

Non-BPS monopoles and dyons via resurgent transseries

Gerald V. Dunne[✉] and Evan Shinn[✉]

Department of Physics, *University of Connecticut*, Storrs, Connecticut 06269, USA



(Received 24 February 2026; accepted 7 July 2026; published 23 July 2026)

Radially symmetric non-BPS 't Hooft-Polyakov monopoles and dyons are constructed as resurgent transseries: infinite sums of exponentially decaying terms, each multiplied by a factorially divergent fluctuation factor. All higher exponential terms are explicitly expressed in terms of the leading order solutions. In the BPS limit all fluctuation terms truncate.

DOI: [10.1103/fhhl-sj3z](https://doi.org/10.1103/fhhl-sj3z)

I. INTRODUCTION

Semiclassical analysis of quantum field theory is a well-developed method built on computing quantum fluctuations about classical saddle solutions [1–4]. This process is most explicit in special limits, achieved by tuning parameters in the Lagrangian, for example with various levels of supersymmetry. In such limits there are dramatic simplifications in the classical equations of motion, and correspondingly also in the fluctuation problem. Beyond such special limits, semiclassical analysis is much more difficult to analyze fully. For example, in Yang-Mills-Higgs theory, the classical equations of motion for monopoles and dyons can be reduced by a symmetric static *Ansatz* to a set of coupled nonlinear ordinary differential equations (ODEs) that involves a parameter β that is the ratio of the Higgs mass to the W-boson mass [5,6]. When $\beta \rightarrow 0$, the BPS limit, these equations reduce to first order equations with simple explicit solutions [7–9], and their fluctuations are also well understood [3,4]. This is an example of a general phenomenon [8]. Away from the BPS limit, the monopole and dyons solutions are considerably more complicated, and even less is known about their fluctuations.

Here we study the 't Hooft-Polyakov monopole and dyon equations away from the BPS limit using ideas from resurgent asymptotics. Previous work on non-BPS monopole solutions has been numerical [10,11] and/or studying small or large deviations from the BPS limit [12–14]. In this paper we apply a method that does not rely on such approximations. This is an application of a general method developed by O. Costin in the context of resurgent asymptotic analysis of coupled nonlinear differential equations [15,16]. We show that the non-BPS monopole equations are solved by a transseries *Ansatz* that expands the solutions in the far field limit as a series of

exponentially suppressed terms, each multiplied by fluctuation expansions. In this approach the nonlinear equations are transformed into an infinite tower of *linear* ODEs, graded by the exponential order of the transseries solutions. The lowest order ODEs are linear and *homogeneous*, while the higher orders are linear and *inhomogeneous*. The homogeneous solutions introduce boundary condition parameters, which become the “transseries parameters,” and the inhomogeneous terms at higher exponential transseries orders propagate the lower order solutions in a systematic manner. The physical consequence of this structure is that all the exponentially suppressed higher transseries terms are expressed entirely in terms of the lowest order homogeneous solutions, an explicit realization of the general resurgent properties of coupled nonlinear ODEs [15,16]. In other words, the formal large r series solutions encode all the information about the beyond-all-orders exponentially suppressed terms in the solution of the non-BPS monopole equations. We also show that this method applies to the more general dyon solutions [9], with a more general transseries structure.

In Sec. II we recall some basic notation and results. Section III describes the transseries structure for non-BPS monopoles for $\beta = 1$, when the Higgs and W-boson masses are equal. Section IV determines the transseries parameters by matching to a solution from small r . Sections V and VI describe the more general transseries structures that arise for general β , and for dyon solutions.

II. 'T HOOFT-POLYAKOV MONOPOLES

We consider $SU(2)$ Yang-Mills-Higgs theory with the Higgs field in the adjoint representation, for which the action is [2–6],

$$S = \int d^4x \left(-\frac{1}{4e^2} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} (D_\mu \Phi^a)(D^\mu \Phi^a) - \frac{\gamma}{8} (\Phi^a \Phi^a - v^2)^2 \right) \quad (1)$$

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

with $a = (1, 2, 3)$. Here, e denotes the gauge coupling constant, with γ the strength of the scalar self-interaction and v the vacuum expectation value of the Higgs field. The mass of the Higgs field is $M_H = v\sqrt{\gamma}$, with the W bosons acquiring a mass $M_W = ve$ via the Higgs mechanism. We define the dimensionless mass ratio $\beta \equiv M_H/M_W$. We also adopt the conventional dimensionless units in which the radial variable r is rescaled as $r \rightarrow r/(ev)$ [14]. The field strength tensor and covariant derivative are

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c; \quad D_\mu \Phi^a = \partial_\mu \Phi^a + \epsilon^{abc} A_\mu^b \Phi^c. \quad (2)$$

For monopoles we adopt the 't Hooft-Polyakov static, purely magnetic, and spherically symmetric *Ansatz* [5,6]

$$A_0^a = 0; \quad A_i^a = \epsilon_{iak} \frac{x_k}{r^2} (W(r) - 1); \quad \Phi^a = H(r) \frac{x_a}{r}. \quad (3)$$

(We discuss dyon solutions in Sec. VI.) In terms of the radial fields, the monopole mass is,

$$\mathcal{M}_\beta = \frac{4\pi v}{e} \int_0^\infty dr \left[W'^2 + \frac{r^2}{2} H'^2 + \frac{(1 - W^2)^2}{2r^2} + \frac{\beta^2 r^2}{8} (H^2 - 1)^2 + W^2 H^2 \right]. \quad (4)$$

The classical equations of motion are

$$W''(r) - W(r)H(r)^2 - \frac{W(r)(W(r)^2 - 1)}{r^2} = 0, \quad (5)$$

$$H''(r) - \frac{2}{r^2} H(r)W(r)^2 - \frac{\beta^2}{2} H(r)(H(r)^2 - 1) + \frac{2H'(r)}{r} = 0. \quad (6)$$

Finiteness of the mass requires the leading behavior as $r \rightarrow \infty$,

$$W(r) \sim e^{-r} + \dots; \quad H(r) \sim 1 + \dots \quad (7)$$

The precise form of the subleading terms is discussed below in Sec. III. In this paper we apply Costin's transseries expansion method [15,16] to analyze the coupled nonlinear ODEs in (5)–(6).

A. BPS monopole: $\beta = 0$

When $\beta = 0$, the monopole mass can be written in factored form

$$\mathcal{M}_{\beta=0} = \frac{4\pi v}{e} \int_0^\infty dr \left[(W' + HW)^2 + \frac{r^2}{2} \left(H' - \frac{1}{r^2} (1 - W^2) \right)^2 + \frac{d}{dr} (H(1 - W^2)) \right]. \quad (8)$$

In the $\beta \rightarrow 0$ limit, the classical equations (5)–(6) reduce to first-order (BPS) equations (retaining the boundary condition that $H(r) \rightarrow 1$ at infinity) [7,8],

$$W'(r) = -H(r)W(r); \quad H'(r) = \frac{1}{r^2} (1 - W(r)^2) \quad (9)$$

This leads to the explicit closed-form BPS solutions [7,8],

$$W_{\text{BPS}}(r) = \frac{r}{\sinh(r)}; \quad H_{\text{BPS}}(r) = \coth(r) - \frac{1}{r}. \quad (10)$$

As $r \rightarrow \infty$ the BPS solutions are expanded in powers of e^{-r} ,

$$W_{\text{BPS}}(r) = 2r \sum_{k=0}^\infty e^{-(2k+1)r}; \quad H_{\text{BPS}}(r) = 1 - \frac{1}{r} + 2 \sum_{k=1}^\infty e^{-2kr}. \quad (11)$$

These are simple, convergent, transseries expansions, without asymptotic fluctuation factors. The expansions at $r = 0$ have a finite radius of convergence, $r_c = \pi$:

$$W_{\text{BPS}}(r) = 1 - \frac{r^2}{6} + \frac{7r^4}{360} + O(r^6);$$

$$H_{\text{BPS}}(r) = \frac{r}{3} - \frac{r^3}{45} + \frac{2r^5}{945} + O(r^7). \quad (12)$$

The BPS mass is determined by the field values at infinity: $W_{\text{BPS}} \rightarrow 0, H_{\text{BPS}} \rightarrow 1 \Rightarrow \mathcal{M}_{\text{BPS}} = \frac{4\pi v}{e}$.

III. NON-BPS MONOPOLES: $\beta = 1$

When $\beta > 0$, the finiteness of the mass requires the leading asymptotic behavior at infinity to be as in (7), with corrections that are exponentially small. This leads to a transseries expansion as $r \rightarrow \infty$, with sums over two independent exponential factors, e^{-r} and $e^{-\beta r}$, and with power law fluctuations. To understand more clearly this resurgent transseries structure, we first specialize to the case where the Higgs and W-boson masses are equal: $\beta = 1$. This is in some sense “intermediate” between the $\beta \rightarrow 0$ BPS limit [7,8] and the $\beta \rightarrow \infty$ limit studied in [12,14].

A. Transseries *Ansatz* for $\beta = 1$

When $\beta = 1$, the classical equations of motion (5)–(6) are

$$W'' = W \left(\frac{W^2 - 1}{r^2} + H^2 \right), \quad (13)$$

$$H'' = 2H \left(\frac{1}{4} (H^2 - 1) + \frac{W^2}{r^2} \right) - \frac{2H'}{r}. \quad (14)$$

We invoke a large r transseries *Ansatz*,

$$W(r) = \sum_{k=1}^{\infty} e^{-kr} w_k(r), \quad (15)$$

$$H(r) = 1 - \sum_{k=1}^{\infty} e^{-kr} h_k(r). \quad (16)$$

Each field is expanded as a sum of powers of the single exponential factor (there is only one exponential factor, because of the choice of $\beta = 1$). Each term in the exponential sum is multiplied by a “fluctuation factor,” $w_k(r)$ or $h_k(r)$, which are generically [in fact, all except $h_1(r)$] formal factorially divergent asymptotic series as $r \rightarrow \infty$. This is very different from the BPS “transseries” solutions in (11), where the $w_k(r)$ and $h_k(r)$ factors truncate to single terms. The non-BPS fluctuation factors $w_k(r)$ and $h_k(r)$ in (15)–(16) are determined recursively by the classical equations of motion (13)–(14).

B. Leading transseries order: Homogeneous linear ODEs

At leading transseries order, $O(e^{-r})$, we obtain *linear and homogeneous* ODEs for $w_1(r)$ and $h_1(r)$,

$$w_1''(r) - 2w_1'(r) + \frac{w_1(r)}{r^2} = 0, \quad (17)$$

$$h_1''(r) + \frac{2h_1'(r)}{r} - 2h_1'(r) - \frac{2h_1(r)}{r} = 0. \quad (18)$$

The two independent real solutions for w_1 can be written in terms of modified Bessel functions with *imaginary index*,

$$w_{(1,\pm)}(r) = \begin{cases} +: \sqrt{\frac{2}{\pi}} \sqrt{r} e^r \left[I_{i\sqrt{3}/2}(r) - \frac{\sinh(\sqrt{3}\pi/2)}{i\pi} K_{i\sqrt{3}/2}(r) \right] \\ -: \sqrt{\frac{2}{\pi}} \sqrt{r} e^r K_{i\sqrt{3}/2}(r) \end{cases}. \quad (19)$$

The homogeneous solution $w_{(1,-)}(r)$ decays at infinity, while $w_{(1,+)}(r)$ grows exponentially fast. So we choose

$$w_1(r) = \sigma_w w_{(1,-)}(r) \quad (20)$$

where σ_w is an arbitrary constant, since (17) is homogeneous. At large r , $w_1(r)$ has a formal asymptotic expansion,

$$w_1(r) \sim \sigma_w \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - \frac{i\sqrt{3}}{2})_n (\frac{1}{2} + \frac{i\sqrt{3}}{2})_n}{(-2)^n n! r^n}, \quad r \rightarrow \infty \\ \sim \sigma_w \left[1 - \frac{1}{2r} + \frac{3}{8r^2} - \frac{7}{16r^3} + \frac{91}{128r^4} - \frac{1911}{1280r^5} + O\left(\frac{1}{r^6}\right) \right]. \quad (21)$$

Here $(a)_n$ denotes Pochhammer’s symbol: $(a)_n := \Gamma(n+a)/\Gamma(a)$. The expansion (21) has factorially divergent coefficients with large order ($n \rightarrow \infty$) behavior,

$$\frac{(\frac{1}{2} - \frac{i\sqrt{3}}{2})_n (\frac{1}{2} + \frac{i\sqrt{3}}{2})_n}{(-2)^n n!} \sim \frac{\cosh(\frac{\sqrt{3}\pi}{2})}{\pi(-2)^n} \Gamma(n) \left[1 + \frac{\frac{1}{2}(-2)}{(n-1)} + \frac{\frac{3}{8}(-2)^2}{(n-1)(n-2)} + \frac{\frac{7}{16}(-2)^3}{(n-1)(n-2)(n-3)} + \dots \right]. \quad (22)$$

Comparing the coefficients, $\{\frac{1}{2}, \frac{3}{8}, \frac{7}{16}, \dots\}$, in (21) and (22), we see the generic resurgent large-order/low-order duality [16–20] between the large n dependence of the expansion coefficients and the large r dependence of the function in (21).

The two independent solutions of the h_1 equation (18) are much simpler,

$$h_{(1,\pm)}(r) = \begin{cases} +: e^{2r}/2r \\ -: 1/r \end{cases} \quad (23)$$

$h_{(1,-)}(r)$ decays at infinity, while $h_{(1,+)}(r)$ grows exponentially fast at infinity. So finiteness selects the solution to be

$$h_1(r) = \sigma_h h_{(1,-)}(r) \quad (24)$$

where σ_h is the other arbitrary constant for the $r \rightarrow \infty$ solutions. Note that $h_1(r)$ does not have a formal

asymptotic series expansion as $r \rightarrow \infty$: the expansion truncates with just a single term.

It is important to point out that the non-BPS monopole equations (13)–(14) are special in the sense that the homogeneous “seed” solutions $w_1(r)$ and $h_1(r)$ for the full transseries in (15)–(16) are obtained in closed form, as a modified Bessel function for $w_1(r)$ in (20) and a simple exponential for $h_1(r)$ in (24). For general ODEs, these homogeneous “seed” solutions, which recursively generate the full transseries, are often known only as formal asymptotic expansions. In this generic situation one proceeds via Borel summation of the seed solutions, and higher order terms are then generated by Borel convolutions [15,16]. For this monopole problem one can effectively bypass this step, taking advantage of the fact that the seed solutions are known special functions, to present a more efficient way to generate the higher exponential orders of the transseries. Resurgent features of the

transseries solution can be seen directly in the fact that lowest order seed solutions generate all higher order terms directly, and are also reflected in the large-order/low-order duality of the nontrivial seed solution $w_1(r)$ in (21) and (22).

C. Higher transseries orders: Inhomogeneous linear ODEs

At higher transseries order, $O(e^{-kr})$ with $k \geq 2$, the transseries *Ansätze* (15)–(16) lead to linear inhomogeneous ODEs for $w_k(r)$ and $h_k(r)$ [15,16].

1. Second transseries order

At second transseries order, $O(e^{-2r})$, we obtain the following linear inhomogeneous equations for $w_2(r)$ and $h_2(r)$:

$$w_2''(r) - 4w_2'(r) + \frac{w_2(r)}{r^2} + 3w_2(r) = \mathcal{W}_2(r), \quad (25)$$

$$h_2''(r) + \frac{2h_2'(r)}{r} - 4h_2'(r) - \frac{4h_2(r)}{r} + 3h_2(r) = \mathcal{H}_2(r). \quad (26)$$

The inhomogeneous terms $\mathcal{W}_2(r)$ and $\mathcal{H}_2(r)$ mix the lower-order solutions $w_1(r)$ and $h_1(r)$,

$$\mathcal{W}_2(r) = -2h_1(r)w_1(r), \quad (27)$$

$$\mathcal{H}_2(r) = -\frac{3}{2}h_1(r)^2 - \frac{2}{r^2}w_1(r)^2. \quad (28)$$

Each of (25)–(26) can be solved straightforwardly using the associated homogeneous solutions [21]. In fact, the homogeneous solutions at second order are related in a very simple way to the leading order (homogeneous) solutions,

$$w_{(2,\pm)}^{\text{homog}}(r) = e^r w_{(1,\pm)}(r), \quad h_{(2,\pm)}^{\text{homog}}(r) = e^r h_{(1,\pm)}(r). \quad (29)$$

Furthermore, the Wronskians are simple,

$$\begin{aligned} \mathfrak{W}[w_{(2,-)}^{\text{homog}}(r), w_{(2,+)}^{\text{homog}}(r)] &= \frac{2}{\pi} e^{4r}, \\ \mathfrak{W}[h_{(2,-)}^{\text{homog}}(r), h_{(2,+)}^{\text{homog}}(r)] &= \frac{e^{4r}}{r^2}. \end{aligned} \quad (30)$$

Therefore the second order solutions can be written explicitly in terms of the leading order solutions,

$$\begin{aligned} w_2(r) &= -e^r w_{(1,+)}(r) \int_r^\infty \frac{\pi}{2} e^{-3t} w_{(1,-)}(t) \mathcal{W}_2(t) dt \\ &\quad + e^r w_{(1,-)}(r) \int_r^\infty \frac{\pi}{2} e^{-3t} w_{(1,+)}(t) \mathcal{W}_2(t) dt, \end{aligned} \quad (31)$$

$$\begin{aligned} h_2(r) &= -e^r h_{(1,+)}(r) \int_r^\infty t^2 e^{-3t} h_{(1,-)}(t) \mathcal{H}_2(t) dt \\ &\quad + e^r h_{(1,-)}(r) \int_r^\infty t^2 e^{-3t} h_{(1,+)}(t) \mathcal{H}_2(t) dt. \end{aligned} \quad (32)$$

The inhomogeneous terms, $\mathcal{W}_2$ and $\mathcal{H}_2$, are also expressed in terms of the first order solutions,

$$\mathcal{W}_2(r) = -2\sigma_w \sigma_h h_{(1,-)}(r) w_{(1,-)}(r), \quad (33)$$

$$\mathcal{H}_2(r) = -\frac{3}{2}\sigma_h^2 [h_{(1,-)}(r)]^2 - \frac{2}{r^2}\sigma_w^2 [w_{(1,-)}(r)]^2. \quad (34)$$

This shows that the fluctuations at second transseries order, $w_2(r)$ and $h_2(r)$ in (31)–(32), are completely determined by the leading transseries order (homogeneous) solutions, $w_{(1,\pm)}(r)$ and $h_{(1,\pm)}(r)$ in (19) and (23), respectively. The constants σ_w and σ_h enter the second order solutions via the inhomogeneous terms inside the integrals, thereby mixing the solutions beyond the leading transseries order.

While the integrals in (31)–(32) cannot be done in closed form, it is straightforward to expand the integrands at large t in order to generate the large r expansions of the fluctuations $w_2(r)$ and $h_2(r)$,

$$w_2(r) \sim \sigma_w \sigma_h \left(-\frac{2}{3r} + \frac{11}{9r^2} - \frac{307}{108r^3} + \frac{1903}{216r^4} - \frac{182185}{5184r^5} + \dots \right), \quad (35)$$

$$h_2(r) \sim -\sigma_h^2 \left(\frac{1}{2r^2} - \frac{2}{3r^3} + \frac{13}{9r^4} - \frac{40}{9r^5} + \frac{484}{27r^6} - \frac{7280}{81r^7} + \dots \right) - \sigma_w^2 \left(\frac{2}{3r^2} - \frac{14}{9r^3} + \frac{118}{27r^4} - \frac{821}{54r^5} + \frac{5260}{81r^6} - \frac{1624441}{4860r^7} + \dots \right). \quad (36)$$

The asymptotic series multiplied by σ_h^2 above in (36) can be written in closed form, and is the sum of two incomplete gamma functions,

$$h_2(r) \sim -\sigma_h^2 \left(\frac{3}{4} \frac{e^r}{r} \Gamma(0, r) - \frac{3}{4} \frac{e^{3r}}{r} \Gamma(0, 3r) \right) - \sigma_w^2(\dots), \quad (37)$$

$$\sim -\sigma_h^2 \frac{3}{4} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{r^{n+2}} \left(1 - \frac{1}{3^{n+1}}\right) - \sigma_w^2(\dots). \quad (38)$$

The σ_h^2 term in (36)–(37) corresponds to Eq. (35) in [14]. This is because they considered a limit in which $W(r)$ decoupled from $H(r)$. But this is only part of the full asymptotic solution. The other terms in (35)–(36), multiplying $\sigma_w \sigma_h$ and σ_w^2 , do not have such simple closed-form expressions, but it is straightforward to generate their large r expansions by expanding products of the modified Bessel functions $I_{i\sqrt{3}/2}(r)$ and $K_{i\sqrt{3}/2}(r)$. This general pattern continues to all higher orders: the large r expansion coefficients are completely determined by the asymptotic expansions of the homogeneous solutions $w_{(1,\pm)}(r)$ and $h_{(1,\pm)}(r)$.

2. General transseries order

At higher order in the transseries, e^{-kr} for $k \geq 2$, the differential equations (13), (14) take the following general linear inhomogeneous form,

$$\mathcal{D}_k^w w_k(r) = \mathcal{W}_k(r), \quad (39)$$

$$\mathcal{D}_k^h h_k(r) = \mathcal{H}_k(r). \quad (40)$$

The differential operators are,

$$\mathcal{D}_k^w := \frac{d^2}{dr^2} - 2k \frac{d}{dr} + \frac{1}{r^2} + k^2 - 1, \quad (41)$$

$$\mathcal{D}_k^h := \frac{d^2}{dr^2} + \left(\frac{2}{r} - 2k\right) \frac{d}{dr} - \frac{2k}{r} + k^2 - 1. \quad (42)$$

These operators have a simple relation to the leading order operators in (17)–(18):

$$\mathcal{D}_k^w[e^{(k-1)r}f(r)] = e^{(k-1)r}\mathcal{D}_1^w f(r), \quad (43)$$

$$\mathcal{D}_k^h[e^{(k-1)r}f(r)] = e^{(k-1)r}\mathcal{D}_1^h f(r). \quad (44)$$

This implies that the homogeneous solutions for (39)–(40) at order k are directly related to the first order homogeneous solutions in (19) and (23),

$$w_{(k,\pm)}^{\text{homog}}(r) = e^{(k-1)r} w_{(1,\pm)}(r), \quad (45)$$

$$h_{(k,\pm)}^{\text{homog}}(r) = e^{(k-1)r} h_{(1,\pm)}(r). \quad (46)$$

Correspondingly, the order k Wronskians are straightforward generalizations of the Wronskians at leading transseries order,

$$\mathfrak{W}_k^w(r) := \mathfrak{W}[w_{(k,-)}^{\text{homog}}(r), w_{(k,+)}^{\text{homog}}(r)] = \frac{2}{\pi} e^{2kr}, \quad (47)$$

$$\mathfrak{W}_k^h(r) := \mathfrak{W}[h_{(k,-)}^{\text{homog}}(r), h_{(k,+)}^{\text{homog}}(r)] = \frac{e^{2kr}}{r^2}. \quad (48)$$

Therefore, we can write the solutions for $w_k(r)$ and $h_k(r)$ in an analogous form to those for $w_2(r)$ and $h_2(r)$ in (31)–(32).

$$w_k(r) = -w_{(k,+)}^{\text{homog}}(r) \int_r^\infty \frac{w_{(k,-)}^{\text{homog}}(t) \mathcal{W}_k(t)}{\mathfrak{W}_k^w(t)} dt + w_{(k,-)}^{\text{homog}}(r) \int_r^\infty \frac{w_{(k,+)}^{\text{homog}}(t) \mathcal{W}_k(t)}{\mathfrak{W}_k^w(t)} dt, \quad (49)$$

$$h_k(r) = -h_{(k,+)}^{\text{homog}}(r) \int_r^\infty \frac{h_{(k,-)}^{\text{homog}}(t) \mathcal{H}_k(t)}{\mathfrak{W}_k^h(t)} dt + h_{(k,-)}^{\text{homog}}(r) \int_r^\infty \frac{h_{(k,+)}^{\text{homog}}(t) \mathcal{H}_k(t)}{\mathfrak{W}_k^h(t)} dt. \quad (50)$$

These expressions make it clear that at any order k the solutions $w_k(t)$ and $h_k(t)$ are completely determined by the first order solutions $w_{(1,\pm)}(r)$ and $h_{(1,\pm)}(r)$ in (19) and (23),

$$w_k(r) = -e^{(k-1)r} w_{(1,+)}(r) \int_r^\infty \frac{\pi}{2} e^{-(k+1)t} w_{(1,-)}(t) \mathcal{W}_k(t) dt + e^{(k-1)r} w_{(1,-)}(r) \int_r^\infty \frac{\pi}{2} e^{-(k+1)t} w_{(1,+)}(t) \mathcal{W}_k(t) dt, \quad (51)$$

$$h_k(r) = -e^{(k-1)r} h_{(1,+)}(r) \int_r^\infty t^2 e^{-(k+1)t} h_{(1,-)}(t) \mathcal{H}_k(t) dt + e^{(k-1)r} h_{(1,-)}(r) \int_r^\infty t^2 e^{-(k+1)t} h_{(1,+)}(t) \mathcal{H}_k(t) dt. \quad (52)$$

Note that the inhomogeneous terms $\mathcal{W}_k(r)$ and $\mathcal{H}_k(r)$ are sums of products of the lower order solutions. For example, the second order inhomogeneous terms are in (27)–(28), while at third order we find,

$$\mathcal{W}_3 = h_1(r)^2 w_1(r) - 2h_2(r) w_1(r) + \frac{1}{r^2} w_1(r)^3 - 2h_1(r) w_2(r), \quad (53)$$

$$\mathcal{H}_3 = \frac{1}{2}h_1(r)^3 - 3h_1(r)h_2(r) + \frac{2}{r^2}h_1(r)w_1(r)^2 - \frac{4}{r^2}w_1(r)w_2(r). \quad (54)$$

This demonstrates that the transseries *Ansätze* in (15)–(16) are consistent, and that all the fluctuation terms $w_k(r)$ and $h_k(r)$, for $k \geq 2$, are determined by the leading order homogeneous solutions.

IV. SOLUTIONS AT THE ORIGIN AND MATCHING

The radial functions $W(r)$ and $H(r)$ in (3) have convergent Taylor series expansions about $r = 0$,

$$W(r) = 1 + br^2 + \frac{r^4}{10}(a^2 + 3b^2) + \frac{r^6}{280}(24a^2b - a^2 + 28b^3) + \dots \quad (55)$$

$$H(r) = ar + \frac{r^3}{20}a(8b - 1) + \frac{r^5}{1120}a(36a^2 + 192b^2 - 16b + 1) + \dots \quad (56)$$

All subsequent coefficients (hence the radius of convergence) are determined by the two parameters, a and b .

To determine the (a, b) parameters we first employed a numerical shooting method from the origin, using the *Mathematica* “StiffnessSwitching” option. We imposed initial conditions at $r_{\min} = 1/10^5$, where we fixed the two functions, $W(r)$ and $H(r)$, and their first derivatives, $W'(r)$ and $H'(r)$, using the first two terms of the small r expansions in (55)–(56). We adjusted the parameters (a, b) to match the large r boundary conditions $W(r) \rightarrow 0$, $H(r) \rightarrow 1$, at large r . Using $r_{\max} \approx 10$ we confirmed that this matching procedure is consistent with the numerical result, $a = 0.7318$ and $b = -0.3409$, in [14], for $\beta = 1$. With these values of a and b , the numerical solutions for the radial profile functions $W(r)$ and $H(r)$ are shown in Fig. 1.

To determine the transseries parameters, σ_w and σ_h , we imposed boundary conditions at $r_{\max}$ given by the leading large r behaviors in (20) and (24). This fixed the two functions, $W(r)$ and $H(r)$, and their first derivatives, $W'(r)$ and $H'(r)$, at $r_{\max}$ in terms of the parameters (σ_w, σ_h) . We then integrated back toward $r = 0$, once again using the *Mathematica* “StiffnessSwitching” option. We adjusted (σ_w, σ_h) to maximize the overlap in the intermediate region $r \approx 1$ with the numerical results obtained starting from the origin. This matching led to matched values of the small and large r boundary condition parameters (a, b) and (σ_w, σ_h) to four digit precision: $a = 0.7318$, $b = -0.3409$, $\sigma_h = 1.905$, and $\sigma_w = 3.336$. If desired, higher precision can be obtained by adjusting $r_{\min}$ and $r_{\max}$, by including

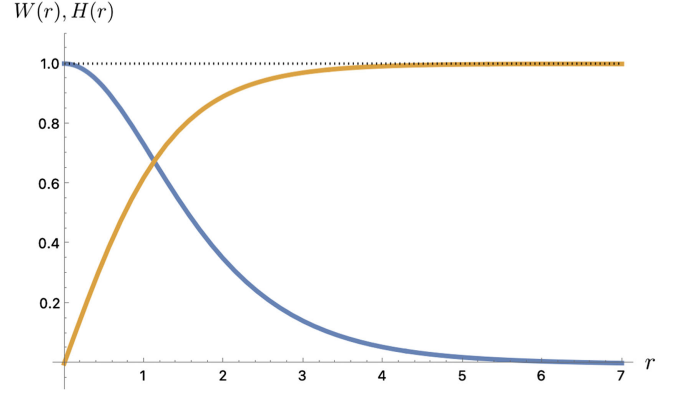


FIG. 1. Numerical solutions generated from the origin. To generate these we use numerical integration in *Mathematica* with the “StiffnessSwitching” option due to the stiffness of the differential equations (13)–(14). $W(r)$ is given by the blue curve, and $H(r)$ by the orange curve. The asymptotic value of 1 is denoted by a dashed black line. The initial conditions at $r = 0$ are explained in the text.

more terms of the small r and large r expansions for the initial conditions, and by successive iterations back and forth between small and large r .

In Figs. 2 and 3 we display the effect of one, two and three terms of the transseries expansions (15)–(16). We observe that increasing the exponential order of the transseries expansions in (15)–(16) leads to better matching with the numerical solution at small r . Given the matched values of the parameters for the small r and large r expansions, we obtain accurate interpolating functions for $W(r)$ and $H(r)$. The resulting integrand of the monopole mass function (4) is plotted in Fig. 4. It matches well with the numerical profile plotted in [12]. The monopole mass can then be evaluated by integrating this interpolated solution. We find the monopole mass, defined in (4), for $\beta = 1$ to be

$$\mathcal{M}_{\beta=1} = 1.237(7) \frac{4\pi v}{e}. \quad (57)$$

This is in good agreement with the numerical results in [10,14].

V. TRANSSERIES STRUCTURE FOR GENERAL β

Having understood the transseries structure for the situation where the Higgs and W-boson mass are equal, we now describe what changes when we consider more general values of the mass ratio, $\beta \neq 1$. Since β enters the H equation but not the W equation, at leading transseries order only the homogeneous $h_{(1,\pm)}$ are affected,

$$e^{-r}w_{(1,\pm)}(r) \rightarrow e^{-r}w_{(1,\pm)}(r) \quad \text{and} \quad e^{-r}h_{(1,\pm)}(r) \rightarrow e^{-\beta r}h_{(1,\pm)}(\beta r). \quad (58)$$

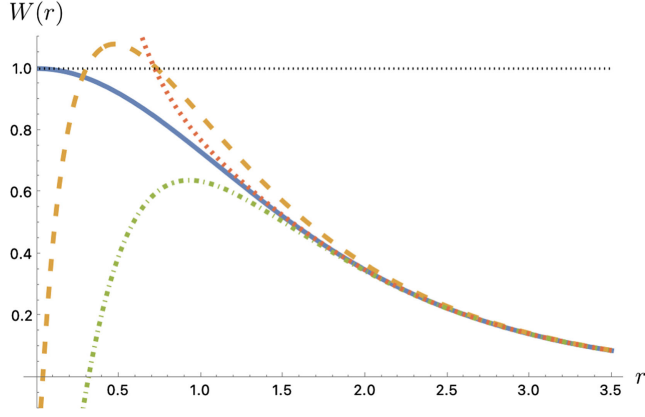


FIG. 2. $W(r)$ solutions for $\sigma_h = 1.905$, $\sigma_w = 3.336$. The numerical solution from $r = 0$ is given by the blue solid curve. First transseries order contributions are given by the yellow dashed curve, with second and third transseries order contributions given by the green dash-dotted and red dotted curves, respectively. The asymptotic value of 1 is denoted by the thinly dotted black line. Note the increased agreement as $r \rightarrow 0$ with the inclusion of additional terms in the transseries.

Here $w_{(1,\pm)}$ and $h_{(1,\pm)}$ are the homogeneous solutions for $\beta = 1$, from Sec. III. B. The parameter β only enters the homogeneous $H(r)$ solutions, which are simple. The more complicated Bessel function for the homogeneous $W(r)$ solutions are not affected. The Wronskians are also only rescaled,

$$\mathfrak{W}_W(r) = \frac{2}{\pi} e^{2r}, \quad \mathfrak{W}_H(r) = \frac{e^{2\beta r}}{\beta r^2}. \quad (59)$$

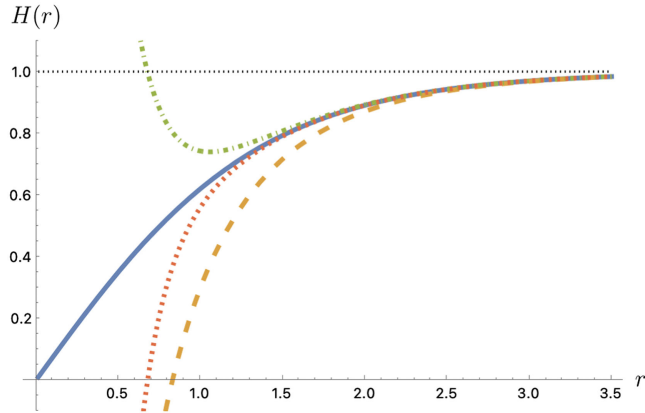


FIG. 3. $H(r)$ solutions for $\sigma_h = 1.905$, $\sigma_w = 3.336$. The numerical solution from $r = 0$ is given by the blue solid curve. First transseries order contributions are given by the yellow dashed curve, with second and third transseries order contributions given by the green dash-dotted and red dotted curves, respectively. The asymptotic value of 1 is denoted by the thinly dotted black line. Note the increased agreement as $r \rightarrow 0$ with the inclusion of additional terms in the transseries.

Therefore we can convert the coupled ODEs to integral equations, as before, and iterate in order to generate the full transseries.

The β dependence enters through the homogeneous $H_{\pm}(r)$ solutions, and this β dependence propagates as one iterates the integral equations, at higher orders also due to the nonlinear coupling. Since the homogeneous $H_{\pm}(r)$ solutions have exponential factors $e^{\pm\beta r}$ involving β in the exponent, while the homogeneous $W_{\pm}(r)$ solutions have exponential factors $e^{\pm r}$ without β in the exponent, the iteration automatically generates a *double series* of exponentials involving all powers of $e^{\pm r}$ and all powers of $e^{\pm\beta r}$. There are still just two transseries parameters σ_w and σ_h , but their values when matching to the expansions at the origin become β dependent.

VI. DYONS

In addition to monopole solutions, the Yang-Mills-Higgs theory (1) supports dyon solutions, which carry both electric and magnetic charge [9,22]. For dyons the radial Ansatz is generalized to include the A_0 field¹:

$$A_0^a = J(r) \frac{x_a}{r^2}; \quad A_i^a = \epsilon_{iak} \frac{x_k}{r^2} (W(r) - 1); \quad \Phi^a = H(r) \frac{x_a}{r}. \quad (60)$$

We now have a system of three coupled nonlinear ODEs,

$$J'' = \frac{2JW^2}{r^2}, \quad (61)$$

$$W'' = W \left(\frac{W^2 - J^2 - 1}{r^2} + H^2 \right), \quad (62)$$

$$H'' = 2H \left(\frac{\beta^2}{4} (H^2 - 1) + \frac{W^2}{r^2} \right) - \frac{2H'}{r}. \quad (63)$$

For simplicity of exposition, and to compare directly with the monopole transseries structure in Sec. III, we consider again the $\beta = 1$ case; noting that the strategy for general β is as described in Sec. V.

The coupled equations (61)–(63) (with $\beta = 1$) are solved at large r by a transseries Ansatz, which is analogous to, but more general than, that for the monopole solution in (15)–(16).

A. Zeroth exponential order

The zeroth exponential order yields the following solutions to (61)–(63):

¹We follow the conventions of [14]; note that the radial profile function of the Higgs field, $H(r)$, differs from that in [9] by a factor of r .

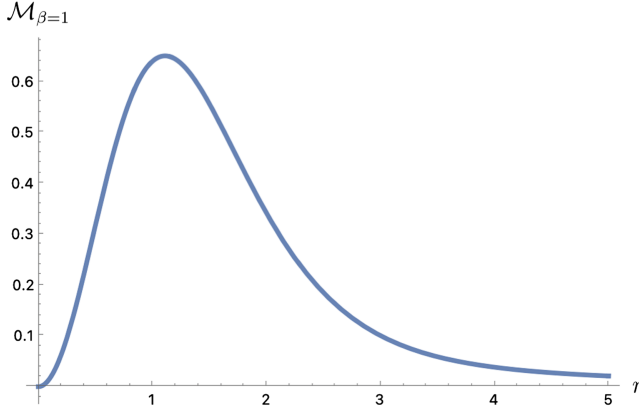


FIG. 4. Integrand of the monopole mass (4) for $\beta = 1$. This is an intermediate curve between the $\beta \rightarrow 0$ and $\beta \rightarrow \infty$ plots presented in Fig. 2 of [12].

$$j_0(r) = \mu + \kappa r; \quad w_0(r) = 0; \quad h_0(r) = 1. \quad (64)$$

Here μ and κ are two new parameters corresponding to the electric charge and mass scale of J , respectively. The fact that $J(r)$ is linear in r follows from the fact that $W(r)$ vanishes at zeroth exponential order, so the J equation (61) implies that $J'' \sim 0$ at this order. The monopole solutions are recovered from the dyon solutions in the limit $\mu \rightarrow 0$ and $\kappa \rightarrow 0$.

B. First exponential order

At first exponential order, the appearance in $j_0(r)$ of the new mass parameter κ , which is restricted by the boundary conditions to the range $0 \leq \kappa < 1$, generates a new exponential term $e^{-r\sqrt{1-\kappa^2}}$, which dominates over e^{-r} at large r in this physical regime. This has the consequence that for generic values of κ the transseries *Ansatz* is generated by two independent exponential factors: $e^{-r\sqrt{1-\kappa^2}}$ and e^{-r} .²

At first exponential order, we find that these two exponential terms enter the solutions of the dyon equations (61)–(63) in a different manner. We first adopt a general one-exponential *Ansatz*

$$J = \mu + \kappa r + e^{-r\sqrt{1-\kappa^2}} j_{1,0}(r) + e^{-r} j_{0,1}(r) + \dots, \quad (65)$$

$$W = e^{-r\sqrt{1-\kappa^2}} w_{1,0}(r) + e^{-r} w_{0,1}(r) + \dots, \quad (66)$$

$$H = 1 - e^{-r\sqrt{1-\kappa^2}} h_{1,0}(r) - e^{-r} h_{0,1}(r) + \dots \quad (67)$$

where the subscripts on the functions $j_{k,\ell}(r)$, $w_{k,\ell}(r)$, and $h_{k,\ell}(r)$ indicate the associated factors of each of the two exponentials as: $e^{-kr\sqrt{1-\kappa^2}-\ell r}$.

²Of course, additional special features may arise in nongeneric resonant cases when $\sqrt{1-\kappa^2}$ is rational.

Interestingly, when the general *Ansatz* in (65)–(67) is inserted into the dyon equations, we find that many of the terms do not actually appear. For example, the J equation yields

$$j_{1,0}(r) = (c_1 + c_2 r) e^{r\sqrt{1-\kappa^2}}, \quad j_{0,1}(r) = (d_1 + d_2 r) e^{-r} \quad (68)$$

for arbitrary constants c_1, c_2, d_1, d_2 . This means that in fact the leading term $j_0(r) = \mu + \kappa r$ is sufficient to first exponential order. Similarly, for $W(r)$ we find that the term $e^{-r} w_{0,1}(r)$ yields exactly the same solution as the term $e^{-r\sqrt{1-\kappa^2}} w_{1,0}(r)$, while for $H(r)$ we find that the term $e^{-r\sqrt{1-\kappa^2}} h_{1,0}(r)$ yields exactly the same solution as the term $e^{-r} h_{0,1}(r)$. This implies that at first exponential order, the transseries structure is in fact much simpler,

$$J = \mu + \kappa r + \dots, \quad (69)$$

$$W = e^{-r\sqrt{1-\kappa^2}} w_{1,0}(r) + \dots, \quad (70)$$

$$H = 1 - e^{-r} h_{0,1}(r) + \dots \quad (71)$$

where the coefficient functions $w_{1,0}(r)$ and $h_{0,1}(r)$ have pure power-law (nonexponential) expansions at large r .

Explicitly, we find that the homogeneous solution $h_{0,1}(r)$ is the same as in the monopole case: recall (24). This can be understood since the coupling term in the H equation is $2HW^2/r^2$, so the other exponential, $e^{-r\sqrt{1-\kappa^2}}$, first enters as $e^{-2r\sqrt{1-\kappa^2}}$, which is at the next exponential order. For the W equation, the new nonlinearity of the dyon system is $-WJ^2/r^2$, which introduces the nonexponential factor $j_0(r)^2 = (\mu + \kappa r)^2$. This has the consequence that the monopole case Bessel equation (17) becomes a confluent hypergeometric equation,

$$w''_{1,0}(r) - 2\sqrt{1-\kappa^2} w'_{1,0}(r) + \frac{1 + (\mu + \kappa r)^2 - \kappa^2 r^2}{r^2} w_{1,0}(r) = 0. \quad (72)$$

The solution can be written as a linear combination of Whittaker functions,

$$w_{1,0}(r) = c_1 e^{r\sqrt{1-\kappa^2}} M_{\alpha,\delta}(2r\sqrt{1-\kappa^2}) + c_2 e^{r\sqrt{1-\kappa^2}} W_{\alpha,\delta}(2r\sqrt{1-\kappa^2}). \quad (73)$$

Here $\alpha = \kappa\mu/\sqrt{1-\kappa^2}$, and $\delta = i\sqrt{3+4\mu^2}/2$. The regular real solution is the second term in (73). In the monopole limit, $\kappa, \mu \rightarrow 0$, the Whittaker functions reduce to the monopole case modified Bessel functions in (19).

C. Second exponential order and beyond

At second exponential order, for generic κ we should allow for three different kinds of exponential factors,

$$e^{-2r\sqrt{1-\kappa^2}}, \quad e^{-r(1+\sqrt{1-\kappa^2})}, \quad e^{-2r} \quad (74)$$

Inserting a transseries *Ansatz* containing these three exponential terms, we find once again that not all exponentials enter each of the functions. Explicitly we find that the required transseries *Ansatz* contains only the following terms:

$$J(r) = \mu + \kappa r + e^{-2r\sqrt{1-\kappa^2}} j_{2,0}(r) + \dots, \quad (75)$$

$$W(r) = e^{-r\sqrt{1-\kappa^2}} w_{1,0}(r) + e^{-2r\sqrt{1-\kappa^2}} w_{2,0}(r) + e^{-r(1+\sqrt{1-\kappa^2})} w_{1,1}(r) + \dots, \quad (76)$$

$$H(r) = 1 - e^{-r} h_{0,1}(r) - e^{-2r\sqrt{1-\kappa^2}} h_{2,0}(r) - e^{-2r} h_{0,2}(r) + \dots. \quad (77)$$

Note that $J(r)$ only contains $e^{-2r\sqrt{1-\kappa^2}}$, but not $e^{-r(1+\sqrt{1-\kappa^2})}$ or e^{-2r} , while $W(r)$ contains $e^{-2r\sqrt{1-\kappa^2}}$ and $e^{-r(1+\sqrt{1-\kappa^2})}$, but not e^{-2r} . And $H(r)$ contains $e^{-2r\sqrt{1-\kappa^2}}$ and e^{-2r} , but not $e^{-r(1+\sqrt{1-\kappa^2})}$. Only $W(r)$ contains the “mixed” exponential term $e^{-r(1+\sqrt{1-\kappa^2})}$. However, because of the cubic nonlinearities in the dyon equations, starting at third exponential order there will appear exponential terms involving powers of both e^{-r} and $e^{-r\sqrt{1-\kappa^2}}$.

Therefore, at all higher orders, each of J , W , and H will contain terms of some mixed powers, resulting in a more intricate general transseries structure,

$$J(r) = \mu + \kappa r + \sum_{k=2}^{\infty} \sum_{\ell=0}^k e^{-kr\sqrt{1-\kappa^2}-\ell r} j_{k,\ell}(r), \quad (78)$$

$$W(r) = \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} e^{-kr\sqrt{1-\kappa^2}-\ell r} w_{k,\ell}(r), \quad (79)$$

$$H(r) = 1 - \sum_{\ell=0}^{\infty} \sum_{k=0, k \neq 1}^{\infty} e^{-kr\sqrt{1-\kappa^2}-\ell r} h_{k,\ell}(r). \quad (80)$$

Inserting this transseries *Ansatz* into the dyon equations (61)–(63), the tower of linear inhomogeneous equations for all exponential orders can be solved iteratively in the same manner as for the monopole case, in terms of lower order solutions and Wronskians.

VII. CONCLUSION

We have shown that non-BPS monopole and dyon solutions to the classical equations of motion for the $SU(2)$ 't Hooft-Polyakov have a natural construction in the far field regime as resurgent transseries. Increasing the exponential order of the transseries extends the precision into the near field regime. This method also applies to other well-known sets of coupled nonlinear equations arising in quantum field theory, away from special parametric limits, such as general kink, vortex, false vacuum decay, and instanton equations, as well as in nonlinear equations in general relativity. An immediate consequence of our result is that the zero-mode solutions, arising for fermions in background monopole fields [23], also have resurgent transseries expansions, involving exponentials of transseries. Current and future work will address the connection problem, relating the far field region to the near field region, as well as the general fluctuation problem. It would also be interesting to study the transseries structure of associated soliton fields [24], and quantum corrections to the monopole mass [25,26].

Note added. Recently, a paper appeared on the arXiv concerning some other resurgent features of the monopole equations, based on the large β limit [27].

ACKNOWLEDGMENTS

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics under Award No. DE-SC0010339.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

- [1] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, International Series of Monographs on Physics (Oxford University Press, New York, 2025), Vol. 77.
- [2] V. A. Rubakov, *Classical Theory of Gauge Fields* (Princeton University Press, Princeton, NJ, 2002).
- [3] N. S. Manton and P. Sutcliffe, *Topological Solitons*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2004).
- [4] E. J. Weinberg, *Classical Solutions in Quantum Field Theory: Solitons and Instantons in High Energy Physics*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2012).
- [5] G. 't Hooft, Magnetic monopoles in unified gauge theories, *Nucl. Phys.* **B79**, 276 (1974).
- [6] A. M. Polyakov, Particle spectrum in quantum field theory, *JETP Lett.* **20**, 194 (1974).
- [7] M. K. Prasad and C. M. Sommerfield, An exact classical solution for the 't Hooft monopole and the Julia-Zee Dyon, *Phys. Rev. Lett.* **35**, 760 (1975).
- [8] E. B. Bogomolny, Stability of classical solutions, *Sov. J. Nucl. Phys.* **24**, 449 (1976).
- [9] B. Julia and A. Zee, Poles with both magnetic and electric charges in non-Abelian gauge theory, *Phys. Rev. D* **11**, 2227 (1975).
- [10] E. B. Bogomolny and M. S. Marinov, Calculation of the monopole mass in gauge theory, *Yad. Fiz.* **23**, 676 (1976).
- [11] F. A. Bais and J. R. Primack, Integral equations for extended solutions in field theory: Monopoles and dyons, *Phys. Rev. D* **13**, 819 (1976).
- [12] T. W. Kirkman and C. K. Zachos, Asymptotic analysis of the monopole structure, *Phys. Rev. D* **24**, 999 (1981).
- [13] C. L. Gardner, The 't Hooft-Polyakov monopole near the Prasad-Sommerfield limit, *Ann. Phys. (Amsterdam)* **146**, 129 (1983).
- [14] P. Forgács, N. Obadia, and S. Reuillon, Numerical and asymptotic analysis of the 't Hooft-Polyakov magnetic monopole, *Phys. Rev. D* **71**, 035002 (2005).
- [15] O. Costin, On Borel summation and Stokes phenomena of nonlinear differential systems, *Duke Math. J.* **93**, 289 (1998).
- [16] O. Costin, *Asymptotics and Borel Summability* (Chapman-Hall, London, 2008).
- [17] M. V. Berry and C. J. Howls, Hyperasymptotics for integrals with saddles, *Proc. R. Soc. A* **434**, 657 (1991).
- [18] M. Mariño, Lectures on non-perturbative effects in large N gauge theories, matrix models and strings, *Fortschr. Phys.* **62**, 455 (2014).
- [19] I. Aniceto, G. Basar, and R. Schiappa, A primer on resurgent transseries and their asymptotics, *Phys. Rep.* **809**, 1 (2019).
- [20] G. V. Dunne, Introductory lectures on resurgence: CERN Summer school 2024, *Eur. Phys. J. Special Topics* **02**, 21 (2026).
- [21] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory* (Springer, New York, 1999), Vol. 1.
- [22] E. Witten, Dyons of charge $e\theta/2\pi$, *Phys. Lett.* **86B**, 283 (1979).
- [23] R. Jackiw and C. Rebbi, Solitons with Fermion number $1/2$, *Phys. Rev. D* **13**, 3398 (1976).
- [24] N. S. Manton, An effective Lagrangian for solitons, *Nucl. Phys.* **B150**, 397 (1979).
- [25] V. G. Kiselev and K. G. Selivanov, Quantum corrections to monopole mass, *Phys. Lett. B* **213**, 165 (1988).
- [26] A. Rajantie, Mass of a quantum 't Hooft-Polyakov monopole, *J. High Energy Phys.* **01** (2006) 088.
- [27] M. Malinský, Resurgent structure of the 't Hooft-Polyakov monopole, [arXiv:2602.14620](https://arxiv.org/abs/2602.14620).