

Generalized Krylov complexity

Amin Faraji Astaneh^{1,2,*} and Niloofar Vardian^{2,†}

¹*Department of Physics, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran*

²*Research Center for High Energy Physics, Department of Physics, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran*



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We extend the concept of Krylov complexity to encompass general unitary evolutions involving multiple generators. This generalization enables us to formulate a framework for generalized Krylov complexity, which serves as a measure of the complexity of states associated with continuous symmetries within a model. Furthermore, we investigate scenarios in which different directions of transformation lead to varying degrees of complexity, in close analogy with geometric approaches to complexity, such as Nielsen's framework. In this context, we introduce a generalized orthogonalization algorithm and delineate its computational structure, which is organized as a network of orthogonal blocks rather than a simple linear chain. Finally, we present explicit evaluations of several illustrative examples to demonstrate the practical application of this framework.

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I. INTRODUCTION

The idea of quantum complexity has attracted significant interest in quantum information science. Researchers from a variety of physics disciplines, ranging from quantum information theory to high-energy physics, have made concerted efforts to rigorously define this concept and calculate it for different quantum systems. The prevailing understanding is that quantum complexity quantifies the size of a quantum circuit or the cost of an algorithm required to evolve an initial reference state into a desired quantum state through unitary operations.

One framework where this concept is rigorously defined is Krylov complexity [1], commonly referred to as spread complexity for quantum states rather than operators [2]. Based on this prescription, we examine a state under study as the result of the time evolution of an initial reference state $|\psi_t\rangle = e^{-iHt}|\psi_0\rangle$. We consider the deviation from the initial state at time zero along a trajectory obtained by solving the Schrödinger equation. This evolution can be expressed by expanding the time evolution operator in time:

$$|\psi_t\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} |\psi_n\rangle, \quad |\psi_n\rangle = H^n |\psi_0\rangle. \quad (1)$$

Intuitively, the rays $|\psi_n\rangle$ can form a vector space whose size is related to the distance between the final state $|\psi_t\rangle$ and the initial reference state $|\psi_0\rangle$. This naturally brings the concept of complexity to mind.

To be more precise, constructing the building blocks within this framework requires selecting only linearly independent vectors. This can be accomplished using the standard Gram-Schmidt orthogonalization procedure, which is referred to as the Lanczos algorithm in this context. By applying this method, we arrive at the Krylov basis, which constitutes a vector space known as the Krylov space. More precisely, one must consider $\text{Span}\{H^n|\psi_0\rangle\}$ and apply the Gram-Schmidt procedure to obtain the corresponding orthonormal basis denoted by $|K_n\rangle$. By collecting these vectors, one constructs the Krylov space of dimension $d_K \leq d_{\mathcal{H}}$, where $\mathcal{H}$ represents the entire Hilbert space. The desired state vector can now be expanded in terms of this basis as follows:

$$|\psi_t\rangle = \sum_{n=0}^{d_K-1} \phi_n(t) |K_n\rangle. \quad (2)$$

The complexity can now be quantified based on the distribution of basis vectors needed to construct the desired state. The more intricate this distribution is, the more complex the state becomes. This motivates us to define the complexity of a state as follows:

$$\mathcal{C}_{|\psi_t\rangle} = \sum n |\phi_n(t)|^2. \quad (3)$$

The intuition behind this definition is that, since $|\phi_n(t)|^2$ represents the probabilities of finding the evolved state in the

* Contact author: faraji@sharif.ir

† Contact author: niloofar.vardian72@sharif.edu

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n th Krylov basis element, the Krylov complexity can be understood as the expectation value of n weighted by these probabilities. This is similar to calculating an average depth in the Krylov chain, which measures how much the state has spread.

Krylov complexity has been widely explored in quantum chaotic systems [2–18], many-body systems, and conformal field theories [19–21]. Beyond chaos, it has proven instrumental in studying topological and quantum phase transitions [22–24], quantum field theory [25–36], thermalization [37], and open quantum systems [38–43]. For a review that compiles many of these studies, see Refs. [44–46].

What we aim to investigate in this paper arises from a natural question: why can we not generalize this concept to other unitary evolutions rather than just time evolution? In this paper, we aim to provide a comprehensive answer to this question. We explore a fully general unitary transformation characterized by multiple generators and transformation parameters. Our goal is to extend the concepts of Krylov complexity and possibly the Lanczos algorithm to accommodate these more general cases. The result of our work is the introduction of a Krylov network, instead of a Krylov chain. In doing so, we define generalized Krylov complexity and compute it for specific examples, focusing on both Abelian and non-Abelian Lie algebras.

II. GENERALIZED KRYLOV COMPLEXITY

To develop our idea of generalizing Krylov complexity, we consider the most general possible case. In this context, we take an orthogonal set of N Hermitian generators $\{T_i\}$ that form a general Lie algebra $[T_i, T_j] = if_{ij}^k T_k$. These operators can be designated as generators of unitary evolution operators, which evolve a given base state $|\psi_0\rangle$ to

$$|\psi\{\alpha_k\}\rangle = \exp\left(\sum_{k=1}^N i\alpha_k T_k\right)|\psi_0\rangle. \quad (4)$$

In a loose sense, this set of unitary operators defines a circuit through which one can build a ray $|\psi\{\alpha_k\}\rangle$ starting from an initial state. The main question is what the Krylov complexity of the final state is, which is naturally closely linked to the complexity of the resulting circuit.

In the following, we briefly introduce how to compute the generalized Krylov complexity. However, we postpone the details and intermediate steps to the next section, where we will proceed algorithmically, starting from the reference state, step by step and block by block.

By expanding the right-hand side of (4), a Krylov basis can be derived as a collection of blocks resulting from block decomposition as

$$\mathcal{K} = \bigoplus_{n=0}^{N_B-1} \mathcal{K}_n, \quad (5)$$

where each of the N_B Krylov blocks, labeled by $\mathcal{K}_n$, is governed by the Gram–Schmidt orthonormalization of its associated span of rays, which reads

$$\text{Span}\{T_1^{k_1} \cdots T_N^{k_N} |\psi_0\rangle | k_1 + \cdots + k_N = n\}, \quad (6)$$

such that $\mathcal{K}_n$ is orthogonal to all $\mathcal{K}_m, m < n$; see Appendix A.

In an appropriate representation, we denote each block as

$$\mathcal{K}_n = \text{Span}\{|K_n^i\rangle, i = 0, \dots, d_{\mathcal{K}_n} - 1\}, \quad (7)$$

where $|K_n^i\rangle$ belongs to the orthonormalized Krylov basis of block n , which forms a vector space of dimension $d_{\mathcal{K}_n}$.

Now, the generic state $|\psi\{\alpha_k\}\rangle$ can be expanded in this basis as

$$|\psi\{\alpha_k\}\rangle = \sum_{n=0}^{N_B-1} \sum_{i=0}^{d_{\mathcal{K}_n}-1} \phi_n^i(\{\alpha_k\}) |K_n^i\rangle. \quad (8)$$

The dimension of the Krylov space as a whole equals $d_{\mathcal{K}} = N_B d_{\mathcal{K}_n}$. We then propose the most natural generalization of the standard Krylov complexity as follows:

$$\mathcal{C}_{|\psi\{\alpha_k\}\rangle} = \sum_{n=0}^{N_B-1} \sum_{i=0}^{d_{\mathcal{K}_n}-1} n |\phi_n^i(\{\alpha_k\})|^2. \quad (9)$$

Note that for the standard Krylov complexity there are N_B blocks of dimension $d_{\mathcal{K}_n} = 1$. Furthermore, since all vectors within a given block are linear combinations of vectors generated by applying the same number of generators to the initial state, we can assume that they are equidistant from the initial state [47]. Thus, only the index n should be present as a coefficient in the definition of generalized Krylov complexity, and the order in which the Gram–Schmidt procedure is applied within each block is irrelevant.

It is important to note that when we generalize the Krylov complexity we do not necessarily mean that we are only enlarging the basis space. In fact, the resulting circuit might not be homogeneous in its generator space. Instead, one might define a weighted generalized Krylov complexity by assigning a weight function to the Krylov space. In this sense, some blocks may contribute more effectively to preparing a state than others. To incorporate this, we start with

$$\text{Span}\{T_1^{k_1} \cdots T_N^{k_N} |\psi_0\rangle | \mu_1 k_1 + \cdots + \mu_n k_N = n\}, \quad (10)$$

to define the weighted Krylov blocks. The distribution of μ_k values defines the geometry of a circuit as a union of blocks, which requires further investigation. An important question then is which distribution of the weight function

leads to an optimal state preparation. We will address this in the following sections.

A crucial step in calculating the generalized Krylov complexity is designing an algorithm to identify the Krylov blocks and to cover the Krylov space. This process is what we refer to as the generalized orthogonalization algorithm. The following section is dedicated to this.

III. GENERALIZED ORTHOGONALIZATION ALGORITHM

In this section, we aim to generalize the orthogonalization algorithm for the problem of general unitary evolution. However, due to the complexity involved, we will not present the algorithm for deriving the Lanczos coefficients in its most general form. Instead, we will focus on formulating an algorithm based on the projection onto the different blocks. We begin with the initial state $|\psi_0\rangle$ and have a set of generators $\{T_k, k = 1, \dots, N\}$. For now, we assume that all generators are equally weighted. What we require is to determine the projections onto different Krylov blocks, step by step.

The zeroth block is generated from the initial state and is always one dimensional, represented as

$$\mathcal{K}_0 = \text{span}\{|K_0^0\rangle = |\psi_0\rangle\}. \quad (11)$$

The projection onto $\mathcal{K}_0$ is denoted by

$$P_0 = |K_0^0\rangle\langle K_0^0| = |\psi_0\rangle\langle\psi_0|. \quad (12)$$

In the next step, we need to construct the first block from the set of states given by

$$\{T_1|\psi_0\rangle, T_2|\psi_0\rangle, \dots, T_N|\psi_0\rangle\}, \quad (13)$$

such that $\mathcal{K}_1 \perp \mathcal{K}_0$. To proceed with building $\mathcal{K}_1$, we first need to remove the components of all vectors in the above set along the subspace $\mathcal{K}_0$. For each vector $|v\rangle$, this can be achieved by replacing it with the vector

$$|v'\rangle = (I - P_0)|v\rangle, \quad (14)$$

since $P_0|v'\rangle = 0$. Consequently, the first block can be constructed from the set

$$\text{Span}\{(I - P_0)T_1|\psi_0\rangle, \dots, (I - P_0)T_N|\psi_0\rangle\}. \quad (15)$$

However, these vectors are not yet orthonormal. Therefore, we need to apply the Gram-Schmidt orthogonalization to obtain

$$\mathcal{K}_1 = \text{Span}\{|K_1^0\rangle, |K_1^1\rangle, \dots, |K_1^{d_{\mathcal{K}_1}-1}\rangle\}. \quad (16)$$

For a generic $\mathcal{K}_n$, we will similarly apply the Gram-Schmidt orthogonalization to

$$\text{Span}\left\{\left(I - \sum_{i=0}^{n-1} P_i\right) T_1^{k_1} T_2^{k_2} \dots T_N^{k_N} |\psi_0\rangle\right\}, \quad (17)$$

where $k_1 + k_2 + \dots + k_N = n$ and P_i is the projection onto the block $\mathcal{K}_i$. This process yields the generalized Krylov basis $|K_n^i\rangle$, ensuring that the blocks remain mutually orthogonal.

We must continue this procedure until we reach the point where

$$\mathcal{K}_i = 0, \quad \forall i \geq N_B. \quad (18)$$

Finally, the generalized Krylov complexity can be derived from Eq. (9), where $\phi_n(\{\alpha_k\}) = \langle K_n^i | \psi\{\alpha_k\} \rangle$. We can express the complexity defined in (9) as

$$\begin{aligned} \mathcal{C}_{|\psi\{\alpha_k\}\rangle} &= \sum_{n=0}^{N_B-1} \sum_{i=0}^{d_{\mathcal{K}_n}-1} n \langle \psi\{\alpha_k\} | K_n^i \rangle \langle K_n^i | \psi\{\alpha_k\} \rangle \\ &= \langle \psi\{\alpha_k\} | \sum_{n=0}^{N_B-1} n P_n | \psi\{\alpha_k\} \rangle, \end{aligned} \quad (19)$$

where $P_n = \sum_{i=0}^{d_{\mathcal{K}_n}-1} |K_n^i\rangle\langle K_n^i|$ is the projection onto the n th block. Therefore, the generalized Krylov complexity can be interpreted as the expectation value of the generalized Krylov operator defined as

$$\hat{K}_{|\psi_0\rangle} = \sum_{n=0}^{N_B-1} n P_n, \quad (20)$$

on the state $|\psi\{\alpha_k\}\rangle$.

It is worth noting that, without using the Gram-Schmidt procedure, one can alternatively determine the projection onto the subspace $V = \text{span}\{|v_i\rangle\}$ as follows (see Ref. [48] for more details). First, we introduce the metric tensor defined by

$$G = [g_{ij}] = [\langle v_i | v_j \rangle], \quad (21)$$

which has an inverse denoted as $G^{-1} = [g^{ij}]$. The projection onto the subspace V can then be expressed as

$$P_V = \sum_{i,j} g^{ij} |v_i\rangle\langle v_j|. \quad (22)$$

In our case, to find the projection onto $\mathcal{K}_n$, we can define the metric tensor for each block as

$$\begin{aligned} G_n &= [g(n)_{\{k_i\};\{k'_i\}}] \\ &= \left[\langle \psi_0 | T_N^{k_N} \dots T_1^{k_1} \left(I - \sum_{i=0}^{n-1} P_i \right) T_1^{k'_1} \dots T_N^{k'_N} | \psi_0 \rangle \right], \end{aligned} \quad (23)$$

where $n = \sum k_i = \sum k'_i$. Therefore, the projections in our problem can be derived from Eq. (22) by substituting Eq. (17) for the constituent vectors. And finally, the generalized Krylov operator reads as

$$\hat{K}_{|\psi_0\rangle} = \sum_{n=0}^{N_B-1} n \sum_{\{k_i\};\{k'_i\}} g(n)^{\{k_i\};\{k'_i\}} T_1^{K_1} \dots T_N^{K_N} |\psi_0\rangle \times \langle \psi_0 | T_N^{K'_N} \dots T_1^{K'_1}, \quad (24)$$

where $G_n^{-1} = [g(n)^{\{k_i\};\{k'_i\}}]$. It is important to note that if there is redundancy in the set described by Eq. (17) we must restrict our analysis to the subspace orthogonal to the kernel of G_n . Lastly, one can find the generalized Krylov operator directly using the form of the projectors obtained here. Additionally, we point out that the procedure of finding the blocks is repeated in this approach until reaching the point where $G_i = 0$ for all $i \geq N_B$. Then, the dimension of the entire Krylov basis can be found as

$$d_K = \sum_{i=0}^{N_B-1} d_{K_i} = \sum_{i=0}^{N_B-1} \text{tr } P_i. \quad (25)$$

IV. GENERALIZED KRYLOV COMPLEXITY AND SYMMETRIES OF THE SYSTEM: CONCRETE EXAMPLES

The generalized Krylov complexity presented in this work offers a valuable framework for analyzing the evolution of complexity under certain symmetries of a model. When a model is invariant under continuous transformations, the infinitesimal generators of these symmetries form a Lie algebra. Consequently, Lie algebras provide a natural basis for classifying and exploring symmetries across various fields of physics, including quantum mechanics, high-energy physics, and condensed matter systems.

In our framework, if T_k are the infinitesimal generators of a symmetry transformation in a model, the proposed generalized quantity measures the differences in complexity among the various states of the model under this symmetry transformation. This approach is applicable to both Abelian and non-Abelian Lie algebras.

To illustrate how our proposed mechanism operates in its most fundamental form, specifically within Abelian groups, we focus on the Abelian subgroups of $SO(n)$ characterized by the structure $U(1) \times U(1)$. Consequently, our aim is to determine the complexity of a state represented as

$$U(\theta_1, \theta_2) |\psi_0\rangle = e^{i\theta_1 T_1 + i\theta_2 T_2} |\psi_0\rangle. \quad (26)$$

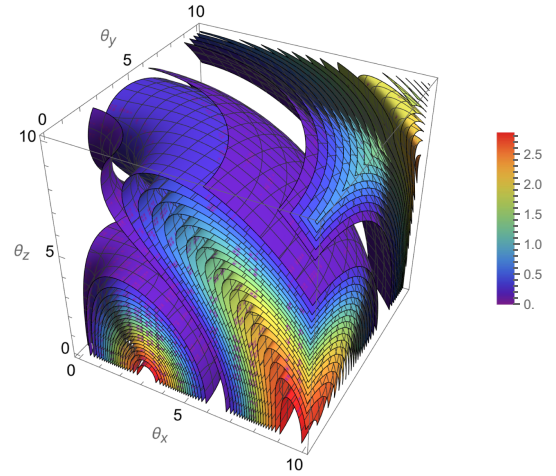
We can simply take $|\psi_0\rangle = \frac{1}{\sqrt{n}}(1\dots 1)$, which is a vector of dimension n [49]. We then proceed to compute the

generalized Krylov complexity, analyzing the Krylov blocks step by step. One can also extend the discussion above to the larger group $U(1)^d$ with d commuting generators.

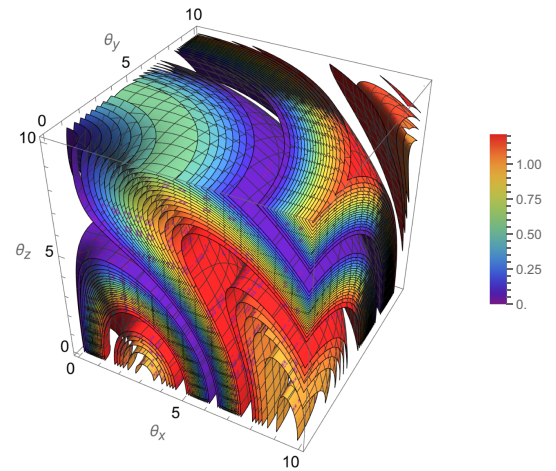
The simplest example involves two independent rotations in two $\mathbb{R}^2$ planes. We consider this symmetry transformation as a subgroup of the rotation group $SO(4)$, parametrized by two angles, θ_1 and θ_2 , acting in two orthogonal planes. The detailed calculations are provided in Appendix B. Here, we only report the final simple result for the generalized Krylov complexity, which reads

$$C_\psi(\theta_1, \theta_2) = 1 - \cos \theta_1 \cos \theta_2. \quad (27)$$

Besides this simple example, to ensure our case is sufficiently illustrative and involves the computation of a significant number of blocks, it is useful to focus on a large



(a) Initial state: $|j = 3/2, m = 3/2\rangle$



(b) Initial state: $|j = 3/2, m = 1/2\rangle$

FIG. 1. The complexity of the initial state in the $j = 3/2$ representation of $SU(2)$, under unitary evolution generated by the $SU(2)$ algebra as a function of $(\theta_x, \theta_y, \theta_z)$.

n case. For this reason, we consider the Abelian subgroup $U(1)_X \times U(1)_Y$ of the group $SO(10)$. In this case, the generators are mathematically identical to those used in Grand Unified Theory (GUT) model building, but we do not propose any specific GUT-related physical interpretation. In Appendix B, one can find the details of the result for this example.

As a non-Abelian symmetry group, we can consider the $SU(2)$ group. In the J representation of $SU(2)$, any initial state $|\psi_0\rangle$ transforms under a general $SU(2)$ transformation as

$$|\psi(\theta_x, \theta_y, \theta_z)\rangle = \exp(-i\vec{\theta} \cdot \vec{J})|\psi_0\rangle. \quad (28)$$

The complexity of $|\psi(\theta_x, \theta_y, \theta_z)\rangle$ in comparison with $|\psi_0\rangle$ can be computed using the generalized Krylov complexity defined here for the generators J_x, J_y , and J_z . More details about this class of symmetry groups can be found in Appendix B. For the particular case where $|\psi_0\rangle = |j, j\rangle$, the generalized Krylov complexity can be obtained as

$$C_\psi(\theta_x, \theta_y, \theta_z) = 2j \sin^2\left(\frac{|\theta|}{2}\right) \frac{|\theta|^2 - \theta_z^2}{|\theta|^2}. \quad (29)$$

In Fig. 1, one can observe the results for the generalized Krylov complexity of the initial states $|j = 3/2, m = 3/2\rangle$ and $|j = 3/2, m = 1/2\rangle$ under evolution generated by the $SU(2)$ algebra.

V. WEIGHTED GENERALIZED KRYLOV COMPLEXITY

The generalized Krylov complexity defined in Sec. II fails to recognize that some unitary transformations are more difficult to implement than others, as it does not discriminate between different directions in the manifold of unitaries. In other words, moving in the direction of T_i might be more complex than moving in the direction of T_j .

This idea is explicitly integrated into the Nielsen complexity framework [50–53]. In this geometric approach to quantum complexity, the central concept is that moving in certain directions on the manifold of unitary operators is more complex than moving in others. A key feature of this framework is the definition of a cost functional along a path in the space of unitaries or states, which assigns different weights to various directions, making some paths more costly than others. In this section, we aim to address this issue in the context of generalized Krylov complexity.

To distinguish between different generators, we assign a factor μ_i to each generator T_i . For a state $U(\{\alpha_i\})|\psi_0\rangle$ on the manifold of the corresponding Lie algebra, we need to block decompose the Krylov space and assign a label to each block. In such a weighted case, this label is modified by the factors μ_i ; recall that in the unweighted case we simply had $n = k_1 + k_2 + \dots + k_N$. It is important to note

that these parameters should be increasing: as we apply a fixed generator more frequently, the complexity of the state should increase.

In terms of the factors μ_i , we can define

$$\tilde{n} := \sum_{i=1}^N \mu_i k_i, \quad (30)$$

and one can choose a normalization for the factors μ_i such that $\sum_{i=1}^N \mu_i = N$.

Now, to find the Krylov blocks and calculate the generalized Krylov complexity, one must first sort all possible values for $\tilde{n}$ while ensuring that $k_i \in \mathbb{N}$. Subsequently, these values are considered in ascending order until the entire Krylov space is covered.

For a fixed $\tilde{n}$, we need to construct the corresponding Krylov block $\mathcal{K}_{\tilde{n}}$ by orthogonalizing the set of vectors given by

$$\left\{ T_1^{k_1} \dots T_N^{k_N} |\psi_0\rangle \mid \sum_{i=1}^N \mu_i k_i = \tilde{n} \right\}. \quad (31)$$

It is required that $\mathcal{K}_{\tilde{n}} \perp \mathcal{K}_{\tilde{m}}$ for all $\tilde{m} < \tilde{n}$. By applying the Gram-Schmidt procedure, one can determine the block decomposition of the Krylov space as $\mathcal{K} = \bigoplus_{\tilde{n}} \mathcal{K}_{\tilde{n}}$, where

$$\mathcal{K}_{\tilde{n}} = \text{Span}\{|K_{\tilde{n}}^i\rangle, i = 0, \dots, d_{\mathcal{K}_{\tilde{n}}} - 1\}. \quad (32)$$

The weighted generalized Krylov complexity can then be expressed as

$$C_{|\psi\{\alpha_k\}\rangle} = \sum_{\tilde{n}} \sum_{i=0}^{d_{\mathcal{K}_{\tilde{n}}}-1} \tilde{n} |\phi_{\tilde{n}}^i(\{\alpha_k\})|^2, \quad (33)$$

where $\phi_{\tilde{n}}^i(\{\alpha_k\}) = \langle K_{\tilde{n}}^i | \psi\{\alpha_k\} \rangle$. The generalized orthogonalization algorithm discussed in Sec. III can be naturally extended to the weighted case.

As a simple example, we revisit the scenario of two independent rotations in four dimensions, this time with two unequal weights. For this case, we select $\mu_1 = \frac{1}{2}$ and $\mu_2 = \frac{3}{2}$. The details for calculating the weighted complexity are provided in Appendix B. Here, we quote the final result, which is given by

$$\tilde{C}_\psi(\theta_1, \theta_2) = \frac{1}{4}(3 - 2 \cos \theta_1 \cos \theta_2 - \cos 2\theta_2). \quad (34)$$

It is evident that the previous symmetry between the two rotations has been disrupted due to the unequal weights assigned in the corresponding directions. Referring to the more complex example introduced earlier, we can compare the generalized Krylov complexity results for the $U(1)_X \times U(1)_Y$ group in both weighted and non-weighted cases. A more detailed discussion of these cases can also be found in Appendix B.

VI. CONCLUSIONS

In this paper, we have expanded the concept of Krylov complexity to include a general unitary transformation with multiple generators. Our goal has been to demonstrate, inspired by Krylov complexity, how complex a state can become when constructed from a completely general unitary transformation of a reference state. Our analysis is conducted using a fully general Lie algebra, allowing us to measure the difference in complexity between states related by the continuous symmetry of the corresponding Lie algebra in a model. We have introduced the generalized Krylov complexity in a broad context and tested it with specific Abelian and non-Abelian groups. Additionally, we explore an interesting generalization by considering weighted complexity, taking into account different weights and contributions in various directions of transformation.

Specifically, by utilizing this weighted complexity, one can develop a geometric representation of Krylov complexity. A general unitary transformation in the parameter space, which acts in a weighted manner along various directions, resembles the concept of a quantum circuit that enables the preparation of a general state from a reference state. While this connection appears intuitive and is closely related to established concepts of circuit complexity, enhancing its precision could yield particularly interesting insights. See Refs. [50–60] for relevant references.

Moreover, our approach provides a natural generalization of Krylov complexity to open quantum systems described by Lindblad dynamics [40,61,62]. When the Lindblad master equation can be formulated using the generators of a Lie algebra, this structure offers a cohesive perspective on operator growth and complexity within dissipative environments.

Our approach may facilitate the exploration of previously unexamined Krylov complexity in systems with time-dependent Hamiltonians, which is an important direction of study. The concept can be naturally applied to cases where the time-dependent Hamiltonian can be expressed as a linear combination of the generators of a Lie algebra. It is essential to consider that all coefficients α_i are functions of time, which means that the generalized Krylov complexity can be expressed as a function of time as well. It is important to be cautious about the time ordering in the definition of time-evolved states when calculating complexity. Furthermore, our approach can be extended to cases where $H = f(\{T_i\})$, allowing us to analyze a broader class of models. For a similar discussion on using Lie algebras to address the complexity of time-dependent Hamiltonians, one can see Ref. [63]. We plan to explore these topics in our future research.

We also note the potential to interpret the generalized Liouvillian operator from two Hamiltonians, such as in a two-sided anti-de Sitter black hole with matter, where it is essential to clarify how to implement evolution with respect to the asymptotic boundaries. This interpretation has been

explored in some papers, including Refs. [64–66], using this framework.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: KRYLOV BLOCKS

In Sec. II, we consider an orthonormal set of N Hermitian generators T_i such that the trace satisfies $\text{Tr}(T_i^\dagger T_j) = \delta_{ij}$. These generators form a general Lie algebra, characterized by the commutation relation $[T_i, T_j] = if_{ij}^k T_k$. We begin with a generic initial state $|\psi_0\rangle$ located at the origin of the manifold of vectors, which are derived from the initial state through the unitary evolution associated with the corresponding Lie group

$$|\psi\{\alpha_k\}\rangle = \exp\left(\sum_{k=1}^N i\alpha_k T_k\right)|\psi_0\rangle. \quad (\text{A1})$$

We can expand the evolved state by using the Taylor expansion as

$$\begin{aligned} |\psi\{\alpha_k\}\rangle &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\sum_k \alpha_k T_k\right)^n |\psi_0\rangle \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \alpha_{i_1} \dots \alpha_{i_n} T_{i_1} \dots T_{i_n} |\psi_0\rangle. \end{aligned} \quad (\text{A2})$$

Using the commutation relation between the generators of the algebra, one can rewrite it as

$$\begin{aligned} |\psi\{\alpha_k\}\rangle &= \sum_{n=0}^{\infty} \sum_{k_1 + \dots + k_N = n} \frac{i^n}{n!} A(i_1, \dots, i_n; \{T_j\}) \\ &\quad \times \alpha_1^{k_1} \dots \alpha_N^{k_N} T_1^{k_1} \dots T_N^{k_N} |\psi_0\rangle, \end{aligned} \quad (\text{A3})$$

where $A(i_1, \dots, i_n; \{T_j\})$ comes from the commutation relation between the generators.

As an example, consider a scenario involving three generators, denoted as (T_1, T_2, T_3) . At the $n = 0$ term in the Taylor expansion, we have only the state $|\psi_0\rangle$, which contributes to $\mathcal{K}_0$. Proceeding to the next term for $n = 1$, we obtain the set of states $\{T_1|\psi_0\rangle, T_2|\psi_0\rangle, T_3|\psi_0\rangle\}$. The first Krylov block can be constructed by applying the Gram-Schmidt procedure to this set, ensuring that the resulting block is orthogonal to $\mathcal{K}_0$.

For the next term corresponding to $n = 2$ in the deformed version of the Taylor expansion of the evolved

state, we need to determine the set of states that will form the second Krylov block. To do this, we only need to consider at most the following vectors:

$$\{T_1^2|\psi_0\rangle, T_1T_2|\psi_0\rangle, T_1T_3|\psi_0\rangle, T_2^2|\psi_0\rangle, T_2T_3|\psi_0\rangle, T_3^2|\psi_0\rangle\}. \quad (\text{A4})$$

It is essential to check that these terms are orthogonal not only to $\mathcal{K}_0$ but also to $\mathcal{K}_1$. For example, the vector $T_2T_1|\psi_0\rangle$ need not be evaluated separately, as it can be expressed as

$$T_2T_1|\psi_0\rangle = T_1T_2|\psi_0\rangle - if_{12}^3T_3|\psi_0\rangle. \quad (\text{A5})$$

Here, $T_1T_2|\psi_0\rangle$ is already included in our set of vectors, while $T_3|\psi_0\rangle$ belongs to $\mathcal{K}_1$. Hence, we can confidently remove it during the orthogonalization process.

Consequently, to construct the n th block $\mathcal{K}_n$, it suffices to consider the set of vectors given by

$$\text{Span}\{T_1^{k_1} \cdots T_N^{k_N}|\psi_0\rangle | k_1 + k_2 + \cdots + k_N = n\}. \quad (\text{A6})$$

We derive $\mathcal{K}_n$ through the Gram-Schmidt procedure applied to this set of vectors while maintaining the criterion that $\mathcal{K}_n \perp \mathcal{K}_m$ for all $m < n$. For more detailed information regarding the construction of the various Krylov blocks, refer to Sec. III.

APPENDIX B: EXPLICIT EXAMPLES

In this appendix, we analyze the Krylov complexity under the action of an Abelian group, the elements of $U(1) \times U(1)$. In the case of two rotations in two $\mathbb{R}^2$ planes, the transformation matrices are defined as follows:

$$T_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \quad (\text{B1})$$

The initial state vector is chosen as $|\psi_0\rangle = \frac{1}{2}(1111)$. After a straightforward calculation, the Krylov operator is determined to be

$$\hat{K}_{|\psi_0\rangle} = \frac{1}{2} \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}. \quad (\text{B2})$$

Thus, the complexity function can be expressed as

$$\mathcal{C}_\psi(\theta_1, \theta_2) = 1 - \cos \theta_1 \cos \theta_2. \quad (\text{B3})$$

Now, let us consider that moving in the direction of these two generators does not create the same amount of complexity. First, let us consider the corresponding weights for the generators to be $\mu_1 = \frac{1}{2}, \mu_2 = \frac{3}{2}$. In this case, the weighted generalized Krylov complexity is determined to be

$$\tilde{\mathcal{C}}_\psi(\theta_1, \theta_2) = \frac{1}{4}(3 - 2 \cos \theta_1 \cos \theta_2 - \cos 2\theta_2). \quad (\text{B4})$$

If we set the weights to $\mu_1 = \frac{3}{2}$ and $\mu_2 = \frac{1}{2}$, we need to swap θ_1 and θ_2 in the expression for the weighted generalized Krylov complexity. In Fig. 2, the plot of the generalized Krylov complexity as a function of θ_1 and θ_2 is displayed. For a more complicated example, we consider the $U(1)_X \times U(1)_Y$ subgroup of $SO(10)$ [67–69]. It is crucial to highlight that the specific physical details of this model are not essential for our discussion. What is significant is the presence of a sufficiently large $U(1) \times U(1)$ subgroup, which provides a relevant example to effectively illustrate the subtleties involved in our calculations. Since numerical analyses with different groups and subgroup embeddings [including $SO(4)$ and $SO(10)$] show qualitatively similar behavior, $SO(10)$ is used merely as a representative higher-rank example.

In **10** representation, one has [70,71]

$$X = 2\mathbb{I}_{5 \times 5} \otimes \sigma_2 \quad \text{and} \quad Y = \text{diag}\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -1, -1\right) \otimes \sigma_2, \quad (\text{B5})$$

where $\mathbb{I}_{5 \times 5}$ is the identity matrix and σ_2 is the second Pauli matrix. We also consider the initial state in this basis to be $|\psi_0\rangle = \frac{1}{\sqrt{10}}(111111111)$. We aim to study the generalized Krylov complexity of the states related to $|\psi_0\rangle$ by the unitary elements of the group $U(1)_X \times U(1)_Y$, i.e.,

$$|\psi(\theta_X, \theta_Y)\rangle = \exp[i(\theta_X X + \theta_Y Y)]|\psi_0\rangle. \quad (\text{B6})$$

By performing a similar calculation as was done previously, we find for the equally weighted case where $\mu_X = \mu_Y = 1$

$$\begin{aligned} \mathcal{C}_\psi(\theta_X, \theta_Y) = \frac{1}{50} \bigg\{ & 49 + 2 \cos(4\theta_X - 2\theta_Y) \\ & - 24 \cos\left(4\theta_X - \frac{\theta_Y}{3}\right) - 24 \cos\left(\frac{5\theta_Y}{3}\right) \\ & - 3 \cos\left[\frac{4}{3}(3\theta_X + \theta_Y)\right] \bigg\}. \end{aligned} \quad (\text{B7})$$

In the case of weighted parameters where $\mu_X = \frac{1}{2}, \mu_Y = \frac{3}{2}$, we find that

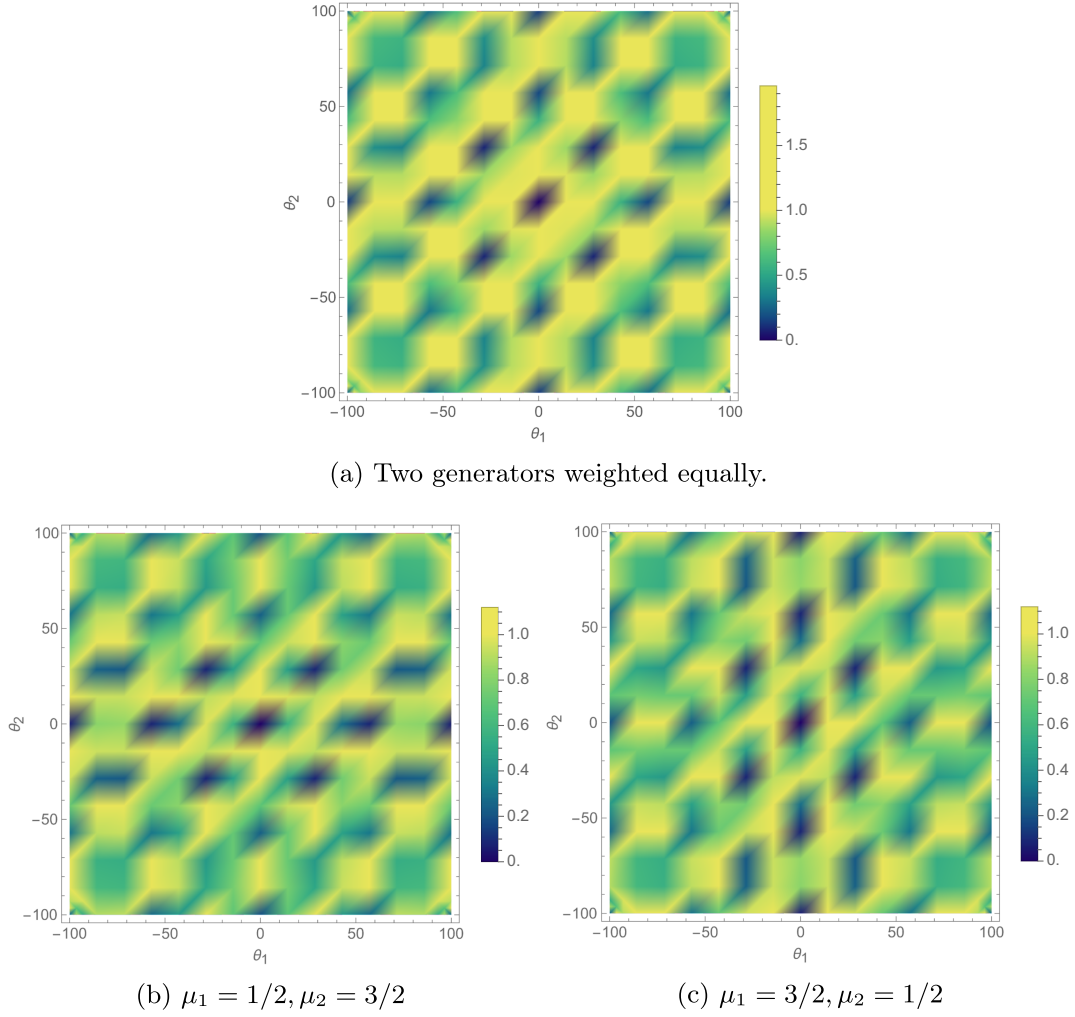


FIG. 2. The generalized Krylov complexity of the initial state $|\psi_0\rangle = \frac{1}{2}(1111)$ under the evolution by the two rotations as a $U(1) \times U(1)$ subgroup of $SO(4)$, as a function of θ_1 and θ_2 .

$$\begin{aligned} \tilde{C}_\psi(\theta_x, \theta_y) = \frac{1}{100} \bigg\{ & 97 + 2 \cos(4\theta_x - 2\theta_y) \\ & - 24 \cos\left(4\theta_x - \frac{\theta_y}{3}\right) - 72 \cos\left(\frac{5\theta_y}{3}\right) \\ & - 3 \cos\left[\frac{4}{3}(3\theta_x + \theta_y)\right] \bigg\}. \end{aligned} \quad (\text{B8})$$

For another example, one can also investigate the $SU(2)$ non-Abelian algebra, whose generators form the Lie algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$. We consider an initial state $|\psi_0\rangle \in \mathcal{H}_j$ and compute its Krylov complexity with respect to the $SU(2)$ generators. The state transformed by a general $SU(2)$ rotation is

$$|\psi(\theta_x, \theta_y, \theta_z)\rangle = \exp(-i\vec{\theta} \cdot \vec{J})|\psi_0\rangle, \quad (\text{B9})$$

where $\vec{J} = (J_x, J_y, J_z)$ and $\vec{\theta} = (\theta_x, \theta_y, \theta_z)$. For simplicity, we choose the initial state to be the highest-weight state $|\psi_0\rangle = |j, j\rangle$ and assume equal weights for all generators

($\mu_x = \mu_y = \mu_z = 1$). Performing similar calculations as before, we obtain the generalized Krylov complexity,

$$\begin{aligned} C_\psi(\theta_x, \theta_y, \theta_z) &= 2j \sin^2\left(\frac{|\theta|}{2}\right) \frac{\theta_x^2 + \theta_y^2}{|\theta|^2} \\ &= 2j \sin^2\left(\frac{|\theta|}{2}\right) \frac{|\theta|^2 - \theta_z^2}{|\theta|^2}, \end{aligned} \quad (\text{B10})$$

where $|\theta| = \sqrt{\theta_x^2 + \theta_y^2 + \theta_z^2}$. In principle, one could derive the generalized Krylov complexity for a general state $|\psi_0\rangle = |j, m\rangle$ or for unequal weights that parametrize rotations about the x , y , and z axes. Although this analysis is quite tedious, the approach is not fundamentally different from what we have demonstrated in the simpler example. Therefore, we will not provide an explicit derivation to keep the discussion concise. Just for a quick comparison, we have plotted the complexity for the state $|j = 3/2, m = 1/2\rangle$ in Fig. 1(b).

- [1] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, A universal operator growth hypothesis, *Phys. Rev. X* **9**, 041017 (2019).
- [2] V. Balasubramanian, P. Caputa, J. M. Magan, and Q. Wu, Quantum chaos and the complexity of spread of states, *Phys. Rev. D* **106**, 046007 (2022).
- [3] A. Faraji Astaneh and N. Vardian, Average spread complexity and the higher-order level spacing, *J. High Energy Phys.* **12** (2025) 128.
- [4] V. Balasubramanian, J. M. Magan, and Q. Wu, Quantum chaos, integrability, and late times in the Krylov basis, *Phys. Rev. E* **111**, 014218 (2025).
- [5] H. Tang, Operator Krylov complexity in random matrix theory, *arXiv:2312.17416*.
- [6] B. Bhattacharjee and P. Nandy, Krylov fractality and complexity in generic random matrix ensembles, *Phys. Rev. B* **111**, L060202 (2025).
- [7] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, Krylov localization and suppression of complexity, *J. High Energy Phys.* **03** (2022) 211.
- [8] G. F. Scialchi, A. J. Roncaglia, and D. A. Wisniacki, Integrability-to-chaos transition through the Krylov approach for state evolution, *Phys. Rev. E* **109**, 054209 (2024).
- [9] A. Gill, K. Pal, K. Pal, and T. Sarkar, Complexity in two-point measurement schemes, *Phys. Rev. B* **109**, 104303 (2024).
- [10] A. Bhattacharya, P. P. Nath, and H. Sahu, Krylov complexity for nonlocal spin chains, *Phys. Rev. D* **109**, 066010 (2024).
- [11] H. A. Camargo, K.-B. Huh, V. Jahnke, H.-S. Jeong, K.-Y. Kim, and M. Nishida, Spread and spectral complexity in quantum spin chains: From integrability to chaos, *J. High Energy Phys.* **08** (2024) 241.
- [12] G. F. Scialchi, A. J. Roncaglia, C. Pineda, and D. A. Wisniacki, Exploring quantum ergodicity of unitary evolution through the Krylov approach, *Phys. Rev. E* **111**, 014220 (2025).
- [13] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, Operator complexity: A journey to the edge of Krylov space, *J. High Energy Phys.* **06** (2021) 062.
- [14] B. Bhattacharjee, P. Nandy, and T. Pathak, Krylov complexity in large q and double-scaled SYK model, *J. High Energy Phys.* **08** (2023) 099.
- [15] N. Hörnedal, N. Carabba, A. S. Matsoukas-Roubeas, and A. del Campo, Ultimate speed limits to the growth of operator complexity, *Commun. Phys.* **5**, 207 (2022).
- [16] J. Erdmenger, S.-K. Jian, and Z.-Y. Xian, Universal chaotic dynamics from Krylov space, *J. High Energy Phys.* **08** (2023) 176.
- [17] S. Chapman, S. Demulder, D. A. Galante, S. U. Sheorey, and O. Shoval, Krylov complexity and chaos in deformed SYK models, *Phys. Rev. B* **111**, 035141 (2025).
- [18] M. Baggioli, K.-B. Huh, H.-S. Jeong, K.-Y. Kim, and J. F. Pedraza, Krylov complexity as an order parameter for quantum chaotic-integrable transitions, *Phys. Rev. Res.* **7**, 023028 (2025).
- [19] B. Bhattacharjee, S. Sur, and P. Nandy, Probing quantum scars and weak ergodicity breaking through quantum complexity, *Phys. Rev. B* **106**, 205150 (2022).
- [20] S. Nandy, B. Mukherjee, A. Bhattacharyya, and A. Banerjee, Quantum state complexity meets many-body scars, *J. Phys. Condens. Matter* **36**, 155601 (2024).
- [21] A. Banerjee, A. Bhattacharyya, P. Drashni, and S. Pawar, From CFTs to theories with Bondi-Metzner-Sachs symmetries: Complexity and out-of-time-ordered correlators, *Phys. Rev. D* **106**, 126022 (2022).
- [22] M. Afrasiar, J. K. Basak, B. Dey, K. Pal, and K. Pal, Time evolution of spread complexity in quenched Lipkin–Meshkov–Glick model, *J. Stat. Mech.* (2023) 103101.
- [23] K. Pal, K. Pal, A. Gill, and T. Sarkar, Time evolution of spread complexity and statistics of work done in quantum quenches, *Phys. Rev. B* **108**, 104311 (2023).
- [24] N. Chakrabarti, N. Nirbhan, and A. Bhattacharyya, Dynamics of monitored SSH model in Krylov space: From complexity to quantum fisher information, *J. High Energy Phys.* **07** (2025) 203.
- [25] H. R. Imani, K. Babaei Velni, and M. R. Mohammadi Mozaffar, Krylov complexity in Lifshitz-type Dirac field theories, *Eur. Phys. J. C* **85**, 958 (2025).
- [26] M. Alishahiha and S. Banerjee, A universal approach to Krylov state and operator complexities, *SciPost Phys.* **15**, 080 (2023).
- [27] N. Vardian, Krylov complexity for 1-matrix quantum mechanics, *arXiv:2407.00155*.
- [28] N. Iizuka and M. Nishida, Krylov complexity in the IP matrix model, *J. High Energy Phys.* **11** (2024) 065.
- [29] N. Iizuka and M. Nishida, Krylov complexity in the IP matrix model. Part II, *J. High Energy Phys.* **11** (2023) 096.
- [30] V. Malvimat, S. Porey, and B. Roy, Krylov complexity in 2d CFTs with $SL(2, \mathbb{R})$ deformed Hamiltonians, *J. High Energy Phys.* **02** (2025) 035.
- [31] M. J. Vasli, K. Babaei Velni, M. R. Mohammadi Mozaffar, A. Mollabashi, and M. Alishahiha, Krylov complexity in Lifshitz-type scalar field theories, *Eur. Phys. J. C* **84**, 235 (2024).
- [32] H. A. Camargo, V. Jahnke, K.-Y. Kim, and M. Nishida, Krylov complexity in free and interacting scalar field theories with bounded power spectrum, *J. High Energy Phys.* **05** (2023) 226.
- [33] A. Kundu, V. Malvimat, and R. Sinha, State dependence of Krylov complexity in 2d CFTs, *J. High Energy Phys.* **09** (2023) 011.
- [34] K. Adhikari, S. Choudhury, and A. Roy, Krylov complexity in quantum field theory, *Nucl. Phys. B* **993**, 116263 (2023).
- [35] A. Avdoshkin, A. Dymarsky, and M. Smolkin, Krylov complexity in quantum field theory, and beyond, *J. High Energy Phys.* **06** (2024) 066.
- [36] A. Dymarsky and M. Smolkin, Krylov complexity in conformal field theory, *Phys. Rev. D* **104**, L081702 (2021).
- [37] M. Alishahiha and M. J. Vasli, Thermalization in Krylov basis, *Eur. Phys. J. C* **85**, 39 (2025).
- [38] A. Bhattacharya, P. Nandy, P. P. Nath, and H. Sahu, Operator growth and Krylov construction in dissipative open quantum systems, *J. High Energy Phys.* **12** (2022) 081.
- [39] B. Bhattacharjee, X. Cao, P. Nandy, and T. Pathak, Operator growth in open quantum systems: Lessons from the dissipative SYK, *J. High Energy Phys.* **03** (2023) 054.

- [40] V. Mohan, Krylov complexity of open quantum systems: From hard spheres to black holes, *J. High Energy Phys.* **11** (2023) 222.
- [41] A. Bhattacharya, P. Nandy, P.P. Nath, and H. Sahu, On Krylov complexity in open systems: An approach via bi-Lanczos algorithm, *J. High Energy Phys.* **12** (2023) 066.
- [42] B. Bhattacharjee, P. Nandy, and T. Pathak, Operator dynamics in Lindbladian SYK: A Krylov complexity perspective, *J. High Energy Phys.* **01** (2024) 094.
- [43] E. Carolan, A. Kiely, S. Campbell, and S. Deffner, Operator growth and spread complexity in open quantum systems, *Europhys. Lett.* **147**, 38002 (2024).
- [44] P. Nandy, A. S. Matsoukas-Roubeas, P. Martínez-Azcona, A. Dymarsky, and A. del Campo, Quantum dynamics in Krylov space: Methods and applications, *Phys. Rep.* **1125**, 1 (2025).
- [45] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, Krylov complexity, [arXiv:2507.06286](https://arxiv.org/abs/2507.06286).
- [46] S. Baiguera, V. Balasubramanian, P. Caputa, S. Chapman, J. Haferkamp, M. P. Heller, and N. Y. Halpern, Quantum complexity in gravity, quantum field theory, and quantum information science, *Phys. Rep.* **1159**, 1 (2026).
- [47] Distance in the sense which is defined for $|\psi_n\rangle$ in the context of Krylov complexity.
- [48] E. Bahiru and N. Vardian, Explicit reconstruction of the entanglement wedge via the Petz map, *J. High Energy Phys.* **07** (2023) 025.
- [49] In general, this construction applies to any initial state in an n -dimensional vector space. However, because the group is Abelian and its generators commute, we can choose a basis in which the generators are diagonal. In this basis, we can also express the matrix elements of the initial state. If the initial state has some zero entries in this basis, it is equivalent to projecting the full vector space onto the subspace where the state has nonzero components. This effectively reduces the problem to working within a smaller Hilbert space from the outset.
- [50] M. A. Nielsen, M. R. Dowling, M. Gu, and A. C. Doherty, Quantum computation as geometry, *Science* **311**, 1133 (2006).
- [51] A. R. Brown and L. Susskind, Complexity geometry of a single qubit, *Phys. Rev. D* **100**, 046020 (2019).
- [52] M. A. Nielsen, A geometric approach to quantum circuit lower bounds, *Quantum Inf. Comput.* **6**, 213 (2006).
- [53] M. R. Dowling and M. A. Nielsen, The geometry of quantum computation, *Quantum Inf. Comput.* **8**, 0861 (2008).
- [54] S. Chapman, M. P. Heller, H. Marrochio, and F. Pastawski, Toward a definition of complexity for quantum field theory states, *Phys. Rev. Lett.* **120**, 121602 (2018).
- [55] L. Hackl and R. C. Myers, Circuit complexity for free fermions, *J. High Energy Phys.* **08** (2018) 139.
- [56] S. Chapman, J. Eisert, L. Hackl, M. P. Heller, R. Jefferson, H. Marrochio, and R. Myers, Complexity and entanglement for thermofield double states, *SciPost Phys.* **6**, 034 (2019).
- [57] P. Caputa and J. M. Magan, Quantum computation as gravity, *Phys. Rev. Lett.* **122**, 231302 (2019).
- [58] M. Guo, J. Hernandez, R. C. Myers, and S.-M. Ruan, Circuit complexity for coherent states, *J. High Energy Phys.* **10** (2018) 011.
- [59] P. Caputa, J. M. Magan, and D. Patramanis, Geometry of Krylov complexity, *Phys. Rev. Res.* **4**, 013041 (2022).
- [60] C. Lv, R. Zhang, and Q. Zhou, Building Krylov complexity from circuit complexity, *Phys. Rev. Res.* **6**, L042001 (2024).
- [61] C. Liu, H. Tang, and H. Zhai, Krylov complexity in open quantum systems, *Phys. Rev. Res.* **5**, 033085 (2023).
- [62] A. Bhattacharya, P. P. Nath, and H. Sahu, Speed limits to the growth of Krylov complexity in open quantum systems, *Phys. Rev. D* **109**, L121902 (2024).
- [63] A. Chowdhury and A. Mahapatra, Krylov complexity of optical Hamiltonians, [arXiv:2409.04156](https://arxiv.org/abs/2409.04156).
- [64] S. E. Aguilar-Gutierrez and J. Xu, Geometry of chord intertwiner, multiple shocks and switchback in double-scaled SYK, [arXiv:2506.19013](https://arxiv.org/abs/2506.19013).
- [65] S. E. Aguilar-Gutierrez, From chords to dynamical wormholes with matter: Towards a bulk double-scaled (SYK) algebra, *J. High Energy Phys.* **10** (2025) 221.
- [66] M. Ambrosini, E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, Operator K-complexity in DSSYK: Krylov complexity equals bulk length, *J. High Energy Phys.* **08** (2025) 059.
- [67] J. Baez and J. Huerta, The algebra of grand unified theories, *Bull. Am. Math. Soc.* **47**, 483 (2010).
- [68] H. Fritzsch and P. Minkowski, Unified interactions of leptons and hadrons, *Ann. Phys. (N.Y.)* **93**, 193 (1975).
- [69] H. Georgi, *The State of the Art—Gauge Theories in Particles and Fields—1974: Proceedings of the Williamsburg Meeting of APS/DPF* (1975).
- [70] S. Raby, Introduction to theories of fermion masses, [arXiv:hep-ph/9501349](https://arxiv.org/abs/hep-ph/9501349).
- [71] B. Bajc, A. Melfo, G. Senjanovic, and F. Vissani, The minimal supersymmetric grand unified theory. 1. Symmetry breaking and the particle spectrum, *Phys. Rev. D* **70**, 035007 (2004).