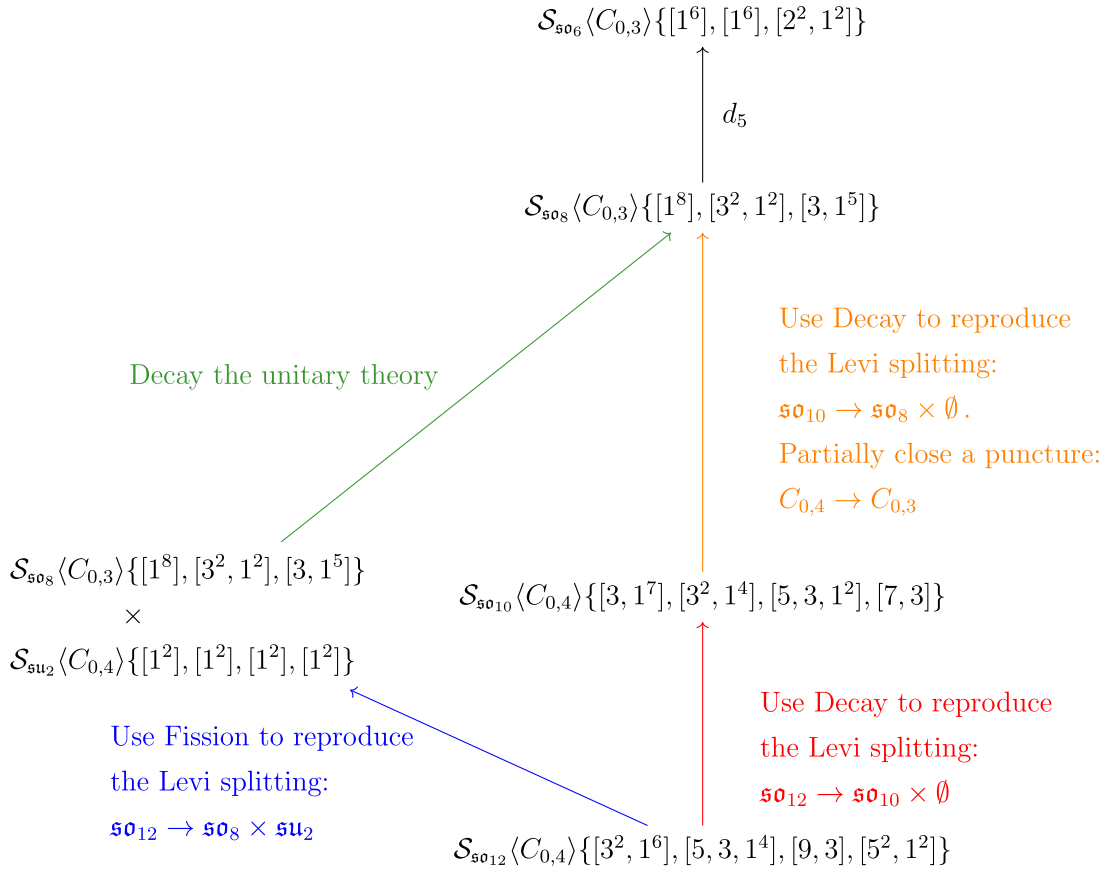


FIG. 9. Some of the Higgs branch RG flows between the class $\mathcal{S}$ theories of different types conjectured to exist from the application of the orthosymplectic decay and fission algorithm applied to the 3D mirror of the class $\mathcal{S}$ theory at the bottom. As explained in the text, the orange arrow does not represent a minimal transition, whereas all other arrows are minimal.

$$\mathcal{S}_{\mathfrak{so}_6}\langle C_{0,3} \rangle \{[1^6], [1^6], [2^2, 1^2]\}, \quad (5.23)$$

are not manifest from the class $\mathcal{S}$ perspective. We leave the construction of the explicit Higgsing on the purely class $\mathcal{S}$ side for future work. However, such a pattern matches the expected behavior from the notions of simultaneous deletion in class $\mathcal{S}$ theories proposed in [45]. We remark that this result was, before our algorithm, inaccessible from the perspective of the 3D mirror.

VI. DISCUSSION

Quiver gauge theories in three dimensions have proven themselves a fruitful and enriching object of study, both for understanding their intrinsic physical phenomena and also as a portal to access information difficult to extract from higher-dimensional theories. Exemplary, the physics of tensionless strings or Argyres-Douglas points contributing to the Higgs branch of SQFTs in, respectively, six, five, and four dimensions becomes easily accessible via the magnetic quiver technique [9,10]. Alongside this result, a follow-up question emerged: what does the Higgs branch structure entail, and what is the possible pattern of partial Higgsing of these theories? In this spirit, the long-standing problem of understanding the Higgs branch, by now known to be a symplectic singularity [1,7], the associated renormalization group flows, and Higgsing patterns of supersymmetric quantum field theories in $d \geq 4$ posed in the last question was tackled in this paper. Leveraging the development of the magnetic quiver technology [9,10], the problem is traded with the study of Higgsing of the Coulomb branches of 3D $\mathcal{N} = 4$ quiver gauge theories. While multiple efforts have been proved successful in extracting partial Higgsing patterns [1] from unitary quiver gauge theories [19,21,22], analogous results have been only partially found for orthosymplectic theories [20].

Our work extends the previous results, proposing an algorithm that predicts the Higgsing pattern for simply laced orthosymplectic quiver theories (with an edge multiplicity one). Multiple features, both expected and unexpected, emerge from the algorithm. Similar to what was shown in the unitary decay and fission case, an orthosymplectic theory can only Higgs to a theory with a gauge group of lower ranks or fission to a product of theories. Unexpectedly, we found that an orthosymplectic quiver theory can undergo fission into an orthosymplectic theory and a unitary one or, in the extreme case, a purely unitary theory when it unitarizes. For instance, using the known existence of a 3D mirror, we can now predict the full Higgs branch Hasse diagram for type $\mathfrak{so}(2n)$ class $\mathcal{S}$ theories. Similarly, we demonstrate that certain 6D (1,0) D-type orbi-instanton SCFTs have Higgs branch RG flows that force them to split into specific product theories, consistent with the idea of separating stacks of M5-branes in their construction via the M-theory.

The algorithm developed in this paper applies to all orthosymplectic quivers. Therefore, equipped with this technology, we can expand our knowledge of the Higgs mechanism across dimensions. In general, orthosymplectic quivers that are magnetic quivers of higher dimensional SCFTs do not have known unitary counterparts. Therefore, our algorithm now provides a method of Higgsing along the Higgs branch for these SCFTs. In this spirit, a direction to pursue is extending the algorithm to encompass more exotic matter, such as rank-two antisymmetric hypermultiplets, and quivers with nonsimply laced edges which are key features of known magnetic quivers for the Higgs branches of certain higher dimensional SCFTs. Moreover, our work lays the groundwork for a more detailed justification of the algorithm from a purely mathematical study of the landscape of symplectic singularities that can be built via Coulomb branches of orthosymplectic quivers.

Aside from the mathematical perspective opened by our work, more interesting research directions may be inspired by our findings. The decay and fission algorithm constitutes an example of a technique descending from the underlying geometry of the Coulomb Branch moduli space that interprets the Higgs mechanism from a purely algebraic-geometric point of view. In this spirit, a parallel transport of ideas to lower supercharges may be viable: for each amount of supercharges, the moduli space of vacua has a specific geometry property enforced by the Super-Poincaré algebra. In each case, an interpretation of Higgsing may be given as a mathematical action on said space, opening an alternative probe of the Higgs mechanism on theories that do not have Lagrangian descriptions.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: HIGGSING 3D $\mathcal{N}=4$ $T_\rho^\sigma(G)$ THEORIES

In this paper, we have proposed an orthosymplectic decay and fission algorithm for extracting the structure of

the Coulomb branch of orthosymplectic quivers. So far, the quivers we have studied, inspired by class $\mathcal{S}$, are star-shaped unframed/flavorless orthosymplectic quivers. These are theories that are well known to be magnetic quivers for the Higgs branch/3D mirrors of class $\mathcal{S}$ theories and other higher dimensional SCFTs.

In this Appendix, we would like to go beyond this paradigm and consider the orthosymplectic decay and fission as applied to *framed* orthosymplectic quivers, that is, those with flavor nodes. Examples of such theories are the $T_\rho^\sigma(G)$ from [17], where G is orthosymplectic.

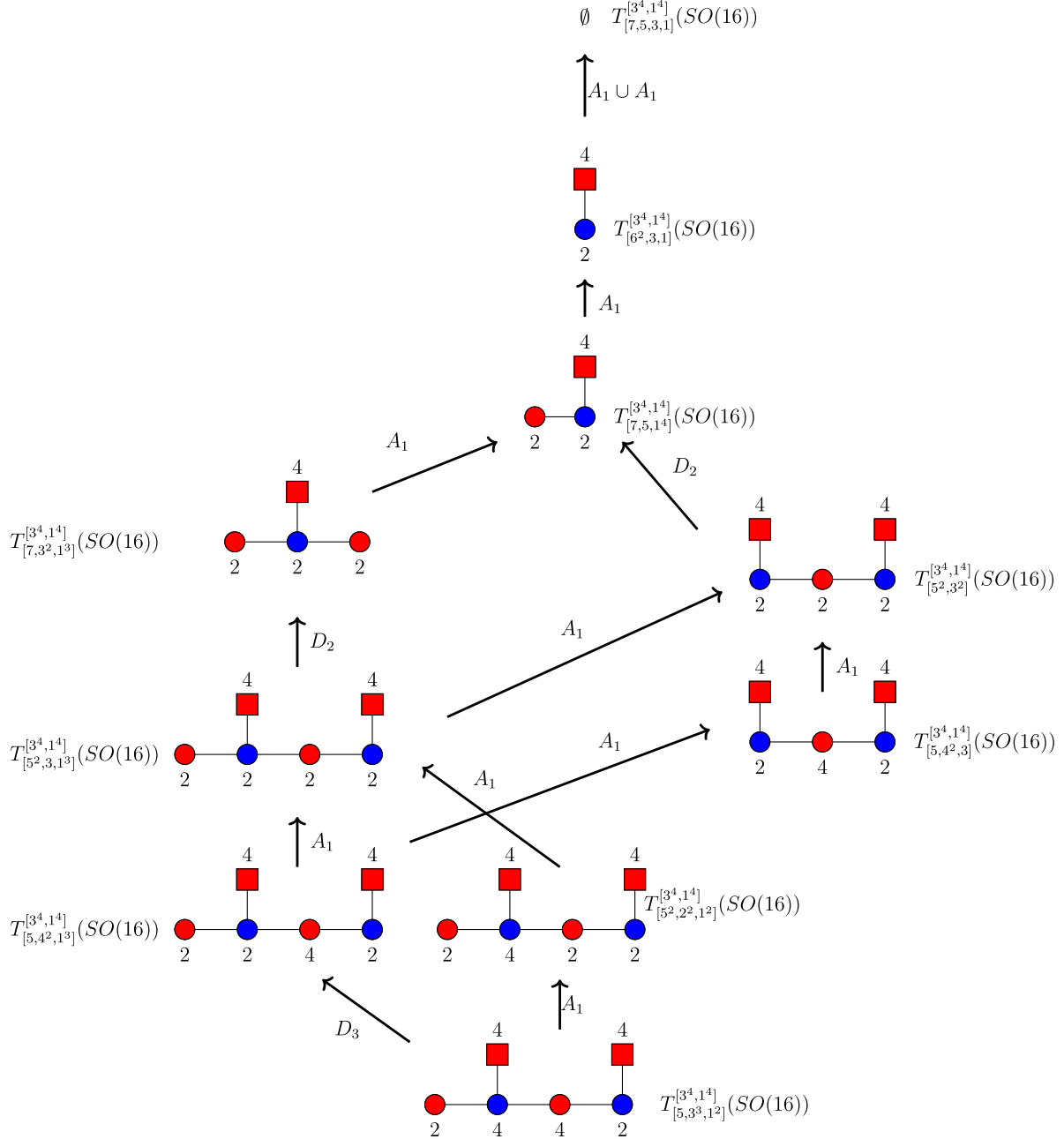


FIG. 10. Coulomb branch Hasse diagram of the 3D $\mathcal{N}=4$ SCFT $T_{[5,3^3,1^2]}^{[3^4,1^4]}(SO(16))$ obtained from the orthosymplectic decay and fission algorithm.

As an example, we consider the orthosymplectic quiver $T_{[5,3^3,1^2]}^{[3^4,1^4]}(SO(16))$. Applying the orthosymplectic decay and fission algorithm yields the Coulomb branch Hasse diagram that is depicted in Fig. 10. We note that this reproduces the dominance ordering of the partitions

associated to the nilpotent orbits ρ , as desired. In this figure, the transverse slices can be obtained either through brane dynamics similar to the analysis in [73] or through subtracting the quivers before/after decay and rebalancing the gauge nodes along the same lines as in [21].

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