

From on-shell amplitude in AdS to cosmological correlators: Gluons and gravitons

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We recently introduced a recursive bootstrap method for constructing n -point gluon and graviton Mellin-momentum amplitudes in (A)dS spacetime. The power of this approach was illustrated by our successful computation of the first five-point graviton amplitude in (A)dS. In this work, we provide further details of these calculations and start with a more in-depth review of the formalism by offering detailed insights into the motivation for defining on-shell amplitudes in AdS, along with various explicit examples. Furthermore, we demonstrate how cosmological correlators can be effortlessly obtained once the amplitude is determined.

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I. INTRODUCTION

In recent decades, mounting evidence suggests that directly understanding observables can be more insightful than focusing on the underlying theory's Lagrangian. The simplicity of a theory may be obscured by its Lagrangian formulation, whereas a direct approach to observables can reveal fundamental insights [1]. Significant progress has been made in bootstrapping the S-matrix using fundamental physical principles such as Lorentz invariance, locality, and unitarity [2–4].

Gravity stands out as one of the most compelling examples of this shift in perspective. Einstein gravity is notoriously difficult to compute, even at the perturbative level. In contrast, the modern scattering amplitudes approach in flat space has achieved remarkable success by using the on-shell methods to compute gravity amplitudes [5–8].

Despite the success of these methods in flat space, many mysteries remain. One might wonder whether these successes are merely coincidences of flat space, especially considering that our universe is not flat. A natural question arises: Can these approaches be implemented or understood in curved backgrounds? (Anti-)de Sitter space provides a

simple yet meaningful playground for exploring these questions. In particular, de Sitter (dS) space is relevant to cosmology, as it can approximate the spacetime during inflation. This area of study, known as the cosmological bootstrap, has been an active research area [9–16].

Our next question is: What are the observables? In cosmology, the known observables are the cosmological correlators (in-in correlators). However, these correlators are not invariant under field redefinition or gauge transformation,¹ which has limited our understanding of higher point amplitude, especially regarding spinning particles [17–28]. To achieve similar success as seen with the S-matrix, we employ a new representation [29], where the cosmological correlators can be directly obtained from computations involving AdS Witten diagrams as we will explain in the paper.

The first goal of this paper is to demonstrate that the Mellin-momentum amplitude [29] exhibits similar analytic properties to the S-matrix in flat space, remaining invariant under field redefinition and gauge transformation. This simple structure makes use of the differential representation [30–37] which allows us to mimic the calculations of scattering amplitudes in flat space. Along the way, we introduce a novel and efficient algorithm for bootstrapping n -point amplitudes, incorporating the modern on-shell amplitude approach. The key concept involves recycling lower-point on-shell amplitudes to recursively construct higher-point amplitudes. By taking the residue of operator product expansion (OPE) poles, we fix the amplitude up to

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¹The wave-function coefficients also suffer the same problem.

contact terms. Finally, by comparing with the soft limit and flat space limit, we determine all the contact terms. It is crucial to emphasize that our algorithm is entirely automated and requires no guesswork, making it possible to explore unknown higher-point functions. Our method proves valuable not only for understanding the structure of higher-point Mellin-momentum amplitudes but also for easily mapping the results from amplitudes to cosmological correlators.

As the most important and compelling example, we consider the gravity correlator. In [38], the four-point gravity correlator in dS_4 was first determined in full generality. Using this new method, as shown in [39], we present the first five-graviton amplitude in (A)dS. This paper provides further illustrations of the bootstrap technique and offers more examples to manifest the properties of Mellin-momentum amplitudes.

The paper is organized as follows. In Sec. II, we review Mellin-momentum formalism in detail, covering its motivation, definition, and some simple scalar examples. Section III introduces a general setup for the spinning particles and provides flat-spacelike Feynman rules for gluons. We then use the Feynman rules to compute all the Witten diagrams up to five-point and discuss the general pole structure for gluon amplitudes. In Sec. IV we give a detailed explanation of the amplitude bootstrap procedure and explicitly demonstrate the bootstrap process for four- and five-point Yang-Mills (YM) and general relativity (GR) Mellin-momentum amplitude. In Sec. V, we present an efficient and recursive method for computing the remaining scalar integrals to obtain wave-function coefficients for YM and GR. Finally, we offer graphical rules for deriving cosmological correlators from the wave-function coefficients. In Sec. VI we present our conclusions with potential future directions. There are appendixes giving technical details on field redefinition, conformal ward identities, the derivation of YM and GR propagators, and specific bootstrapping process, such as determining double OPE poles for five-point YM and single OPE poles for five-point GR amplitudes.

A. Notation and conventions

Our boundary momentum is $k_\mu = \mathbf{k}$, and μ runs over the boundary coordinates, while m runs over both the boundary and radial coordinates z, μ . All the dot products are defined by boundary coordinates: $\eta_{\mu\nu} k^\mu \varepsilon^\nu = \mathbf{k} \cdot \boldsymbol{\varepsilon}$.

We will often use the following shorthand notation to write the dot product of polarizations:

$$\varepsilon_{ij,kl} := \varepsilon_i \cdot \varepsilon_j \varepsilon_k \cdot \varepsilon_l \quad \mathcal{K}_{ij,kl} := k_i \cdot \varepsilon_j k_k \cdot \varepsilon_l. \quad (1)$$

B. Bootstrap codes and data

The data and codes used to implement the bootstrap procedure and reproduce the results in this work, including those for the four-point and five-point Yang-Mills and

general relativity amplitudes, are publicly available in Ref. [40].

II. MELLIN-MOMENTUM SPACE

A. Motivation for on-shell amplitude

Classically, a conserved spin-1 current obeys $\partial^\mu J_\mu = 0$, but when inserted inside the correlation function it holds only on the separated points which leads to the well-known Ward-Takahashi identity:

$$\begin{aligned} \partial^\mu \langle J_\mu(\mathbf{x}_1) O(\mathbf{x}_2) \dots O(\mathbf{x}_n) \rangle \\ = - \sum_{a=2}^n \delta(\mathbf{x}_1 - \mathbf{x}_a) \langle O(\mathbf{x}_2) \dots \delta O(\mathbf{x}_a) \dots O(\mathbf{x}_n) \rangle. \end{aligned} \quad (2)$$

The right-hand side, characterized by a delta function support, will be referred to as local terms, as it represents a local effect that is nonzero only when two operators are close to each other.²

Taking $U(1)$ symmetry as a simple example after Fourier transforms, the Ward identity becomes

$$\begin{aligned} k_1^\mu \langle J_\mu(\mathbf{k}_1) O(\mathbf{k}_2) \dots O(\mathbf{k}_n) \rangle \\ = - \sum_{a=2}^n e_a \langle O(\mathbf{x}_2) \dots O(\mathbf{k}_1 + \mathbf{k}_a) \dots O(\mathbf{x}_n) \rangle. \end{aligned} \quad (3)$$

So unlike the scattering amplitude or S-matrix in flat space, the correlator (cosmological correlator or wave-function coefficient) is not invariant under gauge ward identity but only up to local terms.

Moreover, the correlator is not a unique object. Under a field redefinition, the correlator remains invariant only up to local terms. For example, consider the wave-function coefficient or boundary correlator in momentum space, which is not invariant under field redefinition [12,41]. Specifically, let us examine a massless free scalar theory in $d = 3$. Under a field redefinition:

$$\phi \rightarrow \phi + \alpha \phi^3, \quad (4)$$

the four-point correlator changes to

$$\begin{aligned} \langle \phi(\mathbf{k}_1) \phi(\mathbf{k}_2) \phi(\mathbf{k}_3) \phi(\mathbf{k}_4) \rangle \\ \rightarrow \langle \phi(\mathbf{k}_1) \phi(\mathbf{k}_2) \phi(\mathbf{k}_3) \phi(\mathbf{k}_4) \rangle - \frac{1}{3} \alpha \sum_{i=1}^4 \langle k_i^3 \rangle. \end{aligned} \quad (5)$$

These are the local terms or boundary contact terms in momentum space. More generally, computing the same correlator using different coordinates, gauges, or employing free equations of motion during the derivation, can

²These are also referred to as contact terms, but to distinguish them from the bulk contact interaction terms, we will refer to them as local terms.

TABLE I. Comparison between amplitudes in Minkowski and AdS.

Amplitude in Minkowski space	Mellin-momentum amplitude in AdS
Lorentz invariance	AdS isometries
Translation symmetry $e^{ik_\mu x^\mu}$	Boundary translation + dilatation $e^{ik_\mu x^\mu} z^{-2s+d/2}$
$(2\pi)^{d+1} \delta^{d+1}(\sum_{i=1}^n k_i^\mu)$	$(2\pi)^d \delta^d(\sum_{i=1}^n k_i^\mu) \times 2\pi i \delta(d+b + \sum_{i=1}^n (2s_i - d/2))$
On-shell condition $k^2 + m^2 = 0$	On-shell condition $z^2 k^2 + (d/2 - \Delta)^2 - 4s^2 = 0$
Pole structure $\frac{1}{k_I^2 + m^2}$	Factorization + OPE $\frac{1}{\mathcal{D}_k^\Delta} \frac{1}{(k_I)^2}$

yield different results, up to local terms. In simpler, lower-point examples, identifying such local terms is feasible. However, in higher-point or more complex scenarios, this process becomes considerably more challenging. In this paper, we aim to construct higher-point correlators recursively using lower-point data. This approach raises the question of which local terms should be retained and which should be disregarded when constructing the higher-point correlators.

Another crucial observation is that in a theory with shift symmetry, the correlator should exhibit enhanced soft limits, meaning it should vanish when one of the legs becomes soft [42]. However, in cases such as the simplest nonlinear sigma model (NLSM), there exist local terms that do not vanish in the soft limit. It is important to note that the presence of nonzero terms in the soft limit does not imply a lack of protection by shift symmetry. Instead, this discrepancy arises because the Noether current associated with the shift symmetry conserves only up to local terms for correlators, precisely due to the Ward identity mentioned above Eq. (2).

None of these problems arise in flat space when working with the S-matrix, thanks to the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. In a similar spirit, to overcome all the challenges mentioned above, we introduce the on-shell amplitude Mellin-momentum amplitude below.

Significant progress has been made in this topic on defining such invariant observables. See defining the S-matrix in AdS boundary [43], on-shell correlators in maximally symmetric space [32], and refining the S-matrix in de Sitter space to enhance crossing and positivity [44,45]. While much of this research focuses on scalar theory, here in this work we focus on spinning particles, and how to compute spinning correlators recursively and efficiently. Our approach shares the same spirit with these prior works, so it would be very interesting to combine these works to better improve other properties of the observables.

The definition of the Mellin-momentum amplitude is largely inspired by on-shell correlators [32]. The key aspect

in proving that the on-shell correlator remains invariant under field redefinitions is that the external legs are on-shell and satisfy Ward identities. The first condition is ensured by using a differential representation, allowing for the amputation of any Witten diagrams, while the second is achieved through the Mellin transform [46–48].

It has been argued that, for AdS correlators, a natural counterpart to momentum conservation in the S-matrix is provided by the conformal Ward identities first suggested by the study of scattering equation in AdS [30,49–52]. However, as an operator process, implementing this in momentum space is nontrivial, as these identities do not yield zero on the right-hand side. Nonetheless, the conformal Ward identity can be manifested through boundary momentum conservation and the Mellin delta function, as demonstrated in the examples in Appendix B 1 and B 2. Hence, the boundary momentum conservation and Mellin delta function serve as natural counterparts to momentum conservation in the S-matrix. The comparison between amplitudes in Minkowski and AdS is summarized in Table I.

B. Definition of Mellin-momentum amplitude

We will be working on the Poincaré patch of AdS_{d+1} space with metric

$$\tilde{g}_{mn} dx^m dx^n = \frac{\mathcal{R}^2}{z^2} (dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu), \quad (6)$$

with $0 < z < \infty$ and $\mathcal{R} = 1$. The equation of motion (EoM) operator in momentum space is defined as

$$\mathcal{D}_k^\Delta \phi_\Delta(k, z) = 0,$$

$$\mathcal{D}_k^\Delta \equiv z^2 k^2 - z^2 \partial_z^2 - (1-d)z \partial_z + \Delta(\Delta-d), \quad (7)$$

with Δ the scaling dimension and $k = |\mathbf{k}|$ the norm of the boundary momentum. The solution for the bulk-to-boundary propagator is given by

$$\phi_\Delta(k, z) = \sqrt{\frac{2}{\pi}} z^{d/2} k^{\Delta-\frac{d}{2}} \mathcal{K}_{\Delta-\frac{d}{2}}(zk) \quad (8)$$

with $\mathcal{K}_{\Delta-\frac{d}{2}}(zk)$ being the Bessel K function of the second kind. In the next step, we will consider the Mellin-Fourier transform $\phi_\Delta(x, z) \sim e^{ik \cdot x} z^{-2s+d/2} \phi_\Delta(s, k)$:

$$\phi_\Delta(k, z) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} z^{-2s+d/2} \phi_\Delta(s, k), \quad (9)$$

$$\phi_\Delta(s, k) = \frac{\Gamma\left(s + \frac{1}{2}\left(\frac{d}{2} - \Delta\right)\right) \Gamma\left(s - \frac{1}{2}\left(\frac{d}{2} - \Delta\right)\right)}{2\Gamma\left(\Delta - \frac{d}{2} + 1\right)} \left(\frac{k}{2}\right)^{-2s+\Delta-\frac{d}{2}}. \quad (10)$$

We can now start to define our observables. The observables in AdS-boundary correlators or wave-function coefficients in dS (after Wick rotation) can be treated as the boundary correlators of conformal field theory (CFT) in momentum space,

$$\Psi_n = \delta^d(\mathbf{k}_T) \langle \mathcal{O}(\mathbf{k}_1) \dots \mathcal{O}(\mathbf{k}_n) \rangle, \quad (11)$$

where $\mathbf{k}_T = \mathbf{k}_1 + \dots + \mathbf{k}_n$. The definition of Mellin-momentum amplitude $\mathcal{A}(zk, s)$ is given as

$$\Psi_n = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^n \phi(s_i, k_i) z^{-2s_i+d/2}, \quad (12)$$

with $\int [ds_i] = \prod_{i=1}^n \int_{-i\infty}^{+i\infty} \frac{ds_i}{(2\pi i)}$.

The amplitude obeys the following on-shell condition [the Mellin transform of the EoM equation (7) above]:

$$(z^2 k^2 + (d/2 - \Delta)^2 - 4s^2) \phi_\Delta(s, k) = 0. \quad (13)$$

It is worth emphasizing that momentum k will always be associated with a factor z to capture scale invariance, and the amplitude will also depend on the differential operator of z as we will see later.

We can also integrate out the z variable at every vertex [46]:

$$\int_0^\infty \frac{dz}{z^{d+1}} z^{\sum_{i=1}^n (-2s_i+d/2)} z^a = \delta\left(d - a + \sum_{i=1}^n (2s_i - d/2)\right), \quad (14)$$

where a is counting the extra factor of z due to scale invariance and we can easily read it off from the Mellin-momentum amplitude. The right-hand side is referred to as the Mellin delta function, similar to the momentum conservation that captures the translation invariant.

C. Local terms and LSZ reduction

As shown in (5) for correlators, under the field redefinition the wave-function coefficients/boundary correlators are not invariant. However, we can consider the same field redefinition for the Mellin-momentum amplitude,

$$\phi \rightarrow \phi + \alpha \phi^3. \quad (15)$$

The amplitude will transform as

$$\mathcal{A} \rightarrow \mathcal{A} + \alpha \sum_{i=2}^4 (z^2 k_1 \cdot k_i + (d/2 - 2s_1)(d/2 - 2s_i)). \quad (16)$$

Using boundary momentum conservation, the Mellin delta function and on-shell condition Eq. (13), we can easily show that the second term is zero and hence the Mellin-momentum amplitude is invariant under field redefinition. We leave the details of computation in Appendix A.

These observations can be explained by the generalized LSZ reduction formula in AdS [32,43,44] with a similar argument as QFT in Minkowski space: *the local term is not singular on the on-shell poles*. The local term of the boundary correlator in position space takes the following general form:

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle \delta^d(x_i - x_j), \quad (17)$$

which vanishes unless the two operators collide. We now consider the Mellin-Fourier transform $\phi(x, z) \sim e^{ik \cdot x} z^{-2s+d/2} \phi(s, k)$. Due to the delta function in (17), it becomes an $n - 1$ point correlator. Hence, the contact term can not be written as Eq. (12) where the definition needs $\prod_{i=1}^n \phi(s_i, k_i)$.

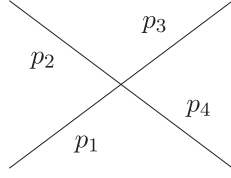
Therefore, contact terms in a boundary correlator do not contribute to the Mellin-momentum amplitude $\mathcal{A}_n$.

More rigorously, one can follow the proof in [32] which closely mimics the proof of the S-matrix to show that the on-shell amplitude is indeed invariant under field redefinition. As emphasized before, the key point is to enforce the external leg on-shell and apply conformal ward identities (which is achieved by boundary momentum conservation and the Mellin delta function, otherwise conformal ward identities for the correlator are usually not zero).

D. Scalar examples

We will start with a few simple examples to see how our definition of amplitude leads to a similar analytic structure as the S-matrix in Minkowski space.

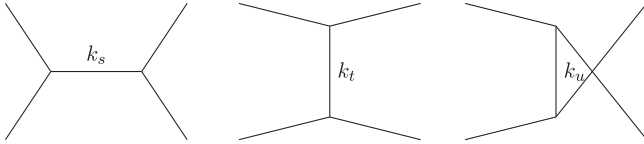
1. Four-point contact diagram for $\lambda\phi^4$ theory



$$\mathcal{A}_{4,c}^{\phi^4} = \lambda \times (2\pi)^d \delta^d \left(\sum_{i=1}^4 k_i^\mu \right) \times 2\pi i \delta(d + \sum_{i=1}^4 (2s_i - d/2)). \quad (18)$$

The amplitude is simply a coupling constant with momentum and Mellin conservation.

2. Four-point exchange diagram for $\lambda\phi^3$ theory



$$\mathcal{A}_4^{\phi^3} = \lambda^2 (2\pi)^d \delta^d \left(\sum_{i=1}^4 k_i^\mu \right) \left(\frac{1}{\mathcal{D}_{k_s}^\Delta} + \frac{1}{\mathcal{D}_{k_t}^\Delta} + \frac{1}{\mathcal{D}_{k_u}^\Delta} \right), \quad (19)$$

where $\mathcal{D}_k^\Delta$ is the EoM operator defined in Eq. (7), whose inverse is defined as

$$(\mathcal{D}(z))^{-1} \mathcal{O}(z) = \int \frac{dy}{y^{d+1}} G(z, y) \mathcal{O}(y), \quad (20)$$

$$\mathcal{D}_k^\Delta G(z, y) = z^{d+1} \delta(z - y). \quad (21)$$

This representation closely mimics the flat space structure.³ Another subtlety is that we can not replace the z derivative in terms of Mellin variables in the EoM operator.⁴ For the same reason there is no overall Mellin delta function. One could only employ the Mellin delta function in contact

³However, one could argue that for scalar exchange, this is a useless setup since that is nothing but the insertion of the bulk-to-bulk propagator. But the power of this representation is coming from particles with spin, whereas we will see later that we can easily determine the rest of the propagator by simply taking residue.

⁴This can be seen that after the Mellin transform this is still not the eigenvalue of the operator and hence we cannot invert the Mellin variables.

interaction or only in the subdiagram at the three-point vertex. But as we will see later in the OPE limits $k^2 \rightarrow 0$, we can then replace the Mellin variables with the inverse operator since in the limits they are the eigenvalues.

III. SPINNING PARTICLES

A. Propagators

In this section, we derive the bulk-to-bulk and bulk-to-boundary propagators for Yang-Mills theory in AdS. Gravity can be treated similarly but is more computationally involved; we therefore relegate the derivation to Appendix C and only present the results here.

We follow the derivation and notation of [53] for Yang-Mills theory, with the action

$$S^{\text{YM}} = \int d^{d+1}z \sqrt{-g} \left(\frac{1}{4} \mathbf{F}_{mn} \mathbf{F}^{mn} + \mathbf{A}_m \mathbf{J}^m \right)$$

and the usual field strength

$$\mathbf{F}_{mn} = \partial_m \mathbf{A}_n - \partial_n \mathbf{A}_m - i[\mathbf{A}_m, \mathbf{A}_n].$$

We rescale the gauge field as $\mathbf{A}_m = (1/z)A_m$ and the source $\mathbf{J}^m = (1/z)J^m$ and impose the transverse boundary gauge condition $k_\mu A^\mu = 0$. The full equation of motion reads:

$$\begin{aligned} \mathcal{D}_k^{d-1} A_\mu &= izk_\mu [(d-2) - z\partial_z] A_z + J_\mu, \\ z^2 k^2 A_z &= J_z. \end{aligned} \quad (22)$$

Note that by contracting k_μ with the EoM and then applying the gauge condition $k_\mu A^\mu = 0$, it implies the usual current conservation

$$zk^\mu J_\mu = i(z\partial_z - d)J_z. \quad (23)$$

We can use the current conservation to reorganize the equations into the following:

$$\begin{aligned} \mathcal{D}_k^{d-1} A_\mu &= \Pi_{\mu\nu} J^\nu, \\ z^2 k^2 A_z &= J_z, \end{aligned} \quad (24)$$

where the spin-1 projection tensor reads $\Pi_{\mu\nu} = (\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})$. So now we are looking for the solution of the bulk-to-bulk propagator

$$A_m(z) = \int \frac{dy}{y^{d+1}} G_{mn}(z, y) J^n(y), \quad (25)$$

and then comparing with (24), the bulk-to-bulk propagator is given by

$$G_{\mu\nu} = \frac{\Pi_{\mu\nu}}{\mathcal{D}_k^{d-1}},$$

$$G_{zz} = \frac{1}{z^2 k^2}. \quad (26)$$

Equivalently, the action that determines the propagator is given by

$$S_p^{\text{YM}} = \int \frac{dz}{z^{d+1}} (A_\mu J^\mu + A_z J^z)$$

$$= \int \frac{dz}{z^{d+1}} \left(J^\mu \frac{\Pi_{\mu\nu}}{\mathcal{D}_k^{d-1}} J^\nu + J^z \frac{1}{z^2 k^2} J^z \right). \quad (27)$$

Next, consider the Einstein gravity action

$$S^{\text{GR}} = \int d^{d+1}z \sqrt{-g} \left(R - 2\Lambda + \frac{1}{2} h_{mn} T^{mn} \right),$$

$$\Lambda = \frac{d(1-d)}{2}.$$

We parametrize the metric fluctuations as $g_{mn} = \tilde{g}_{mn} + (1/z^2)h_{mn}$, where $\tilde{g}_{mn}$ is the background metric given in Eq. (6).

Through computation in Appendix C, we find the action that determines the bulk-to-bulk propagator is

$$S_p^{\text{EG}} = \int \frac{dz}{z^{d+1}} \left(T_{\mu\nu} \frac{\Pi_{2,2}^{\mu\nu,\rho\sigma}}{\mathcal{D}_k^d} T_{\rho\sigma} - 2 \frac{T_{z\mu} \Pi_{\mu\nu} T_{z\nu}}{z^2 k^2} \right.$$

$$+ \frac{2((d-2)T_{zz}T_{zz} - T_{zz}T - TT_{zz})}{(d-1)z^2 k^2}$$

$$\left. + \frac{2d(T + (2-d)T_{zz} + zT'_{zz})(T + (2-d)T_{zz} + zT'_{zz})}{(d-1)z^4 k^4} \right). \quad (28)$$

Last but not least, we should present the bulk-to-boundary propagator. For Yang-Mills theory it is the solutions to Eq. (24) in the free (source-free) case, namely

$$\mathcal{D}_k^{d-1} A_\mu(k, z) = 0, \quad (29)$$

and from the source-free gravity equation of motion (C2) we also find

$$\mathcal{D}_k^d h_{\mu\nu}(k, z) = 0, \quad (30)$$

The solution to Eqs. (29) and (30) are simply scalar bulk-to-boundary propagators dressed with boundary polarization data:

$$A_\mu(k, z) = \varepsilon_\mu \phi_{d-1}(k, z), \quad (31)$$

$$h_{\mu\nu}(k, z) = \varepsilon_{\mu\nu} \phi_d(k, z). \quad (32)$$

With spinning particles being simply scalar dressed up with polarization, we have disentangled scalar integral and tensor structure in the amplitude and so we can now easily use the same definition Eq. (12) to define our spinning Mellin-momentum amplitude.

B. Feynman rules for Yang-Mills Mellin-momentum amplitude

In this section, we write down a new set of Feynman rules for gluons that makes the flat space structure manifest. We will employ the boundary transverse gauge, $k_\mu \cdot A^\mu = 0$, which allows us to have only scalarlike propagators [53]. By directly solving the equation of motion, we found the Feynman rules are identical to flat space in the Coulomb gauge with the simple replacement of the EoM operator $\frac{1}{s}$ with $\frac{1}{\mathcal{D}_k^{d-1}}$; details can be found in Appendix C:

$$\begin{array}{c} \mu \\ \text{~~~~~} \end{array} \begin{array}{c} \nu \\ \text{~~~~~} \end{array} : G_{\mu\nu} = \frac{\Pi_{\mu\nu}}{\mathcal{D}_k^{d-1}}, \quad (33)$$

$$\begin{array}{c} z \\ \text{~~~~~} \end{array} \begin{array}{c} z \\ \text{~~~~~} \end{array} : G_{zz} = \frac{1}{z^2 k^2},$$

where $\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is the spin-1 projection tensor. Similarly, after employing the Mellin-Fourier transform $\phi(x, z) \sim e^{ik \cdot x} z^{-2s+d/2} \phi(k, s)$, it is easy to derive the color ordered vertex [54] and see that they are the same as flat space with

$$\begin{array}{c} p_2 \\ \text{~~~~~} \end{array} \begin{array}{c} n \\ \text{~~~~~} \end{array} \begin{array}{c} p_1 \\ \text{~~~~~} \end{array} \begin{array}{c} m \\ \text{~~~~~} \end{array} : V_{mnq}(p_1, p_2, p_3) = \frac{1}{2} (\eta_{mn}(p_1 - p_2)_q + \eta_{nq}(p_2 - p_3)_m + \eta_{qm}(p_3 - p_1)_n),$$

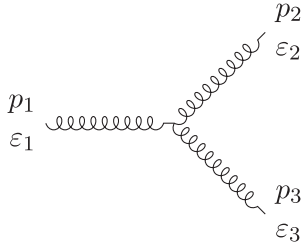
$$\begin{array}{c} m \\ \text{~~~~~} \end{array} \begin{array}{c} n \\ \text{~~~~~} \end{array} \begin{array}{c} o \\ \text{~~~~~} \end{array} \begin{array}{c} q \\ \text{~~~~~} \end{array} : V_{mnqo} = \left(\frac{1}{2} \eta_{mq} \eta_{no} - \frac{1}{4} (\eta_{mn} \eta_{qo} + \eta_{mo} \eta_{nq}) \right), \quad (34)$$

where $p^m = (zk^\mu, i(2s - d/2))$. So we can easily uplift any Feynman diagrams from flat space to AdS amplitude. The crucial distinction between flat space and AdS is the k_I pole, which is nonphysical in flat space. However, this pole in AdS is precisely the signal of a CFT, which is necessary for the CFT to have an infinite expansion (corresponding to infinite number of descendent operators) in the OPE limit [55].

C. Examples of Yang-Mills Witten diagrams

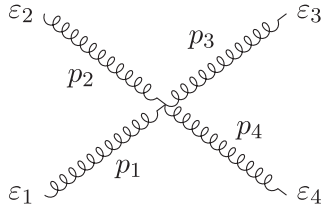
1. Three-point contact diagram

We directly contract the three-point vertex in (34) with polarizations⁵



$$\mathcal{A}_3 = z (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot k_1 + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot k_2 + \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot k_3). \quad (35)$$

2. Four-point contact diagram

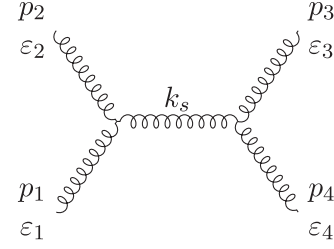


This diagram contributes to the amplitude via direct contraction of polarizations with the Feynman rule

$$\mathcal{A}_4^c = \left(\frac{1}{2} \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \frac{1}{4} \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 - \frac{1}{2} \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 \right). \quad (36)$$

⁵Note that we have stripped off the momentum and Mellin conservation, where unlike scalar contact (18) and four-point gluon contact below (36), the Mellin delta gets shifted in this case because of extract factors of z and should be $2\pi i \delta(d-1 + \sum_{i=1}^3 (2s_i - d/2))$.

3. Four-point gluon s-channel diagram



We first rewrite the three-point vertices in components,

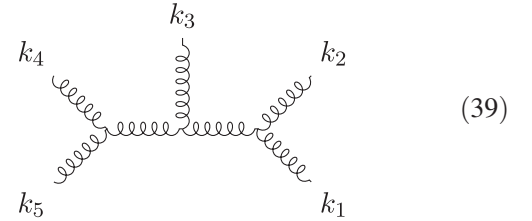
$$\begin{aligned} V_{\mu\nu\rho}(k_1, k_2, k_3) &= \frac{z}{2} (\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_2 - k_3)_\mu \\ &\quad + \eta_{\rho\mu}(k_3 - k_1)_\nu), \\ V_{\mu\nu z}(s_1, s_2, s_3) &= i\eta_{\mu\nu}(s_1 - s_2), \\ V_{\mu z z}(k_1, k_2, k_3) &= \frac{z}{2} \eta_{zz}(k_2 - k_3)_\mu. \end{aligned} \quad (37)$$

Then from the Feynman rule, we should connect the two three-point vertices with the propagator (33), and this gives a transverse part and a longitudinal part:

$$\mathcal{A}_4^s = \left(\frac{V^{12\mu} \Pi_{\mu\nu} V^{\nu 34}}{\mathcal{D}_{k_s}^{d-1}} + \frac{V^{12z} V^{z 34}}{z^2 k_s^2} \right), \quad (38)$$

where we use shorthand notation: $V_{12m} = \epsilon_1^\mu \epsilon_2^\nu V_{\mu\nu m}$ with m could be a boundary coordinate μ or radial coordinate z .

Starting from five-points, we will have two different topologies, the first one is the following diagram:



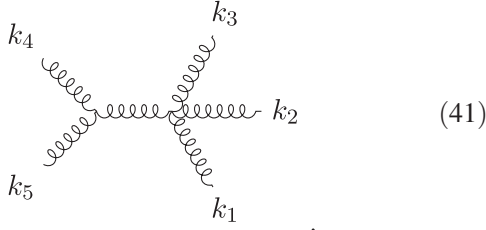
and we label this as $\mathcal{A}_5^a$. There are four possible combinations of indices for the propagator in the middle:

$$\begin{aligned} \mathcal{A}_5^a &= V^{12z} \frac{1}{z^2 k_{12}^2} V^{z 3z} \frac{1}{z^2 k_{45}^2} V^{z 45} + V^{12z} \frac{1}{z^2 k_{12}^2} \frac{V^{z 3\rho} \Pi_{\rho\sigma} V^{\sigma 45}}{\mathcal{D}_{k_{45}}^{d-1}} \\ &\quad + \frac{V^{12\mu} \Pi_{\mu\nu} V^{\nu 3z}}{\mathcal{D}_{k_{12}}^{d-1}} \frac{1}{z^2 k_{45}^2} V^{z 45} \\ &\quad + \frac{V_{12}^\mu \Pi_{\mu\nu} V_3^{\nu\sigma} \Pi_{\sigma\rho} V_{45}^\rho}{\mathcal{D}_{k_{12}}^{d-1} \mathcal{D}_{k_{45}}^{d-1}}, \end{aligned} \quad (40)$$

where $V^{z 3z} = \frac{1}{2} z \epsilon_3 \cdot (k_{45} - k_{12})$, and $V^{z 3\rho} = i \epsilon_3^\rho (s_3 - u)$ contains internal Mellin variables u [48] serving as

derivatives acting on the bulk-to-bulk propagator. We can either keep them or remove them by the Mellin delta function on the vertex. Both give the same result in momentum space as detailed in Appendix E.

For another topology, we label this diagram as $\mathcal{A}_5^b$



We have

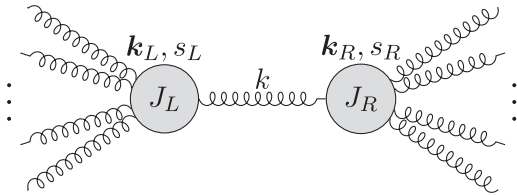
$$\mathcal{A}_5^b = V^{12\mu} \frac{\Pi_{\mu\nu}}{\mathcal{D}_{k_{12}}^{d-1}} V^{\nu 345}. \quad (42)$$

As the polarization only has boundary indices, there will only be transverse propagations, i.e., μ, ν rules in (33).

After summing over channels, the Feynman rule results agree with the bootstrap result; see files 4ym.nb and 5ym.nb in [40].

D. Pole structure

It will also be useful to understand the pole structure of the amplitude using the propagator above. Especially later on we will use this structure to bootstrap the amplitude. We will use current conservation to show that in the OPE limit, the residue of the OPE pole should vanish. Consider the following general exchange diagram for a gluon:



where $k_L^2 = k_R^2 = k^2$ due to boundary momentum conservation. This diagram will contribute to the amplitudes as follows:

$$\begin{aligned} \mathcal{A}_n &= J_L^\mu G_{\mu\nu} J_R^\nu + J_L^z G_{zz} J_R^z \\ &= J_L^\mu \frac{\Pi_{\mu\nu}}{\mathcal{D}_k^{d-1}} J_R^\nu + J_L^z \frac{1}{z^2 k^2} J_R^z. \end{aligned} \quad (43)$$

Considering the following OPE limits, $k^2 \rightarrow 0$, we see the leading contributions correspond to terms with pole $\frac{1}{k^2}$ and taking the residue gives

$$\begin{aligned} \text{Res}_{k^2 \rightarrow 0} \mathcal{A}_n &= k_L^\mu J_L^\mu \frac{1}{\mathcal{D}_k^{d-1}|_{k^2 \rightarrow 0}} k_R^\nu J_R^\nu + J_L^z \frac{1}{z^2} J_R^z \\ &= k_L^\mu J_L^\mu \frac{1}{\frac{1}{4}(d-4s_R-4)(d+4s_R)} k_R^\nu J_R^\nu \\ &\quad + J_L^z \frac{1}{z^2} J_R^z. \end{aligned} \quad (44)$$

To show the above equation is zero, we first recall that the current conservation is

$$\nabla_m J^m = 0, \quad (45)$$

which after applying Mellin-Fourier transform $J_m(x, z) \sim e^{ik \cdot x} z^{-2s+d/2} J_m(s, k)$ yields

$$zk^\mu J_\mu + i(2s + d/2)J_z = 0. \quad (46)$$

Hence by applying current conservation and on the support of the Mellin delta function $\delta(s_R + s_L + 1)$ we have

$$\begin{aligned} \text{Res}_{k^2 \rightarrow 0} \mathcal{A}_n &= -\frac{1}{z^2 k^2} (4s_L + d) J_L^z \frac{1}{(d-4s_R-4)(d+4s_R)} \\ &\quad \times (4s_R + d) J_R^z + J_L^z \frac{1}{z^2 k^2} J_R^z, \\ &= 0. \end{aligned} \quad (47)$$

So we have demonstrated here that by considering a general n -point diagram with gluon exchange, in the OPE limit $k^2 \rightarrow 0$, the vanishing of residue is from current conservation.

With conservation of the stress-energy tensor, an analogous argument should also apply to gravity, which contains $1/k^2$ and $1/k^4$ OPE poles [cf. Eq. (28) as derived in Appendix C]. This will be useful for later sections as we derive the bootstrap conditions.

IV. BOOTSTRAP FROM UNITARITY AND SOFT LIMITS

A. Amplitude bootstrap

In this section, we will use the pole structure of the Mellin-momentum amplitude to bootstrap gluon and graviton amplitudes. To start, our only input for the AdS amplitudes is the following three-point on-shell Yang-Mills amplitude:

$$\mathcal{A}_3 = z(\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot k_1 + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot k_2 + \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot k_3). \quad (48)$$

Based on the Feynman rules in Sec. III B, we see that the amplitude can only have two types of poles which we will refer to as factorization pole $\frac{1}{\mathcal{D}_k}$ and OPE pole $\frac{1}{k^2}$, and we remind the reader that both poles are coming from the kinetic terms in the action. This leads us to the following

ansatz for the n -point amplitude based on their pole structure:

$$\mathcal{A}_n = \sum_{\text{Channels}} \frac{a_1(12\dots n)}{\mathcal{D}_{k_I}^\Delta \mathcal{D}_{k_J}^\Delta \dots \mathcal{D}_{k_M}^\Delta} + \frac{a_2(12\dots n)}{\mathcal{D}_{k_J}^\Delta \dots \mathcal{D}_{k_M}^\Delta} + \dots \\ + \frac{b_{\ell_I, \dots, \ell_M}(12\dots n)}{k_I^{2\ell_I} k_J^{2\ell_J} \dots k_M^{2\ell_M}} + \dots + c(12\dots n). \quad (49)$$

Here the I refers to a general subset of external momenta, $\mathcal{D}_{k_I}^\Delta$ is a bulk-to-bulk propagator with the sum of momenta in the set I flowing through it, and $1/k_I^2 \equiv 1/|\sum_{x \in I} \mathbf{k}_x|^2$. As we will see later, poles in k_I are required by the OPE. The order of these poles can go up to $\ell_{\max} = 1$ for gluons and $\ell_{\max} = 2$ for gravitons. The coefficients appearing in the numerators of this ansatz are then fixed by imposing various constraints coming from factorization, the OPE, and generalized dimensional reduction. We briefly describe each of them below.

1. Factorization

Unitarity implies that the amplitude will factorize into lower-point amplitudes when the exchanged particles are on-shell [13,56,57]

$$\mathcal{A}_n \rightarrow \sum_h \mathcal{A}_a^h \frac{1}{\mathcal{D}_{k_I}^\Delta} \mathcal{A}_{n-a+2}^{-h}. \quad (50)$$

In particular in our examples:

$$\text{YM: } \mathcal{A}_n \rightarrow \mathcal{A}_a^\mu \frac{\sum_h \epsilon_\mu^h \epsilon_\nu^{*h}}{\mathcal{D}_{k_I}^{d-1}} \mathcal{A}_{n-a+2}^\nu, \\ \text{GR: } \mathcal{M}_n \rightarrow \mathcal{M}_a^{\mu_1 \mu_2} \frac{\sum_h \epsilon_{\mu_1 \mu_2}^h \epsilon_{\nu_1 \nu_2}^{*h}}{\mathcal{D}_{k_I}^d} \mathcal{M}_{n-a+2}^{\nu_1 \nu_2}, \quad (51)$$

where the polarization sums are

$$\sum_{h=\pm} \epsilon_\mu(k, h) \epsilon_\nu(k, h)^* = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \equiv \Pi_{\mu\nu}, \\ \sum_{h=\pm} \epsilon_{\mu\nu}(k, h) \epsilon_{\rho\sigma}(k, h)^* = \frac{1}{2} \Pi_{\mu\rho} \Pi_{\nu\sigma} + \frac{1}{2} \Pi_{\mu\sigma} \Pi_{\nu\rho} - \frac{1}{d-1} \Pi_{\mu\nu} \Pi_{\rho\sigma}. \quad (52)$$

This step will fix all the a_i terms in our ansatz.

Current conservation (OPE limit). Consider two operators get close to each other, and they are far from other operators in position space. This corresponds to taking internal momentum $k_I^2 \rightarrow 0$. By conservation (see Sec. III D) in the OPE limit the residue of the OPE poles for amplitudes must be zero:

$$\text{Res}_{k_I^2 \rightarrow 0} \mathcal{A}_n = 0, \quad (53)$$

where I labels internal momentum. Remarkably, with this condition, all the b_i terms in our ansatz are simply determined by taking the residue of a_i :

$$b(12\dots n) = -\text{Res}_{k_I^2 \rightarrow 0} \left(\frac{a_1(12\dots n)}{\mathcal{D}_{k_I}^\Delta \mathcal{D}_{k_J}^\Delta \dots \mathcal{D}_{k_M}^\Delta} + \dots \right). \quad (54)$$

Flat space limit. In the flat space limit, the particles are ignoring the curvature correction, and the Lorentz symmetry will emerge and give us the S-matrix in flat space [58–61]:

$$\lim_{z \rightarrow \infty} \mathcal{A}_n \rightarrow z^\# \mathcal{A}_n, \quad (55)$$

with $z^\#$ being the leading power in flat space limit, and it is easy to see that as $z \rightarrow \infty$:

$$s_i \rightarrow \frac{z k_i}{2}, \quad \mathcal{D}_{k_{ij}} \rightarrow z^2 S_{ij}, \quad (56)$$

where the EoM operator reduced to the Mandelstam variables S_{ij} in flat space and hence the flat space amplitude A_n .

This step fixes all the contact terms c_i in our ansatz for gluons while partly for gravitons.

In particular, for $n \geq 5$ gluon amplitudes, the n -point functions are fully determined by the $(n-1)$ point amplitudes through factorization and the residue of the OPE poles. There are no contact terms beyond four points, as this would result in an incorrect flat space limit.

However, gravity is slightly more subtle, since as a two-derivative theory, the contact terms can potentially receive curvature correction simply from dimensional analysis, but the same procedure applies, and contact terms are fully determined by the flat space limit and external soft limit as follows.

Adler zero. To elaborate, we reduce external legs i, j to scalars by setting $\epsilon_i \cdot \epsilon_j = 1, \epsilon_i \cdot k_a = 0, \epsilon_j \cdot k_a = 0$, where a is any leg. After doing so we obtain $\langle \phi \phi h h h \dots \rangle$ where the scalar will couple with graviton as $\nabla^m \phi \nabla^n \phi h_{mn}$. It follows that the amplitude must vanish when taking a soft limit of a scalar momentum, which is an Adler zero in curved space [42], so for an n -point graviton amplitude:

$$\mathcal{M}_n|_{\epsilon_i \cdot \epsilon_j = 1, \epsilon_i \cdot k_a = 0, \epsilon_j \cdot k_a = 0} \xrightarrow{k_i \rightarrow 0} 0. \quad (57)$$

This will be our last constraint on the gravity amplitude to fix the contact terms $c(12\dots n)$.

Now we have outlined all the conditions used in the Mellin-momentum amplitude bootstrap in AdS for both Yang-Mills and general relativity. To conclude this section, Table II compares the bootstrap conditions for gauge fields

TABLE II. Comparison between on-shell bootstraps of Minkowski and AdS.

	S-matrix	AdS
Space-time symmetry	Manifest	Not manifest
Unitarity	Manifest	Manifest
Gauge condition	$\varepsilon_\mu \cdot k^\mu = 0$ to preserve Lorentz symmetry	Boundary observable forces $\varepsilon^z = 0$

between the flat-space S-matrix and Mellin-momentum amplitude.⁶

B. Gluon amplitudes

1. Four-point

Following our amplitude bootstrap procedure, we now start with three-point Yang-Mills amplitude in Eq. (48) to bootstrap the color-ordered four-point amplitude:

$$\mathcal{A}_4 = \frac{a(1, 2, 3, 4)}{\mathcal{D}_{k_s}^{d-1}} + \frac{b(1, 2, 3, 4)}{k_s^2} + c(1, 2, 3, 4) + [2 \leftrightarrow 4], \quad (58)$$

where the $[2 \leftrightarrow 4]$ is to obtain t -channel and $k_s^2 = k_{12}^2 = (k_1 + k_2)^2$. Then by factorization:

$$a(1, 2, 3, 4) = \sum_{h=\pm} \mathcal{A}_3(1, 2, -k_s^h) \mathcal{A}_3(k_s^{-h}, 3, 4). \quad (59)$$

It can then be written as

$$a(1, 2, 3, 4) = z^2(\Pi_{1,1}\varepsilon_{12,34} + W_s) \quad (60)$$

where $\Pi_{1,1}$ is the spin-1 polarization sum, and W_s are defined as

$$\begin{aligned} \Pi_{1,1} &\equiv \frac{1}{4}(k_1^\mu - k_2^\mu)\Pi_{\mu\nu}(k_3^\nu - k_4^\nu) \\ &= \frac{1}{4}(k_1 - k_2) \cdot (k_3 - k_4) + \frac{(k_1^2 - k_2^2)(k_3^2 - k_4^2)}{4k_s^2}, \end{aligned} \quad (61)$$

$$\begin{aligned} 4W_s &= \varepsilon_1 \cdot \varepsilon_2 (k_1 \cdot \varepsilon_3 k_2 \cdot \varepsilon_4 - k_2 \cdot \varepsilon_3 k_1 \cdot \varepsilon_4) \\ &\quad + \varepsilon_3 \cdot \varepsilon_4 (k_3 \cdot \varepsilon_1 k_4 \cdot \varepsilon_2 - k_4 \cdot \varepsilon_1 k_3 \cdot \varepsilon_2) \\ &\quad + (k_2 \cdot \varepsilon_1 \varepsilon_2 - k_1 \cdot \varepsilon_2 \varepsilon_1) \cdot (k_4 \cdot \varepsilon_3 \varepsilon_4 - k_3 \cdot \varepsilon_4 \varepsilon_3). \end{aligned} \quad (62)$$

It is worth noting that there is a $1/k_s^2$ pole in $\Pi_{1,1}$. This structure is what we need in the second step of our bootstrap procedure, which we demand for the amplitude, the residue of this k_s^2 pole must be zero, so

⁶One could also consider covariant gauge condition in AdS [62,63]. However, for boundary observables we still need to take the external leg $\varepsilon_z \rightarrow 0$ which leaves us with no more gauge redundancy like Coulomb gauge.

$$b(1, 2, 3, 4) = -\text{Res}_{k_s^2 \rightarrow 0} \frac{a(1, 2, 3, 4)}{\mathcal{D}_{k_s}^{d-1}}. \quad (63)$$

To be more precise, the leading term with k_s^2 above is given by

$$\begin{aligned} \frac{a(1, 2, 3, 4)}{\mathcal{D}_{k_s}^{d-1}} \xrightarrow{k_s^2 \rightarrow 0} & \frac{z^2(k_1^2 - k_2^2)(k_3^2 - k_4^2)\varepsilon_{12,34}}{16k_s^2(s_1 + s_2 - 1)(s_3 + s_4 - 1)} \\ &= \frac{(s_1 - s_2)(s_3 - s_4)\varepsilon_{12,34}}{z^2 k_s^2}. \end{aligned} \quad (64)$$

Now we explain the derivation for the calculation above. First, we remind the reader that the inverse EoM operator is acting on the lower point amplitude, and without taking any limits the Mellin transform is not an eigenfunction of the EoM operator (unlike Fourier transform in flat space), we have to keep the inverse operator as the insertion of the bulk-to-bulk propagator. However, in the OPE limits, the EoM operator $\mathcal{D}_k^\Delta|_{k^2 \rightarrow 0} \equiv -z\partial_z z\partial_z + dz\partial_z + \Delta(\Delta - d)$ acts on the lower three-point structure as

$$\begin{aligned} &\frac{1}{\mathcal{D}_{k_s}^{d-1}|_{k_s^2 \rightarrow 0}} (z z^{-2s_1+d/2} z^{-2s_2+d/2}) \\ &= \frac{1}{2(d-2s_1-2s_2)(s_1+s_2-1)} z z^{-2s_1+d/2} z^{-2s_2+d/2}. \end{aligned} \quad (65)$$

In the OPE limits, the Mellin transform z^x is then the eigenfunction of $z\partial_z$ and hence it is now indeed the eigenfunction of the EoM operator in OPE limits. So we can write s in the denominator as the eigenvalue. All the s_i pole here are spurious, and indeed using the on-shell condition (66), which takes $z^4(k_1^2 - k_2^2)(k_3^2 - k_4^2) \rightarrow (s_1^2 - s_2^2)(s_3^2 - s_4^2)$ and shifts the Mellin variables s in the denominator by +1 when applying the on-shell condition.⁷

⁷The subtlety of shifting s here is that when we use the on-shell condition for Mellin-momentum amplitudes,

$$f(s)k^2 \rightarrow f(s+1) \left(\frac{(d/2 - \Delta)^2 - 4s^2}{z^2} \right) \quad (66)$$

where $f(s)$ is any rational function involving the Mellin variable s , we have to shift $s \rightarrow s+1$. And one can derive this by using the definition of (10).

Then we arrive at the result in Eq. (64). Using the Mellin delta function one can eliminate s_1 dependence which will be useful in constructing the five-point in the next section.

In summary, our second bootstrap step gives

$$\begin{aligned} b(1, 2, 3, 4) &= -\text{Res}_{k_s^2 \rightarrow 0} \frac{a(1, 2, 3, 4)}{\mathcal{D}_{k_s}^{d-1}} \\ &= -4(s_1 - s_2)(s_3 - s_4)\varepsilon_{12,34}. \end{aligned} \quad (67)$$

Finally, we are only left with the $c(1, 2, 3, 4)$ term in our ansatz, which by dimensional analysis must be $\varepsilon_{ij,kl}$ with unfixed coefficients. Hence we can easily compare with the flat space limit result and obtain

$$c(1, 2, 3, 4) = \frac{1}{4}(\varepsilon_{13,24} - \varepsilon_{14,23}). \quad (68)$$

So adding up all the bootstrap results together and summing over other channels, we have the four-point color-ordered Yang-Mills amplitude,

$$\begin{aligned} \mathcal{A}_4 &= \frac{z^2(\Pi_{1,1}\varepsilon_{12,34} + W_s)}{\mathcal{D}_{k_s}^{d-1}} - \frac{(s_1 - s_2)(s_3 - s_4)\varepsilon_{12,34}}{z^2 k_s^2} \\ &\quad + \frac{1}{4}(\varepsilon_{13,24} - \varepsilon_{14,23}) + [2 \leftrightarrow 4] \end{aligned} \quad (69)$$

where $[2 \leftrightarrow 4]$ is from summing over the t -channel. To sum up, at four points, we use factorization, the OPE (internal soft) limit, and the flat space limit to fully fix gluon tree-level amplitudes.

This matches our Feynman rules result in Sec. III B. We will translate the result back to momentum space in Sec. V, and it agrees with literature [17]. All processes are given in the file `4ym.nb` in [40].

2. Five-point

We now turn to the five-point ansatz:

$$\begin{aligned} \mathcal{A}_5 &= \frac{a_1(1, 2, 3, 4, 5)}{\mathcal{D}_{k_{12}}^{d-1}\mathcal{D}_{k_{45}}^{d-1}} + \frac{a_2(1, 2, 3, 4, 5)}{\mathcal{D}_{k_{12}}^{d-1}} \\ &\quad + \frac{b(1, 2, 3, 4, 5)}{k_{12}^2 k_{45}^2} + \text{Cyc}, \end{aligned} \quad (70)$$

where Cyc means to get a color-ordered gluon amplitude we have to sum over cyclic permutations for diagrams in the usual trace basis:

$$\begin{aligned} &\{(1, 2, 3, 4, 5), (2, 3, 4, 5, 1), (3, 4, 5, 1, 2), \\ &\quad (4, 5, 1, 2, 3), (5, 1, 2, 3, 4)\}, \end{aligned} \quad (71)$$

and factorization fixes all the a terms by recycling the four-point Eq. (58):

$$\begin{aligned} a_1(1, 2, 3, 4, 5) &= \sum_h a(1, 2, 3, k_{45}) \cdot \mathcal{A}_3(-k_{45}, 4, 5), \\ a_2(1, 2, 3, 4, 5) &= \sum_h \mathcal{A}_3(1, 2, -k_{12}) \cdot \left(\frac{b(k_{12}, 3, 4, 5)}{k_{45}^2} + c(k_{12}, 3, 4, 5) + [3 \leftrightarrow 5] \right). \end{aligned} \quad (72)$$

Their explicit forms are given as

$$\begin{aligned} \frac{a_1(1, 2, 3, 4, 5)}{\mathcal{D}_{k_{12}}^{d-1}\mathcal{D}_{k_{45}}^{d-1}} &= \frac{z^3(k_1^2 - k_2^2)k_3^2(k_4^2 - k_5^2)\varepsilon_1 \cdot \varepsilon_2(k_1 \cdot \varepsilon_3 + k_2 \cdot \varepsilon_3)\varepsilon_4 \cdot \varepsilon_5}{8k_{12}^2 k_{45}^2 \mathcal{D}_{k_{12}}^{d-1}\mathcal{D}_{k_{45}}^{d-1}} \\ &\quad + \frac{z^3(k_1^2 - k_2^2)\varepsilon_1 \cdot \varepsilon_2 X_1^{(54321)}}{8k_{12}^2 \mathcal{D}_{k_{12}}^{d-1}\mathcal{D}_{k_{45}}^{d-1}} + \frac{z^3(k_4^2 - k_5^2)\varepsilon_4 \cdot \varepsilon_5 X_1^{(12345)}}{8k_{45}^2 \mathcal{D}_{k_{12}}^{d-1}\mathcal{D}_{k_{45}}^{d-1}}, \end{aligned} \quad (73)$$

and

$$\begin{aligned} \frac{a_2(1, 2, 3, 4, 5)}{\mathcal{D}_{k_{12}}^{d-1}} &= -\frac{z(k_1^2 - k_2^2)\varepsilon_1 \cdot \varepsilon_2(\varepsilon_3 \cdot \varepsilon_5 \varepsilon_4 \cdot k_{12} - \varepsilon_3 \cdot \varepsilon_4 \varepsilon_5 \cdot k_{12})}{8k_{12}^2 \mathcal{D}_{k_{12}}^{d-1}} \\ &\quad - \frac{(s_4 - s_5)(-d + 2(1 + 2s_3 + s_4 + s_5))(k_1^2 - k_2^2)\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot k_{12} \varepsilon_4 \cdot \varepsilon_5}{4zk_{12}^2 k_{45}^2 \mathcal{D}_{k_{12}}^{d-1}} \\ &\quad + \frac{(s_4 - s_5)(-d + 2(1 + 2s_3 + s_4 + s_5))\varepsilon_4 \cdot \varepsilon_5}{4zk_{45}^2 \mathcal{D}_{k_{12}}^{d-1}} W_{s_1}^{(12345)} + \frac{zW_{s_2}^{(12345)}}{8\mathcal{D}_{k_{12}}^{d-1}} + (3 \leftrightarrow 5) \end{aligned} \quad (74)$$

where $(3 \leftrightarrow 5)$ is from the t -channel contribution in four-point kinematic and X_1 , W_{s_1} , and W_{s_2} are given as

$$X_1^{(12345)} = 2\varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot k_1 k_3^2 - 2\varepsilon_1 \cdot k_2 \varepsilon_2 \cdot \varepsilon_3 k_3^2 + \varepsilon_1 \cdot \varepsilon_2 (2\varepsilon_3 \cdot k_2 k_3^2 + (\varepsilon_3 \cdot k_4 + \varepsilon_3 \cdot k_5)(k_1^2 - k_2^2 + k_3^2)), \quad (75)$$

$$W_{s_1}^{(12345)} = -2\varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot k_1 + 2\varepsilon_1 \cdot k_2 \varepsilon_2 \cdot \varepsilon_3 + \varepsilon_1 \cdot \varepsilon_2 (\varepsilon_3 \cdot k_1 - \varepsilon_3 \cdot k_2),$$

$$W_{s_2}^{(12345)} = 2\varepsilon_1 \cdot \varepsilon_5 \varepsilon_2 \cdot k_1 \varepsilon_3 \cdot \varepsilon_4 - 2\varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot k_1 \varepsilon_3 \cdot \varepsilon_5 + \varepsilon_1 \cdot k_2 (-2\varepsilon_2 \cdot \varepsilon_5 \varepsilon_3 \cdot \varepsilon_4 + 2\varepsilon_2 \cdot \varepsilon_4 \varepsilon_3 \cdot \varepsilon_5) + \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_5 \varepsilon_4 \cdot k_1 \\ - \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_5 \varepsilon_4 \cdot k_2 - \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 \varepsilon_5 \cdot k_1 + \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 \varepsilon_5 \cdot k_2. \quad (76)$$

Having fixed a_1 and a_2 , the next step is to fix the b 's which is determined by the vanishing residue of the amplitudes in OPE limits. Following the same step as the four-point, we can easily determine it by taking the residue of the OPE poles for a_1 and a_2 . We can first determine the b 's with two OPE poles,

$$\begin{aligned} -b(1, 2, 3, 4, 5) &= \text{Res}_{\substack{k_{12}^2 \rightarrow 0 \\ k_{45}^2 \rightarrow 0}} \left(\frac{a_1(1, 2, 3, 4, 5)}{\mathcal{D}_{k_{12}}^{d-1} \mathcal{D}_{k_{45}}^{d-1}} + \frac{a_2(1, 2, 3, 4, 5)}{\mathcal{D}_{k_{12}}^{d-1}} + \frac{a_2(4, 5, 1, 2, 3)}{\mathcal{D}_{k_{45}}^{d-1}} \right) \\ &= -\frac{z^3 \varepsilon_1 \cdot \varepsilon_2 (k_4 \cdot \varepsilon_3 + k_5 \cdot \varepsilon_3) \varepsilon_4 \cdot \varepsilon_5 (k_1^2 - k_2^2) k_3^2 (k_4^2 - k_5^2)}{8 \mathcal{D}_{k_{12}}^{d-1} \mathcal{D}_{k_{45}}^{d-1}} \\ &\quad - \frac{\varepsilon_1 \cdot \varepsilon_2 (k_1 \cdot \varepsilon_3 + k_2 \cdot \varepsilon_3) \varepsilon_4 \cdot \varepsilon_5 (k_1^2 - k_2^2) (s_4 - s_5) (-d + 2(1 + 2s_3 + s_4 + s_5))}{4z \mathcal{D}_{k_{12}}^{d-1}} + (1 \leftrightarrow 5, 4 \leftrightarrow 2). \end{aligned} \quad (77)$$

To proceed, we first use as in (65) to convert $\mathcal{D}$ to s for $\mathcal{D}_{k_{12}}^{d-1}$ and $\mathcal{D}_{k_{45}}^{d-1}$

$$\frac{1}{\mathcal{D}_{k_{ij}}^{d-1}} \rightarrow \frac{1}{2(d - 2s_i - 2s_j)(s_i + s_j - 1)}. \quad (78)$$

After that, we use the on-shell condition to eliminate all k^2 dependence with $\Delta = d - 1$,

$$f(s)k^2 \rightarrow f(s+1) \left(\frac{(d/2 - \Delta)^2 - 4s^2}{z^2} \right).$$

In the double OPE limit, the whole expression of b is a contact interaction and we can use the overall Mellin delta function. Then finally evaluating the support of the Mellin delta function

$$2 + 4s_1 + 4s_2 + 4s_3 + 4s_4 + 4s_5 = 3d, \quad (79)$$

we get

$$\begin{aligned} -b(1, 2, 3, 4, 5) &= \text{Res}_{\substack{k_{12}^2 \rightarrow 0 \\ k_{45}^2 \rightarrow 0}} \left(\frac{a_1(1, 2, 3, 4, 5)}{\mathcal{D}_{k_{12}}^{d-1} \mathcal{D}_{k_{45}}^{d-1}} + \frac{a_2(1, 2, 3, 4, 5)}{\mathcal{D}_{k_{12}}^{d-1}} + \frac{a_2(4, 5, 1, 2, 3)}{\mathcal{D}_{k_{45}}^{d-1}} \right) \\ &= \frac{(s_1 - s_2)(s_4 - s_5)}{z^3} \varepsilon_1 \cdot \varepsilon_2 \varepsilon_4 \cdot \varepsilon_5 \varepsilon_3 \cdot (k_4 + k_5). \end{aligned} \quad (80)$$

After getting the double OPE poles, we find that the amplitude already has the correct flat space limit and has no terms with a single OPE pole. After summing over permutations, we obtained the full color-ordered five-point amplitude by using only the known four-point data and demanding the vanishing of OPE poles. It is also worth pointing out that simply by dimensional analysis there are no contact terms c in $n \geq 5$ points.

All processes are given in 5ym.nb in [40]. Additionally, we have confirmed that our five-point result matches the

results from the new Feynman rules calculation and the literature [17,23] after applying the mapping in Sec. V. Furthermore, the result has been checked to ensure the correct soft limit [64].

C. Graviton amplitude

1. Four-point

Our starting point is the three graviton amplitude, which takes a very simple form in Mellin-momentum amplitude

as manifestly the square of the Yang-Mills one [35,65,66],

$$\mathcal{M}_3 = \mathcal{A}_3^2 = z^2(\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot k_1 + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot k_2 + \varepsilon_3 \cdot \varepsilon_1 \varepsilon_2 \cdot k_3)^2. \quad (81)$$

Our four-point ansatz is given as

$$\mathcal{M}_4 = \frac{a^{\text{GR}}(1, 2, 3, 4)}{\mathcal{D}_{k_s}^d} + \frac{b^{(2,\text{GR})}(1, 2, 3, 4)}{k_s^2} + \frac{b^{(4,\text{GR})}(1, 2, 3, 4)}{k_s^4} + c^{\text{GR}}(1, 2, 3, 4) + \mathcal{P}(2, 3, 4), \quad (82)$$

where the $\mathcal{P}(2, 3, 4)$ denotes permutation to obtain other channels. Similar to the spin-1 case, we first determine a from factorization

$$a^{\text{GR}}(1, 2, 3, 4) = \sum_{h=\pm} \mathcal{M}_3(1, 2, -k_s^h) \cdot \mathcal{M}_3(k_s^{-h}, 3, 4), \quad (83)$$

where the spin-2 polarization sum is given by

$$\sum_{h=\pm} \varepsilon_{\mu\nu}(k, h) \varepsilon_{\rho\sigma}(k, h)^* = \frac{1}{2} \Pi_{\mu\rho} \Pi_{\nu\sigma} + \frac{1}{2} \Pi_{\mu\sigma} \Pi_{\nu\rho} - \frac{1}{d-1} \Pi_{\mu\nu} \Pi_{\rho\sigma}. \quad (84)$$

We get

$$\frac{a^{\text{GR}}(1, 2, 3, 4)}{\mathcal{D}_{k_s}^d} = \frac{(\varepsilon_{12,34} z^2 \Pi_{1,1} + 4z^2 W_s)^2 - \varepsilon_{12,34}^2 z^4 \Pi_{2,2}^{\text{Tr}}}{\mathcal{D}_{k_s}^d}, \quad (85)$$

where the $\Pi_{2,2}^{\text{Tr}}$ is the trace part of the spin-2 polarization sum and given in Appendix D. Next, we can easily see that for gravity we have two OPE poles k_s^{-2} and k_s^{-4} coming from the spin-2 polarization sum. Hence in the OPE limit, we should eliminate k_s^{-2} and k_s^{-4} poles, so b should be

$$b^{(2,\text{GR})}(1, 2, 3, 4) = -\text{Res}_{k_s^2 \rightarrow 0} \frac{a(1, 2, 3, 4)}{\mathcal{D}_{k_s}^d},$$

$$b^{(4,\text{GR})}(1, 2, 3, 4) = -\text{Res}_{k_s^4 \rightarrow 0} \frac{a(1, 2, 3, 4)}{\mathcal{D}_{k_s}^d}. \quad (86)$$

Determining the b terms is very similar to the spin-1 case, except that this time we should expand the inverse operator to higher order for the k_s^2 pole to capture full contributions⁸:

$$\frac{1}{\mathcal{D}_{k_s}^d} \Big|_{k_s^2 \rightarrow 0} = \frac{1}{2(s_{12}-1)(d-2s_{12}+2)} - \frac{z^2 k_s^2}{4(s_{12}-2)(s_{12}-1)(d-2s_{12}+2)(d-2s_{12}+4)} + O(k_s^4). \quad (87)$$

Plugging (87) into (85), collecting the coefficients of k_s^{-4} and k_s^{-2} , using the on-shell condition (66), and massaging the expression using Mellin delta (14), we get k_s^{-4} contribution

$$b^{(4,\text{GR})}(1, 2, 3, 4) = -\frac{(d-2)(s_1-s_2)(s_3-s_4)(k_1^2-k_2^2)(k_3^2-k_4^2)\varepsilon_{12,34}^2}{4(d-1)}. \quad (88)$$

Similarly for the k_s^{-2} contribution

$$b^{(2,\text{GR})}(1, 2, 3, 4) = [W_s + ((k_1 - k_2) \cdot (k_3 - k_4) + k_s^2) \varepsilon_{12,34}] \Pi_{1,0} \varepsilon_{12,34} + \frac{\left(\frac{4(d-2)z^2(k_1^2-k_2^2)(k_3^2-k_4^2)(s_1-s_2)(s_3-s_4)}{(d-2(s_1+s_2))(d-2(s_3+s_4))} + \frac{2(k_3^2-k_4^2)(d^2-8(s_1^2+s_2^2))(s_3-s_4)}{d-2(s_3+s_4)} + 1, 2 \leftrightarrow 3, 4 \right)}{16(d-1)} \varepsilon_{12,34}^2. \quad (89)$$

Poles like $d-2(s_3+s_4)$ in the expression are spurious, which can be easily shown to go away by the on-shell condition (66) and Mellin delta (14), and the spurious pole-free expression can be found in 4gr.nb in [40].

Finally, we need to determine the terms with no poles that is the c terms in our ansatz which can be split into two terms $c^{\text{GR}}(1, 2, 3, 4) = c_0^{\text{GR}}(1, 2, 3, 4) + c_1^{\text{GR}}(1, 2, 3, 4)$. We would first like to make our expression Lorentz invariant in the flat

⁸We use operator expansion $\frac{1}{\mathcal{D}_{k_s}^d} \Big|_{k_s^2 \rightarrow 0} = \frac{1}{-z\partial_z z\partial_z + dz\partial_z} - \frac{1}{-z\partial_z z\partial_z + dz\partial_z} z^2 k_s^2 \frac{1}{-z\partial_z z\partial_z + dz\partial_z} + O(k_s^4)$. And it acts on $z^2 z^{-2s_1+d/2} z^{-2s_2+d/2}$ gives (87).

space limit so we add $c_0^{\text{GR}}(1, 2, 3, 4)$ to cancel out the non-Lorentz-invariant part. Furthermore, the expression c_0^{GR} should also contain curvature correction. This can be probed by Alder's zero condition in curved space when we do dimension reduction: $\varepsilon_1 \cdot \varepsilon_2 = 1, \varepsilon_1 \cdot k = 0, \varepsilon_2 \cdot k = 0$.

The procedure for fixing $c_0^{\text{GR}}(1, 2, 3, 4)$ is as follows. For gravity, i.e., $\frac{a(1,2,3,4)}{\mathcal{D}_{k_s}^d} + \frac{b(1,2,3,4)}{k_s^{2m}}$, it is not Lorentz

invariant when we consider the flat space limit using Eq. (56). This corresponds to the fact that we miss the contribution of the graviton propagator in (28) with no OPE poles. We can bootstrap out the missing term as follows. Adding up $\frac{a(1,2,3,4)}{\mathcal{D}_{k_s}^d} + \frac{b(1,2,3,4)}{k_s^{2m}} + c_0^{\text{GR}}(1, 2, 3, 4)$ together, the result will be Lorentz invariant in flat space limit. This fixes

$$c_0^{\text{GR}}(1, 2, 3, 4)|_{\text{flat space limit}} = \frac{z^2(-((d-2)k_1^2) - 2(d-2)k_3k_1 + dk_2^2 + 2dk_2k_3 + 2k_3^2)}{8(d-1)} \varepsilon_{12,34}^2.$$

To fill up with the AdS corrections, we first uplift $c_0^{\text{GR}}(1, 2, 3, 4)|_{\text{flat space limit}}$ to AdS by $k_i \rightarrow \frac{2s_i + x_i}{z}$,

$$c_0^{\text{GR}}(1, 2, 3, 4) = -\frac{((d-2)s_1^2 + 2(d-2)s_3s_1 - ds_2(s_2 + 2s_3) - 2s_3^2)}{2(d-1)} \varepsilon_{12,34}^2 + f(s, x) \varepsilon_{12,34}^2,$$

where the $f(s, x)$ captures subleading terms in the flat space limit which corresponds to AdS curvature corrections.

There should not be any corrections involving cubic power of s (dual to zdz in coordinate) because we only consider 2 derivative gravity. The quadratic power of s is already fixed by the flat space limit.

Next, to determine curvature corrections that are linear and zeroth order in s 's we need the Adler zero condition: we do a generalized dimension reduction for 1 and 2: $\varepsilon_1 \cdot \varepsilon_2 = 1, \varepsilon_1 \cdot k = 0, \varepsilon_2 \cdot k = 0$ and after dimension reduction the amplitude vanishes in the soft limit of leg one—taking $k_1 \rightarrow 0$ along with $s_1 \rightarrow \frac{d}{4}$ [64]:

$$\lim_{k_1 \rightarrow 0} \left[\frac{a^{\text{GR}}(1, 2, 3, 4)}{\mathcal{D}_{k_s}^d} + \sum_{m=1}^2 \frac{b^{(m, \text{GR})}(1, 2, 3, 4)}{k_s^{2m}} + c_0^{\text{GR}}(1, 2, 3, 4) \right] \Big|_{\varepsilon_1 \cdot \varepsilon_2 = 1, \varepsilon_1 \cdot k = 0, \varepsilon_2 \cdot k = 0} = 0.$$

After some linear algebra,⁹ we find

$$c_0(1, 2, 3, 4)^{\text{GR}} = \frac{(2(d-2)s_1^2 - (d-2)s_1(d-4s_3) - 2ds_2^2 + ds_2(d-4s_3) + 2s_3(d-2s_3))}{4(d-1)} \varepsilon_{12,34}^2. \quad (90)$$

Putting it in a more compact form by Mellin delta, we have

$$c_0^{\text{GR}}(1, 2, 3, 4) = \varepsilon_{12,34}^2 \frac{8d(s_1 - s_2)(s_3 - s_4) - 4(s_1 + s_3 - s_2 - s_4)^2 + d^2}{16(d-1)}. \quad (91)$$

Finally, for the contact term c_1 , dimensional analysis leads us to write down the most general two-derivative ansatz that is Lorentz invariant in the flat space limit:

$$c_1^{\text{GR}}(1, 2, 3, 4) = \varepsilon_{ab,cd,ef} (C_1 z^2 \varepsilon_m \cdot k_i \varepsilon_n \cdot k_j + C_2 \varepsilon_m \cdot \varepsilon_n \mathcal{D}_{k_s}^d) \quad (92)$$

where the EoM operator directly acts on the bulk-to-boundary propagator and gives $\mathcal{D}_{k_s}^d = z^2 k_s^2 + 4(s_1 + s_2)(s_3 + s_4)$,¹⁰ and comparing with the flat space amplitude determines the unknown coefficient C_i and gives

$$c_1^{\text{GR}}(1, 2, 3, 4) = \left[\left(V_s^c - \frac{\varepsilon_{12,34}}{4} \right)^2 - \frac{d-2}{d-1} \frac{\varepsilon_{12,34}^2}{16} \right] \mathcal{D}_{k_s}^d + V_s^c \varepsilon_{12,34} \mathcal{D}_{k_t}^d - 2z^2 V_s^c W_s. \quad (93)$$

⁹Technically we take terms quadratic in x as uniformly a constant: $x_i x_k = f$. Then we get a set of linear equations in x 's after extracting the coefficients of s 's.

¹⁰This is also simply two covariant derivatives acting on a scalar field.

The full process is given in 4gr.nb in [40].¹¹ After adding different channels the four-graviton amplitude is symmetric under particle exchange. This completely determines the four-graviton amplitude and matches with [29], and [38,67] after using the mapping in Sec. V.

To sum up, we have used factorization, internal (OPE) and external soft limit, and flat space limit to fix the four-point graviton Mellin-momentum amplitudes in AdS_{d+1} . We should emphasize that our bootstrap algorithm is quite automatic with no guesswork.

2. Five-point

To demonstrate the power of our bootstrap algorithm, we derive the first five-graviton amplitude in AdS. We first write down our ansatz according to the pole structure:

$$\begin{aligned} \mathcal{M}_5 = & \frac{a_1^{\text{GR}}(1,2;3;4,5)}{\mathcal{D}_{k_{12}}^d \mathcal{D}_{k_{45}}^d} + \frac{a_2^{\text{GR}}(1,2;3,4,5)}{\mathcal{D}_{k_{12}}^d} \\ & + \sum_{m_1, m_2=1}^2 \frac{b_1^{(m_1, m_2, \text{GR})}(1,2;3;4,5)}{k_{12}^{2m_1} k_{45}^{2m_2}} \\ & + \sum_{m=1}^2 \frac{b_2^{(m, \text{GR})}(1,2;3,4,5)}{k_{12}^{2m}} + c^{\text{GR}}(1,2,3,4,5) + \text{perm.} \end{aligned} \quad (94)$$

To proceed, we should first choose the channel that we need to sum over. Following the permutations in [68], the topology of the Feynman diagram with two propagators (39) has 15 channels. We pick our 15 channels for a_1 , b_1 as

$$\begin{aligned} a_1 \text{ and } b_1: & \{1,2,3,4,5\}, \{1,2,4,5,3\}, \{1,2,5,3,4\}, \{2,3,1,4,5\}, \{1,5,4,3,2\}, \\ & \{2,3,5,1,4\}, \{4,3,2,1,5\}, \{2,5,1,3,4\}, \{3,1,2,4,5\}, \{1,5,3,2,4\}, \\ & \{2,5,4,1,3\}, \{3,1,5,2,4\}, \{4,1,3,2,5\}, \{3,5,2,1,4\}, \{4,2,1,3,5\}. \end{aligned} \quad (95)$$

While a_2 has only one factorization pole and hence 10 permutations with topology like (41)

$$\begin{aligned} a_2 \text{ and } c_0: & \{1,2,3,4,5\}, \{2,3,1,4,5\}, \{4,3,2,1,5\}, \{5,4,3,2,1\}, \{5,1,2,3,4\}, \\ & \{3,1,2,4,5\}, \{4,1,3,2,5\}, \{4,2,3,5,1\}, \{2,5,1,3,4\}, \{5,3,1,2,4\}, \end{aligned} \quad (96)$$

while c_0 shares the same list.

The b_2 is the single OPE contribution, again with 10 channels but with a slightly different list due to the derivation and which we will explain later,

$$\begin{aligned} b_2^{\text{Type I}}: & \{1,2,3,4,5\}, \{2,3,1,4,5\}, \{3,1,2,4,5\}, \{4,1,3,2,5\}, \{4,2,1,3,5\}, \{4,3,2,1,5\}, \{5,4,3,2,1\} \\ b_2^{\text{Type II}}: & \{1,5,4,3,2\}, \{2,5,4,1,3\}, \{3,5,4,2,1\}. \end{aligned} \quad (97)$$

Now we start to use factorization to determine the a 's. We can first directly determine a_1 and a_2 by recycling our four-point results to obtain all the terms with factorization $1/\mathcal{D}$ poles:

$$\begin{aligned} a_1^{\text{GR}}(1,2;3;4,5) &= \sum_h a^{\text{GR}}(1,2,3,-k_I) \cdot \mathcal{M}_3(k_I,4,5), \\ a_2^{\text{GR}}(1,2;3,4,5) &= \sum_h \mathcal{M}_3(1,2,-k_I) \cdot \left(\sum_{m=1}^2 \frac{b^{(m, \text{GR})}(k_I,3,4,5)}{k_{45}^{2m}} + c^{\text{GR}}(k_I,3,4,5) + \text{Cyc}(3,4,5) \right), \end{aligned} \quad (98)$$

where $\text{Cyc}(3,4,5)$ means $\{3,4,5\}, \{5,3,4\}, \{4,5,3\}$ and the $\cdot$ contains polarization sums (84). Inheriting from three- and four-point symmetric property, a_1^{GR} is symmetric under

particle exchange of $1 \leftrightarrow 2$ and $4 \leftrightarrow 5$, and invariant under reversal:

$$\begin{aligned} a_1^{\text{GR}}(A,B;C;D,E) &= a_1^{\text{GR}}(\{B,A\}; \\ & C; \{E,D\}) = a_1^{\text{GR}}(E,D;C;B,A). \end{aligned} \quad (99)$$

By construction, a_2 has the following symmetry:

¹¹One might be concerned that the two-derivative contact interaction could have $1/R^2$ correction, which vanishes in the flat space limit. However, such term will violate the Alder zero when taking external soft limit before, hence such terms should not be written down.

$$a_2^{\text{GR}}(A, B; C, D, E) = a_2^{\text{GR}}(\{A, B\}, \{C, D, E\}). \quad (100)$$

This will be important in fixing the OPE poles and simplifying computation. Next collecting the double OPE pole structure, we have

$$-b_1^{(m_1, m_2, \text{GR})}(1, 2; 3; 4, 5) = \text{Res}_{\substack{k_{12}^{2m_1} \rightarrow 0 \\ k_{45}^{2m_2} \rightarrow 0}} \left(\frac{a_1^{\text{GR}}(1, 2; 3; 4, 5)}{\mathcal{D}_{k_{12}}^d \mathcal{D}_{k_{45}}^d} + \frac{a_2^{\text{GR}}(1, 2; 3, 4, 5)}{\mathcal{D}_{k_{12}}^d} + \frac{a_2^{\text{GR}}(4, 5; 1, 2, 3)}{\mathcal{D}_{k_{45}}^d} \right). \quad (101)$$

The $1/\mathcal{D}_{k_{ij}}$ should expand to quadratic order of k_{ij} as in (87). We find that because of symmetry in a_1 and a_2 , b_1 has reversal symmetry $b_1^{(m_1, m_2, \text{GR})}(1, 2; 3; 4, 5) = b_1^{(m_1, m_2, \text{GR})}(5, 4; 3; 2, 1)$. The detailed expression for b is in 5gr.nb in [40].

Unlike YM, gravity also has terms with a single OPE pole. The terms seem complicated, but we can still simply take residue to determine them:

$$\begin{aligned} -b_2^{m, \text{GR}}(1, 2; 3, 4, 5)^{\text{Type I}} = & \text{Res}_{k_{12}^{2m} \rightarrow 0} \left\{ \left(\frac{a_1^{\text{GR}}(1, 2; 3; 4, 5)}{\mathcal{D}_{k_{12}}^d \mathcal{D}_{k_{45}}^d} + \sum_{m_1, m_2=1}^2 \frac{b_1^{(m_1, m_2, \text{GR})}(1, 2; 3; 4, 5)}{k_{12}^{2m_1} k_{45}^{2m_2}} + \text{Cyc}(3, 4, 5) \right) \right. \\ & \left. + \frac{a_2^{\text{GR}}(1, 2; 3, 4, 5)}{\mathcal{D}_{k_{12}}^d} + \left(\frac{a_2^{\text{GR}}(3, 4; 5, 1, 2)}{\mathcal{D}_{k_{34}}^d} + \text{Cyc}(3, 4, 5) \right) \right\}, \end{aligned} \quad (102)$$

where $\text{Cyc}(3, 4, 5)$ means $\{3, 4, 5\}, \{5, 3, 4\}, \{4, 5, 3\}$ which is defined as canonical permutation, i.e., generally for C, D, E , we have

$$\sum_{m_1, m_2=1}^2 \frac{b_1^{(m_1, m_2, \text{GR})}(A, B; C, D, E)}{k_{AB}^{2m_1} k_{DE}^{2m_2}} + \frac{b_1^{(m_1, m_2, \text{GR})}(A, B; D, E, C)}{k_{AB}^{2m_1} k_{CE}^{2m_2}} + \frac{b_1^{(m_1, m_2, \text{GR})}(A, B; E, C, D)}{k_{AB}^{2m_1} k_{CD}^{2m_2}}. \quad (103)$$

This combination of b_1 gives the b_2 of type I. Computational details are given in Appendix F 1. At this point, one is tempted to add up all cyclic permutations to get full b_2 . But it turns out that we can at best do 7 permutations out of 10 as the first line in (97).

The reason is that the fixed lists a_1 a_2 and b_1 in Eqs. (95) and (96) are sufficient to determine the first line of b_2 in Eq. (97); no new b_1 terms beyond the 15 listed in Eq. (95) are needed.¹² However, if one keeps going with b_2 determined from canonical permutation in (103), one will require new b_1 that the ordering of 1, 2, 3, 4, 5 is different from any one in (95).

The solution is to have a second way of determining b_2 from a new b_1 combination as shown below. With this, we can determine all b_2 using only b_1 within the list of (95),

$$\begin{aligned} -b_2^m(1, 5; 4, 3, 2)^{\text{Type II}} = & \text{Res}_{k_{12}^{2m} \rightarrow 0} \left\{ \frac{a_1(5, 1; 2; 4, 3)}{\mathcal{D}_{k_{15}}^d \mathcal{D}_{k_{34}}^d} + \frac{a_1(1, 5; 4; 3, 2)}{\mathcal{D}_{k_{15}}^d \mathcal{D}_{k_{23}}^d} + \frac{a_1(1, 5; 3; 2, 4)}{\mathcal{D}_{k_{15}}^d \mathcal{D}_{k_{24}}^d} \right. \\ & + \left(\frac{b_1(5, 1, 2, 3, 4)}{k_{15}^{2m_1} k_{34}^{2m_2}} + \frac{b_1(1, 5, 4, 3, 2)}{k_{15}^{2m_1} k_{23}^{2m_2}} + \frac{b_1(1, 5, 3, 2, 4)}{k_{15}^{2m_1} k_{24}^{2m_2}} \right) \\ & \left. + \frac{a_2(5, 1; 2, 3, 4)}{\mathcal{D}_{k_{15}}^d} + \left(\frac{a_2(2, 3; 4, 5, 1)}{\mathcal{D}_{k_{23}}^d} + \frac{a_2(4, 2; 3, 5, 1)}{\mathcal{D}_{k_{24}}^d} + \frac{a_2(3, 4; 2, 1, 5)}{\mathcal{D}_{k_{34}}^d} \right) \right\}. \end{aligned} \quad (104)$$

Some computational details are given in Appendix F 2. With the list (97), one finally has full b_2 contributions.

Having a_1 a_2 b_1 b_2 's and sum over all channels, we now have the total contribution that has no k_I^2 poles in the OPE limit. As a consistency check there should not be any k_I^2 pole in the flat space limit.

As before, we now need to determine the contact terms c with no poles. The procedure is basically the same as four-point by using the flat space limit and Alder zero condition and gives

$$c_0(1, 2; 3, 4, 5) = \frac{d^2 - 4(s_1 - s_2)^2 - 2d(s_1 + s_2)}{32(d-1)} \varepsilon_{12, 12, 34, 35, 45}. \quad (105)$$

¹²The symmetry of a_1 and a_2 usually save one from requiring new permutations of indices in a_1 , a_2 to determine b_2 in a given channel.

As a consistency check summing over all the contributions we have now, the expression is Lorentz invariant in the flat space limit and vanishing soft limits after dimension reduction.

Finally, there is only one contribution left which corresponds to five graviton contact interactions c_1 and it takes the following form. We are only left with unfixed coefficients that can be readily determined by comparison with the flat space amplitude,¹³

$$c_1 = \varepsilon_{ab,cd,ef,rs} (C_1 z^2 \varepsilon_m \cdot k_i \varepsilon_n \cdot k_j + C_2 \varepsilon_m \cdot \varepsilon_n \mathcal{D}_{k_{ij}}^d). \quad (106)$$

Summing over permutation for all the terms bootstrapped above, we obtained the first five-graviton amplitude in AdS_{d+1} and it shares the similar analytic structure of the S-matrix in flat space. We have checked the symmetry properties by swapping particles. The final result is given in `GR5fullADS.txt` and the computation process is coded in *Mathematica* file `5gr.nb` in [40].

So in the end, we bootstrap five-point graviton amplitudes in AdS_{d+1} from factorization soft limit (internal OPE limit, external soft limit) and high energy (flat space) limit,

$$\begin{aligned} \mathcal{M}_5^{\text{GR}} = & c_1 + \sum_{X \in \text{list}_{a_1, b_1}} a_1(X) + b_1(X) \\ & + \sum_{X \in \text{list}_{a_2, c_0}} a_2(X) + c_0(X) + \sum_{X \in \text{list}_{b_2^{\text{Type I}}}} b_2^{\text{Type I}}(X) \\ & + \sum_{X \in \text{list}_{b_2^{\text{Type II}}}} b_2^{\text{Type II}}(X). \end{aligned} \quad (107)$$

V. SCALAR INTEGRALS AND COSMOLOGICAL CORRELATORS IN MOMENTUM SPACE

Ultimately, our interest still lies in correlators composed of pure kinematic momentum. Mellin-momentum amplitude not only serves as a convenient framework for understanding amplitude structure but also serves as a useful computational tool for cosmological correlators. In this section, we will translate the Mellin-momentum amplitude results above back into momentum space by computing scalar integrals and provide a straightforward algebraic algorithm to demonstrate that this transition is transparent and simple for n -point. Finally, we will show that this directly gives us the cosmological correlators without any further computation. The overall picture of our strategy is summarized in Fig. 1.

The wave-function coefficients in dS space can be computed from Witten diagrams in AdS after wick rotation $z \rightarrow i\eta$ [41,69–72] which is precisely the AdS correlators

¹³The full result for c_1 is in `GR5flatminusGR5ADS-flat.txt` [40].

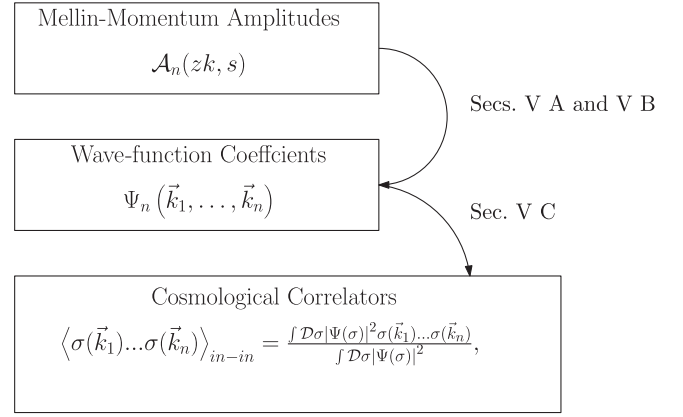


FIG. 1. From on-shell Mellin-momentum amplitude in AdS to cosmological correlators.

we studied here. This is particularly straightforward for gluons and gravitons, whose correlators are simply rational functions without IR divergences. In this case, the Wick rotation is trivial, with $z \rightarrow i\eta$ equivalent to $k_i \rightarrow ik_i$ in the bulk-to-boundary propagator. Thus, after performing the bulk integral over z , our result matches the wave-function coefficients, up to an overall normalization factor. For the more careful readers, we refer the detailed discussion and derivation in [73] which we also record the formula here:

$$\begin{aligned} \psi_n^{\text{dS}}(q_1, \dots, q_n) \\ = (-i)^{n(d-\Delta)} \langle\langle \mathcal{O}(q_1) \dots \mathcal{O}(q_n) \rangle\rangle^{\text{AdS}}|_{L_{\text{AdS}} \rightarrow iL_{\text{dS}}}. \end{aligned} \quad (108)$$

A. Yang-Mills

Using the definition of Mellin-momentum amplitude (12), we can easily obtain the following map for the scalar integral/form factor in $d = 3$:

amplitudes $\rightarrow$ wave-function coefficients

$$\begin{aligned} A(k, \varepsilon) & \rightarrow A(k, \varepsilon) \mathcal{C}_{\text{YM}} \\ \frac{A(k, \varepsilon)}{\mathcal{D}_{k_I}^{d-1}} & \rightarrow A(k, \varepsilon) \mathcal{I}_I^{\text{YM}} \\ \frac{A(k, \varepsilon)}{\mathcal{D}_{k_J}^{d-1} \dots \mathcal{D}_{k_M}^{d-1}} & \rightarrow A(k, \varepsilon) \mathcal{I}_{J \dots M}^{\text{YM}} \\ s_i & \rightarrow \frac{zk_i}{2} + \frac{1}{4}. \end{aligned} \quad (109)$$

The mapping of the Mellin variable is simply coming from the definition that the z derivative acts on the bulk-to-boundary propagator.¹⁴

¹⁴This mapping for YM is slightly different from our previous work, as it turns out that at six points and beyond there will be internal Mellin variables, and the previous mapping is not enough.

1. Contact integrals

Since the scalar part of the YM bulk-to-boundary propagator is similar to the conformally coupled scalar, which is just a plane wave in $d = 3$, the scalar integral could be rather trivial as just pulling down the total energy in the exponent. Considering the following n -points contact integrals, we have

$$\begin{aligned} C_{\text{YM}}^{n;1} &= \int_0^\infty \frac{dz}{z^4} z^{4-n} \prod_{i=1}^n \phi_{\Delta=2}(z, k_i) = \frac{1}{E_t}, \\ C_{\text{YM}}^{n;2} &= \int_0^\infty \frac{dz}{z^4} z^{3-n} (2s_i - 1/2) \prod_{i=1}^n \phi_{\Delta=2}(z, k_i) = \frac{k_i}{E_t}, \end{aligned} \quad (110)$$

where we denote total energy as $E_t = \sum_{i=1}^n k_i$, and the second integral is a simple example that involves Mellin variables. The Mellin variable s_i should be understood as $\frac{1}{4}(d - 2z\partial_z)$ acting on the bulk-to-boundary propagator ϕ_i .¹⁵ Then by definition when s acts on the bulk-to-boundary propagator we have $s\phi_{\Delta=2}(z, k) \equiv \frac{1}{4}(3 - 2z\partial_z)ze^{-zk} = (\frac{3}{2} + \frac{1}{4})ze^{-zk}$. So to sum up, we have $s_i \rightarrow \frac{z k_i}{2} + \frac{1}{4}$.

2. Exchange integrals

We will be using the following representation of the bulk-to-bulk propagator,¹⁶

$$\begin{aligned} G_\Delta(\mathbf{k}, z_1, z_2) &= -i \int_0^\infty \frac{dp}{2\pi i} \frac{p^{d+1-2\Delta}}{k^2 + p^2} (\phi_\Delta(z_1, ip) \\ &\quad - \phi_\Delta(z_1, -ip)) (\phi_\Delta(z_2, ip) - \phi_\Delta(z_2, -ip)), \end{aligned} \quad (112)$$

where ϕ_Δ is the bulk-to-boundary propagator $\phi_\Delta(z, k) = \sqrt{\frac{2}{\pi}} z^{d/2} k^{\Delta-d/2} K_{\Delta-d/2}(kz)$ with $k = |\mathbf{k}|$. For n -point amplitude with only one inverse operator $\frac{1}{\mathcal{D}_{k_I}}$, the scalar exchange integral involves for exchange diagram with n_I legs at left is

¹⁵This is because a single Mellin variable s is pulled down as $\frac{1}{4}(d - 2z\partial_z)z^{-2s+d/2}$.
¹⁶The usual bulk-to-bulk propagator is [74]

$$\begin{aligned} G_\Delta(\mathbf{k}, z_1, z_2) &= (z_1 z_2)^{d/2} \int_0^\infty dp \frac{p}{p^2 + k^2} J_{\Delta-d/2}(pz_1) J_{\Delta-d/2}(pz_2). \end{aligned} \quad (111)$$

This can also be shown to be equivalent by using the Mellin representation of Bessel function J and K and $\Gamma(x)\Gamma(1-x) = \csc(\pi x)$.

$$\begin{aligned} \mathcal{I}_I^{\text{YM}} &= \int \frac{dz_1}{z_1^4} \frac{dz_2}{z_2^4} \left(z_1^{4-n_I} \prod_{i=1}^{n_I} \phi_{\Delta=2}(z_1, k_i) \right) \\ &\quad \times G_{\Delta=2}(\mathbf{k}_I, z_1, z_2) \left(z_2^{4-(n-n_I)} \prod_{i=n_I+1}^n \phi_{\Delta=2}(z_2, k_i) \right) \\ &= \int_0^\infty \frac{dp}{2\pi i} \frac{-i}{k_I^2 + p^2} \bar{\mathcal{C}}_{\text{YM}}(k_1, \dots, ip) \bar{\mathcal{C}}_{\text{YM}}(ip, \dots, k_n) \end{aligned} \quad (113)$$

where $\mathbf{k}_I = \mathbf{k}_1 + \dots + \mathbf{k}_{n_I}$ and we define shifted functions: $\bar{\mathcal{C}} = \mathcal{C}(k_1, \dots, ip) - \mathcal{C}(k_1, \dots, -ip)$ and $\bar{\mathcal{I}} = \mathcal{I}(ip, \dots, k_n) - \mathcal{I}(-ip, \dots, k_n)$. The integral for p is from minus infinity to plus infinity and can be regarded as the sum of all residues above the real line [54]. Summing up those residues, one will end up with the final line which coincides with the usual result. We will give an example later.

Similarly, when we have more than one inverse operator/bulk-to-bulk propagator $\frac{1}{\mathcal{D}_{k_I} \dots \mathcal{D}_{k_M}}$, we have the following recursive integrals:

$$\mathcal{I}_{IJ\dots M}^{\text{YM}} \equiv \int_0^\infty \frac{dp}{2\pi i} \frac{-i}{k_I^2 + p^2} \bar{\mathcal{C}}_{\text{YM}}(k_1, \dots, ip) \bar{\mathcal{I}}_{J\dots M}^{\text{YM}}(ip, \dots, k_n). \quad (114)$$

Finally we should note that, in the $n < 6$ point, we can just use the recursion in [39]. However, we find that at six-points we will have internal Mellin variables, so we need this new version to make things work.

3. Examples on recursive integral

We start with a four-point YM exchange scalar integral which corresponds to mapping a single inverse operator to momentum space

$$\frac{1}{\mathcal{D}_{k_{12}}^{d-1}} \rightarrow \mathcal{I}_{\{12\}}^{\text{YM}}(k_1, k_2, k_3, k_4, k_{12}). \quad (115)$$

With this, by definition

$$\begin{aligned} \mathcal{I}_{\{12\}}^{\text{YM}}(k_1, k_2, k_3, k_4, k_{12}) &= \int \frac{dz_1}{z_1^4} \frac{dz_2}{z_2^4} \prod_{i=1}^2 \phi_{\Delta=2}(z_i, k_i) z_1 G_{\Delta=2}(k_{12}, z_1, z_2) z_2 \\ &\quad \times \prod_{j=3}^4 \phi_{\Delta=2}(z_2, k_j) \end{aligned} \quad (116)$$

where the additional z_1 and z_2 are from scaling of the three-point YM vertex. The alternative way of evaluating it that we advocate here is as follows: starting with the three-point contact

$$\mathcal{C}_{\text{YM}}^{3;1} = \int_0^\infty \frac{dz}{z^4} z^{4-n} \prod_{i=1}^3 \phi_{\Delta=2}(z, k_i) = \frac{1}{k_1 + k_2 + k_3}, \quad (117)$$

then using (112), the integral is just taking residues of all the poles on the upper half-plane $p \rightarrow ik_{12}, i(k_1 + k_2), i(k_3 + k_4)$ of the shifted three-point contact:

$$\begin{aligned} \mathcal{I}_{\{12\}}^{\text{YM}}(k_1, k_2, k_3, k_4, k_{12}) &= \text{Res}_p \frac{1}{(k_{12}^2 + p^2)} \left(\frac{1}{k_1 + k_2 + ip} - \frac{1}{k_1 + k_2 - ip} \right) \left(\frac{1}{k_3 + k_4 + ip} - \frac{1}{k_3 + k_4 - ip} \right) \\ &= \frac{1}{(k_1 + k_2 + k_3 + k_4)(k_{12} + k_1 + k_2)(k_{12} + k_3 + k_4)}. \end{aligned} \quad (118)$$

Now we can move to more complicated example:

$$\frac{1}{\mathcal{D}_{k_{12}}^{d-1} \mathcal{D}_{k_{45}}^{d-1}} \rightarrow \mathcal{I}_{\{12\}\{45\}}^{\text{YM}}(k_1, k_2, k_3, k_4, k_5, k_{12}, k_{45}). \quad (119)$$

Recycling four-point and three-point data, we can compute a five-point exchange diagram with two propagators as

$$\mathcal{I}_{\{12\}\{45\}}^{\text{YM}} = \int_0^\infty \frac{dp}{2\pi i} \frac{-i}{k_{12}^2 + p^2} \bar{\mathcal{C}}_{\text{YM}}^{3,1}(k_1, k_2, ip) \bar{\mathcal{I}}_{\{45\}}^{\text{YM}}(ip, k_3, k_4, k_5). \quad (120)$$

Summing up the residues at $p \rightarrow ik_{12}, i(k_1 + k_2), i(k_3 + k_4 + k_5), i(k_3 + k_{45})$, one gets the result matched with literature

$$\mathcal{I}_{\{12\}\{45\}}^{\text{YM}} = \frac{k_{12} + k_{45} + k_1 + k_2 + 2k_3 + k_4 + k_5}{E_t(k_{12} + k_1 + k_2)(k_{12} + k_3 + k_4 + k_5)(k_{45} + k_1 + k_2 + k_3)(k_{45} + k_4 + k_5)(k_{12} + k_{45} + k_3)}, \quad (121)$$

with $E_t = k_1 + k_2 + k_3 + k_4 + k_5$.

4. Examples with Mellin variable

Apart from purely exchange diagrams, one may have additional s variables enter the game. In proactive, this occurs when we convert a_2 terms back to momentum space. And one will encounter terms that have the Mellin variable of quadratic power

$$\frac{(\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot \varepsilon_3)(\varepsilon_4 \cdot \varepsilon_5)(s_4 - s_5)(2(1 + 2s_3 + s_4 + s_5) - d)}{2z \mathcal{D}_{k_{12}} k_{45}^2}. \quad (122)$$

In this case, one should first use on-shell condition $s_a^2 = \frac{1}{16}((d - 2\Delta)^2 + 4z^2 k_a^2)$ to lower the explicit Mellin variable power to 1 and then use the map $s_i \rightarrow \frac{z k_i}{2} + \frac{1}{4}$. Eventually for this example, one gets back to momentum space as

$$\frac{(\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot \varepsilon_3)(\varepsilon_4 \cdot \varepsilon_5)(k_4 - k_5)(2k_3 + k_4 + k_5)}{4(k_1 + k_2 + k_3 + k_4 + k_5)(k_1 + k_2 + k_{12})(k_3 + k_4 + k_5 + k_{12})k_{45}^2}. \quad (123)$$

B. Gravity

For gravity in $d = 3$, we found a similar recursion for the n -point scalar integral as well.

amplitudes $\rightarrow$ wave-function coefficients

$$\begin{aligned} z^2 M(k, \varepsilon) &\rightarrow M(k, \varepsilon) \mathcal{C}_{\text{GR}}^{n;1} \\ \prod_{m=1}^l (-2s_m + d/2) M(k, \varepsilon) &\rightarrow M(k, \varepsilon) \mathcal{C}_{\text{GR}}^{n;2} \\ \frac{M(k, \varepsilon)}{\mathcal{D}_{k_i}^d} &\rightarrow M(k, \varepsilon) \mathcal{I}_I^{\text{GR}} \\ \frac{M(k, \varepsilon)}{\mathcal{D}_{k_j}^d \cdots \mathcal{D}_{k_M}^d} &\rightarrow M(k, \varepsilon) \mathcal{I}_{J \dots M}^{\text{GR}}. \end{aligned} \quad (124)$$

1. Contact integrals

The scalar part of the GR bulk-to-boundary propagator is similar to the massless scalar with additional scalings. The possible contact integrals one would encounter are

$$\mathcal{C}_{\text{GR}}^{n;1} = \int \frac{dz}{z^4} z^2 \prod_i^n \phi_{\Delta=3}(z, k_i) = \left(\sum_{m=0}^{n-2} m! \sum_{1 \leq i_1 < \dots < i_{m+2}}^n \frac{k_{i_1} \dots k_{i_{m+2}}}{E_t^{m+1}} \right) - E_t, \quad (125)$$

$$\mathcal{C}_{\text{GR}}^{n;2} = \int \frac{dz}{z^4} \prod_{m=1}^l (-2s_m + d/2) \phi_{\Delta=3}(z, k_i) = \left(\sum_{m=0}^{n-l} (2l-4)! \sum_{l+1 \leq i_{l+1} < \dots < i_{m+l}}^n \frac{k_{i_{l+1}} \dots k_{i_{m+l}}}{E_t^{(2l-4)+m+1}} \right) (-1)^l k_1^2 \dots k_l^2, \quad (126)$$

and we refer the reader to Appendix H for the derivation.

2. Gravity exchange integral

We again use the following representation of the bulk-to-bulk propagator with $\Delta = d = 3$,

$$G_{\Delta}(\mathbf{k}, z_1, z_2) = -i \int_0^\infty \frac{dp}{2\pi i} \frac{p^{d+1-2\Delta}}{k^2 + p^2} (\phi_{\Delta}(z_1, ip) - \phi_{\Delta}(z_1, -ip)) (\phi_{\Delta}(z_2, ip) - \phi_{\Delta}(z_2, -ip)), \quad (127)$$

where $\phi_{\Delta}(z, k) = \sqrt{\frac{2}{\pi}} z^{d/2} k^{\Delta-d/2} K_{\Delta-d/2}(kz)$. Consider an exchange diagram with n_l legs at the left. Replacing the propagator with (127), one will end up with the two contact scalar integrals we determined earlier:

$$\begin{aligned} \mathcal{I}_I^{\text{GR}} &= \int \frac{dz_1}{z_1^4} \frac{dz_2}{z_2^4} \left(z_1^2 \prod_{i=1}^{n_l} \phi_{\Delta=3}(z_1, k_i) \right) G_{\Delta=3}(\mathbf{k}_I, z_1, z_2) \left(z_2^2 \prod_{i=n_l+1}^n \phi_{\Delta=3}(z_2, k_i) \right) \\ &= \int_0^\infty \frac{dp}{2\pi i} \frac{-ip^{-2}}{k_I^2 + p^2} \bar{\mathcal{C}}_{\text{GR}}^{n;1}(k_1, \dots, ip) \bar{\mathcal{C}}_{\text{GR}}^{n;1}(ip, \dots, k_n), \end{aligned} \quad (128)$$

where $\bar{\mathcal{C}}_{\text{GR}}^{n;1}(k_1, \dots, ip) = \mathcal{C}_{\text{GR}}^{n;1}(k_1, \dots, ip) - \mathcal{C}_{\text{GR}}^{n;1}(k_1, \dots, -ip)$. Furthermore, we can recursively build up exchange diagrams with more propagators as

$$\mathcal{I}_{IJ\dots M}^{\text{GR}} \equiv \int_0^\infty \frac{dp}{2\pi i} \frac{-ip^{-2}}{k_I^2 + p^2} \bar{\mathcal{C}}_{\text{GR}}(k_1, \dots, ip) \bar{\mathcal{I}}_{J\dots M}^{\text{GR}}(ip, \dots, k_n). \quad (129)$$

These cover all n -point scalar integrals for the gravity scalar integral in $d = 3$. Therefore, if one is provided with an n -point Mellin-momentum amplitude, one can simply follow the map to obtain the wave-function coefficients, requiring only the computation of a finite number of residues without the need for any new time integrals.

3. Examples on recursive integral

We start with a four-point GR s -channel exchange scalar integral

$$\frac{1}{\mathcal{D}_{k_{12}}^d} \rightarrow \mathcal{I}_{\{12\}}^{\text{GR}}(k_1, k_2, k_3, k_4, k_{12}), \quad (130)$$

which has the form

$$\mathcal{I}_{\{12\}}^{\text{GR}}(k_1, k_2, k_3, k_4, k_{12}) = \int \frac{dz_1}{z_1^4} \frac{dz_2}{z_2^4} z_1^2 z_2^2 \prod_{i=1}^2 \phi_{\Delta=3}(z_i, k_i) G_{\Delta=3}(\mathbf{k}_{12}, z_1, z_2) \prod_{j=3}^4 \phi_{\Delta=3}(z_j, k_j). \quad (131)$$

With representation (127), any scalar integral in $d = 3$ can be derived recursively starting from the three-point contact integral:

$$\begin{aligned}
\mathcal{C}_3(k_1, k_2, k_3) &= \int \frac{dz_1}{z_1^4} z_1^2 \phi_{\Delta=3}(z_1, k_1) \phi_{\Delta=3}(z_1, k_2) \phi_{\Delta=3}(z_1, k_3) \\
&= \frac{k_2 k_3 k_1}{(k_1 + k_2 + k_3)^2} - k_1 - k_2 - k_3 + \frac{k_1 k_2 + k_3 k_2 + k_1 k_3}{k_1 + k_2 + k_3}.
\end{aligned} \tag{132}$$

In this case, we have

$$\begin{aligned}
\mathcal{I}_{\{12\}}^{\text{GR}}(k_1, k_2, k_3, k_4, k_{12}) &= \frac{1}{2i} \text{Res}_p \frac{p^{-2}}{k_{12}^2 + p^2} (\mathcal{C}_3(k_1, k_2, ip) - \mathcal{C}_3(k_1, k_2, -ip)) (\mathcal{C}_3(k_3, k_4, ip) - \mathcal{C}_3(k_3, k_4, -ip)) \\
&= \frac{2k_1 k_2 k_3 k_4 (E_L E_R + E k_{12})}{E_L^2 E^3 E_R^2} + \frac{k_1 k_2 (E_L (k_3 + k_4) + E k_{12})}{E_L^2 E^2 E_R} + \frac{k_3 k_4 (E k_{12} + E_R (k_1 + k_2))}{E_L E^2 E_R^2} \\
&\quad + \frac{E_L E_R - k_{12}^2}{E_L E E_R},
\end{aligned} \tag{133}$$

where the residues are the same as YM and $E_L = k_1 + k_2 + k_s$, $E_R = k_3 + k_4 + k_s$, and $E = k_1 + k_2 + k_3 + k_4$. We should emphasize that there is no integral to do just taking residues of the integral variable p . This agrees with the conventional scalar integral computation using (111) in $d = 3$. Once we have it, the five-point exchange with two propagators

$$\frac{1}{\mathcal{D}_{k_{12}}^d \mathcal{D}_{k_{45}}^d} \rightarrow \mathcal{I}_{\{12\}\{45\}}^{\text{GR}}(k_1, k_2, k_3, k_4, k_5, k_{12}, k_{45}) \tag{134}$$

can be expressed by

$$\begin{aligned}
\mathcal{I}_{\{12\}\{45\}}^{\text{GR}}(k_1, k_2, k_3, k_4, k_5, k_{12}, k_{45}) &= \int_{-\infty}^{\infty} \frac{dp_1}{2\pi i} \frac{-ip_1^{-2}}{k_{12}^2 + p_1^2} \bar{\mathcal{C}}_1(k_1, k_2, ip_1) \bar{\mathcal{I}}_{\{45\}}^{\text{GR}}(ip_1, k_3, k_4, k_5), \\
&= \text{Res}_{p_1} \text{Res}_{p_2} \frac{64k_3^3 p_1^4 p_2^4 (k_1^2 + 4k_2 k_1 + k_2^2 + p_1^2) (k_4^2 + 4k_5 k_4 + k_5^2 + p_2^2)}{(k_3^2 + (p_1 - p_2)^2)^2 (k_3^2 + (p_1 + p_2)^2)^2 ((k_1 + k_2)^2 + p_1^2)^2 (k_{12}^2 + p_1^2)^2 ((k_4 + k_5)^2 + p_2^2)^2 (k_{45}^2 + p_2^2)^2},
\end{aligned} \tag{135}$$

where we define a shifted function: $\bar{\mathcal{C}} = \mathcal{C}(k_1, \dots, ip) - \mathcal{C}(k_1, \dots, -ip)$ and $\bar{\mathcal{I}} = \mathcal{I}(ip, \dots, k_n) - \mathcal{I}(-ip, \dots, k_n)$.

C. Cosmological correlators

After finishing discussing the scalar integral we can get the wave function coefficients for YM and GR explicitly. Then in the following, we will review how to get cosmological correlators starting from wave-function coefficients of arbitrary spins. We use subscript *in-in* to distinguish the CFT correlator with the cosmological correlator (*in-in* correlator)

$$\langle \sigma(\mathbf{k}_1) \dots \sigma(\mathbf{k}_n) \rangle_{\text{in-in}} = \frac{\int \mathcal{D}\sigma |\Psi(\sigma)|^2 \sigma(\mathbf{k}_1) \dots \sigma(\mathbf{k}_n)}{\int \mathcal{D}\sigma |\Psi(\sigma)|^2}, \tag{136}$$

where the wave function expanding in field profiles on the future spacelike boundary takes the form

$$\Psi(\sigma) \propto \exp \left[-\frac{1}{2} \int \sigma_1 \sigma_2 \Psi_2 - \frac{1}{3!} \int \sigma_1 \dots \sigma_3 \Psi_3 - \frac{1}{4!} \int \sigma_1 \dots \sigma_4 \Psi_4 + \dots \right], \tag{137}$$

and we use the shorthand notation

$$\int \sigma_1 \dots \sigma_n = \int \prod_{i=1}^n \frac{d^3 k_i}{(2\pi)^3} \sigma(\mathbf{k}_i) \delta^3 \left(\sum_{i=1}^n \mathbf{k}_i \right), \tag{138}$$

and the wave-function coefficients are

$$\Psi_n = \Psi_n(\mathbf{k}_1, \dots, \mathbf{k}_n), \tag{139}$$

which are given in (12). Hence once we determine the Mellin-momentum amplitude we can use the map in Sec. V, which computes all the scalar integrals to obtain the wave-function coefficients.

In order to get an intuitive relation between wave-function coefficients and cosmological correlators, we can rephrase the path integral in (136) in terms of an effective action:

$$\langle \sigma(\mathbf{k}_1) \dots \sigma(\mathbf{k}_n) \rangle_{\text{in-in}} = \frac{\int \mathcal{D}\sigma e^{-S(\sigma)} \sigma(\mathbf{k}_1) \dots \sigma(\mathbf{k}_n)}{\int \mathcal{D}\sigma e^{-S(\sigma)}}, \quad (140)$$

where

$$S(\sigma) = \frac{1}{2} \int \sigma_1 \sigma_2 \text{Re}\Psi_2 + \frac{1}{3!} \int \sigma_1 \sigma_2 \sigma_3 \text{Re}\Psi_3 + \frac{1}{4!} \int \sigma_1 \dots \sigma_4 \text{Re}\Psi_4 + \frac{1}{6!} \int \sigma_1 \dots \sigma_6 \text{Re}\Psi_6 + \dots \quad (141)$$

The Feynman rules of this action are easy to read off:

$$\text{---} \underset{k}{\text{---}} = \frac{1}{\text{Re}\Psi_2}, \quad \text{---} \text{---} \text{---} = \text{Re}\Psi_3, \quad \text{---} \text{---} \text{---} = \text{Re}\Psi_4, \quad \dots \quad (142)$$

The effective action used to compute the *in-in* correlators has an infinite number of vertices, each of which has a perturbative expansion in the bulk coupling. We can then compute the *in-in* correlators using the effective action, whose vertices will contain different powers of the bulk coupling and then keep all contributions that contain a given order in the coupling. In practice, we can form Feynman diagrams out of the rules written in (142). And the additional rule is to also dress the diagrams with external propagators. As an example, at tree-level, the three-point *in-in* correlator comes from a three-point contact diagram where we only keep the tree-level contribution to the coefficient $\text{Re}\Psi_3$,

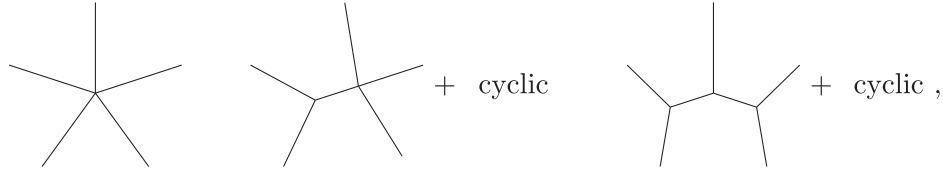
$$\langle \sigma(\mathbf{k}_1) \sigma(\mathbf{k}_2) \sigma(\mathbf{k}_3) \rangle_{\text{in-in}} = \frac{\text{Re}\Psi_3}{\prod_{a=1}^3 \text{Re}\Psi_2(\mathbf{k}_a)} \quad (143)$$

As a slightly interesting example, a tree-level four-point *in-in* correlator comes from the four-point wave function coefficient $\text{Re}\Psi_4$, together with exchange diagram in three channels with a propagator as $1/\text{Re}\Psi_2(k)$,

$$\langle \sigma(\mathbf{k}_1) \sigma(\mathbf{k}_2) \sigma(\mathbf{k}_3) \sigma(\mathbf{k}_4) \rangle_{\text{in-in}} = \left[4 \sum_{s,t,u} \sum_h \frac{1}{\text{Re}\Psi_2(\mathbf{k}_s)} \text{Re}\Psi_3(\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_s) \text{Re}\Psi_3(\mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_s) - 2 \text{Re}\Psi_4 \right] \prod_{a=1}^4 \frac{1}{\text{Re}\Psi_2(\mathbf{k}_a)}, \quad (144)$$

where $\sum_h$ means we also need to sum over the internal helicity for spinning particles.

Finally for the purpose of this paper, the tree-level five-point cosmological correlator comes from a five-point wave-function coefficient $\text{Re}\Psi_5$ together with exchange diagrams of two topologies:



with the expression given as

$$\begin{aligned}
 & \langle \sigma(k_1) \sigma(k_2) \sigma(k_3) \sigma(k_4) \sigma(k_5) \rangle_{\text{in-in}} \\
 &= \left[\text{Re}\Psi_5 + \sum_h \frac{1}{\text{Re}\Psi_2(k_{12})} \text{Re}\Psi_3(k_1, k_2, -k_{12}) \text{Re}\Psi_4(k_{12}, k_3, k_4, k_5) + \text{cyclic} \right. \\
 & \quad \times \left. \sum_{h_1, h_2} \frac{1}{\text{Re}\Psi_2(k_{12})} \frac{1}{\text{Re}\Psi_2(k_{45})} \text{Re}\Psi_3(k_1, k_2, -k_{12}) \text{Re}\Psi_3(k_{12}, k_3, k_{45}) \text{Re}\Psi_3(-k_{45}, k_4, k_5) + \text{cyclic} \right] \prod_{a=1}^5 \frac{1}{\text{Re}\Psi_2(k_a)}. \quad (145)
 \end{aligned}$$

Such graphical rules also work at loop level; see the example in Appendix A in [75].

VI. CONCLUSION

In this paper, we build upon the concept of the on-shell Mellin-momentum amplitudes for cosmological correlators, an idea first introduced in [29]. To demonstrate the power of this approach, we have conducted a detailed bootstrap derivation of the full Mellin-momentum amplitudes for both Yang-Mills and general relativity at the five-point in anti-de Sitter space [39]. Our findings reveal that the analytic structure of Mellin-momentum amplitudes for n -points is exceptionally straightforward and recursively computable, similar to the process for amplitudes in flat space. This finding supports the notion that these amplitudes in (A)dS spaces serve a role analogous to that of the S-matrix in flat space.

Practically, the accuracy of our bootstrapped result for the five-graviton amplitude is inherently verifiable, as any discrepancies would appear as nonphysical poles. Moreover, the structure of the five-graviton amplitude supports the extension of the double copy construction, as proposed in [29], to higher-point amplitudes—a prospect that warrants further investigation.

In flat space, the bootstrap method effectively makes Lorentz invariance, locality, and unitarity manifest while leaving a gauge redundancy. However, the situation is slightly different in cosmology, where our observables are boundary correlators. This necessitates setting the bulk coordinate $\varepsilon_z \rightarrow 0$, so our bootstrap strategy would be making locality and unitarity manifest with no gauge redundancy while leaving the spacetime symmetry not manifest.

Although the amplitude serves as the integrand for AdS boundary correlators or wave-function coefficients for cosmological correlators, as demonstrated in the main

text, it is still convenient to go back to momentum space. In three dimensions ($d = 3$), we show that higher-point scalar integrals can be iteratively constructed from their lower-point counterparts. This capability simplifies the translation of our amplitude results into the Yang-Mills and, most crucially, the gravity quadrispectrum for cosmological correlators. We also hope that this paper will act as an introductory guide for the amplitude community interested in cosmological correlators. In this formalism, newcomers can easily adopt Feynman rules, closely resembling those in flat space, to compute their favorite amplitudes and then map these to the corresponding correlators. Alternatively, they can follow the amplitude bootstrap approach detailed in this paper to build their own favorite amplitude.

There are several directions we can explore. Our first dream for the future is to discover an all multiplicity formula like the Park-Taylor formula [76] for Yang-Mills in (A)dS. Especially, in this paper we discovered that the above five-point factorization is indeed enough to fix the Yang-Mills amplitude. So for such a highly constrained theory, it is natural to expect that some form of an all multiplicity formula exists. It would be very interesting to combine the recent development of twistors in (A)dS space [77], which makes conformal symmetry manifest. We are also keen to extend our gravity calculations to loop level. Once employing our bootstrap approach to determine the Mellin-momentum amplitude, we are then left with scalar loop integrals. Particularly, it was shown in [75] that the scalar loop integral for the in-in correlator is closer to the S-matrix in flat space. More broadly, our bootstrap strategy does not rely on spacetime symmetry [12], positioning our AdS study as a foundational example applicable to various curved backgrounds, including FLRW spacetime and black hole scenarios. Lastly, utilizing the techniques developed herein, we have successfully derived the soft limit of

YM and GR [64].¹⁷ It would also be interesting to extend our understanding into the massive case [79–81].

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DATA AVAILABILITY

The data and code that support the findings of this article are openly available [40].

APPENDIX A: EXAMPLE FOR FIELD REDEFINITION

Consider field redefinition for a massless free field in the bulk with conformal dimension $\Delta = 3$ and $d = 3$.

$$\begin{aligned} & \delta\langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle \\ &= 6\alpha \int_0^\infty \frac{dz}{z^4} \left\{ -z^2 \prod_{i=1}^4 \phi(k_i, z) (\mathbf{k}_1 \cdot (\mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) + \mathbf{k}_2 \cdot (\mathbf{k}_3 + \mathbf{k}_4) + \mathbf{k}_3 \cdot \mathbf{k}_4) + z^2 \sum_{1 \leq i < j \leq 4} \partial_{z_i} \partial_{z_j} \prod_{i=1}^4 \phi(k_i, z) \right\} \\ &= -3\alpha(k_1^3 + k_2^3 + k_3^3 + k_4^3). \end{aligned} \quad (\text{A4})$$

So we have shown that

$$\begin{aligned} & \langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle \\ & \rightarrow \langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle - 3\alpha(k_1^3 + k_2^3 + k_3^3 + k_4^3) \end{aligned} \quad (\text{A5})$$

and k_i^3 is analytic in three other momenta and so is a local term. The correlator is not strictly invariant under field

The original Lagrangian is given by

$$\mathcal{L} = \partial_m \phi \partial^m \phi \quad (\text{A1})$$

where m runs over (z, x) coordinates. And then under field redefinition $\phi \rightarrow \phi + \alpha\phi^3$ we have the Lagrangian changing to

$$\mathcal{L} \rightarrow \mathcal{L} + 6\alpha\phi^2 \partial_m \phi \partial^m \phi + O(\phi^6). \quad (\text{A2})$$

1. Correlator under field redefinition

The new interaction $6\alpha\phi^2 \partial_m \phi \partial^m \phi$ will change the four-point correlator as

$$\begin{aligned} & \langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle \\ & \rightarrow \langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle + \delta\langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle \end{aligned} \quad (\text{A3})$$

where $\delta\langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle$ should be a local term. Considering Fourier transformation, the field in the bulk is $\phi(x, z) = \int d^d k e^{-ik \cdot x} \phi(k, z)$ with $\phi(k, z) = (1 + kz)e^{-kz}$ and under field redefinition, the new Feynman rule directly gives

redefinition or say it is only invariant up to boundary local terms.

2. Amplitude under field redefinition

Let us suppose additional $6\alpha\phi^2 \partial_m \phi \partial^m \phi$ produces changes in Mellin-momentum amplitude as

$$\mathcal{A} \rightarrow \mathcal{A} + \delta\mathcal{A}. \quad (\text{A6})$$

From directly reading off the Feynman rule,

¹⁷There is a similar result for fermions [78] in the correlator version.

$$\begin{aligned}
\delta\mathcal{A} &= 6\alpha \left\{ -z^2 \left(\frac{1}{2} (k_1^2 + k_2^2 + k_3^2 + k_4^2) \right) \right. \\
&\quad \left. + \sum_{1 \leq i < j \leq 4} (-2s_i + d/2)(-2s_j + d/2) \right\} \delta(d - 2(s_{1234})) \\
&= 6\alpha \{ d^2 - 3d(s_{1234}) + 2(s_{1234})^2 \} \delta(d - 2(s_{1234})) \\
&= 0,
\end{aligned} \tag{A7}$$

where in the second equation we have used the on-shell condition for massless field $k_i^2 z^2 = \frac{16s_i^2 - d^2}{4}$. And on the support of the Mellin delta function, we see that the change of the Mellin-momentum amplitude is indeed zero.

APPENDIX B: EXAMPLES ON CONFORMAL WARD IDENTITY

For AdS correlators, the conformal Ward identities provide a natural analog to momentum conservation in the S-matrix, as mentioned in the main text. In the first part of this section, using the example of a massless ϕ^4 theory, we will illustrate how these identities do not lead to a vanishing right-hand side. We will then demonstrate how this issue is resolved through the enforcement of an on-shell condition, boundary momentum conservation, and the use of the Mellin delta function.

The second part of the section is to show how the Mellin-momentum amplitude is exactly invariant under special conformal identity. We use the example of $\mathcal{A}_{(J_1\varphi_2\varphi_3)}$ to demonstrate and at the same time it is compared with $\langle J_1\varphi_2\varphi_3 \rangle$ as computed in [11].

1. Correlator: Conformal ward identity on contact diagram

We start by defining [49]

$$\mathcal{C}_n^\Delta \equiv \int \frac{dz}{z^{d+1}} \prod_{a=1}^n \phi_\Delta(k_a, \eta), \tag{B1}$$

where $\mathcal{C}_n^\Delta$ is the scalar contact integral with conformal dimension Δ and $\phi_\Delta(k_a, \eta)$ is the bulk-to-boundary propagator. The generators in the conformal group are

$$\begin{aligned}
P^\mu &= k^\mu, \\
D &= k^\mu \partial_\mu + (d - \Delta), \\
K_\mu &= k_\mu \partial^\nu \partial_\nu - 2k^\nu \partial_\nu \partial_\mu - 2(d - \Delta) \partial_\mu,
\end{aligned} \tag{B2}$$

which act on the bulk-to-boundary propagator and gives

$$\begin{aligned}
P^\mu \phi_\Delta &= k^\mu \phi_\Delta, \\
D \phi_\Delta &= z \frac{\partial}{\partial z} \phi_\Delta, \\
K_\mu \phi_\Delta &= -z^2 k_\mu \phi_\Delta.
\end{aligned} \tag{B3}$$

So defining

$$\mathcal{D}_a \cdot \mathcal{D}_b = \frac{1}{2} (P_a^i K_{bi} + K_{ai} P_b^i) + D_a D_b, \tag{B4}$$

using

$$(\mathcal{D}_a \cdot \mathcal{D}_b) \phi_\Delta^a \phi_\Delta^b = z^2 [\partial_z \phi_\Delta^a \partial_z \phi_\Delta^b - (\mathbf{k}_a \cdot \mathbf{k}_b) \phi_\Delta^a \phi_\Delta^b]. \tag{B5}$$

One can immediately see that the calculation for massless and $d = 3$ is basically the same as our previous field redefinition examples:

$$(\mathcal{D}_1 \cdot \mathcal{D}_1 + \mathcal{D}_1 \cdot \mathcal{D}_2 + \mathcal{D}_1 \cdot \mathcal{D}_3 + \mathcal{D}_1 \cdot \mathcal{D}_4) \mathcal{C}_1 = -k_1^3, \tag{B6}$$

which is a boundary local term. So one can see that the conformal ward identity is not vanishing but up to local terms.

2. Amplitude: Conformal ward identity on contact diagram

Using the Mellin transformation of the bulk-to-boundary propagator in (10), we have

$$\begin{aligned}
P^\mu \phi_\Delta &= k^\mu \phi_\Delta, \\
D \phi_\Delta &= \left(\frac{d}{2} - 2s \right) \phi_\Delta, \\
K_\mu \phi_\Delta &= -z^2 k_\mu \phi_\Delta.
\end{aligned} \tag{B7}$$

Then (B5) becomes

$$\begin{aligned}
(\mathcal{D}_a \cdot \mathcal{D}_b) \phi_\Delta^a \phi_\Delta^b &= \left[\left(\frac{d}{2} - 2s_a \right) \left(\frac{d}{2} - 2s_b \right) - z^2 (\mathbf{k}_a \cdot \mathbf{k}_b) \right] \phi_\Delta^a \phi_\Delta^b.
\end{aligned} \tag{B8}$$

That gives the Mellin-momentum amplitude version of (B6):

$$\begin{aligned}
&(\mathcal{D}_1 \cdot \mathcal{D}_1 + \mathcal{D}_1 \cdot \mathcal{D}_2 + \mathcal{D}_1 \cdot \mathcal{D}_3 + \mathcal{D}_1 \cdot \mathcal{D}_4) \mathcal{C}_4 \\
&= \left[\Delta_1 (\Delta_1 - d) \left(\frac{d}{2} - 2s_1 \right) + \left(\frac{d}{2} - 2s_2 \right) - z^2 (\mathbf{k}_1 \cdot \mathbf{k}_2) \right. \\
&\quad \left. + \left(\frac{d}{2} - 2s_1 \right) \left(\frac{d}{2} - 2s_3 \right) - z^2 (\mathbf{k}_1 \cdot \mathbf{k}_3) + \left(\frac{d}{2} - 2s_1 \right) \left(\frac{d}{2} - 2s_4 \right) - z^2 (\mathbf{k}_1 \cdot \mathbf{k}_4) \right] \mathcal{A}_4.
\end{aligned} \tag{B9}$$

For direct comparison, we can restrict ourselves to massless scalar field $\Delta = d$, and get

$$\begin{aligned}
 & (\mathcal{D}_1 \cdot \mathcal{D}_1 + \mathcal{D}_1 \cdot \mathcal{D}_2 + \mathcal{D}_1 \cdot \mathcal{D}_3 + \mathcal{D}_1 \cdot \mathcal{D}_4) \mathcal{A}_4 \\
 &= \left[\frac{3d^2}{4} - d(3s_1 + s_2 + s_3 + s_4) + z^2 k_1^2 + 4s_1(s_2 + s_3 + s_4) \right] \mathcal{A}_4 \\
 &= \frac{1}{2} (d - 4s_1)(d - 2(s_1 + s_2 + s_3 + s_4)) \mathcal{A}_4 \\
 &= 0.
 \end{aligned} \tag{B10}$$

In the second line, we applied momentum conservation, while in the third line, we utilized the massless on-shell condition (13) for $z^2 k_1^2$. The final line follows from the fact that this contact Mellin-momentum amplitude is evaluated on the support of the Mellin delta function (14), given by $\delta(d - 2(s_1 + s_2 + s_3 + s_4))$. Thus, the on-shell condition, boundary momentum conservation, and the Mellin delta function imply that the AdS analogs of the Mandelstam invariants are exactly zero.

3. Correlator: Review on special conformal ward identity with spin

The special conformal generator $\tilde{K}$ on the boundary spacetime acts on conformal scalar φ and spin-1 field J as [11]

$$\begin{aligned}
 \tilde{K}_\varphi^\mu \varphi &= -K^\mu \varphi, \\
 \tilde{K}_J^\mu J_\nu \varepsilon^\nu &= \left(-\varepsilon^\nu K^\mu + 2\varepsilon^\mu \frac{k^\nu}{k^2} \right) J_\nu,
 \end{aligned} \tag{B11}$$

where K_μ acting on explicit tensor indices is given as

$$\begin{aligned}
 K^\mu O_{\nu_1 \dots \nu_\ell} &= [2(\Delta - 3)\partial_{k_\mu} + k^\mu \partial_{k^\rho} \partial_{k_\rho} - 2k^\rho \partial_{k^\rho} \partial_{k_\mu} \\
 &\quad - 2\partial_{k_\rho} \Sigma^\mu_\rho] O_{\nu_1 \dots \nu_\ell},
 \end{aligned} \tag{B12}$$

with

$$\Sigma_{\mu\rho} O_{\nu_1 \dots \nu_\ell} = \ell(O_{\mu(\nu_1 \dots \nu_{\ell-1}} \eta_{\nu_\ell)\rho} - O_{\rho(\nu_1 \dots \nu_{\ell-1}} \eta_{\nu_\ell)\mu}). \tag{B13}$$

Now we consider the boundary correlator $\langle J_1 \varphi_2 \varphi_3 \rangle$ which is

$$\langle J_1 \varphi_2 \varphi_3 \rangle = \frac{\varepsilon_1 \cdot k_2}{k_1 + k_2 + k_3}. \tag{B14}$$

After working out the algebra, the change $\delta \langle J_1 \varphi_2 \varphi_3 \rangle$ under special conformal transformation is

$$(\tilde{K}_{\varphi_3}^\mu + \tilde{K}_{\varphi_2}^\mu + \tilde{K}_{J_1}^\mu) \langle J_1 \varphi_2 \varphi_3 \rangle = -\frac{(k_2 - k_3)\varepsilon_1^\mu}{k_1^2} \tag{B15}$$

which is a boundary local term.

4. Amplitude: Special conformal ward identity with spin

Starting from the definition of amplitude (12), we first record how the generators cross the bulk-to-boundary propagator, and then the generators $\hat{K}$ directly acting on the amplitudes have the form

$$\begin{aligned}
 \hat{K}_\varphi^\mu \mathcal{A} &= -z^2 k^\mu \mathcal{A}|_{s \rightarrow s-1} + \{2(2s - d/2)\partial_{k_\mu} + k^\mu \partial_{k^\rho} \partial_{k_\rho} - 2k^\rho \partial_{k^\rho} \partial_{k_\mu}\} \mathcal{A} \\
 \hat{K}_J^\mu \mathcal{A}_\nu \varepsilon^\nu &= \varepsilon^\nu z^2 k^\mu \mathcal{A}_\nu|_{s \rightarrow s-1} + \left[-\varepsilon^\nu \{2(2s - d/2)\partial_{k_\mu} + k^\mu \partial_{k^\rho} \partial_{k_\rho} - 2k^\rho \partial_{k^\rho} \partial_{k_\mu} - 2(v^\rho + \partial_{k_\rho})\Sigma^\mu_\rho\} + 2\varepsilon^\mu \frac{k^\nu}{k^2} \right] \mathcal{A}_\nu,
 \end{aligned} \tag{B16}$$

where s is the Mellin variable associated to the external leg and $v^\mu = k^\mu \frac{(\Delta - d/2 - 2s)}{k^2}$ from $\partial_{k^\mu} \phi_\Delta(s, zk)$.

Starting from the Mellin-momentum amplitude corresponds to $\langle J_1 \varphi_2 \varphi_3 \rangle$:

$$\mathcal{A}_{\langle J_1 \varphi_2 \varphi_3 \rangle} = z \varepsilon_1 \cdot k_2, \tag{B17}$$

where each generator acting on the amplitude gives

$$\begin{aligned}
z\hat{K}_{\varphi_3}^\mu \varepsilon_1 \cdot k_2 &= z^3 k_3^\mu \varepsilon_1 \cdot k_2, \\
z\hat{K}_{\varphi_2}^\mu \varepsilon_1 \cdot k_2 &= z^3 k_2^\mu \varepsilon_1 \cdot k_2 - 2z(2s_2 - d/2)\varepsilon_1^\mu, \\
z\hat{K}_{J_1}^\mu \varepsilon_1 \cdot k_2 &= z^3 k_1^\mu \varepsilon_1 \cdot k_2 + 2z\varepsilon_1^\nu v_1^\rho \Sigma_\rho^\mu k_{2\nu} + 2z\varepsilon_1^\mu \frac{k_1 \cdot k_2}{k_1^2}.
\end{aligned} \tag{B18}$$

Here we know from (B13) that $\Sigma_{\mu\rho} k^\nu = k_\mu \delta_\rho^\nu - k_\rho \delta_\mu^\nu$. Plugging this in the third line of (B18) we find

$$z\hat{K}_{J_1}^\mu \varepsilon_1 \cdot k_2 = z^3 k_1^\mu \varepsilon_1 \cdot k_2 - z\varepsilon_1^\nu (-k_1^2 - k_2^2 + k_3^2) \frac{(d/2 - 2s_1 - 2)}{k_1^2}. \tag{B19}$$

Adding (B19) and the first two lines of (B18) together gives

$$(\hat{K}_{\varphi_3}^\mu + \hat{K}_{\varphi_2}^\mu + \hat{K}_{J_1}^\mu) \mathcal{A}_{\langle J_1 \varphi_2 \varphi_3 \rangle} = -2z(2s_2 - d/2)\varepsilon_1^\mu - z\varepsilon_1^\mu (d/2 - 2s_1 - 2) + 2z\varepsilon_1^\mu (k_2^2 - k_3^2) \frac{(d/2 - 2s_1 - 2)}{k_1^2}. \tag{B20}$$

Using on-shell condition on the numerator [see Eq. (66)] and on the denominator i.e.,

$$\frac{X}{k^2} = \frac{z^2 X|_{s \rightarrow s-1}}{4(s-1)^2 - (\Delta - d/2)^2} \tag{B21}$$

and in $d = 3$, we have

$$\begin{aligned}
(\tilde{K}_{\varphi_3}^\mu + \tilde{K}_{\varphi_2}^\mu + \tilde{K}_{J_1}^\mu) \mathcal{A}_{\langle J_1 \varphi_2 \varphi_3 \rangle} &= z\varepsilon_1^\mu (-2s_1 - 1/2 + 2(3/2 - 2s_2)) - 8z\varepsilon_1^\mu \frac{(s_2^2 - s_3^2)}{4s_1 - 5} \\
&= z\varepsilon_1^\mu \left(\frac{(4s_1 + 4s_2 - 4s_3 - 5)(5 - (4s_1 + 4s_2 + 4s_3))}{2(4s_1 - 5)} \right).
\end{aligned}$$

Noticing there is an additional power of z on the support of the Mellin delta (14), and thus we have

$$5 - 4(s_1 + s_2 + s_3) = 0.$$

Finally we find

$$(\tilde{K}_{\varphi_3}^\mu + \tilde{K}_{\varphi_2}^\mu + \tilde{K}_{J_1}^\mu) \mathcal{A}_{\langle J_1 \varphi_2 \varphi_3 \rangle} = 0. \tag{B22}$$

So using the on-shell condition, three-momentum conservation, and Mellin delta function, we show that the three-point Mellin-momentum amplitude $\mathcal{A}_{\langle J_1 \varphi_2 \varphi_3 \rangle}$ is exactly invariant under special conformal transformation.

APPENDIX C: DERIVATION OF GRAVITY PROPAGATORS

The graviton will be parametrized as $g_{mn} = \tilde{g}_{mn} + \frac{\mathcal{R}^2}{z^2} h_{mn}$, and we will use the following gauge condition¹⁸:

$$\begin{aligned}
i\eta^{\mu\nu} k_\mu h_{z\nu} &= \frac{1}{2} \eta^{\mu\nu} \partial_z h_{\mu\nu} + \frac{d}{2z} h_{zz}, \\
\eta^{\nu\rho} k_\rho h_{\mu\nu} &= \frac{1}{2} k_\mu (\eta^{\nu\rho} h_{\nu\rho} + \beta h_{zz}).
\end{aligned} \tag{C1}$$

We can then expand it in the Einstein field equation,

¹⁸Here β is just a parameter that will not appear in the final answer but serve as a consistency check.

$$\begin{aligned}
k^2 h_{zz} &= \frac{2\kappa}{d-1} [(d-2)T_{zz} - \eta^{\mu\nu} T_{\mu\nu}], \\
k^2 h_{z\mu} &= 2\kappa T_{z\mu} + \frac{i}{2z} (d-2 - \beta z \partial_z) k_\mu h_{zz}, \\
\mathcal{D}_k^d h_{\mu\nu} &= 2\kappa z^2 T_{\mu\nu} - \frac{2\kappa z^2}{(d-1)} \eta_{\mu\nu} (T_{zz} + \eta^{\rho\sigma} T_{\rho\sigma}) + [(\beta-1)z^2 k_\mu k_\nu + \eta_{\mu\nu} (z\partial_z - d)] h_{zz} + iz[(d-1) - z\partial_z] (k_\mu h_{\nu z} + k_\nu h_{\mu z}), \\
k^2 h_\mu^\mu &= \frac{2\kappa}{d-1} ((-\beta)T_\mu^\mu + (2-2d-2\beta+d\beta)T_{zz}).
\end{aligned} \tag{C2}$$

As before, applying the gauge condition implies the energy-momentum conservation:

$$\begin{aligned}
g^{mn} \nabla_m T_{nz} &= 0 \Rightarrow (z\partial_z - d + 2)T_{zz} + izk^\mu T_{\mu z} + T_{\mu\nu} \eta^{\mu\nu} = 0, \\
g^{mn} \nabla_m T_{n\mu} &= 0 \Rightarrow (z\partial_z - d + 1)T_{z\mu} + izk^\nu T_{\nu\mu} = 0.
\end{aligned} \tag{C3}$$

We will do the tensor decomposition on the spin-1 and spin-2 fields to obtain the transverse and traceless part [74],

$$\begin{aligned}
h_{z\mu} &= h_{z\mu}^\perp - \frac{ik_\mu}{2k^2} \left(\partial_z h + \frac{d}{z} h_{zz} \right), \\
h_{\mu\nu} &= h_{\mu\nu}^\perp + \eta_{\mu\nu} \frac{h - \beta h_{zz}}{2(d-1)} + \frac{k_\mu k_\nu}{k^2} \frac{(d-2)h + d\beta h_{zz}}{2(d-1)},
\end{aligned} \tag{C4}$$

where $k^\mu h_{z\mu}^\perp = 0$ and $k^\mu h_{\mu\nu}^\perp = 0 = \eta^{\mu\nu} h_{\mu\nu}^\perp$. So this means only the following equations are kinematic, everything else is just constraint equations:

$$\begin{aligned}
k^2 h_{z\mu}^\perp &= 2\kappa \Pi_{\mu\nu} T_z^\nu, \\
\mathcal{D}_k^d h_{\mu\nu}^\perp &= 2\kappa z^2 \Pi_{2,2}^{\mu\nu,\rho\sigma} T_{\rho\sigma}, \\
k^2 h_{zz} &= \frac{2\kappa}{d-1} [(d-2)T_{zz} - \eta^{\mu\nu} T_{\mu\nu}], \\
k^2 h_\mu^\mu &= \frac{2\kappa}{d-1} ((-\beta)T_\mu^\mu + (2-2d-2\beta+d\beta)T_{zz}).
\end{aligned} \tag{C5}$$

Therefore, the action that determines the bulk-to-bulk propagator is now given by

$$\begin{aligned}
S^{EG} &= \int \frac{dz}{z^{d+1}} (h_{\mu\nu} T^{\mu\nu} - 2h_{z\mu} T^{z\mu} + h_{zz} T^{zz}) \\
&= \int \frac{dz}{z^{d+1}} \left(T_{\mu\nu} \frac{\Pi_{2,2}^{\mu\nu,\rho\sigma}}{\mathcal{D}_k^d} T_{\rho\sigma} - 2 \frac{T_{z\mu} \Pi_{\mu\nu} T_{z\nu}}{z^2 k^2} + \frac{2((d-2)T_{zz} T_{zz} - T_{zz} T - T T_{zz})}{(d-1)z^2 k^2} \right. \\
&\quad \left. + \frac{2d(T + (2-d)T_{zz} + zT'_{zz})(T + (2-d)T_{zz} + zT'_{zz})}{(d-1)z^4 k^4} \right).
\end{aligned} \tag{C6}$$

When the graviton is coupled to the conformally coupled scalar, the stress tensor obeys the traceless condition $T_{zz} + T = 0$, which simplifies the expression and in $d = 3$

$$\begin{aligned}
S^{\text{conformal}} &= \int \frac{dz}{z^4} \left(\frac{T_{\mu\nu} \Pi_{2,2}^{\mu\nu,\rho\sigma}}{\mathcal{D}_k^{\Delta=3}} - 2 \frac{T_{z\mu} \Pi_{\mu\nu} T_{z\nu}}{z^2 k^2} + 3 \frac{TT}{k^2} + \frac{3}{z^2} \frac{(2T - zT')(2T - zT')}{z^4 k^4} \right) \\
&= \int \frac{dz}{z^4} \left(\frac{T_{\mu\nu} \Pi_{2,2}^{\mu\nu,\rho\sigma}}{\mathcal{D}_k^{\Delta=3}} - 2 \frac{T_{z\mu} \Pi_{\mu\nu} T_{z\nu}}{z^2 k^2} - 3 \frac{T \mathcal{D}_k^{\Delta=2} T}{z^4 k^4} \right),
\end{aligned} \tag{C7}$$

where $\mathcal{D}_k^{\Delta=2}$ is the EoM operator for the $\Delta = 2$ scalar acting on T .¹⁹

¹⁹Note for a conformally coupled scalar, the $\frac{TT}{k^4} = \Pi_{2,0}$ in [11].

APPENDIX D: POLARIZATION SUMS

In this appendix, we provide the details of the polarization sums employed in this paper. Following the boundary transverse gauge [53] (this is the same as in the QFT textbook [82] with Coulomb gauge):

$$\sum_{h=\pm} \varepsilon_\mu(k, h) \varepsilon_\nu(k, h)^* = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \equiv \Pi_{\mu\nu}, \quad (\text{D1})$$

$$\begin{aligned} \sum_{h=\pm} \varepsilon_{\mu\nu}(k, h) \varepsilon_{\rho\sigma}(k, h)^* \\ = \frac{1}{2} \Pi_{\mu\rho} \Pi_{\nu\sigma} + \frac{1}{2} \Pi_{\mu\sigma} \Pi_{\nu\rho} - \frac{1}{d-1} \Pi_{\mu\nu} \Pi_{\rho\sigma}, \end{aligned} \quad (\text{D2})$$

which are transverse and traceless projection tensors. Let us return to QED in Coulomb gauge for a moment. The polarization tensor above which appears in the photon propagator is not Lorentz invariant on its own, but we can

restore Lorentz invariance to obtain the covariant photon propagator. This is the same logic that we use to derive all of the polarization sums below by demanding conformal invariance.

Let us explicitly write out the polarization sums at four-point; see also Refs. [10,11,83] for the case of conformally coupled scalar:

$$\begin{aligned} \Pi_{1,1} &\equiv \frac{1}{4} (k_1^\mu - k_2^\mu) \Pi_{\mu\nu} (k_3^\nu - k_4^\nu) \\ &= \frac{1}{4} (k_1 - k_2) \cdot (k_3 - k_4) + \frac{(k_1^2 - k_2^2)(k_3^2 - k_4^2)}{4k_s^2}, \end{aligned} \quad (\text{D3})$$

$$\Pi_{1,0} \equiv -\frac{(s_1 - s_2)(s_3 - s_4)}{z^2 k_s^2}. \quad (\text{D4})$$

Next, we write the spin-2 polarization sums in a way that makes its double copy structure clear:

$$\Pi_{2,2} \equiv \frac{1}{16} (k_1^\mu - k_2^\mu)(k_3^\nu - k_4^\nu) \left(\frac{1}{2} \Pi_{\mu\rho} \Pi_{\nu\sigma} + \frac{1}{2} \Pi_{\mu\sigma} \Pi_{\nu\rho} - \frac{1}{d-1} \Pi_{\mu\nu} \Pi_{\rho\sigma} \right) (k_3^\rho - k_4^\rho)(k_3^\sigma - k_4^\sigma) \quad (\text{D5})$$

$$= \Pi_{1,1}^2 - \Pi_{2,2}^{\text{Tr}}, \quad (\text{D6})$$

$$\Pi_{2,1} \equiv 2\Pi_{1,1}\Pi_{1,0}, \quad (\text{D7})$$

$$\Pi_{2,2}^{\text{Tr}} \equiv \frac{(k_1^\mu - k_2^\mu) \Pi_{\mu\nu} (k_3^\nu - k_4^\nu) (k_3^\rho - k_4^\rho) \Pi_{\rho\sigma} (k_3^\sigma - k_4^\sigma)}{16(d-1)}, \quad (\text{D8})$$

$$\begin{aligned} \Pi_{2,0} &\equiv -\frac{d(k_1^2 - k_2^2)(k_3^2 - k_4^2)(s_1 - s_2)(s_3 - s_4)}{4(d-1)k_s^4} + \frac{(d-2)z^2(k_1^2 - k_2^2)(k_3^2 - k_4^2)(s_1 - s_2)(s_3 - s_4)}{4(d-1)k_s^2(d-2s_{12})(d-2s_{34})} \\ &+ \frac{(k_1^2 - k_2^2)(s_1 - s_2)(d^2 - 8(s_3^2 + s_4^2))}{8(d-1)k_s^2(d-2s_{12})} + \frac{(k_3^2 - k_4^2)(s_3 - s_4)(d^2 - 8(s_1^2 + s_2^2))}{8(d-1)k_s^2(d-2s_{34})} \\ &+ \frac{(z^2 k_s^2 + 4s_{12}s_{34})}{(d-1)} + \frac{4(s_1 - s_2)^2 + 4(s_3 - s_4)^2 - d^2}{16(d-1)}, \end{aligned} \quad (\text{D9})$$

where $s_{ij} = s_i + s_j$.

Note that there are still terms with Mellin variables in the denominator that naively violate locality. However, they will all cancel after using the Mellin delta function.

APPENDIX E: EXAMPLE WITH INTERNAL MELLIN VARIABLE FOR YM AMPLITUDE

When computing the YM amplitude at a higher point, one will encounter an internal Mellin variable which corresponds to the z derivative of the bulk-to-bulk propagator. We will now show how to deal with this.

In YM five-point Mellin-momentum amplitudes, using the Feynman rule we encounter the following contribution in the main text (40):

$$\frac{V^{12\mu} \Pi_{\mu\nu} V^{3\nu z}}{\mathcal{D}_{k_{12}}^{d-1}} \frac{1}{z^2 k_{45}^2} V^{45z}.$$

The nontrivial part is

$$z V^{3\nu z} \frac{1}{z^2 k_{45}^2 \mathcal{D}_{k_{12}}^{d-1}} V^{45z} = \frac{(s_4 - s_5)(u - s_3)}{z k_{45}^2 \mathcal{D}_{k_{12}}^{d-1}}, \quad (\text{E1})$$

where the additional z factor comes from the scaling in vertex $V^{12\mu}$. Recall that each Mellin variable is $z\partial_z$.²⁰ Then this represents the scalar integral

²⁰The u notation is from the $z\partial_z$ acting on the Mellin representation of the bulk-to-bulk propagator [84].

$$\frac{1}{4} \int_0^\infty dz_2 \int_0^\infty dz_1 z_1 \phi_1(z_1) \phi_2(z_1) (\partial_4^{z_2} - \partial_5^{z_2}) (\partial_3^{z_2} - \partial_G^{z_2}) \frac{G_{k_{12}}(z_1, z_2)}{k_{45}^2} \phi_3(z_2) \phi_4(z_2) \phi_5(z_2), \quad (\text{E2})$$

where the remaining z factor will contribute to the three-point vertex for particles 1 and 2. Using exchange to contact representation of bulk-to-bulk propagator in Sec. V, we have

$$\begin{aligned} & \frac{1}{4k_{45}^2} \int_0^\infty dz_2 \int_0^\infty dz_1 z_1 \phi_1(z_1) \phi_2(z_1) (\partial_4^{z_2} - \partial_5^{z_2}) (\partial_3^{z_2} - \partial_G^{z_2}) \\ & \times \int_0^\infty \frac{dp}{2\pi i} \frac{-p^{d+1-2\Delta}}{k^2 + p^2} (\phi_\Delta(z_1, ip) - \phi_\Delta(z_1, -ip)) (\phi_\Delta(z_2, ip) - \phi_\Delta(z_2, -ip)) \phi_3(z_2) \phi_4(z_2) \phi_5(z_2). \end{aligned} \quad (\text{E3})$$

Using the map

$$\partial_z \phi_{\Delta=2}(z, k) = \left(\frac{1}{z} - k \right) \phi_{\Delta=2}(z, k). \quad (\text{E4})$$

After some algebra, we have

$$\begin{aligned} & \frac{1}{4k_{45}^2} \int_0^\infty \frac{dp}{2\pi i} \frac{-1}{k_{12}^2 + p^2} (\mathcal{C}_3(ip, k_1, k_2) - \mathcal{C}_3(-ip, k_1, k_2)) \\ & \times ((k_4 - k_5)(k_3 - ip) \mathcal{C}_4(ip, k_3, k_4, k_5) - (k_4 - k_5)(k_3 + ip) \mathcal{C}_4(-ip, k_3, k_4, k_5)) \\ & = - \frac{(k_4 - k_5)(2k_3 + k_4 + k_5)}{4k_{45}^2(k_1 + k_2 + k_3 + k_4 + k_5)(k_1 + k_2 + k_{12})(k_3 + k_4 + k_5 + k_{12})}. \end{aligned} \quad (\text{E5})$$

This exactly matches with explicit computation when using the Mellin subdelta function to convert u to $s_3 s_4 s_5$. In particular, noticing the sub-Mellin delta function $2(s_1 + s_2 + s_3 + \bar{u} + 1) = d$ on subdiagram $V^{12z} \frac{1}{z^2 k_{12}^2} V^{3z\rho}$, we have

$$\begin{aligned} & \frac{(s_4 - s_5)(u - s_3)}{zk_{45}^2 \mathcal{D}_{k_{12}}^{d-1}} = - \frac{(s_4 - s_5)(-d + 2(1 + 2s_3 + s_4 + s_5))}{2zk_{45}^2 \mathcal{D}_{k_{12}}^{d-1}} \\ & \xrightarrow{\text{to momentum space (109)}} - \frac{(k_4 - k_5)(2k_3 + k_4 + k_5)}{4k_{45}^2(k_1 + k_2 + k_3 + k_4 + k_5)(k_1 + k_2 + k_{12})(k_3 + k_4 + k_5 + k_{12})}. \end{aligned} \quad (\text{E6})$$

where the second line follows from Eq. (109). One final comment would be that when we go to six points, it is not always possible to turn all the internal Mellin variables into external ones and hence the method here of dealing with internal Mellin variables is more general.

APPENDIX F: FIVE POINT GR SINGLE OPE POLE COMPUTATIONAL DETAILS

1. $b_2^{\text{type I}}$

In practice, instead of (102) which determines the single OPE pole contribution of type I, one can just pick a single channel

$$-b_{21}^{(m),I,\text{GR}}(1,2;3,4,5) = \text{Res}_{k_{12}^{2m} \rightarrow 0} \left\{ \frac{a_1^{\text{GR}}(1,2;3,4,5)}{\mathcal{D}_{k_{12}}^d \mathcal{D}_{k_{45}}^d} + \frac{a_{2s}^{\text{GR}}(1,2;3,4,5)}{\mathcal{D}_{k_{12}}^d} + \frac{a_{2s}^{\text{GR}}(5,4;3,2,1)}{\mathcal{D}_{k_{45}}^d} + \sum_{m_1, m_2=1}^2 \frac{b_1^{(m_1, m_2, \text{GR})}(1,2,3,4,5)}{k_{12}^{2m_1} k_{45}^{2m_2}} \right\}, \quad (\text{F1})$$

where $a_{2s, \text{GR}}(1,2;3,4,5) = \sum_h \mathcal{M}_3(1,2, -k_I) \cdot \left(\frac{b^{\text{GR}}(k_I, 3, 4, 5)}{k_{45}^{2m}} + c^{\text{GR}}(k_I, 3, 4, 5) \right)$.

Then one can get the single channel single OPE contribution of type I by doing a sum of the canonical order of the last three labels

$$\begin{aligned}
b_2^{(m, \text{GR})}(1, 2; 3, 4, 5)^{\text{Type I}} \\
= b_{21}^{(m, I, \text{GR})}(1, 2, 3, 4, 5) + b_{21}^{(m, I, \text{GR})}(1, 2, 4, 5, 3) \\
+ b_{21}^{(m, I, \text{GR})}(1, 2, 5, 3, 4), \tag{F2}
\end{aligned}$$

and one should use the Mellin delta function and on-shell condition to eliminate spurious poles.

2. $b_2^{\text{type II}}$

In practice, instead of (104) which determines the single OPE pole contribution of type II, we compute a new b_{21}

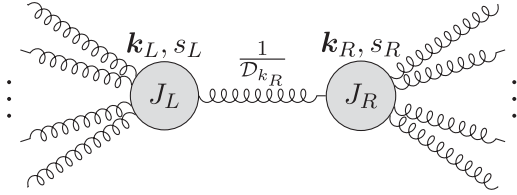
$$\begin{aligned}
& -b_{21}^{(m, II, \text{GR})}(1, 5; 4, 3, 2) \\
& = \text{Res}_{k_{12}^2 \rightarrow 0} \left\{ \frac{a_1(1, 5; 4, 3, 2)}{\mathcal{D}_{k_{15}}^d \mathcal{D}_{k_{23}}^d} + \frac{a_{2s}(5, 1; 4, 2, 3)}{\mathcal{D}_{k_{15}}^d} \right. \\
& \quad \left. + \frac{a_{2s}(2, 3; 4, 5, 1)}{\mathcal{D}_{k_{23}}^d} + \sum_{m_1, m_2=1}^2 \frac{b_1^{(m_1, m_2, \text{GR})}(1, 5, 4, 3, 2)}{k_{15}^{2m_1} k_{23}^{2m_2}} \right\}.
\end{aligned}$$

To get the single OPE contribution of type II, we need the following combination:

$$\begin{aligned}
b_2^{(m, \text{GR})}(1, 5; 4, 3, 2)^{\text{Type II}} \\
= b_{21}^{(m, I, \text{GR})}(5, 1, 2, 3, 4) + b_{21}^{(m, II, \text{GR})}(1, 5, 4, 3, 2) \\
+ b_{21}^{(m, II, \text{GR})}(1, 5, 3, 2, 4),
\end{aligned}$$

with $b_{21}^{(m, I, \text{GR})}(5, 1, 2, 3, 4)$ given in the former section.

APPENDIX G: IDENTITY OPERATOR



In higher point computation in the Mellin-momentum amplitude, due to boundary momentum conservation, we find that when taking limits e.g., OPE or soft limits, different channels of diagrams will combine together. Some exchange terms would turn out to be contact. The usual form that may be suspected to be an identity operator would be

$$z_L^a \frac{z_L^2 k_L^2 - \left(a + \frac{n_L d}{2} - \Delta - 2s_L\right) \left(a + \frac{(n_L - 2)d}{2} - \Delta - 2s_L\right)}{\mathcal{D}_{k_R}^\Delta} \tag{G1}$$

where $k_L^2 = k_R^2$, a is the power of z , n_L denotes the number of legs on the left, and s_L sums all the left Mellin variables. And we should think of the inverse operator as acting on the right-hand side of the subamplitude. This can also be understood from integration by parts (IBP), as we explain in next section and we provide examples in G 2

1. Proof

Consider having a numerator which is of the form in general $f(s_L, k_L)$ and denominator $\frac{1}{\mathcal{D}_{k_R}^\Delta}$. We have the Mellin-momentum amplitude as

$$\mathcal{A} = f(s_L, k_L) z_L^a A_L \frac{1}{\mathcal{D}_{k_R}^\Delta} A_R z_R^b \tag{G2}$$

where we have taken out explicit z dependence for Mellin-momentum amplitudes and write it as $\mathcal{A}_L = z_L^a A_L$. Here, the split of z_L and z_R are determined by primitive amplitudes, e.g., $\mathcal{A}_3^{21}$ that $\mathcal{A}_R$ and $\mathcal{A}_L$ are built upon. And the s_L is directly acting on the bulk-to-boundary propagator.

We claim that

$$f(s_L, k_L) z_L^a A_L \frac{1}{\mathcal{D}_{k_R}^\Delta} A_R z_R^b = A_L A_R z^{a+b} \tag{G3}$$

if and only if

$$f(s_L, k_L) = z^2 k_L^2 - \left(a + \frac{n_L d}{2} - \Delta - 2s_L\right) \left(a + \frac{(n_L - 2)d}{2} - \Delta - 2s_L\right), \tag{G4}$$

where $s_L = s_{1\dots n_L} \equiv \sum_{j=1}^{n_L} s_j$ and there is no splitting of z on the right-hand side in (G3) as it is already a contact term.

To show this, we can first prove

$$\int \frac{dz_L}{z_L^{d+1}} \Phi(z_L, k_i) [\mathcal{D}_{k_L}^{z_L \Delta}] [G_\Delta(k_R, z_L, z_R)] = \int \frac{dz_L}{z_L^{d+1}} [G_\Delta(k_R, z_L, z_R)] [\mathcal{D}_{k_L}^{z_L \Delta}] \Phi(z_L, k_i), \tag{G5}$$

²¹For YM $\mathcal{A}_3$ carries one power of z while power of two for GR.

where

$$\mathcal{D}_k^\Delta \equiv z^2 k^2 - z^2 \partial_z^2 - (1-d)z\partial_z + \Delta(\Delta-d). \quad (\text{G6})$$

Proof. Adding two total derivative terms and after some algebra, we get

$$\begin{aligned} & \int \frac{dz_L}{z_L^{d+1}} \Phi(z_L, k_i) \mathcal{D}_{k_L}^{\Delta} G_\Delta(k_R, z_L, z_R) + \partial_{z_L} \left[z_L^2 \frac{1}{z_L^{d+1}} \Phi(z_L, k_i) \partial_{z_L} G(z_R, z_L, k_R) \right] - \partial_{z_L} \left[z_L^2 \frac{1}{z_L^{d+1}} \partial_{z_L} \Phi(z_L, k_i) G(z_R, z_L, k_R) \right] \\ &= \int \frac{dz_L}{z_L^{d+1}} G_\Delta(k_R, z_L, z_R) \mathcal{D}_{k_L}^{\Delta} \Phi(z_L, k_i). \end{aligned} \quad (\text{G7})$$

■

Now we move back to proof (G4). Starting from (G3), we have equivalently

$$\mathcal{A} = z_L^a A_L \mathcal{D}_{k_L}^{\Delta} \frac{1}{\mathcal{D}_{k_R}^{\Delta}} A_R z_R^b. \quad (\text{G8})$$

To determine $f(s_L, k_L)$ we can introduce back the z integrals when we go back to momentum space from (G8),

$$\begin{aligned} & \int \frac{dz_L}{z_L^{d+1}} \frac{dz_R}{z_R^{d+1}} z_L^a A_L \left(\prod_{i=1}^{n_l} \phi_\Delta(z_L, k_i) \right) \mathcal{D}_{z_L, k_L}^{\Delta} G_\Delta(k_R, z_L, z_R) \left(\prod_{i=n_l+1}^n \phi_\Delta(z_R, k_i) \right) A_R z_R^b \\ &= \int \frac{dz_L}{z_L^{d+1}} \frac{dz_R}{z_R^{d+1}} G_\Delta(k_R, z_L, z_R) \mathcal{D}_{z_L, k_L}^{\Delta} \left(z_L^a \prod_{i=1}^{n_l} \phi_\Delta(z_L, k_i) \right) \left(\prod_{i=n_l+1}^n \phi_\Delta(z_R, k_i) \right) z_R^b A_L A_R \end{aligned} \quad (\text{G9})$$

where in the last line we used (G5). Writing $z_L^a \prod_{i=1}^{n_l} \phi_\Delta(z_L, k_i) \propto z_L^a \prod_{i=1}^{n_l} z^{-2s_i+d/2}$, we found that $\mathcal{D}_{z_L, k_L}^{\Delta}$ creates the following f in (G2):

$$f(s_L, k_L) = z^2 k_L^2 - \left(a + \frac{n_L d}{2} - \Delta - 2 \sum_{j=1}^{n_L} s_j \right) \left(a + \frac{(n_L - 2)d}{2} - \Delta - 2 \sum_{j=1}^{n_L} s_j \right). \quad \blacksquare$$

2. Examples

a. Yang-Mills

Identity operator for four-point YM amplitude in the s -channel is

$$1 = \frac{z^2 k_{12}^2 + 2(s_1 + s_2)(d - 2 - 2s_1 - 2s_2)}{\mathcal{D}_{k_{34}}^{d-1}}. \quad (\text{G10})$$

This corresponds to $f(s_L, k_L)|_{a=-1, n_L=2, \Delta=d-1}$. The reason $a = -1$ is that the numerator has z to power 2 while originally the YM vertex on the 12 side has only z to power 1. One has to write a $1/z$ to compensate the z power.

In the special case of $d = 3$, this identity turns into

$$1 = \frac{z^2 k_{12}^2 - z^2 (k_1 + k_2)^2}{\mathcal{D}_{k_{34}}^{d-1}}. \quad (\text{G11})$$

After using the mapping to correlator we can see that the result is not just $\frac{1}{E_i}$ but also contains a local term $\frac{1}{k_{12}+k_3+k_4}$. This is as expected because in momentum space the form of correlators is not unique but up to local terms.

b. Gravity

Identity operator for four-point GR amplitude in the s -channel is

$$1 = \frac{z^2 k_{12}^2 + 2(s_1 + s_2)(d - 2s_1 - 2s_2)}{\mathcal{D}_{k_{34}}^d}. \quad (\text{G12})$$

This corresponds to $f(s_L, k_L)|_{a=0, n_L=2, \Delta=d}$. The reason $a=0$ is that the numerator has z to power 2 while originally the three-point GR vertex on the 12 side has the same power. So no compensation of z is needed.

APPENDIX H: DERIVATION FOR GR SCALAR INTEGRALS

1. Derivation of Eq. (125)

$$C_{\text{GR}}^{n;1} = \int \frac{dz}{z^4} z^2 \prod_i^n \phi_{\Delta=3}(z, k_i). \quad (\text{H1})$$

In $d=3$, we can express the contact integration as

$$C_{\text{GR}}^{n;1} = \int_0^\infty \frac{dz}{z^4} z^2 e^{-z \sum_{j=1}^n k_n} \prod_{j=1}^n (k_j z + 1), \quad (\text{H2})$$

where the z^2 comes from the additional scaling factor for gravity contact interaction vertex.

The first two inverse powers of z diverge separately. However power divergences would be considered as pure phases in a Bunch-Davies wave function and it has no logarithmic growth for a pure graviton sector at tree level

[85]. So we just use a regulator to extract the finite term

$$\lim_{\epsilon \rightarrow 0} \int_0^\infty \frac{dz}{z^{4+\epsilon}} z^2 \left(1 + z \sum_{j=1}^n k_j\right) e^{-z \sum_{j=1}^n k_n} = -E_t + O(\epsilon). \quad (\text{H3})$$

This accounts for the second term in (125). For other finite integration, we have

$$\int_0^\infty z^m e^{-E_t z} = \frac{m!}{E_t^{m+1}}, \quad (\text{H4})$$

whose coefficients are the corresponding cyclic sum of k 's: $\sum_{1 \leq i_1 < \dots < i_{m+2}} = k_{i_1} \dots k_{i_{m+2}}$.²²

2. Derivation of Eq. (126)

$$C_{\text{GR}}^{n;2} = \int \frac{dz}{z^4} \prod_{m=1}^l (-2s_m + d/2) \phi_{\Delta=3}(z, k_i). \quad (\text{H5})$$

By definition

$$\left(-2s_m + \frac{d}{2}\right) = z \frac{\partial}{\partial z} \Big|_{\text{acting on only the } m \text{ th bulk-to-boundary propagator}},$$

$$\left(-2s_m + \frac{d}{2}\right) e^{-z k_m} (1 + z k_m) = -z^2 k_m^2 e^{-z k_m}, \quad (\text{H6})$$

where the second line is the first line acting on scalar part of a graviton bulk-to-boundary propagator in $d=3$.

So the Mellin s variable $\prod_{m=1}^l (-2s_m + d/2) M(k, \epsilon) \rightarrow M(k, \epsilon) C_{\text{GR}}^{n;2}$ changes the $(kz + 1)$ to $z^2 k^2$ in the integrand in (H2):

$$C_{\text{GR}}^{n;2} = \int_0^\infty \frac{dz}{z^4} z^{2l} e^{-z \sum_{i=1}^n k_n} \prod_{j=l+1}^n (k_j z + 1) (-1)^l k_1^2 \dots k_l^2. \quad (\text{H7})$$

For the number of derivatives greater than two ($l \geq 2$), the power of z in the integrand gives $\int_0^\infty z^{(2l-4)+m} e^{-E_t z} = \frac{(2l-4+m)!}{E_t^{(2l-4)+m+1}}$. The power of m can be at

most $n-l$, so we eventually end up with

$$C_{\text{GR}}^{n;2} = \left(\sum_{m=0}^{n-l} (2l-4+m)! \sum_{l+1 \leq i_{l+1} < \dots < i_{m+l}} \frac{k_{i_{l+1}} \dots k_{i_{m+l}}}{E_t^{(2l-4)+m+1}} \right) \times (-1)^l k_1^2 \dots k_l^2, \quad (\text{H8})$$

where when $m=0$, the numerator above E_t is just 1. This identity holds for all the Mellin variables with power 1, as in YM, if we have a Mellin variable of s power greater than 2, one has to use the on-shell condition to lower the power.

²²For example, for $m=0$ at contact three-point it is $k_1 k_2 + k_1 k_3 + k_2 k_3$.

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