

Symmetries and anomalies of Hamiltonian staggered fermions

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We review the shift (translation) and time reversal symmetries of Hamiltonian staggered fermions and their connection to continuum symmetries concentrating in particular on the case of massless fermions and $(3 + 1)$ dimensions. We construct operators using the staggered fields that implement these symmetries on finite lattices. We show that shifts composed of an odd multiple of the elementary shift anticommute with time reversal and are related to continuum axial transformations. We argue that the presence of these nontrivial commutation relations implies the existence of lattice 't Hooft anomalies. From the shifts we also construct a set of conserved, quantized charges that generate continuous symmetries of the lattice theory. In general these do not commute with the vector charge signaling further 't Hooft anomalies.

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I. INTRODUCTION

In this paper, we focus on the symmetries of staggered fermions. Although the symmetries of the Euclidean formulation, in which both time and space are discretized, are well known [1–6], the Hamiltonian formalism developed in [7] has received less attention in the case when the spatial dimension is greater than one.¹ In our work we focus on the structure of the shift and time reversal symmetries for Hamiltonian staggered fermions for arbitrary spatial dimension. We are particularly interested in understanding the connection between the anomalies seen in Euclidean formulations of staggered or Kähler-Dirac fermions [10,11] and the structure of these theories as viewed from a Hamiltonian perspective. In particular, we would like to understand whether we can build chiral lattice gauge theories by gauging certain discrete translation symmetries of staggered fermions along the lines proposed in [12]. These discrete symmetries are called shift symmetries in the literature, and, as we will discuss later, can be thought of as a finite subgroup of the axial-flavor symmetry of the continuum theory. In particular, our focus will be on understanding whether such symmetries break in response to gauging other symmetries signaling the presence of

mixed 't Hooft anomalies—a phenomenon that has been observed in other lattice systems [13–15]. In path integral approaches to quantum field theory, anomalies including mixed anomalies, arise from a noninvariance of the fermion measure.² In contrast within the canonical formalism mixed anomalies are realized when operators representing distinct symmetries do not commute.

Following the procedure given in [14] we construct explicit operators that implement the elementary shifts S_k , time reversal $\mathcal{T}$, and a global $U(1)$ phase symmetry on a finite lattice. Our work can be seen as an extension of recent work on Majorana chains in one spatial dimension [14] and the Schwinger model [8,9]. Motivation for our work can also be found in the phenomenon of symmetric mass generation which requires the cancellation of lattice 't Hooft anomalies [10,11,16–19] and the formulation of certain lattice chiral gauge theories using mirror fermions [20,21]. Many of these studies start from the Hamiltonian formulation which provides a motivation for this work.

We start by reviewing the staggering procedure for the Hamiltonian formalism and then discuss the symmetries of the system focusing on the shift and time reversal symmetries. We then construct finite operators that implement these symmetries on the lattice and examine their commutator structure. We examine in particular the case of three spatial dimensions showing the relationship of shift invariance to continuum axial-flavor symmetries. Finally, we are able to construct a series of exact continuous symmetries of the model by combining the phase symmetry with the shift symmetries. These symmetries obey a nontrivial algebra

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¹Our work has been influenced, however, by recent theoretical work on staggered fermions in $(1 + 1)$ dimensions given in [8,9].

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²Notice that the usual Adler-Bell-Jackiw anomaly can be thought of as a mixed anomaly since it corresponds to the breaking of a global axial symmetry in the presence of the background gauge field for a vector symmetry.

which we conjecture encodes the continuum anomaly structure for vanishing lattice spacing.

II. HAMILTONIAN STAGGERED FERMIONS

Our starting point is the continuum Dirac Hamiltonian given by

$$H = \int d^3x \bar{\Psi}(x)(i\gamma_i \partial_i + m)\Psi(x), \quad (1)$$

where i are spatial indices running from $1 \cdots d$. The lattice Hamiltonian is obtained by first introducing a cubic spatial lattice and replacing the derivative with a symmetric finite difference

$$H = \sum_x \bar{\Psi}(x)[i\gamma_i \Delta_i(x, y) + m\delta_{x,y}]\Psi(y), \quad (2)$$

where x now labels an integer position vector on the lattice with components $\{x_i\}$ and the symmetric difference operator is defined by

$$\Delta_i(x, y) = \frac{1}{2}(\delta_{y, x+i} - \delta_{y, x-i}). \quad (3)$$

We indicate a shift of the site x by one lattice spacing in the i th direction by $(x+i)$. It is convenient to introduce the Hermitian matrices $\alpha_i = \gamma_0 \gamma_i$ and $\beta = \gamma_0$ and rewrite this as

$$H = \sum_{x,y} \Psi^\dagger(x)[i\alpha_i \Delta_i(x, y) + m\beta \delta_{x,y}]\Psi(y). \quad (4)$$

As is well known, this lattice Hamiltonian suffers from fermion doubling and describes 2^d degenerate Dirac fermions in the continuum limit. Staggered fermions represent efforts to reduce this degeneracy. To obtain the staggered Hamiltonian, we first perform a unitary transformation on Ψ to a new basis χ as follows:

$$\begin{aligned} \Psi(x) &= \alpha^x \chi(x), \\ \Psi^\dagger(x) &= \chi^\dagger(x) (\alpha^x)^\dagger, \end{aligned} \quad (5)$$

where

$$\alpha^x = \alpha_1^{x_1} \alpha_2^{x_2} \cdots \alpha_d^{x_d}. \quad (6)$$

The Hamiltonian in this basis is then given by

$$H = \sum_{x,y} \chi^\dagger(x)[i\eta_i(x)\Delta_i(x, y) + m\epsilon(x)\delta_{x,y}]\chi(y), \quad (7)$$

where $\eta_i(x) = (-1)^{x_1+x_2+\cdots+x_{i-1}}$ and $\epsilon(x) = (-1)^{\sum_i x_i}$. Unlike the analogous situation in Euclidean space the resultant operator for $m \neq 0$ is not proportional to the unit matrix in spinor space. So we cannot stagger the field by

merely discarding all but one component of χ as one would do in that case. Instead, we can go back to Eq. (4) and decompose $\Psi(x)$ into two components $\Psi_\pm(x)$ that are eigenstates of β ,

$$\Psi(x) = \Psi_+(x) + \Psi_-(x), \quad (8)$$

where $P_\pm = \frac{1}{2}(1 \pm \beta)$. The Hamiltonian is then

$$\begin{aligned} H &= \sum_{x,y} [\Psi_+^\dagger(x) i\alpha_i \Delta_i(x, y) \Psi_-(y) \\ &\quad + \Psi_-^\dagger(x) i\alpha_i \Delta_i(x, y) \Psi_+(y)] \\ &\quad + m \sum_x (\Psi_+^\dagger(x) \Psi_+(x) - \Psi_-^\dagger(x) \Psi_-(x)), \end{aligned} \quad (9)$$

and we write $\Psi_\pm$ as

$$\begin{aligned} \Psi_\pm(x) &= P_\pm \alpha^x \chi(x) = \alpha^x \frac{1}{2} (1 \pm \epsilon(x)\beta) \chi(x) \\ &= \alpha^x \frac{1}{2} (1 \pm \epsilon(x)\beta) [\chi_+(x) + \chi_-(x)] \\ &= \alpha^x \frac{1}{2} [(1 \pm \epsilon(x))\chi_+(x) + (1 \mp \epsilon(x))\chi_-(x)]. \end{aligned} \quad (10)$$

Thus,

$$\begin{aligned} \Psi_+ &= \alpha^x (\chi_{+e} + \chi_{-o}), \\ \Psi_- &= \alpha^x (\chi_{+o} + \chi_{-e}). \end{aligned} \quad (11)$$

It is now possible to truncate the system by setting $\chi_{-o} = \chi_{-e} = 0$. On substitution into Eq. (9) one obtains

$$\begin{aligned} H &= \sum_{x,i} \chi_{+e}^\dagger i\eta_i \Delta_i \chi_{+o} + \chi_{+o}^\dagger i\eta_i \Delta_i \chi_{+e} \\ &\quad + m(\chi_{+e}^\dagger \chi_{+e} - \chi_{+o}^\dagger \chi_{+o}). \end{aligned} \quad (12)$$

Since the action is now diagonal in spinor indices one can further truncate $\chi_+(x) \equiv \chi(x)$ to a single component field and in this way obtain the final staggered fermion Hamiltonian:

$$H = \sum_{x,y,i} \chi^\dagger(x) i\eta_i(x) \Delta_i(x, y) \chi(y) + m \sum_x \epsilon(x) \chi^\dagger(x) \chi(x) \quad (13)$$

This final step thins the degrees of freedom by another factor of 2^3 and the resulting staggered Hamiltonian describes two Dirac fermions in three (spatial) dimensions [7,22] and one

³For simplicity we will continue to use the notation $\chi^\dagger$ going forward even though χ is a single component field. To show that this Hamiltonian is Hermitian one must employ the result $(ab)^\dagger = b^\dagger a^\dagger$ for fermion operators.

Dirac fermion in one dimension in the continuum limit [8,9]. The canonical anticommutators of the staggered fields are given by

$$\{\chi^\dagger(x, t), \chi(x', t)\} = \delta_{x,x'} \quad (14)$$

with all other anticommutators vanishing. The equation of motion is

$$i \frac{\partial \chi(x)}{\partial t} = [H, \chi(x)] = i\eta_i(x)\Delta_i(x, y)\chi(y) + m\epsilon(x)\chi(x). \quad (15)$$

It is not hard to verify that

$$\eta_i(x)\eta_j(x+i) + \eta_j(x)\eta_i(x+j) = 2\delta_{ij}. \quad (16)$$

This result, together with the fact that the site parity operator $\epsilon(x)$ anticommutes with the symmetric difference operator Δ_i , ensures that the field χ satisfies a discrete Klein-Gordon equation

$$\frac{\partial^2 \chi(x)}{\partial t^2} = \frac{1}{2} \sum_i [\chi(x+2i) + \chi(x-2i) - 2\chi(x)] + m^2 \chi(x). \quad (17)$$

Notice the appearance of a discrete Laplacian operator on a block lattice with twice the lattice spacing. This equation implies that there are 2^d degenerate solutions for every site on the block lattice. These solutions can be constructed by performing a translation accompanied by a phase shift $\xi_i(x)$ of the block lattice solution for every site in the unit cell of the original lattice. These translations within the unit cell, or shifts as they are commonly called, play a crucial role in our analysis and will be discussed later.

Let us try to understand how this works in more detail. For simplicity let us restrict the following discussion to odd d . The naive fermion on a d -dimensional spatial lattice gives rise to 2^d Dirac fermions in the continuum limit because of doubling. After staggering one expects the continuum theory to correspond to $2^{\frac{d-1}{2}}$ Dirac fermions with a total of 2^d complex components. These can be identified with the staggered fields $\chi(x)$ living at the corners of the unit hypercube in the original lattice or equivalently the unit cell in the block lattice. To expose the spin-flavor structure of the continuum fermion we can build a matrix fermion Λ according to the rule

$$\Lambda(x) = \frac{1}{8} \sum_{\{b\}} \chi(x+b) \alpha^{x+b}, \quad (18)$$

where $\{b\}$ is a set of 2^d vectors with components $b_i = \{0, 1\}$ corresponding to points in a unit cell. Clearly, as written, Λ carries more degrees of freedom

than the original staggered field. But the low momentum components can be taken as independent and are enough to expose the spin-flavor structure—the continuum spinors can then be read off from the columns of this matrix as the lattice spacing is sent to zero [23]. As we will show, it also allows us to see the connection between shift symmetries and the continuum axial-flavor symmetry. A matrix mapping which preserves the number of degrees of freedom is called the spin-taste basis and is given in terms of a distinct matrix field ψ living on the block lattice only. This is reviewed in Appendix B.

Finally, we note that the Hamiltonian is also clearly invariant under a $U(1)$ phase symmetry in which $\chi(x) \rightarrow e^{i\theta} \chi(x)$ which will play an important role in our later discussion. We now turn to a discussion of important additional symmetries of the staggered Hamiltonian.

III. CONTINUUM FLAVOR AND LATTICE SHIFT SYMMETRIES

In this section we will focus on staggered fermions in $(3+1)$ dimensions but the arguments can easily be generalized to any dimension. We start from a chiral basis for the Dirac gamma matrices given by

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \quad (19)$$

where $\sigma_\mu = (I, \sigma_i)$ and $\bar{\sigma}_\mu = (I, -\sigma_i)$. In this case

$$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}. \quad (20)$$

For later reference we also list $i\alpha_j\alpha_k$ and $i\alpha_1\alpha_2\alpha_3$ in this basis:

$$i\alpha_j\alpha_k = \begin{pmatrix} \epsilon_{jki}\sigma_i & 0 \\ 0 & \epsilon_{jki}\sigma_i \end{pmatrix}, \quad (21)$$

$$i\alpha_1\alpha_2\alpha_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (22)$$

The matrix Ψ in Eq. (18) then takes the form

$$\Lambda = \begin{pmatrix} \lambda_R & 0 \\ 0 & \lambda_L \end{pmatrix}, \quad (23)$$

where

$$\begin{aligned} \lambda_R &= \chi(x)I - \chi(x+i)\sigma_i - \frac{i}{2}\chi(x+i+j)\epsilon_{ijk}\sigma_k \\ &\quad + iI\chi(x+\hat{1}+\hat{2}+\hat{3}), \\ \lambda_L &= \chi(x)I + \chi(x+i)\sigma_i - \frac{i}{2}\chi(x+i+j)\epsilon_{ijk}\sigma_k \\ &\quad - iI\chi(x+\hat{1}+\hat{2}+\hat{3}). \end{aligned} \quad (24)$$

In the naive continuum limit the massless staggered field thus gives rise to doublets of left- and right-handed Weyl fields transforming under two independent $SU(2)$ symmetries. These symmetries protect the continuum theory from developing bilinear mass terms.⁴ In addition, the theory is invariant under both the vector $U(1)$ symmetry discussed earlier and a singlet axial symmetry $U(1)_A$ in which the λ_R and λ_L carry opposite charges. Of course, the crucial question is whether sufficient lattice symmetries exist that guarantee that these continuum symmetries emerge as the lattice spacing is sent to zero. Part of the answer lies in the existence of exact translation-by-one or *shift* symmetries of the lattice Hamiltonian. We now turn to these symmetries and their relation to continuum symmetries.

The continuum symmetries act by right multiplication of a continuum matrix fermion given by Eq. (18) as $a \rightarrow 0$ by an axial-flavor transformation matrix F . This takes the form $F = e^{i\theta_A \alpha_A}$ where the Hermitian basis α_A is given in terms of products of the individual α_i matrices:

$$\alpha_A = \{\alpha_i, i\alpha_i\alpha_j, i\alpha_1\alpha_2\alpha_3\} \quad \text{where } i = 1 \dots 3. \quad (25)$$

It can be seen that this yields the continuum flavor group $SU(2) \times SU(2) \times U_A(1)$ as described above.

Staggered fermions, being discretizations of Kähler-Dirac fermions [11], are invariant under a twisted rotation group corresponding to the diagonal subgroup of the flavor and rotation symmetries [3,6,12,24,25]. Upon discretization, the rotational symmetries are restricted to the cubic group, and hence the flavor rotation angles are similarly restricted to odd multiples of $\frac{\pi}{2}$. Then the remaining elementary discrete flavor rotation acting on the continuum matrix field becomes (see [23])

$$\Lambda(x) \rightarrow \Lambda(x) e^{i\frac{\pi}{2}\alpha_j} = \Lambda(x) i\alpha_j. \quad (26)$$

The lattice equivalent is

$$\Lambda(x) \rightarrow \Lambda(x+j) i\alpha_j. \quad (27)$$

We will now show that this induces a unit translation or *shift* on the staggered fields. Acting on the lattice fermion matrix given in Eq. (18), the transformed matrix is given by

$$\begin{aligned} \Lambda'(x) &= \sum_{\{b\}} \chi(x+b+j) \alpha^{x+b+j} i\alpha_j \\ &= i \sum_{\{b\}} \xi_j(x+b) \chi(x+b+j) \alpha^{x+b}, \end{aligned} \quad (28)$$

where one must anticommute the α_j matrix from the right which produces the phase factor $\xi_j(x+b)$ where

⁴A Majorana mass for either a left- or right-handed doublet vanishes identically.

$\xi_j(x) = (-1)^{\sum_{k=j+1}^d x_k}$. The net effect is clearly to just produce an elementary shift

$$\begin{aligned} \chi(x) &\xrightarrow{S_j} i\xi_j(x) \chi(x+j), \\ \chi^\dagger(x) &\xrightarrow{S_j} -i\xi_j(x) \chi^\dagger(x+j). \end{aligned} \quad (29)$$

Thus, there is a direct connection between the continuum transformation $e^{i\frac{\pi}{2}\alpha_j}$ and the elementary shift S_j .

It is straightforward to examine the invariance of the Hamiltonian under such a shift S_k ,

$$\begin{aligned} H &\rightarrow \sum_{x,i} i\eta_i(x) \chi^\dagger(x+k) \xi_k(x) \xi_k(x+i) \frac{1}{2} [\chi(x+i+k) \\ &\quad - \chi(x-i+k)] + \sum_x m\epsilon(x) \xi_k(x)^2 \chi^\dagger(x+k) \chi(x+k) \\ &= \sum_{x,i} i\eta_i(x-k) \chi^\dagger(x) \xi_k(x-k) \xi_k(x+i-k) \frac{1}{2} [\chi(x+i) \\ &\quad - \chi(x-i)] + \sum_x m\epsilon(x-k) \chi^\dagger(x) \chi(x) \\ &= \sum_x i\eta(x) \chi^\dagger(x) \frac{1}{2} [\chi(x+i) - \chi(x-i)] \\ &\quad - \sum_x m\epsilon(x) \chi^\dagger(x) \chi(x), \end{aligned} \quad (30)$$

where we have used the result $\eta_i(k) \xi_k(i) = 1$. Notice that the Hamiltonian is only invariant under S_k if $m = 0$. It should be clear that elementary shifts can be applied consecutively to yield additional symmetries. For example, the double shift S_{ij} :

$$\chi(x) \xrightarrow{S_{ij}} S_i S_j \chi(x) = -\xi_j(x) \xi_i(x+j) \chi(x+i+j) \quad i \neq j. \quad (31)$$

It is trivial to see that $S_i S_j = -S_j S_i$. Thus, just as S_k is associated with the continuum symmetry generator α_k , the properties of the double shift S_{ij} mimic those of the generator $i\alpha_i \alpha_j$. It is important to notice that in the case of the double shift the Hamiltonian is invariant even for a nonzero mass. This suggests that even lattice shifts are associated with vector transformations in the continuum theory. If one goes to a chiral basis in the continuum, it is easy to verify that the generators $\alpha_i \alpha_j$ indeed act as vector symmetries.

Conversely the Hamiltonian is only invariant under an odd number of shifts if the mass is zero, suggesting odd shifts are associated with continuum axial transformations. This can also be explicitly verified by going to a chiral basis. An explicit example of this is the triple shift $S_1 S_2 S_3$ which corresponds to the generator of the singlet axial $U_A(1)$ symmetry $\gamma_5 = i\alpha_1 \alpha_2 \alpha_3$.

It is important to recognize that the connection between lattice shifts and continuum flavor is only one to one up to

ordinary translations. For example, a double shift along the same direction yields a simple translation T on the block lattice:

$$\begin{aligned}\chi(x) &\xrightarrow{S_i^2} -\xi_i(x)\xi_i(x+i)\chi(x+2i) \\ &= -\chi(x+2i) = -T_i[\chi(x)].\end{aligned}\quad (32)$$

Similarly, performing a S_{ij} shift followed by a S_j shift yields

$$\begin{aligned}S_{ij}S_j\chi(x) &\rightarrow S_{ij}i\xi_j(x)\chi(x+j) \\ &= -i\xi_j(x+j)\xi_i(x+2j)\xi_j(x)\chi(x+2j+i) \\ &= -i\xi_i(x)T_j\chi(x+i) \\ &= -T_jS_i\chi(x)\end{aligned}\quad (33)$$

or more generally $[S_{ij}, S_j] = -2T_jS_i$. That is, the combination of two shifts generates another shift up to a block translation. In a similar fashion the double shifts satisfy the relation

$$[S_{ij}, S_{jk}] = 2S_{ik}T_j. \quad (34)$$

The shift symmetries hold even in the presence of gauge interactions provided the gauge link field $U_i(x)$ transforms similarly under shifts

$$U_i(x) \xrightarrow{S_j} U_i(x+j). \quad (35)$$

It should now be clear that the staggered fermion shift symmetries form a discrete subgroup of the continuum symmetries corresponding to discrete axial-flavor transformations.⁵ This group comprises the elements given in Eq. (25) together with their negatives and the identity. When the lattice mass is nonzero, only the even shifts are good lattice symmetries. These form a discrete subgroup of the $SU(2)_V$ symmetry of the continuum theory. In Sec. V we show that the exact symmetries of the lattice theory are sufficient to protect the theory against developing relevant mass terms, and these continuum symmetries are hence restored automatically as the lattice spacing is sent to zero. These conclusions parallel similar arguments that can be made in the Euclidean theory—see [1–5] for a discussion of these issues in the context of the Euclidean theory.

IV. TIME REVERSAL ON AND OFF THE LATTICE

In $(3+1)$ dimensions the discrete symmetries of charge conjugation C and time reversal T (where $t \rightarrow -t$) act on a continuum spinor field Ψ in the following manner:

$$\Psi \xrightarrow{C} \gamma_2 \Psi^* = \beta \alpha_2 \Psi^* \quad (36)$$

and

⁵The word “flavor” is often called “taste” in the lattice gauge theory literature.

$$\begin{aligned}\Psi &\xrightarrow{T} \gamma_1 \gamma_3 \Psi = -\alpha_1 \alpha_3 \Psi, \\ i &\xrightarrow{T} -i.\end{aligned}\quad (37)$$

In particular, the combination $\mathcal{T} = CT$ acts as follows:

$$\begin{aligned}\Psi &\xrightarrow{\mathcal{T}} \Gamma \Psi^*, \\ i &\xrightarrow{\mathcal{T}} -i,\end{aligned}\quad (38)$$

where $\Gamma = \beta \alpha_2 \alpha_1 \alpha_3$. A standard calculation shows that if $\Psi(x, t)$ is a solution of the equations of motion, then so is $\Gamma \Psi^*(-t, x)$. Performing the CT transformation on the Hamiltonian after the unitary transformation given in Eq. (5) shows that the symmetry operation $\mathcal{T}$ acts on staggered fields as follows:

$$\begin{aligned}\chi(x) &\xrightarrow{\mathcal{T}} \epsilon(x) \chi^*(x) = \epsilon(x) (\chi^\dagger(x))^T, \\ i &\xrightarrow{\mathcal{T}} -i.\end{aligned}\quad (39)$$

Under $\mathcal{T}$ the kinetic term $K \rightarrow K'$,

$$\begin{aligned}K' &= \sum_x \chi(x) \epsilon(x) [-i\eta_i(x) \Delta_i(x, y)] \epsilon(y) \chi^*(y) \\ &= \sum_x \chi(x) i\eta_i(x) \Delta_i(x, y) \chi^*(y) \\ &= \sum_x -\chi^*(y) i\eta_i(x) \Delta_i(x, y) \chi(x) \\ &= K,\end{aligned}\quad (40)$$

where the last line follows from the fact that Δ_i is an antisymmetric matrix and χ, χ^* anticommute. Thus, the kinetic operator is time reversal invariant. In contrast, it is easy to see that the mass term $\sum_x \epsilon(x) \chi^*(x) \chi(x)$ is not invariant under $\mathcal{T}$. For brevity we will refer to this product of charge conjugation and time reversal symmetry as simply time reversal $\mathcal{T}$ in the rest of the paper.⁶ At this point it is important to notice that the elementary shift symmetry does *not* commute with time reversal:

$$\begin{aligned}\chi(x) &\xrightarrow{\mathcal{T}} \epsilon(x) \chi^*(x), \\ \chi(x) &\xrightarrow{S_k \mathcal{T}} -i\epsilon(x) \xi_k(x) \chi^*(x+k), \\ \chi(x) &\xrightarrow{S_k} i\xi_k(x) \chi(x+k), \\ \chi(x) &\xrightarrow{\mathcal{T} S_k} -i\xi_k(x) \epsilon(x+k) \chi^*(x+k) = i\epsilon(x) \xi_k(x) \chi^*(x+k).\end{aligned}\quad (41)$$

Clearly, the two symmetries anticommute.⁷ Indeed, this statement is true for any shift composed of an odd number of elementary shifts.

⁶Clearly $\mathcal{T}^2 = 1$.

⁷Notice that we can multiply the elementary shift symmetry by an arbitrary phase α . But this does not change the anticommutation property with $\mathcal{T}$.

V. RENORMALIZATION

Let us summarize our conclusions so far. The staggered Hamiltonian with zero mass is invariant under a $U(1)$ phase symmetry, a time reversal, and a set of shift symmetries that form a discrete subgroup of the continuum axial-flavor group. The lattice theory is also invariant under discrete rotations and axis inversion as described in Appendix B and a charge conjugation symmetry C :

$$\chi(x) \rightarrow (\chi^\dagger(x))^T, \quad \chi^\dagger(x) \rightarrow \chi^T(x). \quad (42)$$

To understand whether these symmetries enhance to yield a Lorentz and time reversal invariant theory equipped with the full $SU(2) \times SU(2) \times U(1)$ continuum axial-flavor symmetry, we need to write down all relevant and marginal lattice operators that are invariant under the lattice symmetries and determine whether any of these correspond to relevant (or marginal) operators that break the continuum symmetries. The only candidate terms one can construct correspond to fermion bilinears coupling sites within the unit cell of the lattice. The only operators that are both shift and $U(1)$ invariant correspond to fermion fields connected by a string of $\eta_i(x)$ link phases along a path between x and $x + \vec{n}$:

$$\chi^\dagger(x) \left[\prod_{n_i \neq 0} \eta_{n_i} \left(x + \sum_{j < i} n_j \right) \right] \chi(x + \vec{n}) + \text{H.c.} \quad (43)$$

We have already shown the shift invariance of the kinetic term which takes this form. As another example consider the following operator:

$$\sum_x \chi^\dagger(x) i\eta_i(x) \eta_j(x+i) \chi(x+i+j), \quad i \neq j. \quad (44)$$

Under a shift S_k it becomes

$$\begin{aligned} & \sum_x \chi^\dagger(x+k) \xi_k(x) i\eta_i(x) \eta_j(x+i) \xi_k(x+i+j) \\ & \times \chi(x+i+j+k) \\ & = \sum_x \chi^\dagger(x) i\eta_i(x) \eta_i(k) \eta_j(x+i) \eta_j(k) \xi_k(i) \xi_k(j) \chi(x+i+j) \\ & = \sum_x \chi^\dagger(x) i\eta_i(x) \eta_j(x+i) \chi(x+i+j), \end{aligned} \quad (45)$$

where we have used the result $\eta_i(k) \xi_k(i) = 1$ twice. It should be clear that a similar result will be obtained for any string of η -link phases terminated by fermion fields. Notice that any insertion of $\xi_k(x)$ or $\epsilon(x)$ into this expression will break the shift symmetry. For example, the term

$$\sum_x \chi^\dagger(x) \xi_i(x) \chi(x+i) \quad (46)$$

transforms to

$$\begin{aligned} & \sum_x \chi^\dagger(x+k) \xi_k(x) \xi_i(x) \xi_k(x+i) \chi(x+i+k) \\ & = \sum_x \chi^\dagger(x) \xi_i(x) \chi(x+i) [\xi_k(i) \xi_i(k)]. \end{aligned} \quad (47)$$

The factor in square brackets is $2\delta_{ik} - 1$.

The goal of this section is to write down all such terms and then decide whether they are invariant under both $\mathcal{T}$ and C . We will do this systematically according to the number of nonzero elements or links in the vector $\vec{n}$.

A. Zero link operators

There are just two Hermitian operators of this kind:

$$(1) \epsilon(x) \chi^\dagger(x) \chi(x);$$

$$(2) \chi^\dagger(x) \chi(x).$$

Both lead to relevant mass terms in the continuum limit. We have already shown that the former is not invariant under single shifts or time reversal. It is also not invariant under C . While the second operator is invariant under shifts, it is easy to see that it is not invariant under $\mathcal{T}$ or C since $\{\chi(x), \chi^\dagger(y)\} = \delta_{xy}$.

B. One link operators

The following Hermitian lattice operators are possible:

$$(1) \chi^\dagger(x) i\eta_i(x) [\chi(x+i) - \chi(x-i)];$$

$$(2) \chi^\dagger(x) \eta_i(x) [\chi(x+i) + \chi(x-i)].$$

These expressions should be summed over the index i to enforce rotational invariance, but we will suppress this aspect for simplicity. The first one is just the original kinetic term, and we have shown that it is invariant under $\mathcal{T}$ and C . The second is a mass term. Under time reversal it becomes

$$\begin{aligned} & - \sum_x \epsilon(x) \chi^T(x) \eta_i(x) \epsilon(x) [(\chi^\dagger(x+i))^T + (\chi^\dagger(x-i))^T] \\ & = \sum_x [(\chi^\dagger(x+i))^T + (\chi^\dagger(x-i))^T] \eta_i(x) \chi(x) \\ & = \sum_x \chi^\dagger(x) \eta_i(x) [\chi(x+i) + \chi(x-i)]. \end{aligned} \quad (48)$$

It is thus $\mathcal{T}$ invariant. Under C it becomes

$$\begin{aligned} & \sum_x \chi^T(x) \eta_i(x) (\chi^\dagger(x+i) + \chi^\dagger(x-i))^T \\ & = \sum_x -\chi^\dagger(x) \eta_i(x) [\chi(x+i) + \chi(x-i)]. \end{aligned} \quad (49)$$

It is hence not C invariant.

C. Two link operators

We consider

$$\chi^\dagger(x) i\eta_i(x) \eta_j(x+i) [\chi(x+i+j) + \chi(x-i-j)]. \quad (50)$$

In principle, to enforce rotational invariance this expression

should be summed over all sets of neighbor points $x \pm i \pm j$, but again we will ignore this requirement for the purpose of testing $\mathcal{T}$ and $\mathcal{C}$ invariance other than requiring that the term be Hermitian. Notice that if we left off the factor of i we could seemingly construct a term involving $[\chi(x+i+j) - \chi(x-i-j)]$. However, this clearly gives rise to a second derivative term in the continuum, and so we neglect such irrelevant operators in our analysis. The first half of this term transforms under $\mathcal{T}$ as

$$\begin{aligned} & \sum_x -i\epsilon(x)\chi^T(x)\eta_i(x)\eta_j(x+i)\epsilon(x+i+j)(\chi^\dagger(x+i+j))^T \\ &= \sum_x i\chi^\dagger(x+i+j)\eta_i(x)\eta_j(x+i)\chi(x) \\ &= \sum_x i\chi^\dagger(x)\eta_i(x)\eta_j(x+i)\chi(x-i-j)[\eta_i(j)\eta_j(i)]. \end{aligned} \quad (51)$$

But $[\eta_i(j)\eta_j(i)] = -1$ for $i \neq j$, and so this term violates $\mathcal{T}$. It can be shown to be invariant under $\mathcal{C}$.

D. Three link operators

We consider the following masslike operator:

$$\chi^\dagger(x)i\eta_i(x)\eta_j(x+i)\eta_k(x+i+j)\chi(x+i+j+k) + \text{H.c.} \quad (52)$$

Following the same strategy as above it can be shown to be $\mathcal{T}$ invariant but not $\mathcal{C}$ invariant.

Let us summarize our conclusions. The original massless Hamiltonian is invariant under $U(1)$, shift, $\mathcal{T}$, and $\mathcal{C}$ symmetries. The only marginal or relevant fermion bilinear constructed from fields within the unit cell that is invariant under all these symmetries is the original kinetic operator. In our arguments we have left out any gauge field. However, it is straightforward to extend the analysis to gauged fermion bilinears. The transformation of any gauge field under the shift and $U(1)$ symmetries has already been discussed. Under both $\mathcal{C}$ and the $\mathcal{T}$ symmetry the gauge field is taken to transform as

$$U_i(x) \rightarrow U_i^*(x). \quad (53)$$

One can then repeat the previous analysis with the same conclusion: that the only operator in the massless theory that remains invariant under all the lattice symmetries in the presence of gauge fields remains the kinetic term. Thus, the theory does not suffer from additive mass renormalization, and the continuum symmetries should be restored automatically as the lattice spacing is sent to zero. There is one caveat—clearly the coupling to the kinetic operator can be renormalized due to interactions. This would correspond to a renormalization of the speed of light.

VI. REAL FIELDS AND SYMMETRY OPERATORS

To derive explicit operators that implement shifts and time reversal on a finite lattice it is useful to first re-express the staggered fermion χ in terms of real fields λ^1 and λ^2 :

$$\begin{aligned} \chi(x) &= \frac{1}{2}(\lambda^1(x) + i\lambda^2(x)), \\ \chi^\dagger(x) &= \frac{1}{2}(\lambda^1(x) - i\lambda^2(x)). \end{aligned} \quad (54)$$

The massless Hamiltonian is then

$$H = \frac{1}{4} \sum_{x,j} \sum_{a=1}^2 \lambda^a(x) i\eta_j(x) \Delta_j \lambda^a(x), \quad (55)$$

and the equal time anticommutators become

$$\{\lambda^a(x), \lambda^b(x')\} = 2\delta^{ab}\delta(x, x'). \quad (56)$$

The $U(1)$ symmetry discussed earlier yields an $SO(2)$ symmetry acting on the real fields

$$\lambda^a(x) = e^{\theta R^{ab}} \lambda^b(x), \quad (57)$$

where $R^{ab} = \epsilon^{ab}$. Similarly, time reversal $\mathcal{T}$ acts as

$$\begin{aligned} \lambda^1(x) &\xrightarrow{\mathcal{T}} \epsilon(x)\lambda^1(x), \\ \lambda^2(x) &\xrightarrow{\mathcal{T}} \epsilon(x)\lambda^2(x). \end{aligned} \quad (58)$$

We can write the elementary shift symmetry described in the previous section as $S_k = R\hat{S}_k$ where $\hat{S}_k$ is given by

$$\lambda^a(x) \xrightarrow{\hat{S}_k} \xi_k(x)\lambda^a(x+k). \quad (59)$$

In fact, it should be clear that the massless Hamiltonian is actually invariant under two separate half shifts given by

$$\begin{aligned} \lambda^1(x) &\xrightarrow{A_k} \xi_k(x)\lambda^1(x+k), \\ \lambda^2(x) &\xrightarrow{A_k} \lambda^2(x), \end{aligned} \quad (60)$$

and

$$\begin{aligned} \lambda^2(x) &\xrightarrow{B_k} \xi_k(x)\lambda^2(x+k), \\ \lambda^1(x) &\xrightarrow{B_k} \lambda^1(x), \end{aligned} \quad (61)$$

with

$$\hat{S}_k = A_k B_k. \quad (62)$$

The B half shift is precisely the same symmetry considered in [9] for a $(1+1)$ -dimensional staggered fermion model.

Let us now construct operators that implement A_k , B_k , and R on a finite lattice equipped with periodic boundary conditions. As a warm-up let us start with a one-dimensional lattice with L sites and coordinate $x \equiv x_1 = 0 \cdots L-1$. For staggered fermions L must be even, and for $d=1$ the phase $\xi_1(x) = 1$. The A_1 shift can then be implemented by the action of the shift operator

$$\lambda^{1,2} \rightarrow A_1^{-1} \lambda^{1,2} A_1, \quad (63)$$

where

$$A_1 = 2^{-L/2} \prod_{x=0}^{L-1} (1 - \lambda^1(x) \lambda^1(x+1)) \quad (64)$$

and

$$A_1^{-1} = 2^{-L/2} \prod_{x=0}^{L-1} (1 + \lambda^1(x) \lambda^1(x+1)). \quad (65)$$

To see this one uses the results

$$\begin{aligned} -\lambda^1(x+1) &= \frac{1}{2} [1 + \lambda^1(x) \lambda^1(x+1)] \lambda^1(x) [1 \\ &\quad - \lambda^1(x) \lambda^1(x+1)], \\ \lambda^2(x) &= \frac{1}{2} [1 + \lambda^1(x) \lambda^1(x+1)] \lambda^2(x) [1 \\ &\quad - \lambda^1(x) \lambda^1(x+1)], \\ 1 &= \frac{1}{2} [1 + \lambda^1(x) \lambda^1(x+1)] [1 - \lambda^1(x) \lambda^1(x+1)]. \end{aligned} \quad (66)$$

A similar result follows for B_1 which is given by

$$B_1 = 2^{-L/2} \prod_{x=0}^{L-1} (1 - \lambda^2(x) \lambda^2(x+1)). \quad (67)$$

Combining the A and B shifts, one obtains

$$\hat{S}_1 = 2^{-L} \prod_{a=1}^2 \prod_{x=0}^{L-1} (1 - \lambda^a(x) \lambda^a(x+1)). \quad (68)$$

The time reversal operator can also be implemented in a similar way. When L is a multiple of 4, it can be achieved using the operator $\mathcal{T}$:

$$\mathcal{T} = \mathcal{K} \left(\prod_{x \text{ even}} \lambda^1(x) \right) \left(\prod_{x \text{ odd}} \lambda^2(x) \right), \quad (69)$$

where $\mathcal{K}$ represents complex conjugation. When L is an odd multiple of 2, this gives the wrong site parity, so we use GT instead, where G is the fermion parity operator

$$G \lambda^a(x) G^{-1} = -\lambda^a(x), \quad (70)$$

with $G = \prod_x \prod_{a=1}^2 \lambda^a(x)$.

For a two-dimensional lattice with coordinates (x_1, x_2) with $x_i = 0 \cdots L-1$ the story is similar. An A shift along x_1 is given by the action of a shift operator A_1 ,

$$\begin{aligned} A_1 &= 2^{-L^2/2} \prod_{x_2=0}^{L-1} \prod_{x_1=0}^{L-1} \\ &\quad \times (1 - \xi_1(x) \lambda^1(x_1, x_2) \lambda^1(x_1+1, x_2)), \end{aligned} \quad (71)$$

with $\xi_1(x) = (-1)^{x_2}$. Similarly, a shift along x_2 is generated by the operator

$$\begin{aligned} A_2 &= 2^{-L^2/2} \prod_{x_1=0}^{L-1} \prod_{x_2=0}^{L-1} \\ &\quad \times (1 - \xi_2(x) \lambda^1(x_1, x_2) \lambda^1(x_1, x_2+1)), \end{aligned} \quad (72)$$

with $\xi_2(x) = 1$. The B shifts work in the same way with $\lambda^1(x) \rightarrow \lambda^2(x)$. This allows us to write $\hat{S}_k$ in the form

$$\begin{aligned} \hat{S}_1 &= 2^{-L^2} \prod_{a=1}^2 \prod_{x_2=0}^{L-1} \prod_{x_1=0}^{L-1} (1 - \xi_1(x) \lambda^a(x_1, x_2) \lambda^a(x_1+1, x_2)), \\ \hat{S}_2 &= 2^{-L^2} \prod_{a=1}^2 \prod_{x_1=0}^{L-1} \prod_{x_2=0}^{L-1} (1 - \xi_2(x) \lambda^a(x_1, x_2) \lambda^a(x_1, x_2+1)). \end{aligned} \quad (73)$$

When L is a multiple of 4, time reversal can be achieved using

$$\mathcal{T} = \mathcal{K} \left(\prod_{x \text{ even}} \lambda^1(x_1, x_2) \right) \left(\prod_{x \text{ odd}} \lambda^2(x_1, x_2) \right), \quad (74)$$

while when L is an odd multiple of 2, we again use GT instead. In three dimensions the $\hat{S}$ shifts are

$$\begin{aligned} \hat{S}_1 &= 2^{-L^3} \prod_{a=1}^2 \prod_{x_3=0}^{L-1} \prod_{x_2=0}^{L-1} \prod_{x_1=0}^{L-1} \\ &\quad \times (1 - \xi_1(x) \lambda^a(x_1, x_2, x_3) \lambda^a(x_1+1, x_2, x_3)), \\ \hat{S}_2 &= 2^{-L^3} \prod_{a=1}^2 \prod_{x_1=0}^{L-1} \prod_{x_3=0}^{L-1} \prod_{x_2=0}^{L-1} \\ &\quad \times (1 - \xi_2(x) \lambda^a(x_1, x_2, x_3) \lambda^a(x_1, x_2+1, x_3)), \\ \hat{S}_3 &= 2^{-L^3} \prod_{a=1}^2 \prod_{x_2=0}^{L-1} \prod_{x_1=0}^{L-1} \prod_{x_3=0}^{L-1} \\ &\quad \times (1 - \xi_3(x) \lambda^a(x_1, x_2, x_3) \lambda^a(x_1, x_2, x_3+1)). \end{aligned} \quad (75)$$

Time reversal when $L = 0 \bmod 4$ is given by

$$\mathcal{T} = \mathcal{K} \left(\prod_{\sum_i x_i = \text{even}} \lambda^1(x_1, x_2, x_3) \right) \left(\prod_{\sum_i x_i = \text{odd}} \lambda^2(x_1, x_2, x_3) \right), \quad (76)$$

with the same modification as before for $L = 0 \bmod 2$. We can also write down an operator in terms of the fermion fields that implements the R operation. It is given by

$$\hat{R} = 2^{-L^d} \prod_{x=0}^{L^d} (1 - \lambda^1(x) \lambda^2(x)). \quad (77)$$

To write down operators that correspond to multiple shifts one simply compounds a series of single shift operators as discussed earlier. As observed earlier, one can verify that the odd shift operators anticommute with $\mathcal{T}$:

$$\{S_k, \mathcal{T}\} = 0. \quad (78)$$

Furthermore, the definition of S_k involves an element R of the $U(1)$ symmetry. Notice that R satisfies $R^4 = 1$ and hence R belongs to a Z_4 subgroup. The shifts are clearly symmetries for any element of this Z_4 . The fact that S_k and $\mathcal{T}$ do not commute implies that any attempt to gauge this Z_4 subgroup will break $\mathcal{T}$ —a mixed lattice 't Hooft anomaly. In particular, the singlet $U_A(1)$ symmetry corresponding to the 3-shift S_{123} will be broken if this subgroup of the vector symmetry is gauged. This is analogous to the result obtained in [9]. In the next section we will explore how these 't Hooft anomalies can be canceled.

VII. ANOMALY CANCELLATION

The question we would like to address is whether the mixed 't Hooft anomaly we found can be canceled. One way to approach this question is to ask whether we can design interactions that can gap the fermions without breaking the Z_4 symmetry. One way to do the latter is to add Z_4 invariant four fermion interactions to the theory. The simplest term we can write down takes the form⁸

$$G \sum_x \chi^1(x) \chi^2(x) \chi^3(x) \chi^4(x) + \text{H.c.}, \quad (79)$$

which requires four complex staggered fermions and exhibits an explicit $SU(4) = SO(6)$ global symmetry as well as time reversal and shift invariance. For $G \rightarrow 0$ one expects the ground state to be eightfold degenerate since it

⁸This is the operator that has been the best studied both analytically and numerically in the literature but clearly there are many others corresponding to displacing some of the fields within the unit cell of the lattice.

corresponds to eight noninteracting real staggered fermions. However, for $G \rightarrow \infty$ the ground state is given by diagonalizing the single site Hamiltonian. It was shown in [26,27] that the ground state of this system is in fact a singlet. Indeed, in the latter paper it was shown how to construct a variety of four fermion terms with differing symmetry groups [the maximal symmetry being $SO(7)$] that result in a nondegenerate ground state—generalizing the original result of Fidkowski and Kitaev [28]. The singlet nature of the ground state implies that the system is incapable of undergoing spontaneous symmetry breaking. This fact in turn implies that the system is free of 't Hooft anomalies. This phenomenon of producing a gapped, invariant ground state has been termed symmetric mass generation and has already been observed in staggered fermion models with four fermion interactions [16,18,19,28–32]. Clearly the model can be gapped in this way for eight real staggered fermions each of which yields two Majorana fermions in the naive continuum limit. Thus, this analysis confirms the Z_{16} anomaly cancellation condition for Majorana fermions in $(3+1)$ dimensions [27,33–35].

One might be tempted to ask whether this anomaly cancellation condition can be obtained directly from the algebra of operators. Consider N flavors of massless staggered fermion. The operators needed to implement shifts and time reversal take the form of products of mutually commuting terms for each flavor, e.g.,

$$S_k = \prod_{a=1}^N S_k^a, \quad \mathcal{T} = \prod_{a=1}^N \mathcal{T}^a, \quad (80)$$

where S_k^a and $\mathcal{T}^a$ denote the operators for a single flavor derived in the previous section. Even though $\{S_k^a, \mathcal{T}^a\} = 0$, it is easy to verify that $[S_k, \mathcal{T}] = 0$ for $N = 2k$. Thus, it appears in $d = 3$ one would need four Dirac or eight Majorana fermions to cancel this mixed anomaly. In contrast, the gapping argument tells us symmetric mass generation should only be possible for $N = 4k$, i.e., 16 Majorana fermions in $d = 3$. This discrepancy between the number of fermions needed to gap the system and the number needed to cancel off a naive 't Hooft anomaly has been noted previously in the literature—see, e.g., [36]. The Z_{16} anomaly cancellation condition for the discrete spin- Z_4 symmetry of continuum Weyl fields can not be seen by considering chiral fermions on the torus where only a Z_8 condition is found. One must instead consider fermions propagating on manifolds with different topology to see the Z_{16} classification.

VIII. CONSERVED CHARGES AND CONTINUOUS SYMMETRIES

We have seen that the theory admits a phase (vector) symmetry that manifests as an $SO(2)$ rotation on the

doublet of real fermions $\lambda(x)$ at each site. The explicit expression for the generator of this second quantized symmetry operator was given in Eq. (77). In fact, an arbitrary $SO(2)$ rotation can be generated using the operator $U_V = e^{-i\theta Q_V}$ whose action on the doublet field λ is given by

$$U_V \lambda(x) U_V^\dagger, \quad (81)$$

where the vector charge is given by

$$Q_V = \frac{i}{2} \sum_x \lambda^1(x) \lambda^2(x). \quad (82)$$

For infinitesimal θ the transformation becomes

$$\lambda^a(x) \rightarrow \lambda^a - i\theta [Q_V, \lambda^a]. \quad (83)$$

From the fundamental anticommutators of the fields one deduces

$$\begin{aligned} \lambda^1(x) &\rightarrow \lambda^1(x) - \theta \lambda^2(x), \\ \lambda^2(x) &\rightarrow \lambda^2(x) + \theta \lambda^1(x). \end{aligned} \quad (84)$$

It is easily verified that $[H, Q_V] = [H, U_V] = 0$ as expected.

For the rest of this section we will again focus on the interesting case of $d=3$ although it is not hard to generalize the results to arbitrary dimension d . We have already seen that a discrete $U_A(1)$ transformation corresponds to right multiplication of the matrix fermion Λ by γ_5 and maps to the 3-shift S_{123} operator acting on the staggered field. Its axial character can be made obvious by observing that

$$\Lambda \gamma_5 = (\gamma_5 \Lambda \gamma_5) \gamma_5 = \gamma_5 \Lambda, \quad (85)$$

where we have used the fact that Λ commutes with γ_5 or equivalently has eigenvalue unity under the twisted chiral operator $\gamma_5 \otimes \gamma_5$. It is natural to look for a generalization of the vector charge that incorporates this 3-shift, generates a continuous axial symmetry of the lattice system, and becomes the singlet axial charge in the continuum limit. In terms of the doublet $\lambda(x)$ it is given by

$$\begin{aligned} Q_A &= B_{123} Q_V B_{123}^{-1} \\ &= \frac{i}{2} \sum_x \Xi_{123}(x) \lambda^1(x) \lambda^2(x + \hat{1} + \hat{2} + \hat{3}), \end{aligned} \quad (86)$$

where $\Xi_{123}(x) = \xi_1(x) \xi_2(x + \hat{1}) \xi_3(x + \hat{1} + \hat{2})$ is the phase associated with the 3-shift. It is very important to note that here, following [9], we have employed a 3-shift only on the λ^2 field; i.e., the shift is not a full shift S_{123} but just B_{123} . A simultaneous shift on both fields would leave the vector charge invariant. Since B_{123} is a symmetry of the (massless)

Hamiltonian, the resultant charge Q_A also commutes with the Hamiltonian.⁹ Again, we can exponentiate this charge to produce a continuous lattice symmetry corresponding to the operator $U_A = e^{-i\theta Q_A}$. To see this note that

$$\begin{aligned} U_A &= e^{\theta/2 \sum_x \Xi_{123}(x) \lambda^1(x) \lambda^2(x + \hat{1} + \hat{2} + \hat{3})} \\ &= \prod_x e^{\theta/2 \Xi_{123}(x) \lambda^1(x) \lambda^2(x + \hat{1} + \hat{2} + \hat{3})} \\ &= \prod_x \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \Xi_{123}(x) \lambda^1(x) \lambda^2(x + \hat{1} + \hat{2} + \hat{3}) \right). \end{aligned} \quad (87)$$

This acts as a rotation on the doublet $(U_A s^a U_A^\dagger)$ where $a \in \{1, 2\}$

$$\begin{aligned} s^1(x) &= \lambda^1(x), \\ s^2(x) &= \Xi_{123}(x) \lambda^2(x + \hat{1} + \hat{2} + \hat{3}), \end{aligned} \quad (88)$$

giving

$$\begin{aligned} s^1(x) &\rightarrow \cos(\theta) s^1(x) - \sin(\theta) s^2(x), \\ s^2(x) &\rightarrow \cos(\theta) s^2(x) + \sin(\theta) s^1(x). \end{aligned} \quad (89)$$

However, if we compute the commutator with the vector charge, we find a nonzero result

$$\begin{aligned} [Q_V, Q_A] &= -\frac{1}{2} \sum_x \Xi_{123}(x) [\lambda^1(x) \lambda^1(x + \hat{1} + \hat{2} + \hat{3}) \\ &\quad - \lambda^2(x) \lambda^2(x + \hat{1} + \hat{2} + \hat{3})]. \end{aligned} \quad (90)$$

While this clearly vanishes in the naive continuum limit, it is clearly nonzero on a finite lattice and suggests that the theory suffers from a mixed 't Hooft anomaly—when the lattice vector symmetry is gauged, Q_A is broken.

By following this strategy it should be clear that one can construct a series of conserved and quantized charges for the remaining shift symmetries B_a and B_{ab} :

$$Q_{\hat{a}} = B_a Q_V B_a^{-1}, \quad Q_{\hat{a}+\hat{b}} = B_{ab} Q_V B_{ab}^{-1} (d=3). \quad (91)$$

These, again, can be exponentiated to yield continuous symmetries $U_a = e^{-i\theta Q_{\hat{a}}}$ and $U_{ab} = e^{-i\theta Q_{\hat{a}+\hat{b}}}$. However, these charges will not in general commute with the vector symmetry

$$[Q_V, Q_{\hat{a}}] = G_{\hat{a}} [Q_V, Q_{\hat{a}+\hat{b}}] = G_{\hat{a}+\hat{b}}, \quad (92)$$

where

⁹Start with $H Q_V = Q_V H$. Multiply left and right by B_{123} and B_{123}^{-1} , respectively, and use the fact that B_{123} commutes with H to show that $B_{123} Q_V B_{123}^{-1}$ must also commute with H .

$$G_{\vec{n}} = -\frac{1}{2} \sum_x \Xi_{\vec{n}}(x) (\lambda^1(x) \lambda^1(x + \vec{n}) - \lambda^2(x) \lambda^2(x + \vec{n})),$$

where $\Xi_{\vec{n}}(x)$ is the phase associated with the shift $\vec{n}$. Thus, in three (spatial) dimensions $\Xi_{\vec{n}}(x) = \xi_a(x)$ or $\xi_a(x) \xi_b(x + \vec{a})$ or $\xi_1(x) \xi_2(x + \hat{1}) \xi(x + \hat{1} + \hat{2})$. $G_{\vec{n}}$ commutes with H and satisfies $G_{-\vec{n}} = -G_{\vec{n}}$.

In a similar fashion, one can compute the commutators of these charges with Q_A . For example, using

$$\begin{aligned} Q_A Q_{\hat{1}} &= B_1 B_2 B_3 Q_V B_3^{-1} B_2^{-1} B_1^{-1} B_1 Q_V B_1^{-1} \\ &= B_1 Q_{\hat{2}+\hat{3}} Q_V B_1^{-1}, \end{aligned} \quad (93)$$

we find

$$[Q_A, Q_{\hat{1}}] = -B_1 G_{\hat{2}+\hat{3}} B_1^{-1}. \quad (94)$$

Similarly

$$\begin{aligned} Q_A Q_{\hat{1}+\hat{2}} &= B_1 B_2 B_3 Q_V B_3^{-1} B_2^{-1} B_1^{-1} B_1 B_2 Q_V B_2^{-1} B_1^{-1} \\ &= B_1 B_2 Q_{\hat{3}} Q_V B_1^{-1} B_2^{-1} \end{aligned} \quad (95)$$

leading to

$$[Q_A, Q_{\hat{1}+\hat{2}}] = -B_1 B_2 G_{\hat{3}} B_1^{-1} B_2^{-1}. \quad (96)$$

Other commutators follow a similar pattern, e.g.,

$$[Q_{\hat{1}}, Q_{\hat{3}}] = B_1 G_{\hat{3}-\hat{1}} B_1^{-1} \quad (97)$$

and

$$\begin{aligned} [Q_{\hat{2}+\hat{1}}, Q_{\hat{2}+\hat{3}}] &= [B_2 B_1 Q_V B_1^{-1} B_2^{-1}, B_2 B_3 Q_V B_3^{-1} B_2^{-1}] \\ &= B_2 [Q_{\hat{1}}, Q_{\hat{3}}] B_2^{-1} = B_2 G_{\hat{3}-\hat{1}} B_2^{-1} \end{aligned} \quad (98)$$

and

$$[Q_{\hat{1}}, Q_{\hat{1}+\hat{2}}] = B_1 [Q_V, Q_{\hat{2}}] B_1^{-1} = B_1 G_{\hat{2}} B_1^{-1} \quad (99)$$

and

$$[Q_{\hat{1}}, Q_{\hat{2}+\hat{3}}] = [Q_{\hat{1}}, Q_{\hat{2}+\hat{3}-\hat{1}+\hat{1}}] = B_1 G_{\hat{2}+\hat{3}-\hat{1}} B_1^{-1}. \quad (100)$$

These commutation relations can be summarized as

$$[Q_{\vec{n}}, Q_{\vec{m}}] = \hat{G}_{\vec{m}-\vec{n}}, \quad (101)$$

where $\hat{G}_{\vec{n}} = B_{\vec{M}} G_{\vec{n}} B_{\vec{M}}^{-1}$ and $B_{\vec{M}}$ denotes a half shift corresponding to the vector $\vec{M}$. The latter has nonzero components arising from repeated elementary vectors in $\vec{n}$ and $\vec{m}$. For example, if $\vec{n} = \hat{1} + \hat{2}$ and $\vec{m} = \hat{1} + \hat{2} + \hat{3}$, then $\vec{M} = \hat{1} + \hat{2}$ and $B_{\vec{M}} = B_{12} = B_1 B_2$. In detail

$$\begin{aligned} \hat{G}_{\vec{n}}^M &= -\frac{1}{2} \sum_x \Xi_{\vec{n}}(x) [\lambda^1(x) \lambda^1(x + \vec{n}) \\ &\quad - \Xi_{\vec{M}}(\vec{n}) \lambda^2(x + \vec{M}) \lambda^2(x + \vec{n} + \vec{M})], \end{aligned} \quad (102)$$

which simplifies to

$$\begin{aligned} \hat{G}_{\vec{n}}^M &= -\frac{1}{2} \sum_x \Xi_{\vec{n}}(x) [\lambda^1(x) \lambda^1(x + \vec{n}) \\ &\quad - \Xi_{\vec{n}}(\vec{M}) \Xi_{\vec{M}}(\vec{n}) \lambda^2(x) \lambda^2(x + \vec{n})]. \end{aligned} \quad (103)$$

There are additional nontrivial commutation relations between the Q and $\hat{G}$ operators, and, if one allows for multiples of the elementary shifts, the algebra described here generalizes the Onsager algebra described in [9] to higher dimensional lattices. We postpone a detailed investigation of this algebra to future work and merely note here that the representation theory of this non-Abelian algebra would provide a set of maximally commuting operators and thereby encode any continuum anomaly.

It is interesting to compute the commutator of $G_{\vec{n}}$ with the fields λ^a . One finds

$$\begin{aligned} [G_{\vec{n}}, \lambda^1(x)] &= \Xi_{\vec{n}}(x) (\lambda^1(x + \vec{n}) - \lambda^1(x - \vec{n})), \\ [G_{\vec{n}}, \lambda^2(x)] &= -\Xi_{\vec{n}}(x) (\lambda^2(x + \vec{n}) - \lambda^2(x - \vec{n})). \end{aligned} \quad (104)$$

In momentum space the commutator behaves as

$$[G_{\vec{n}}, \lambda^a(\vec{k})] \sim (e^{i\vec{k} \cdot \vec{n}} - e^{-i\vec{k} \cdot \vec{n}}) \lambda^a(\vec{k}), \quad (105)$$

where $\vec{k} = \frac{2\pi m}{L}$ with $m = -\frac{L}{2} \dots \frac{L}{2}$ for periodic boundary conditions.

This vanishes on zero energy modes of the form

$$\lambda^a(x) \sim \frac{1}{\sqrt{V}} e^{i\pi \vec{A} \cdot x}, \quad (106)$$

where $\vec{A}$ is one of the shift vectors whose components take the values 0,1, and V is the lattice volume. The nontrivial $\vec{A}$ are the doublers. On low energy modes in the vicinity of these zero energy states the right-hand side of Eq. (105) scales like $1/L$ and hence vanishes as $L \rightarrow \infty$. The fact that $G_{\vec{n}}$ commutes with low energy modes of the field operator at large L and annihilates the vacuum can be used to show that the matrix element of $G_{\vec{n}}$ (and $\hat{G}_{\vec{n}}^M$) between any two low energy states goes to zero in the naive continuum limit following the argument given in [9].

IX. CONCLUSIONS

In this paper we have examined the shift, time reversal, and phase symmetries of Hamiltonian staggered fermions on finite spatial lattices focusing on the case of $(3+1)$ dimensions. We have reviewed how the shift symmetries

correspond to a discrete subgroup of the product of the continuum axial-flavor symmetry and translations. In particular, the odd shifts correspond to discrete axial transformations in the continuum theory and are only symmetries in the massless theory. Furthermore, we find that the odd shifts anticommute with time reversal.

We have constructed explicit operators to generate these symmetries along the lines of [14]. To do this we decompose the complex staggered fields into two real fields. This also enlarges the set of shift symmetries of the massless theory—one can apply independent half shifts to each of these two fields. The presence of anticommuting symmetries hints at the presence of 't Hooft anomalies. However, we have argued that the system can be gapped, and hence the 't Hooft anomaly canceled, for four complex staggered fields yielding 16 Majorana fermions in three (spatial) dimensions. We conjecture that canceling these mixed shift-time reversal anomalies in the Hamiltonian formalism may be equivalent to canceling the gravitational anomalies of the Euclidean theory.

We have also constructed a set of local, conserved, and quantized charges by combining each of the half shift symmetries with the vector charge Q_V . The resultant charges generate a set of new continuous global symmetries of the lattice theory. However, these charges do not commute with the vector charge on finite lattices. This implies that gauging the lattice vector symmetry will break these global symmetries—a lattice 't Hooft anomaly. In addition, the commutation relations of the full set of charges form a nontrivial algebra. We plan to investigate the representations of this non-Abelian algebra in future work. This will reveal the maximal set of commuting operators which should determine the continuum anomalies that can arise as the continuum limit is taken.

Note added. Another recent work also elucidates the symmetry structure of Hamiltonian staggered fermions with the goal of classifying the possibilities for symmetric mass generation [37]. Our results are consistent with their conclusions where the two papers overlap.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: CONNECTIONS BETWEEN EUCLIDEAN AND HAMILTONIAN STAGGERED FERMIONS

Since a single Dirac fermion has $2^{\frac{d+1}{2}}$ components, the most general square matrix that can be built from such spinors will possess $2^{\frac{d+1}{2}} \times 2^{\frac{d+1}{2}} = 2 \times 2^d$ elements. This is twice the number of points in the spatial cube employed in our Hamiltonian construction and hence cannot be reproduced with our single Hamiltonian staggered fermion. Clearly the missing degrees of freedom correspond to the fact that we truncated the theory before spin diagonalization by retaining only fields $\chi_+(x)$ whose β -eigenvalue was equal to unity. Equivalently the matrix fermion we construct commutes with γ_5 . It corresponds to a reduced staggered fermion in the Euclidean formulation [2,18,21]. Indeed, the mass term that arises in the Hamiltonian formulation can be seen as the dimensional reduction of the temporal one-link mass term of a reduced fermion.

To achieve the full set of Dirac fermions one can employ a second staggered fermion χ' and expand it on the additional matrix basis given by $\beta\alpha^x$

$$\Lambda'(x) = \frac{1}{8} \sum_{\{b\}} \chi'(x+b) \beta\alpha^{x+b}. \quad (\text{A1})$$

This matrix fermion anticommutes with γ_5 . By adding Λ and Λ' we can then build a theory containing $2^{\frac{d+1}{2}}$ Dirac fermions corresponding to the number of fermions that arise in Euclidean formulations of full staggered fermions where time is also discretized.

APPENDIX B: ADDITIONAL PROPERTIES OF STAGGERED FERMIONS

In this appendix we list the additional symmetries of the free staggered fermion Hamiltonian that are not already covered in the main text [3,38].

1. Rotational invariance

Consider the transformation

$$\chi(x) \rightarrow S_R(R^{-1}x) \chi(R^{-1}x),$$

where the rotation matrix $R \equiv R^{pr}$ acts on the spatial coordinates as

$$x_p \rightarrow x_r, \quad x_r \rightarrow -x_p, \quad x_s \rightarrow x_s \quad (s \neq p, r)$$

and

$$S_R(x) = \frac{1}{2} [1 \pm \eta_p(x) \eta_r(x) \mp \xi_p(x) \xi_r(x)] \quad (\text{B1})$$

$$+ \eta_p(x) \eta_r(x) \xi_p(x) \xi_r(x), \quad p \leq r. \quad (\text{B2})$$

Invariance of the Hamiltonian follows from

$$S_R(R^{-1}x)\eta_i(x)S_R(R^{-1}x + R^{-1}(x+i)) = R_{ij}\eta_j(R^{-1}x). \quad (\text{B3})$$

2. Axis reversal

We define an axis reversal transformation $I \equiv I^{(p)}$, which acts on the spatial coordinates as

$$x'_i = \begin{cases} -x_i & \text{if } i = p \\ x_i & \text{if } i \neq p \end{cases}. \quad (\text{B4})$$

Under this transformation, the field transforms as

$$\chi(x) \rightarrow (-1)^{x_p} \chi(Ix).$$

3. Spin-taste basis

To understand the continuum limit of staggered fermions it is useful to employ what has been termed “taste basis” for Euclidean staggered fermions [38]. For the Hamiltonian theory we can provide an analogous construction by defining two matrix fermions on a lattice with twice the lattice spacing:

$$\begin{aligned} \psi_{ba}^+(y) &= \frac{1}{2^{1/2}} \sum_{A \in A_e} (\alpha^A)_{ba} \chi_e(2y + A), \\ \psi_{ba}^-(y) &= \frac{1}{2^{1/2}} \sum_{A \in A_o} (\alpha^A)_{ba} \chi_o(2y + A), \end{aligned} \quad (\text{B5})$$

where A denotes a unit cell vector with components 0 or 1 and A_e denotes the subset of such vectors with an even number of nonzero components while A_o denotes those with an odd number. As before

$$\alpha^A = \alpha_1^{A_1} \alpha_2^{A_2} \alpha_3^{A_3}. \quad (\text{B6})$$

The matrix index b is interpreted as a “spin” index while a is interpreted as a “taste” index. Notice that both ψ_+ and ψ_- are constructed from just four complex parameters, and hence the field $\psi = \psi_+ + \psi_-$ can contain only two Dirac fermions as noted in the main text. Another way to see this is to note that ψ commutes with $\gamma_5 = i\alpha_1\alpha_2\alpha_3$. Since β commutes with α^{A_e} and anticommutes with α^{A_o} , the matrix fields on the left satisfy

$$\begin{aligned} \beta\psi_+\beta &= \psi_+, \\ \beta\psi_-\beta &= -\psi_-. \end{aligned} \quad (\text{B7})$$

Equation (B5) can be inverted [using $\text{tr}(\alpha^A \alpha^B) = 4\delta^{AB}$] to give

$$\chi(2y + A) = \frac{1}{2^{3/2}} \text{tr} \left(\frac{1}{2} (\psi(y) + \epsilon(A)\beta\psi(y)\beta) \alpha^A \right). \quad (\text{B8})$$

The mass term becomes

$$\begin{aligned} \sum_x \epsilon(x) \chi^\dagger(x) \chi(x) &= \frac{1}{8} \sum_{y,A} \text{tr}(\psi^\dagger(y) \alpha^A) \text{tr}(\beta\psi(y) \beta \alpha^A) \\ &= \frac{1}{2} \sum_y \text{tr}(\psi^\dagger \beta \psi(y) \beta), \end{aligned} \quad (\text{B9})$$

where we have used the completeness relation

$$\sum_A \alpha_{ab}^A \alpha_{cd}^A = 4\delta_{ad} \delta_{bc}, \quad (\text{B10})$$

which is valid for any matrix which can be expanded on the α^A . Using the results

$$\begin{aligned} \sum_{A, A_i=0} \eta_i(A) (\alpha^{A+i})_{aa} (\alpha^A)_{bb} &= 4(\alpha_i)_{ab} \delta_{ab} + 4\beta_{ab} (\beta \alpha_i)_{ba}, \\ \sum_{A, A_i=0} \eta_i(A) (\alpha^A)_{aa} (\alpha^{A+i})_{bb} &= 4(\alpha_i)_{ab} \delta_{ab} - 4\beta_{ab} (\beta \alpha_i)_{ba}, \end{aligned} \quad (\text{B11})$$

one can show that the Hamiltonian in the spin-taste basis is given by

$$\begin{aligned} H &= \frac{i}{4} \sum_{y,i} \text{tr}[\psi^\dagger(y) \alpha_i (\psi(y+i) - \psi(y-i))] \\ &\quad + \text{tr}[\psi^\dagger(y) \beta (\psi(y+i) + \psi(y-i) - 2\psi(y)) \beta \alpha^i] \\ &\quad + \frac{1}{2} \sum_y \text{tr}(\psi^\dagger(y) \beta \psi(y) \beta). \end{aligned} \quad (\text{B12})$$

We see that in this form, the free Hamiltonian describes two naive Dirac fermions, with the addition of a nondiagonal Wilson-like mass term, which removes the doublers.

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